Properties of Triangles

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Chapter 3 Overview

Theorems involving angles and side lengths of triangles are presented. The last two lessons discuss properties and theorems of 45°-45°-90° triangles and 30°-60°-90° triangles.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
3.1	Triangle Sum, Exterior Angle, and Exterior Angle Inequality Theorems	G.CO.10 G.MG.1	1	In this lesson, students prove the Triangle Sum Theorem, Exterior Angle Theorem, and Exterior Angle Inequality Theorem. Questions ask students to investigate the side lengths and angle measures of triangles before proving the theorems.	X			X	
3.2	The Triangle Inequality Theorem	G.CO.10	1	Students complete an activity with pasta strands to investigate the possible side lengths that can form triangles. Students then prove the Triangle Inequality Theorem.	X		X		
3.3	Properties of a 45°–45°–90° Triangle	G.CO.10 G.MG.1	1	Students investigate the properties of 45°–45°–90° triangles in this lesson. Questions ask students to apply the 45°–45°–90° Triangle Theorem and construction to solve problems and verify properties of 45°–45°–90° triangles.	X				
3.4	Properties of a 30°–60°–90° Triangle	G.CO.10 G.MG.1	1	Students investigate the properties of 30°-60°-90° triangles in this lesson. Questions ask students to apply the 30°–60°–90° Triangle Theorem and construction to solve problems and verify properties of 30°–60°–90° triangles. As a culminating activity, students compare the properties of 45°–45°–90° triangles with 30°–60°–90° triangles.	X			X	

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Skills Practice Correlation for Chapter 3

Lesson		Problem Set	Objectives
3.1	Triangle Sum, Exterior Angle, and Exterior Angle Inequality Theorems		Vocabulary
		1 – 6	Determine the measure of missing angle measures in triangles
		7 – 12	Determine the order of side lengths given information in diagrams
		13 – 18	Identify interior, exterior, and remote interior angles of triangles
		19 – 24	Solve for x given triangle diagrams
		25 – 30	Write two inequalities needed to prove the Exterior Angle Inequality Theorem given triangle diagrams
3.2	The Triangle Inequality Theorem		Vocabulary
		1 – 6	Order angle measures of triangles without measuring
		7 – 16	Determine whether it is possible to form a triangle from given side lengths
		17 – 22	Write inequalities to describe possible unknown side lengths of triangles
3.3	Properties of a 45°–45°–90° Triangle		Vocabulary
		1 – 4	Determine the length of the hypotenuse of 45°–45°–90° triangles
		5 – 8	Determine the lengths of the legs of 45°–45°–90° triangles
		9 – 12	Solve problems involving 45°–45°–90° triangles
		13 – 16	Determine the area of 45°–45°–90° triangles
		17 – 20	Solve problems involving 45°–45°–90° triangles
		21 – 24	Construct 45°–45°–90° triangles
3.4	Properties of a 30°–60°–90° Triangle		Vocabulary
		1 – 4	Determine the measure of indicated interior angles
		5 – 8	Determine the length of the long leg and the hypotenuse of 30°–60°–90° triangles
		9 – 12	Determine the lengths of the legs of 30°–60°–90° triangles
		13 – 16	Determine the length of the short leg and the hypotenuse of 30°–60°–90° triangles
		17 – 20	Determine the area of 30°–60°–90° triangles
		21 – 24	Construct 30°–60°–90° triangles

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Inside Out

3.1

Triangle Sum, Exterior Angle, and Exterior Angle Inequality Theorems

LEARNING GOALS

In this lesson, you will:

- Prove the Triangle Sum Theorem.
- Explore the relationship between the interior angle measures and the side lengths of a triangle.
- Identify the remote interior angles of a triangle.
- Identify the exterior angle of a triangle.
- Explore the relationship between the exterior angle measure and two remote interior angles of a triangle.
- Prove the Exterior Angle Theorem.
- Prove the Exterior Angle Inequality Theorem.

KEY TERMS

- Triangle Sum Theorem
- remote interior angles of a triangle
- Exterior Angle Theorem
- Exterior Angle Inequality Theorem

ESSENTIAL IDEAS

- The Triangle Sum Theorem states: "The sum of the measures of the interior angles of a triangle is 180°."
- The longest side of a triangle lies opposite the largest interior angle.
- The shortest side of a triangle lies opposite the smallest interior angle.
- The remote interior angles of a triangle are the two interior angles non-adjacent to the exterior angle.
- The Exterior Angle Theorem states: "The measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles of the triangle."
- The Exterior Angle Inequality Theorem states: "An exterior angle of a triangle is greater than either of the remote interior angles of the triangle."

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Prove geometric theorems

10. Prove theorems about triangles.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

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Overview

Students informally show and formally prove the Triangle Sum Theorem. Next, students explore the effect the angle measure has on the length of the side opposite the angle in a triangle. As the angle measure increases, the length of the side opposite the angle increases. As the angle measure decreases, the length of the side opposite the angle decreases. These relationships are the foundation of the Hinge Theorem introduced in a later chapter. Students prove the Exterior Angle Theorem and the Exterior Angle Inequality Theorem. Maps are used to model triangles, and students answer questions related to the problem situations.

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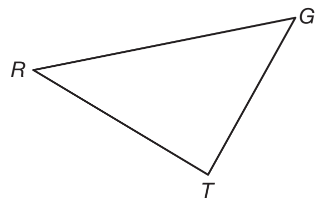
Warm Up

Triangle *RTG* is shown.



1. If we increase $m\angle T$ in triangle *RTG*, what effect will this have on $m\angle R$ and $m\angle G$?

If $m\angle T$ increases in triangle *RTG*, $m\angle R$ or $m\angle G$ will decrease because the sum of the measures of the interior angles of a triangle is 180°.

2. If we decrease $m\angle T$ in triangle *RTG*, what effect will this have on $m\angle R$ and $m\angle G$?

If $m\angle T$ decreases in triangle *RTG*, $m\angle R$ or $m\angle G$ will increase because the sum of the measures of the interior angles of a triangle is 180°.

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3. If we increase the length of side *GT* in triangle *RTG*, what effect will this have on the lengths of sides *RT* and *RG*?

If the length of side *GT* increases in triangle *RTG*, *RT* or *RG* will increase because a triangle is a closed figure.

4. If we decrease the length of side *GT* in triangle *RTG*, what effect will this have on the lengths of sides *RT* and *RG*?

If the length of side *GT* decreases in triangle *RTG*, *RT* or *RG* will decrease because a triangle is a closed figure.

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3.1

Inside Out

Triangle Sum, Exterior Angle, and Exterior Angle Inequality Theorems

LEARNING GOALS

In this lesson, you will:

- Prove the Triangle Sum Theorem.
- Explore the relationship between the interior angle measures and the side lengths of a triangle.
- Identify the remote interior angles of a triangle.
- Identify the exterior angle of a triangle.
- Explore the relationship between the exterior angle measure and two remote interior angles of a triangle.
- Prove the Exterior Angle Theorem.
- Prove the Exterior Angle Inequality Theorem.

KEY TERMS

- Triangle Sum Theorem
- remote interior angles of a triangle
- Exterior Angle Theorem
- Exterior Angle Inequality Theorem

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Easter Island is one of the remotest islands on planet Earth. It is located in the southern Pacific Ocean approximately 2300 miles west of the coast of Chile. It was discovered by a Dutch captain in 1722 on Easter Day. When discovered, this island had few inhabitants other than 877 giant statues, which had been carved out of rock from the top edge of a wall of the island's volcano. Each statue weighs several tons, and some are more than 30 feet tall.

Several questions remain unanswered and are considered mysteries. Who built these statues? Did the statues serve a purpose? How were the statues transported on the island?

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Problem 1

First students manipulate the three interior angles of a triangle to informally show they form a line, thus showing the sum of the measures of the three interior angles of a triangle is 180 degrees. Then students will use a two-column proof to prove the Triangle Sum Theorem.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- What is a straight angle?
- What is the measure of a straight angle?
- Does a line have a degree measure?
- If three angles form a line when arranged in an adjacent configuration, what is the sum of the measures of the three angles?
- What do you know about the sum of the measures of angles 3, 4, and 5?
- What is the relationship between angle 1 and angle 4?
- What is the relationship between angle 2 and angle 5?
- What do you know about the sum of the measures of angles 1, 2, and 3?
- What properties are used to prove this theorem?

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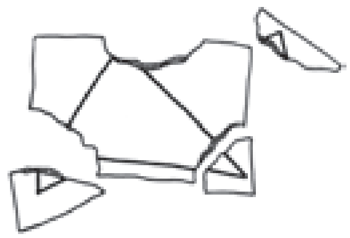
PROBLEM 1 Triangle Interior Angle Sums

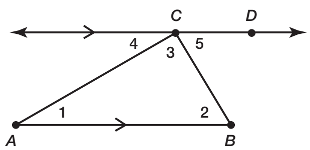
1. Draw any triangle on a piece of paper. Tear off the triangle's three angles. Arrange the angles so that they are adjacent angles. What do you notice about the sum of these three angles?

