PROVING THE LAW OF COSINES

Common Core Standards

MCC9-12.G.SRT.10 Prove the Laws of Sines and Cosines and use them to solve problems.

MCC9-12.G.SRT.11 Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles

Introduction

The purpose of this task is to guide students through the derivation of the Law of Cosines. The teacher should spend the extra time and effort in helping students understand the conceptual foundation for the Law of Cosines, and not just memorizing the formula. A few problems have been provided at the end of the task, but the classroom teacher should provide ample opportunities to practice and improve fluency.

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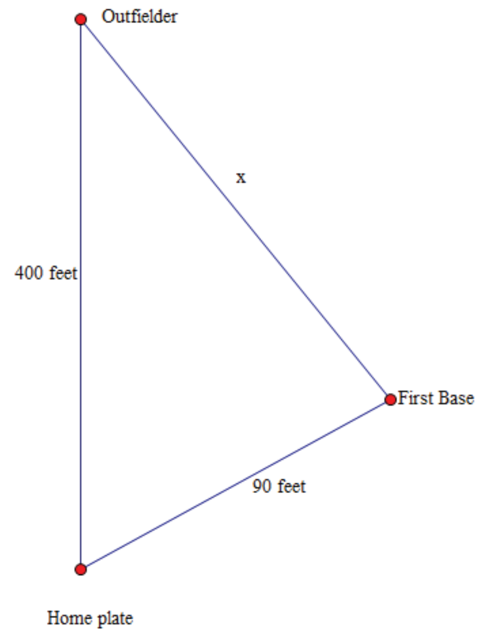
During a baseball game an outfielder caught a ball hit to dead center field, 400 feet from home plate. If the distance from home plate to first base is 90 feet, how far does the outfielder have to throw the ball to get it to first base?

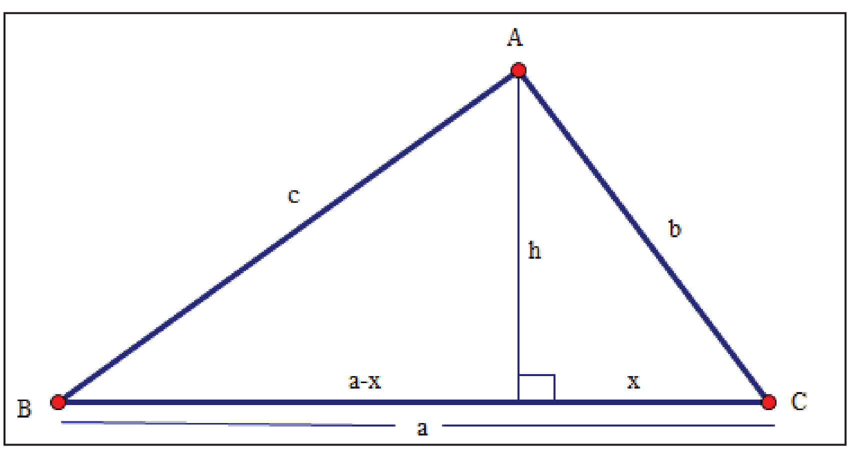
1. Model the problem with a picture. Be sure to label information that you know.

2. Do you have enough information to solve the problem? If not, what is missing?

We know two sides of the triangle. We also know the angle at home base is 45° (The 'diamond' is a square. The diagonal of a square bisects the angle at the vertex.)



Typically, you have solved triangles that are right triangles. This is a case where we do not have a right triangle to solve. We know two sides and one included angle. In this task, you will develop a method for solving triangles like this using trigonometry. We will come back to the baseball example later. For now, consider the triangle below. Follow these steps to derive a way to solve for c knowing just that much information. For this example, assume we know measurements for segments *a*, *b*, and angle *C*.



3. What does segment h represent? What are its properties and what does it do to the large triangle?

   h is the altitude of the triangle. It is perpendicular to the base and divides the larger triangle into two right triangles.

   Comment:
   Get students to realize that the altitude does not necessarily bisect the angle!

4. Write an equation that represents c^2 . Explain the method you used.

   According to the Pythagorean Theorem:
   $c^2 = h^2 + (a-x)^2 = h^2 + a^2 - 2ax + x^2$

5. Now write an equation that represents h^2 in terms of b and x. Substitute this expression into the expression you wrote in #4. Expand and simplify.

   According to the Pythagorean Theorem:

   $h^2 = b^2 - x^2$

   When you substitute this into the c^2 equation you get:

   $c^2 = b^2 + a^2 - 2ax$

6. Now write an expression that represents x in terms of the angle C. Substitute this expression into the equation you wrote in #5. Simplify completely.