   The sum of the angles is 180° because they form a straight line.



The **Triangle Sum Theorem** states: "the sum of the measures of the interior angles of a triangle is 180°."

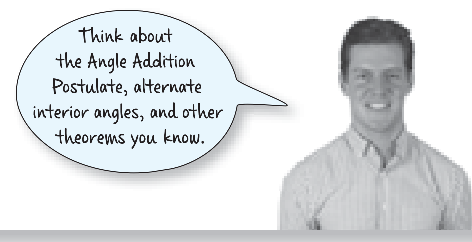
2. Prove the Triangle Sum Theorem using the diagram shown.



Given: Triangle *ABC* with $\overline{AB} \parallel \overline{CD}$

Prove: $m\angle 1 + m\angle 2 + m\angle 3 = 180°$

Statements	Reasons
1. Triangle *ABC* with $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	1. Given
2. $m\angle 4 + m\angle 3 + m\angle 5 = 180°$	2. Angle Addition Postulate and Definition of straight angle
3. $\angle 1 \cong \angle 4$	3. Alternate Interior Angle Theorem
4. $m\angle 1 = m\angle 4$	4. Definition of congruent angles
5. $\angle 2 \cong \angle 5$	5. Alternate Interior Angle Theorem
6. $m\angle 2 = m\angle 5$	6. Definition of congruent angles
7. $m\angle 1 + m\angle 3 + m\angle 2 = 180°$	7. Substitution Property of Equality
8. $m\angle 1 + m\angle 2 + m\angle 3 = 180°$	8. Associative Property of Addition



- How many steps is your proof?
- Do your classmates have the same number of steps?
- Did your classmates use different properties to prove this theorem?

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Problem 2

Students use measuring tools to draw several triangles. Through answering a series of questions, students realize the measure of an interior angle in a triangle is directly related to the length of the side of the triangle opposite that angle. As an angle increases in measure, the opposite side increases in length to accommodate the angle. And as a side increases in length, the angle opposite the side increases in measure to accommodate the length of the side.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

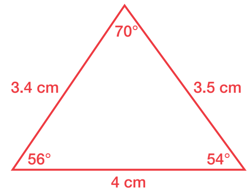
- What is an acute triangle?
- What is an obtuse triangle?
- What is a right triangle?
- Is the shortest side of your triangle opposite the angle of smallest measure?
- Do you think this relationship is the same in all triangles? Why or why not?

PROBLEM 2 Analyzing Triangles

1. Consider the side lengths and angle measures of an acute triangle.

   a. Draw an acute scalene triangle. Measure each interior angle and label the angle measures in your diagram.

   Answers will vary.



   b. Measure the length of each side of the triangle. Label the side lengths in your diagram.

   See diagram.

   c. Which interior angle is opposite the longest side of the triangle?

   The largest interior angle is opposite the longest side of the triangle. In my diagram, the largest interior angle is 70° and the longest side is 4 centimeters.

   d. Which interior angle lies opposite the shortest side of the triangle?

   The smallest interior angle is opposite the shortest side of the triangle. In my diagram, the smallest interior angle is 54° and the shortest side is 3.4 centimeters.

- Is the longest side of your triangle opposite the angle of largest measure?
- Do you think this relationship is the same in all triangles? Why or why not?

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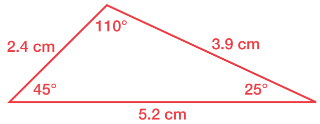
2. Consider the side lengths and angle measures of an obtuse triangle.

a. Draw an obtuse scalene triangle. Measure each interior angle and label the angle measures in your diagram.

Answers will vary.



b. Measure the length of each side of the triangle. Label the side lengths in your diagram.

See diagram.

c. Which interior angle lies opposite the longest side of the triangle?

The largest interior angle lies opposite the longest side of the triangle. In my diagram, the largest interior angle is 110° and the longest side is 5.2 centimeters.

d. Which interior angle lies opposite the shortest side of the triangle?

The smallest interior angle lies opposite the shortest side of the triangle. In my diagram, the smallest interior angle is 25° and the shortest side is 2.4 centimeters.

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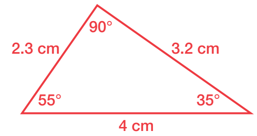
3. Consider the side lengths and angle measures of a right triangle.

   a. Draw a right scalene triangle. Measure each interior angle and label the angle measures in your diagram.

   Answers will vary.



   b. Measure each side length of the triangle. Label the side lengths in your diagram.

   See diagram.

   c. Which interior angle lies opposite the longest side of the triangle?

   The largest interior angle lies opposite the longest side of the triangle. In my diagram, the largest interior angle is 90° and the longest side is 4 centimeters.

   d. Which interior angle lies opposite the shortest side of the triangle?

   The smallest interior angle lies opposite the shortest side of the triangle. In my diagram, the smallest interior angle is 35° and the shortest side is 2.3 centimeters.

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4. The measures of the three interior angles of a triangle are 57°, 62°, and 61°. Describe the location of each side with respect to the measures of the opposite interior angles without drawing or measuring any part of the triangle.

a. longest side of the triangle

The longest side of the triangle is opposite the largest interior angle; therefore, the longest side of the triangle lies opposite the 62° angle.

b. shortest side of the triangle

The shortest side of the triangle is opposite the smallest interior angle; therefore, the shortest side of the triangle lies opposite the 57° angle.

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5. One angle of a triangle decreases in measure, but the sides of the angle remain the same length. Describe what happens to the side opposite the angle.

As an interior angle of a triangle decreases in measure, the sides of that angle are forced to move closer together, creating an opposite side of the triangle that decreases in length.

6. An angle of a triangle increases in measure, but the sides of the angle remain the same length. Describe what happens to the side opposite the angle.

As an interior angle of a triangle increases in measure, the sides of that angle are forced to move farther apart, creating an opposite side of the triangle that increases in length.

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Grouping

Have students complete Question 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 7

- What is the first step when solving this problem?
- Which side is the shortest side of the triangle? How do you know?
- Which side is the longest side of the triangle? How do you know?
- How did you determine the measure of the third interior angle of the triangle?

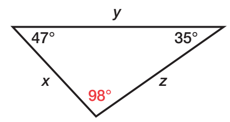
Problem 3

Students prove two theorems:

- For the two-column proof of the Exterior Angle Theorem, students are required to write both the statements and the reasons.
- The Exterior Angle Inequality Theorem has two Prove statements so it must be done in two separate parts. In the first part of this two-column proof, the reasons are provided and students are required to write only the statements. This was done because students may need some support using the Inequality Property. In the second part, students are expected to write both the statements and reasons. They can use the first part as a model.

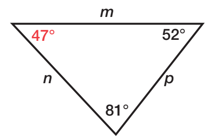
7. List the sides from shortest to longest for each diagram.

a.



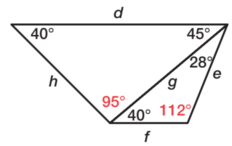
x, z, y

b.



p, n, m

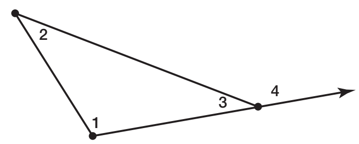
c.



f, e, g, h, d

PROBLEM 3 Exterior Angles

Use the diagram shown to answer Questions 1 through 12.



1. Name the interior angles of the triangle.
   The interior angles are ∠1, ∠2, and ∠3.

2. Name the exterior angles of the triangle.
   The exterior angle is ∠4.

3. What did you need to know to answer Questions 1 and 2?
   I needed to know the definitions of interior and exterior angles.

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Grouping

Have students complete Questions 1 through 12 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 12

- Where are interior angles located?
- Where are exterior angles located?
- What is the difference between an interior angle and an exterior angle?
- How can the Triangle Sum Theorem be applied to this situation?
- What is the relationship between angle 3 and angle 4?
- How many exterior angles can be drawn on a given triangle?
- How are exterior angles formed?
- How many remote interior angles are associated with each exterior angle of a triangle?

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4. What does $m\angle 1 + m\angle 2 + m\angle 3$ equal? Explain your reasoning.

$m\angle 1 + m\angle 2 + m\angle 3 = 180°$

The Triangle Sum Theorem states that the sum of the measures of the interior angles of a triangle is equal to 180°.

5. What does $m\angle 3 + m\angle 4$ equal? Explain your reasoning.

$m\angle 3 + m\angle 4 = 180°$

A linear pair of angles is formed by $\angle 3$ and $\angle 4$. The sum of any linear pair's angle measures is equal to 180°.

6. Why does $m\angle 1 + m\angle 2 = m\angle 4$? Explain your reasoning.

If $m\angle 1 + m\angle 2 + m\angle 3$ and $m\angle 3 + m\angle 4$ are both equal to 180°, then $m\angle 1 + m\angle 2 + m\angle 3 = m\angle 3 + m\angle 4$ by substitution. Subtracting $m\angle 3$ from both sides of the equation results in $m\angle 1 + m\angle 2 = m\angle 4$.

7. Consider the sentence "The buried treasure is located on a remote island." What does the word *remote* mean?

The word *remote* means far away.

8. The exterior angle of a triangle is $\angle 4$, and $\angle 1$ and $\angle 2$ are interior angles of the same triangle. Why would $\angle 1$ and $\angle 2$ be referred to as "remote" interior angles with respect to the exterior angle?

Considering all three interior angles of the triangle, $\angle 1$ and $\angle 2$ are the two interior angles that are farthest away from, or not adjacent to, $\angle 4$.

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The **remote interior angles of a triangle** are the two angles that are non-adjacent to the specified exterior angle.

9. Write a sentence explaining $m\angle 4 = m\angle 1 + m\angle 2$ using the words *sum, remote interior angles of a triangle*, and *exterior angle of a triangle*.

The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles of the triangle.

10. Is the sentence in Question 9 considered a postulate or a theorem? Explain your reasoning.

It would be considered a theorem because it can be proved using definitions, facts, or other proven theorems.

11. The diagram was drawn as an obtuse triangle with one exterior angle. If the triangle had been drawn as an acute triangle, would this have changed the relationship between the measure of the exterior angle and the sum of the measures of the two remote interior angles? Explain your reasoning.

No. If $\angle 1$ and $\angle 2$ were both acute, the sum of their measures would still be equal to $m\angle 4$.

12. If the triangle had been drawn as a right triangle, would this have changed the relationship between the measure of the exterior angle and the sum of the measures of the two remote interior angles? Explain your reasoning.

No. If $\angle 1$ or $\angle 2$ were a right angle, the sum of their measures would still be equal to $m\angle 4$.

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Grouping

Have students complete Question 13 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 13

- Are any theorems used to prove this theorem? If so, which theorems?
- Are any definitions used to prove this theorem? If so, which definitions?
- Are any properties used to prove this theorem? If so, which properties?

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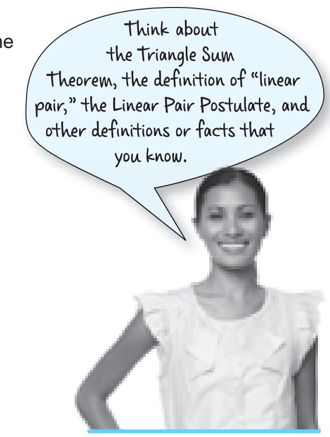
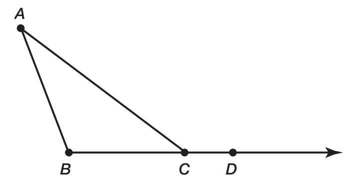
The **Exterior Angle Theorem** states: "the measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles of the triangle."



13. Prove the Exterior Angle Theorem using the diagram shown.



Given: Triangle *ABC* with exterior $\angle ACD$

Prove: $m\angle A + m\angle B = m\angle ACD$

Statements	Reasons
1. Triangle *ABC* with exterior $\angle ACD$	1. Given
2. $m\angle A + m\angle B + m\angle BCA = 180°$	2. Triangle Sum Theorem
3. $\angle BCA$ and $\angle ACD$ are a linear pair	3. Definition of linear pair
4. $\angle BCA$ and $\angle ACD$ are supplementary	4. Linear Pair Postulate
5. $m\angle BCA + m\angle ACD = 180°$	5. Definition of supplementary angles
6. $m\angle A + m\angle B + m\angle BCA = m\angle BCA + m\angle ACD$	6. Substitution Property using step 2 and step 5
7. $m\angle A + m\angle B = m\angle ACD$	7. Subtraction Property of Equality

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Grouping

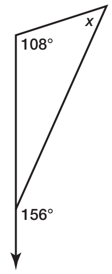
Have students complete Question 14 with a partner. Then have students discuss their responses as a class.

Guiding Questions for Share Phase, Question 14

- What is the first step when solving for the value of x?
- What information in the diagram helps you determine additional information?
- If you know the measure of the exterior angle, what else can be determined?
- Is an equation needed to solve for the value of x? Why or why not?
- What equation was used to solve for the value of x?

14. Solve for x in each diagram.

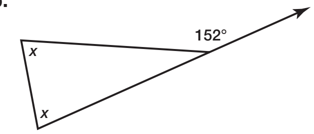
a.



$x + 108° = 156°$

$x = 48°$

b.

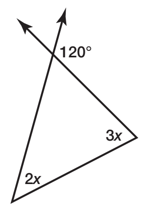


$x + x = 152°$

$2x = 152°$

$x = 76°$

c.

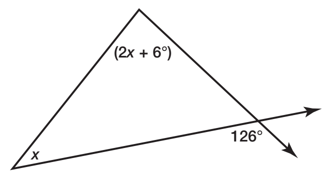


$2x + 3x = 120°$

$5x = 120°$

$x = 24°$

d.



$x + 2x + 6° = 126°$

$3x + 6° = 126°$

$3x = 120°$

$x = 40°$

The **Exterior Angle Inequality Theorem** states: "the measure of an exterior angle of a triangle is greater than the measure of either of the remote interior angles of the triangle."

15. Why is it necessary to prove two different statements to completely prove this theorem?

I must prove that the exterior angle is greater than each remote interior angle separately.

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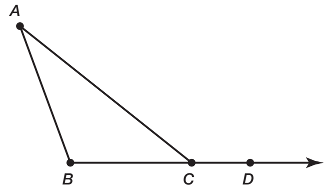
Grouping

Have students complete Questions 15 through 16 part (a) with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 15 through 16 part (a)

- Which theorems were used to prove this theorem?
- Which definitions were used to prove this theorem?
- Which properties were used to prove this theorem?
- Which angles are the three interior angles?
- Which angle is the exterior angle?
- Which angles are the two remote interior angles?
- Which angles form a linear pair of angles?
- What substitution is necessary to prove this theorem?
- Is it possible for the measure of an angle to be equal to 0 degrees? Why or why not?

16. Prove both parts of the Exterior Angle Inequality Theorem using the diagram shown.



a. Part 1

Given: Triangle *ABC* with exterior $\angle ACD$

Prove: $m\angle ACD > m\angle A$

Statements	Reasons
1. Triangle *ABC* with exterior $\angle ACD$	**1.** Given
2. $m\angle A + m\angle B + m\angle BCA = 180°$	**2.** Triangle Sum Theorem
3. $\angle BCA$ and $\angle ACD$ are a linear pair	**3.** Linear Pair Postulate
4. $m\angle BCA + m\angle ACD = 180°$	**4.** Definition of linear pair
5. $m\angle A + m\angle B + m\angle BCA = m\angle BCA + m\angle ACD$	**5.** Substitution Property using step 2 and step 4
6. $m\angle A + m\angle B = m\angle ACD$	**6.** Subtraction Property of Equality
7. $m\angle B > 0°$	**7.** Definition of an angle measure
8. $m\angle ACD > m\angle A$	**8.** Inequality Property (if $a = b + c$ and $c > 0$, then $a > b$)

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Grouping

Have students complete Question 16, part (b) with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 16 part (b)

- Which theorems were used to prove this theorem?
- Which definitions were used to prove this theorem?
- Which properties were used to prove this theorem?
- How is this proof similar to the proof in part (a)?
- How is this proof different from the proof in part (a)?

b. Part 2

Given: Triangle ABC with exterior $\angle ACD$

Prove: $m\angle ACD > m\angle B$

Statements	Reasons
1. Triangle ABC with exterior $\angle ACD$	**1.** Given
2. $m\angle A + m\angle B + m\angle BCA = 180°$	**2.** Triangle Sum Theorem
3. $\angle BCA$ and $\angle ACD$ are a linear pair	**3.** Linear Pair Postulate
4. $m\angle BCA + m\angle ACD = 180°$	**4.** Definition of linear pair
5. $m\angle A + m\angle B + m\angle BCA = m\angle BCA + m\angle ACD$	**5.** Substitution Property using step 2 and step 4
6. $m\angle A + m\angle B = m\angle ACD$	**6.** Subtraction Property of Equality
7. $m\angle A > 0°$	**7.** Definition of an angle measure
8. $m\angle ACD > m\angle B$	**8.** Inequality Property (if $a = b + c$ and $c > 0$, then $a > b$)

3

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Problem 4

Students are given two different maps of Easter Island. Both maps contain a key in which distance can be measured in miles or kilometers. Easter Island is somewhat triangular in shape and students predict and then compute which side appears to contain the longest coastline in addition to answering questions related to the perimeter of the island.