Students should use the cosine ratio here to arrive at: $\cos C = \dfrac{x}{b}$

$x = b(\cos C)$

Substituting this in gives: $c^2 = b^2 + a^2 - 2ab(\cos C)$***, one of the versions of the Law of Cosines.***

Your answer to #6 is one of three formulas that make up the **Law of Cosines**. Each of the formulas can be derived in the same way you derived this one by working with each vertex and the other heights of the triangle.

Law of Cosines
Let a, b, and c be the lengths of the legs of a triangle opposite angles A, B, and C. Then,

$$a^2 = b^2 + c^2 - 2bc(\cos A)$$
$$b^2 = a^2 + c^2 - 2ac(\cos B)$$
$$c^2 = a^2 + b^2 - 2ab(\cos C)$$

These formulas can be used to solve for unknown lengths and angles in a triangle.

Students should also be able to verbally articulate the generalization of these Laws. It should go something like this: The square of the length of any side of a triangle is equal to the sum of the squares of the other two sides minus two times the product of the lengths of the other two sides and the cosine of their included angle.

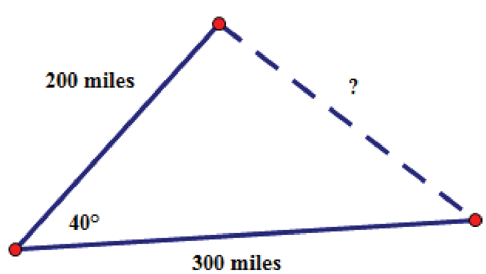
7. Solve the baseball problem at the beginning of this task using the Law of Cosines.

In this problem a = 90, b= 400 and C=45°.
Using the Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos C$
$c^2 = 90^2 + 400^2 - 2(90)(400)\cos 45°$
$c \approx 342$
The distance is about 342 feet from dead centerfield to 1st base.

Here are a few problems to help you apply the Law of Cosines.

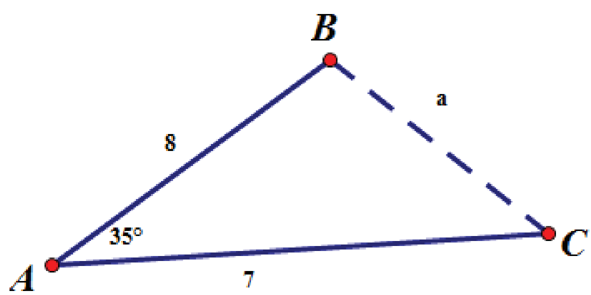
8. Two airplanes leave an airport, and the angle between their flight paths is 40º. An hour later, one plane has traveled 300 miles while the other has traveled 200 miles. How far apart are the planes at this time?

$a^2 = b^2 + c^2 - 2bc\cos A$
Let a be the missing side, b = 300, c = 200 and A = 40.
$a^2 = 300^2 + 200^2 - 2(300)(200)\cos(40)$
$a^2 = 38074.66683$
$a \approx 195.13\ miles$
The planes are approximately 195 miles apart.



9. A triangle has sides of 8 and 7 and the angle between these sides is 35°. Solve the triangle. (Find all missing angles and sides.)

$a^2 = b^2 + c^2 - 2bc\cos A$
b = 7, c = 8 and A = 35.
$a^2 = 7^2 + 8^2 - 2(7)(8)\cos(35)$
$a^2 \approx 21.255$
$a \approx 4.61$

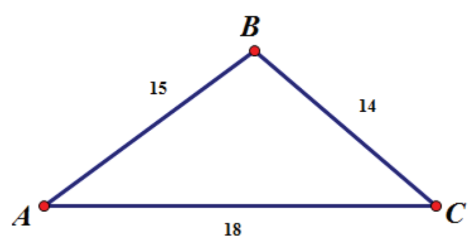


To find angle C
$c^2 = a^2 + b^2 - 2ab\cos C$
$8^2 = 4.61^2 + 7^2 - 2(4.61)(7)(\cos C)$
$64 = 70.2521 - 64.54(\cos C)$
0.0969 = cos C
84.44 =C

B = 180 – 84.44 – 35 = 60.56

10. Three soccer players are practicing on a field. The triangle they create has side lengths of 18, 14, and 15 feet. At what angles are they standing from each other?

To find angle C
$c^2 = a^2 + b^2 - 2ab\cos C$
$15^2 = 14^2 + 18^2 - 2(14)(18)(\cos C)$
$225 = 520 - 504(\cos C)$
0.5853 = cos C
54.17°=C



To find angle B
$b^2 = a^2 + c^2 - 2ac\cos B$
$18^2 = 14^2 + 15^2 - 2(14)(15)(\cos B)$
$324 = 421 - 420(\cos C)$
0.231 = cos C
76.65° =C

To find angle A
A = 180 – 54.17 – 76.65 = 49.18°

11. Is it possible to know two sides of a triangle and the included angle and not be able to solve for the third side?

Using the Law of Cosines, you will always be able to find the third side of the triangle.