3

Grouping

Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

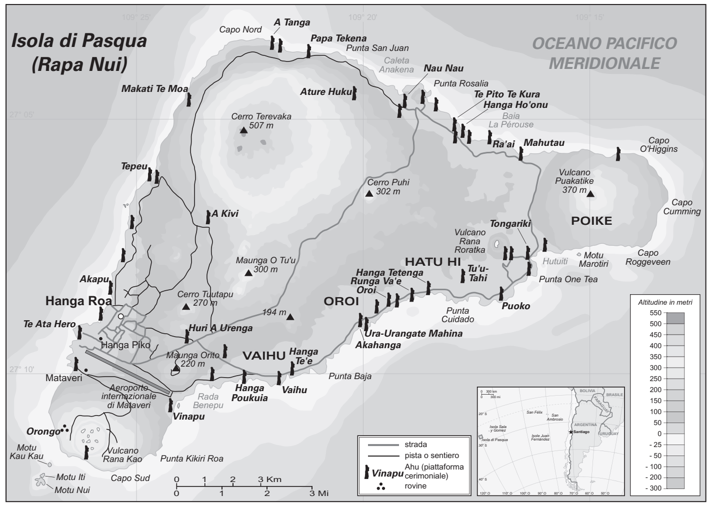
Guiding Questions for Share Phase, Questions 1 through 7

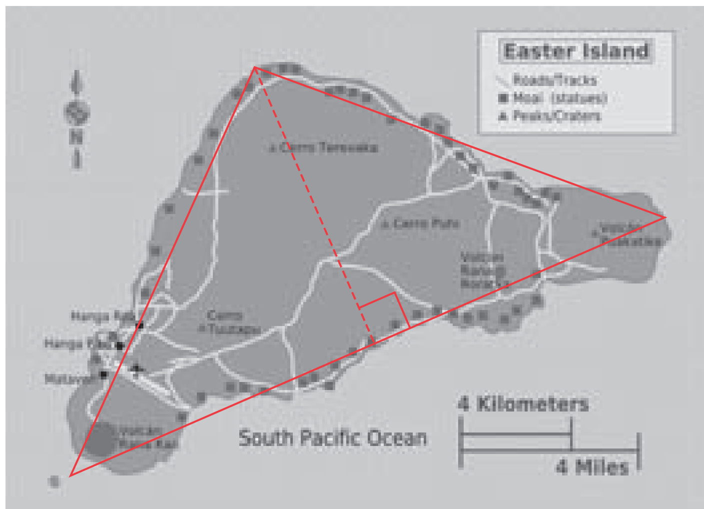
- What is the difference between the two maps?
- Are the distances on each map reasonably close to each other? How do you know?
- If you had to answer questions associated with elevation, which map is most useful?
- How is the map key used?
- Do you suppose all maps contain a map key? Why or why not?
- Which corner of the island appears to be formed by the angle greatest in measure?
- What does this tell you about the coastline opposite the angle of greatest measure?

PROBLEM 4 Easter Island

Easter Island is an island in the southeastern Pacific Ocean, famous for its statues created by the early Rapa Nui people.

Two maps of Easter Island are shown.





- What operation is used to determine the number of statues per square mile?
- What is the approximate perimeter of Easter Island?

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1. What questions could be answering using each map?

   Answers will vary.

   If I need to answer questions about the elevation of various locations, I would need to use the first map.

   If I want to compute linear measurements, both maps contain a map key that I can use.

2. What geometric shape does Easter Island most closely resemble? Draw this shape on one of the maps.

   Easter Island most closely resembles a triangle.

3. Is it necessary to draw Easter Island on a coordinate plane to compute the length of its coastlines? Why or why not?

   No. It is not necessary to use a coordinate plane because a map key is provided.

4. Predict which side of Easter Island appears to have the longest coastline and state your reasoning using a geometric theorem.

   The south side of Easter Island lies opposite what appears to be the angle of greatest measure, so it will have the longest coastline.

5. Use either map to validate your answer to Question 4.

   Answers will vary.

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6. Easter Island has 887 statues. How many statues are there on Easter Island per square mile?

I used the triangle that I drew and the scale to determine that Easter Island is approximately 69 square miles.

There are 887 ÷ 69, or 12.86, statues per square mile.

3

7. Suppose we want to place statues along the entire coastline of the island, and the distance between each statue was 1 mile. Would we need to build additional statues, and if so, how many?

I used the triangle that I drew and the scale to determine that the coastline of Easter Island is less than 887 miles, so we would not need to build additional statues.

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Talk the Talk

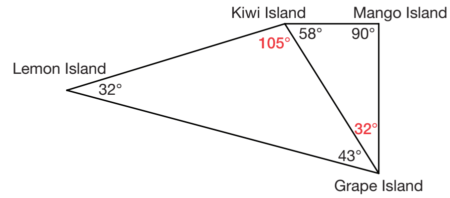
A diagram is given consisting of two triangles sharing a common side. Students determine the longest line segment in a diagram using only the measures of angles provided.

Grouping

Have students complete the Talk the Talk with a partner. Then have students share their responses as a class.

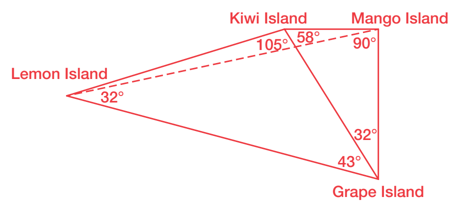
Talk the Talk

Using only the information in the diagram shown, determine which two islands are farthest apart. Use mathematics to justify your reasoning.



Using the Triangle Sum Theorem, the unknown angle measure near Kiwi Island is 105° and the unknown angle measure near Grape Island is 32°. In the triangle on the right, the angle near Mango Island is the largest angle in the triangle, so the side opposite this angle must be the longest side (and is shared by both triangles). In the triangle on the left, the angle near Kiwi Island is the largest angle in the triangle, so the side opposite this angle must be the longest side of the triangle. Therefore, the longest side of the two triangles is the side between Lemon Island and Grape Island.

However, Lemon Island and Mango Island may be the two islands that are farthest apart. To determine whether Lemon Island and Mango Island are the farthest apart, I would need to know the angle measures of the triangle formed by Lemon Island, Mango Island, and Grape Island.



Be prepared to share your solutions and methods.

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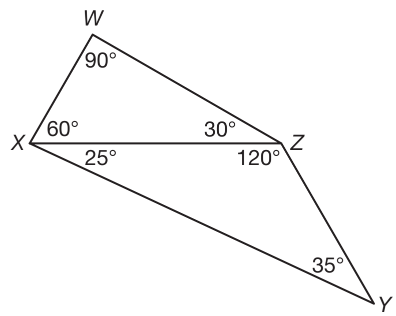
Check for Students' Understanding

Quadrilateral *WXYZ* is shown.

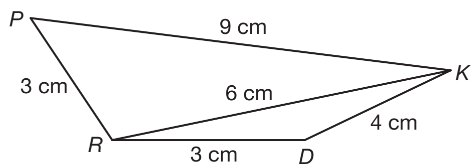


1. Without using a ruler, determine which line segment is the longest in this figure? Explain.

   In triangle *WXZ*, ∠*W* is the largest angle, so the segment opposite this angle in triangle *WXZ* must be the longest side. In triangle *XYZ*, ∠*Z* the largest angle, so the segment opposite this angle in triangle *XYZ* must be the longest side. So line segment *XY* is the longest line segment in the figure.

3

Quadrilateral *KDRP* is shown.



2. Without using a protractor, determine which angle is the largest in triangle *KDR*. Explain.

   ∠*D* is the largest angle in triangle *KDR* because it is opposite the longest side.

3. Without using a protractor, determine which angle is the largest in triangle *PRK*. Explain.

   ∠*R* the largest angle in triangle *PRK* because it is opposite the longest side.

4. Compare the largest angle in triangle *KDR* to the largest angle in triangle *PRK*. Which angle is larger? How do you know?

   ∠*PRK* is larger than ∠*KDR*, because it is opposite the longer side.

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3.2

Trade Routes and Pasta Anyone?

The Triangle Inequality Theorem

LEARNING GOALS

In this lesson, you will:

- Explore the relationship between the side lengths of a triangle and the measures of its interior angles.
- Prove the Triangle Inequality Theorem.

KEY TERM

- Triangle Inequality Theorem

ESSENTIAL IDEAS

- The Triangle Inequality Theorem states: "The sum of the lengths of any two sides of a triangle is greater than the length of the third side."

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Prove geometric theorems

10. Prove theorems about triangles.

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Overview

Students informally show and formally prove the Triangle Sum Theorem. Next, they explore the relationships that exist between the lengths of the sides and the measures of the angles opposite those sides in a triangle.

3

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Warm Up

This list of numbers represents the lengths of three sides of four different triangles.

3, 4, 5
6, 7, 8
6, 8, 11
4, 6, 7

Jan has a secret! She claims she can just look at the numbers and know if the triangle is a right triangle, an acute triangle, or an obtuse triangle. What is Jan's secret?

Jan knows the 3, 4, 5 triangle is a right triangle because of the Converse of the Pythagorean Theorem: $3^2 + 4^2 = 5^2$.

Jan knows the 6, 7, 8 triangle is not a right triangle because $6^2 + 7^2 \neq 8^2$. If it were a right triangle the longest side would have to be equal to $\sqrt{85}$. The longest side is equal to $\sqrt{64}$ which is less than $\sqrt{85}$, so the third side is shorter than a hypotenuse which makes the angle opposite the third side an acute angle. The triangle is an acute triangle.

Jan knows the 6, 8, 11 triangle is not a right triangle because $6^2 + 8^2 \neq 11^2$. If it were a right triangle, the longest side would have to be equal to $\sqrt{100}$. The longest side is equal to $\sqrt{121}$ which is greater than $\sqrt{100}$, so the third side is longer than a hypotenuse which makes the angle opposite the third side an obtuse angle. The triangle is an obtuse triangle.

Jan knows the 4, 6, 7 triangle is not a right triangle because $4^2 + 6^2 \neq 7^2$. If it were a right triangle, the longest side would have to be equal to $\sqrt{52}$. The longest side is equal to $\sqrt{49}$ which is less than $\sqrt{52}$, so the third side is shorter than a hypotenuse which makes the angle opposite the third side an acute angle. The triangle is an acute triangle.

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3.2

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- Prove the Triangle Inequality Theorem.

KEY TERM

- Triangle Inequality Theorem

Triangular trade best describes the Atlantic trade routes among several different destinations in Colonial times. The Triangular Trade Routes connected England, Europe, Africa, the Americas, and the West Indies. The Triangular Trade Routes included the following:

- Trade Route 1: England to Africa to the Americas
- Trade Route 2: England to Africa to the West Indies
- Trade Route 3: Europe to the West Indies to the Americas
- Trade Route 4: Americas to the West Indies to Europe

3

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Problem 1

This activity requires students to use a manipulative. A suggestion is to give each student one piece of raw pasta such as linguine or spaghetti. They will also need a centimeter ruler. Then each student will randomly break their piece of pasta into three pieces and use the pieces to form a triangle, if possible. Students measure the length of each piece and note if the pieces formed a triangle. The data from all students are collected and recorded in a chart provided, and a series of guided questions leads the students to conclude that only when the sum of any two lengths is greater than the third length, a triangle formation is possible.

The Triangle Inequality Theorem is stated. A two-column proof for this theorem is provided and students are required to complete the reasons. Note that students need to draw a line segment to prove this theorem.

Grouping

Complete Questions 1 through 3 as a class, with students working in partners to form triangles for Question 2.

3

PROBLEM 1 Who Is Correct?

1. Sarah claims that any three lengths will determine three sides of a triangle. Sam does not agree. He thinks some combinations will not work. Who is correct?

   Sam is correct. The lengths 4 centimeters, 5 centimeters, and 12 centimeters do not determine a triangle.

2. Sam then claims that he can just look at the three lengths and know immediately if they will work. Sarah is unsure. She decides to explore this for herself.

   Help Sarah by working through the following activity.

   To begin, you will need a piece of strand pasta (like linguine). Break the pasta at two random points so the strand is divided into three pieces. Measure each of your three pieces of pasta in centimeters. Try to form a triangle from your three pieces of pasta. Try several pieces of pasta with different breaking points.

   Answers will vary.

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Guiding Questions for Share Phase, Questions 1 through 3

- What three lengths could Sam be thinking about?
- What three lengths could Sarah be thinking about?
- Do your three pieces of pasta form a triangle?
- Do all of your classmates' three pieces of pasta form a triangle?
- If one piece of pasta was extremely long and the other two pieces were extremely short, would the three pieces of pasta form a triangle?
- If one piece of pasta was extremely short and the other two pieces were extremely long, would the three pieces of pasta form a triangle?
- If the three pieces of pasta were the same length, would the three pieces of pasta form a triangle?
- Did any of your classmates choose the same lengths?
- Did anyone have measurements that did not form a triangle?
- How does the sum of the lengths of two pieces of pasta compare to the length of the third piece of pasta?

3. Collect and record your classmates' measurements.

Piece 1 (cm)	Piece 2 (cm)	Piece 3 (cm)	Forms a Triangle? (yes or no)

3

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Grouping

Have students complete Questions 4 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 7

- If the sum of the lengths of two pieces of pasta is longer than the length of the third piece of pasta, do the three pieces of pasta form a triangle?
- If the sum of the lengths of two pieces of pasta is not longer than the length of the third piece of pasta, do the three pieces of pasta form a triangle?

3

4. Examine the lengths of the pasta pieces that did form a triangle. Compare them with the lengths of the pasta pieces that did not form a triangle. What observations can you make?

Answers will vary.

If the three pieces were of equal length, or if the sum of the lengths of two pieces was longer than the length of the third piece, I could form a triangle.

5. Under what conditions were you able to form a triangle?

A triangle can always be formed if the sum of the lengths of the two shorter pieces is longer than the length of the third piece.

6. Under what conditions were you unable to form a triangle?

A triangle can never be formed if the sum of the lengths of the two shorter pieces is less than or equal to the length of the third piece.

7. Based upon your observations, determine if it is possible to form a triangle using segments with the following measurements. Explain your reasoning.

a. 2 centimeters, 5.1 centimeters, 2.4 centimeters

The measurements would not form a triangle because the sum of the lengths of the shorter pieces is less than the length of the third piece.

b. 9.2 centimeters, 7 centimeters, 1.9 centimeters

The measurements would not form a triangle because the sum of the lengths of the two shorter pieces is less than the length of the third piece.

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Grouping

Have students complete Question 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 8

- Are any theorems used to prove this theorem? If so, which theorems?
- Are any definitions used to prove this theorem? If so, which definitions?
- Are any properties used to prove this theorem? If so, which properties?

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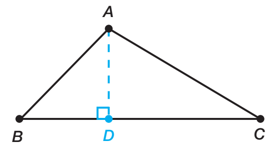
The rule that Sam was using is known as the Triangle Inequality Theorem.

The **Triangle Inequality Theorem** states: "the sum of the lengths of any two sides of a triangle is greater than the length of the third side."

8. Prove the Triangle Inequality Theorem by completing each step.



Given: Triangle *ABC*

Prove: $AB + AC > BC$

A perpendicular line segment *AD* is constructed through point *A* to side *BC*.

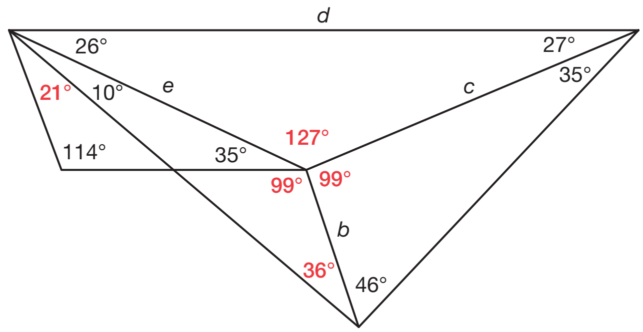
Statements	Reasons
1. Triangle *ABC*	**1.** Given
2. Draw $\overline{AD} \perp \overline{BC}$	**2.** Construction
3. $\angle ADB$ is a right angle.	**3.** Definition of perpendicular.
4. $\angle ADC$ is a right angle.	**4.** Definition of perpendicular.
5. $BD^2 + AD^2 = AB^2$	**5.** Pythagorean Theorem
6. $CD^2 + AD^2 = AC^2$	**6.** Pythagorean Theorem
7. $AB^2 > BD^2$	**7.** Definition of greater than.
8. $AC^2 > DC^2$	**8.** Definition of greater than.
9. $AB > BD$	**9.** Square root of both sides.
10. $AC > DC$	**10.** Square root of both sides.
11. $AB + AC > BD + DC$	**11.** Addition Property of Inequality
12. $BD + DC = BC$	**12.** Segment Addition Postulate
13. $AB + AC > BC$	**13.** Substitution Property using step 11 and step 13.

Be prepared to share your solutions and methods.

3

Check for Students' Understanding

List the unknown lengths from least to greatest.



b, c, e, d

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3.3

Stamps Around the World

Properties of a 45°–45°–90° Triangle

LEARNING GOALS

In this lesson, you will:

- Use the Pythagorean Theorem to explore the relationship between the side lengths of a triangle and the measures of its interior angles.
- Prove the 45°–45°–90° Triangle Theorem.

KEY TERM

- 45°–45°–90° Triangle Theorem

ESSENTIAL IDEAS

- An isosceles right triangle is also known as a 45°–45°–90° triangle.
- The 45°–45°–90° Triangle Theorem states: "The length of the hypotenuse in a 45°–45°–90° triangle is $\sqrt{2}$ times the length of a leg."

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Prove geometric theorems

10. Prove theorems about triangles.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

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Overview

Students measure the sides and angles of a stamp in the shape of a 45°–45°–90° triangle. The 45°–45°–90° Triangle Theorem is stated and students prove the theorem using the Pythagorean Theorem. Several different triangular stamps are given and students solve for the length of the sides using both the Pythagorean Theorem and the 45°–45°–90° Triangle Theorem. Using a compass and straightedge, students construct an isosceles right triangle and discuss alternate methods of construction.

3

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Warm Up

Camping for the weekend with friends, you park the car and hike into the woods, walking 2 miles due south, 1 mile due east, 3 miles due south, and 4 due miles east before setting up camp. Sam thinks the campsite is 10 miles from the car, and Gwen thinks the campsite is 7.24 miles from the car. You prove them both wrong. Explain how each person arrived at their incorrect conclusions and provide the correct solution.

Using the Pythagorean Theorem, the camp is $\sqrt{(5^2 + 5^2)} \approx 7.07$ miles from the car. Sam just added all of the distances. Gwen solved for the hypotenuse of two right triangles and added those answers together.

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3.3

Stamps Around the World

Properties of a 45°–45°–90° Triangle

LEARNING GOALS

In this lesson, you will:

- Use the Pythagorean Theorem to explore the relationship between the side lengths of a triangle and the measures of its interior angles.
- Prove the 45°–45°–90° Triangle Theorem.

KEY TERM

- 45°–45°–90° Triangle Theorem

The first adhesive postage stamp was issued in the United Kingdom in 1840. It is commonly known as the Penny Black which makes sense because it cost one penny and had a black background. It featured a profile of 15-year-old former Princess Victoria.

You may think that the very first stamp is quite rare and valuable. However, that isn't quite true. The total print run was 68,808,000. During this time envelopes were not normally used. The address and the stamp was affixed to the folded letter itself. Many people kept personal letters and ended up keeping the stamp too.

As of 2012, the most valuable stamp was the Treskilling Yellow stamp from Sweden. Only one known copy exists. In 2010 it sold at auction for over three million dollars!

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Problem 1

Students measure the sides and angles of a postage stamp in the shape of a 45°–45°–90° triangle. Using a protractor, students determine that the measure of each acute angle in an isosceles right triangle is equal to 45°. The 45°–45°–90° Triangle Theorem is stated and proven algebraically using the Pythagorean Theorem. The 45°–45°–90° Triangle Theorem is given as a shortcut for determining the length of the hypotenuse, when a length of the leg is known. Triangular stamps from other countries are shown and students use the Pythagorean Theorem in conjunction with the 45°–45°–90° Triangle Theorem to determine unknown lengths. Questions require students to work both forward and backward.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

- If the stamp is an enlargement of the original stamp, is it the same size?
- If the stamp is an enlargement of the original stamp, is it the same shape?
- Does enlarging a triangle change the lengths of the sides of the triangle?
- Does enlarging a triangle change the measures of the angles of the triangle?
- Does dilation preserve shape of the figure?

PROBLEM 1 Stamp Collecting Isn't Always Square!

The first triangle-shaped U.S. stamps were issued on June 8, 1997. The pair of 32-cent commemorative stamps of triangular shape featured a mid-19th-century clipper ship and a U.S. mail stagecoach.

Each image shown is an enlargement of both stamps.

1. Can you use this enlargement to determine the measures of the angles of the actual stamp? Why or why not?

   Yes. I can use this enlargement to determine the measures of the angles of the actual stamp because enlarging the stamp does not change the measures of the angles. The triangles are similar triangles, and when triangles are dilated, the measures of the angles are preserved.

2. Measure the angles of one of the commemorative stamps.

   The measure of each acute angle is 45°, and the measure of the largest angle is 90°.

3. Measure the length of the sides of one of the commemorative stamps and describe the relationship between the length of each side and the measure of the angle located opposite each side.

   The sides opposite the 45° angles are congruent, and the side opposite the 90° angle is the longest side of the triangle.

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The **45°–45°–90° Triangle Theorem** states: "the length of the hypotenuse in a 45°–45°–90° triangle is $\sqrt{2}$ times the length of a leg."

4. Use the Pythagorean Theorem to prove the 45°–45°–90° Triangle Theorem. Let c represent the length of the hypotenuse and let ℓ represent the length of each leg.

$a^2 + b^2 = c^2$

$\ell^2 + \ell^2 = c^2$

$2\ell^2 = c^2$

$c = \sqrt{2\ell^2} = \sqrt{2} \cdot \sqrt{\ell^2}$

$c = \ell\sqrt{2}$

5. Using the 45°–45°–90° Triangle Theorem, what is the length of the longest side of the enlargement of the commemorative stamp?

The length of the longest side is equal to the length of the shortest side times the square root of two.

6. What additional information is needed to determine the length of the longest side of the actual commemorative stamp?

Additional information is needed to determine the length of the longest side of an actual commemorative stamp. We would need to know the ratio of the sides of the two similar triangles.

3

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Grouping

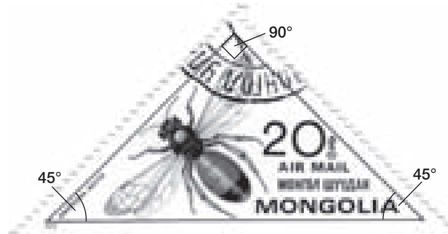
Have students complete Questions 7 through 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 7 through 10

- If two sides of a triangle are congruent, are the angles opposite those sides also congruent? Why or why not?
- What is the square root of l^2?
- Is it easier calculating the length of the shortest side of the 45°–45°–90° triangle or the length of the longest side of the 45°–45°–90° triangle? Why?

3

7. This stamp was issued in Mongolia.



Suppose the longest side of this stamp is 50 millimeters.

a. Use the Pythagorean Theorem to determine the approximate length of the other sides of this stamp. Round your answer to the nearest tenth of a millimeter.

$a^2 + b^2 = c^2$

$\ell^2 + \ell^2 = 50^2$

$2\ell^2 = 2500$

$\ell^2 = 1250$

$\ell = \sqrt{1250} \approx 35.4$

The approximate length of the other two sides is 35.4 millimeters.

b. Use the 45°–45°–90° Triangle Theorem to determine the approximate length of the other sides of this stamp. Round your answer to the nearest tenth of a millimeter.

$h = \ell\sqrt{2}$

$50 = \ell\sqrt{2}$

$\ell = \frac{50}{\sqrt{2}} \approx 35.4$

The approximate length of the other two sides is 35.4 millimeters.

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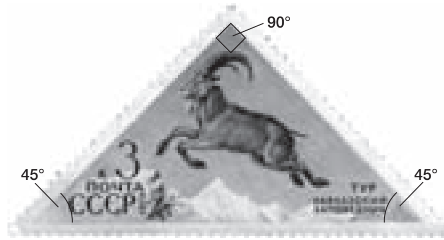
8. This stamp was issued in Russia.



Suppose the longest side of this stamp is 50 millimeters. Use the 45°–45°–90° Triangle Theorem to determine the *actual* length of the shortest side of this stamp.

$h = \ell\sqrt{2}$

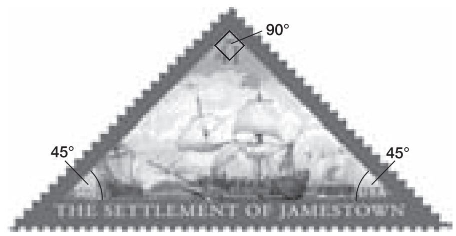
$50 = \ell\sqrt{2}$

$\ell = \frac{50}{\sqrt{2}} = \frac{50}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{50\sqrt{2}}{2} = 25\sqrt{2}$

The actual length of the shortest side of this stamp is $25\sqrt{2}$ millimeters.

9. In 2007, another triangle-shaped stamp was issued in the United States. It was issued to commemorate the 400th anniversary of the Settlement of Jamestown, Virginia, by English colonists in 1607. This stamp features a painting of the three ships that carried the colonists from England to the United States. Was it a coincidence that the first fort built by the settlers was shaped like a triangle?

This is an enlargement of the Jamestown stamp.



Measure the length of the shortest side and use the 45°–45°–90° Triangle Theorem to determine the length of the longest side of the enlargement of the commemorative stamp.

$h = \ell\sqrt{2}$

3

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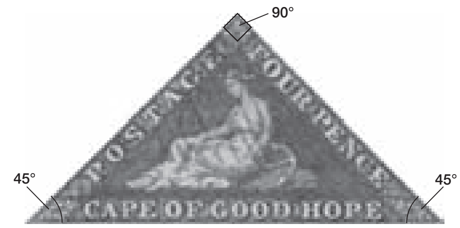
10. The first triangular stamp was issued by the Cape of Good Hope in 1853. This is an enlargement of the Cape of Good Hope stamp.



Measure the length of the longest side and use the 45°–45°–90° Triangle Theorem to determine the length of the shortest side of the enlargement of the commemorative stamp.

$h = \ell\sqrt{2}$

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Problem 2

Students use a compass and straightedge to construct an isosceles right triangle (triangle *ABC*) given the length of leg *CB* and right angle *C*. After constructing the isosceles triangle, students use measuring tools to verify anticipated measurements of angles and lengths of sides. An alternate method to constructing an isosceles right triangle is to construct a square and draw a diagonal.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

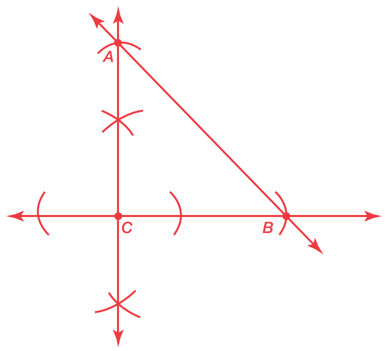
- Could the construction of a rectangle be used to construct a 45°–45°–90° triangle? Why or why not?
- Could the construction of a parallelogram be used to construct a 45°–45°–90° triangle? Why or why not?
- Could the construction of a rhombus be used to construct a 45°–45°–90° triangle? Why or why not?
- Could the construction of an equilateral triangle be used to construct a 45°–45°–90° triangle? Why or why not?

PROBLEM 2 Construction

1. Construct an isosceles right triangle with $\overline{CB}$ as a leg and $\angle C$ as the right angle.

After completing the construction, use a protractor and a ruler to confirm the following:

- $m\angle A = 45°$
- $m\angle B = 45°$
- $AC = BC$
- $AB = AC\sqrt{2}$
- $AB = BC\sqrt{2}$



2. Explain how you can use an alternate method for constructing a 45°–45°–90° triangle by constructing a square first.

If you first construct a square, you could then construct a bisector for a right angle of the square. The bisector divides the square into two 45º–45º–90º triangles.

Be prepared to share your solutions and methods.

- Could the construction of a trapezoid be used to construct a 45°–45°–90° triangle? Why or why not?

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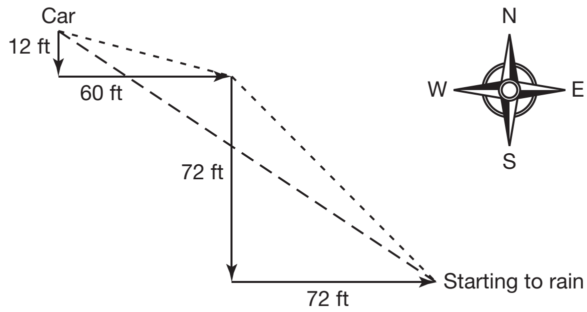
Check for Students' Understanding

Gus and Helen went on a hiking trip. They parked their car and started to hike due south, then due east, and so on. After hiking only 216 feet, it started to rain, so they decided to return to the car. This map shows the directions and distances they hiked before it started to rain.



Both Gus and Helen agree that the shortest path back to the car has something to do with the length of the hypotenuse in a right triangle.

3

- Two routes are on the map indicated by dashed lines.
- Helen believes the shortest route is indicated by small dashes.
- Gus believes the shortest route is indicated by the longer dashes.

1. Can both routes be equal distances? Explain.

No, both routes cannot be the same distance because both routes form a triangle and if the sum of the two shorter sides was equal to the length of the longest side, it would not be a triangle.

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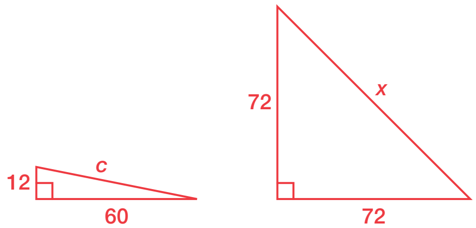
2. Which route is the shortest? Justify your conclusion.

Helen's route:

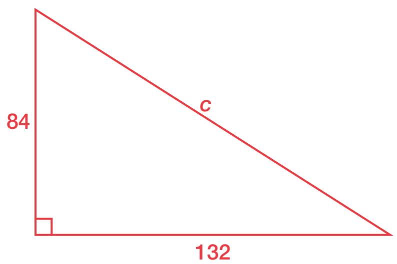


$$c^2 = 12^2 + 60^2$$
$$= 144 + 3600$$
$$= 3744$$
$$c = \sqrt{3744} \approx 61.19 \text{ ft}$$

$$x = 72\sqrt{2}$$
$$\approx 101.82 \text{ ft}$$

Approximate distance of Helen's route: 61.19 + 101.82 = 163.01 ft.

Gus's route:



$$c^2 = 84^2 + 132^2$$
$$= 7056 + 17{,}424$$
$$= 24{,}480$$
$$c = \sqrt{24480} \approx 156.46 \text{ ft}$$

Approximate distance of Gus's route: 156.46 ft.

Gus's route is approximately 6.55 ft shorter than Helen's route.

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3.4

More Stamps, Really?

Properties of a 30°–60°–90° Triangle

LEARNING GOALS

In this lesson, you will:

- Use the Pythagorean Theorem to explore the relationship between the side lengths of a triangle and the measures of its interior angles.
- Prove the 30°–60°–90° Triangle Theorem.

KEY TERM

- 30°–60°–90° Triangle Theorem

ESSENTIAL IDEAS

- The 30°–60°–90° Triangle Theorem states: "The length of the hypotenuse in a 30°–60°–90° triangle is two times the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg."

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Prove geometric theorems

10. Prove theorems about triangles.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

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Overview

Students measure the sides and angles of a stamp in the shape of a 30°–60°–90° triangle. The 30°–60°–90° Triangle Theorem is stated and students prove the theorem using the Pythagorean Theorem. Several different triangular stamps are given and students solve for the length of the sides using both the Pythagorean Theorem and the 30°–60°–90° Triangle Theorem. Using a compass and straightedge, students construct a 30°–60°–90° triangle and discuss alternate methods of construction.

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Warm Up

The value of x is expressed as a ratio.

$$x = \frac{18}{\sqrt{3}}$$

1. Express x by rationalizing the denominator.

   $$\frac{18}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{18\sqrt{3}}{3} = 6\sqrt{3}$$

2. Express x as an approximate answer.

   $$\frac{18}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{18\sqrt{3}}{3} = 6\sqrt{3} \approx 10.39$$

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More Stamps, Really?

Properties of a 30°–60°–90° Triangle

3.4

LEARNING GOALS

In this lesson, you will:

- Use the Pythagorean Theorem to explore the relationship between the side lengths of a triangle and the measures of its interior angles.
- Prove the 30°–60°–90° Triangle Theorem.

KEY TERM

- 30°–60°–90° Triangle Theorem

The US Postal Services doesn't have an official motto but an inscription on the James Farley Post Office in New York is well known. It reads, "Neither snow nor rain nor heat nor gloom of night stays these couriers from the swift completion of their appointed rounds."

There have been many popular characters on television who were mail carriers including Mister McFeely from the children's series *Mister Rogers' Neighborhood*, Cliff Clavin from the comedy series *Cheers*, and Newman from the comedy series *Seinfeld*.

Can you think of any other famous mail carriers? What other professions seem to inspire characters on television, the movies, or in books?

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Problem 1

Students are given a stamp shaped like an equilateral triangle. They draw the altitude to the base and form two 30°–60°–90° triangles. The 30°–60°–90° Triangle Theorem is stated and proven algebraically using the Pythagorean Theorem. The 30°–60°–90° Triangle Theorem is given as a shortcut for determining the length of the hypotenuse, when a length of the leg is known. Triangular stamps from other countries are shown and students use the Pythagorean Theorem in conjunction with the 30°–60°–90° Triangle Theorem to determine unknown lengths. Questions require students to work both forward and backward.

3

Grouping

Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 7

- If three sides of a triangle are congruent, are the angles opposite those sides also congruent? Why or why not?
- If the three angles of an equilateral triangle are congruent, what is the measure of each angle? Why?

PROBLEM 1 Something Is Different About This Stamp!

This stamp was issued in Malaysia.

1. How is this stamp different from the stamps you studied in the previous lesson?
   This stamp is not shaped like a right triangle. It appears to be an equilateral triangle.

2. This Malaysian stamp is shaped like an equilateral triangle. What is the measure of each interior angle of the triangle? Explain your reasoning.
   The measure of each interior angle of an equilateral triangle is 60°. The sum of the measures of the interior angles of any triangle is 180°. Because the three interior angles of an equilateral triangle are congruent, each interior angle must have a measure equal to 60°.

3. Use the diagram of the stamp to draw an altitude to the base of the equilateral triangle. Describe the two triangles formed by the altitude.
   Each of the two triangles formed by the altitude to the base of the equilateral triangle is a right triangle containing a 60° angle. The measure of the third interior angle in each triangle must be equal to 30°.

4. How do you know that the two triangles formed by the altitude drawn to the base of an equilateral triangle are congruent.
   I know that each of the triangles has angles that measure 30°, 60°, and 90°. I also know that one pair of sides are congruent because they are both sides of the equilateral triangle. They share one side in common. I can use the Pythagorean Theorem to show that the third pair of sides are congruent.

- What is an altitude?
- How is an altitude in the equilateral triangle drawn?
- How is an altitude in the equilateral triangle constructed?
- What is the difference between drawing an altitude and constructing an altitude?
- Are the two triangles formed by the altitude right triangles? Why or why not?
- Do the two triangles formed by the altitude share a common side? If so, which side?

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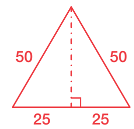
- Is the common side shared by both right triangles the hypotenuse or the leg in each triangle?
- Does the altitude drawn to the base of an equilateral triangle bisect the base? Why or why not?
- Does the altitude drawn to the base of an equilateral triangle bisect the vertex angle? Why or why not?

5. If the length of each side of the Malaysian stamp is 50 millimeters, determine the length of the three sides in each of the two 30°–60°–90° triangles formed by the altitude drawn to the base of the equilateral triangle.



$a^2 + b^2 = c^2$

$25^2 + b^2 = 50^2$

$625 + h^2 = 1875$

$h^2 = \sqrt{1875}$

$h = 25\sqrt{3}$

The length of the hypotenuse in each 30°–60°–90° triangle is 50 millimeters. The length of the shortest leg in each 30°–60°–90° triangle is 25 millimeters. The length of the altitude or longest leg in each 30°–60°–90° triangle is $25\sqrt{3}$ millimeters, or about 43.30 millimeters.

6. How does the length of the hypotenuse in each of the two 30°–60°–90° triangles relate to the length of the shortest leg?

The length of the hypotenuse is twice the length of the shortest leg.

7. How does the length of the longer leg in each of the two 30°–60°–90° triangles relate to the length of the shortest leg?

The length of the longest leg is $\sqrt{3}$ times the length of the shortest leg.

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Grouping

Have students complete Questions 8 through 11 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 8 through 11

- Is it easier calculating the length of a leg of the 30°–60°–90° triangle or the length of the hypotenuse of the 30°–60°–90° triangle? Why?
- Given the length of the hypotenuse, it easier calculating the length of the shortest leg of the 30°–60°–90° triangle or the length of the longest leg of the 30°–60°–90° triangle? Why?

3

The **30°–60°–90° Triangle Theorem** states: "the length of the hypotenuse in a 30°–60°–90° triangle is two times the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg."

8. Use the Pythagorean Theorem to demonstrate the 30°–60°–90° Triangle Theorem. Let *x* represent the length of the shortest leg.

If *x* represents the length of the shorter leg then $x\sqrt{3}$ represents the length of the longer leg and 2*x* represents the length of the hypotenuse. Substitute into the Pythagorean Theorem. If the substituted values result in a true statement then the 30°–60°–90° Triangle Theorem is true.

$$a^2 + b^2 = c^2$$
$$x^2 + (x\sqrt{3})^2 = (2x)^2$$
$$x^2 + 3x^2 = (2x)^2$$
$$4x^2 = 4x^2$$

9. This stamp was issued in the Netherlands.

Suppose the length of each side of the Netherlands stamp is 40 millimeters. Use the 30°–60°–90° Triangle Theorem to determine the height of the stamp.

The length of the shorter leg of the 30°–60°–90° triangle is half the length of the hypotenuse (40 millimeters). The altitude or side opposite the 60° angle is $\sqrt{3}$ times the length of the shorter leg.

$$\textit{altitude} = \left(\frac{1}{2}\textit{hypotenuse}\right)(\sqrt{3})$$

$$\textit{altitude} = \frac{40}{2}(\sqrt{3}) = 20\sqrt{3} \approx 34.64$$

The height of the Netherlands stamp is $20\sqrt{3}$, or approximately 34.64, millimeters.

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10. In 1929, Uruguay issued a triangular parcel post stamp with a picture of wings, implying rapid delivery.

Suppose the height of the Uruguay stamp is 30 millimeters. Use the 30°–60°–90° Triangle Theorem to determine the length of the three sides of the stamp.

The height of the stamp or side opposite the 60° angle (30 millimeters) is $\sqrt{3}$ times the length of the shorter leg. The length of the shorter leg of the 30°–60°–90° triangle is half the length of the hypotenuse, or length of the three sides of the stamp.

$30 = (\frac{1}{2}\textit{hypotenuse})(\sqrt{3})$

$\frac{30}{\sqrt{3}} = (\frac{1}{2}\textit{hypotenuse})$

$\frac{60}{\sqrt{3}} = (\textit{hypotenuse})$

$(\textit{hypotenuse}) = \frac{60}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{60\sqrt{3}}{3} = 20\sqrt{3} \approx 34.64$

The length of the three sides of the Uruguay stamp is $20\sqrt{3}$, or approximately 34.64, millimeters.

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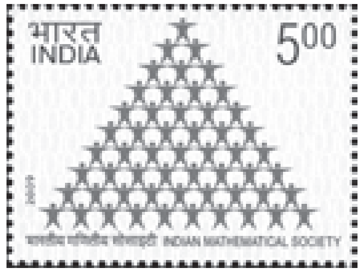
11. A mathematical society in India designed this stamp. The pyramidal design is an equilateral triangle.



Suppose the height of the pyramidal design on the stamp is 42 millimeters. Determine the area of the pyramidal design on the stamp.

$42 = (\frac{1}{2}\textit{hypotenuse})(\sqrt{3})$

$\frac{42}{\sqrt{3}} = (\frac{1}{2}\textit{hypotenuse})$

$\frac{84}{\sqrt{3}} = (\textit{hypotenuse})$

$(\textit{hypotenuse}) = \frac{84}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{84\sqrt{3}}{3} = 28\sqrt{3} \approx 48.5$

$A = \frac{1}{2}bh$

$A = \frac{1}{2}(28\sqrt{3})(42)$

$A = 1018.5$

The area of the pyramidal design on the stamp is $588\sqrt{3}$, or approximately 1018.5, square millimeters.

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Problem 2

Students use a compass and straightedge to construct an equilateral triangle with an altitude. After constructing an equilateral triangle with an altitude, students use measuring tools to verify anticipated measurements of angles and lengths of sides of a 30°–60°–90° triangle.

Grouping

Have students complete the problem with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Problem 2

- Could the construction of a rectangle be used to construct a 30°–60°–90° triangle? Why or why not?
- Could the construction of a parallelogram be used to construct a 30°–60°–90° triangle? Why or why not?
- Could the construction of a rhombus be used to construct a 30°–60°–90° triangle? Why or why not?
- Could the construction of an equilateral triangle be used to construct a 30°–60°–90° triangle? Why or why not?
- Could the construction of a trapezoid be used to construct a 30°–60°–90° triangle? Why or why not?

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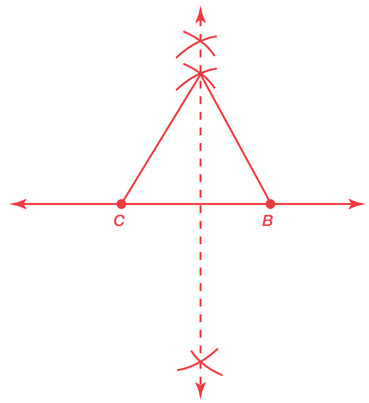
PROBLEM 2 Construction

30°–60°–90° Triangle

Construct a 30°–60°–90° triangle by constructing an equilateral triangle and one altitude.

After completing the construction, use a protractor and a ruler to confirm that:

- one angle measure is 30°.
- one angle measure is 60°.
- one angle measure is 90°.
- the side opposite the 30° angle is one-half the length of the hypotenuse.
- the side opposite the 60° angle is one-half the hypotenuse times $\sqrt{3}$.



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Talk the Talk

Students end the lesson by organizing what they have learned about 45°–45°–90° triangles and 30°–60°–90° triangles. Students can compare the different side-length ratios of each special triangle.

Grouping

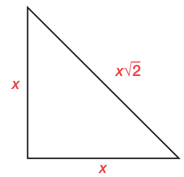
Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

3

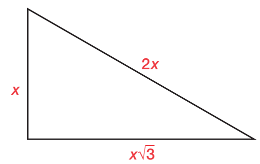
Talk the Talk

1. Label the shortest side of each triangle as x. Then label the remaining sides of each triangle in terms of x.

   a. 45°–45°–90° triangle



   b. 30°–60°–90° triangle



2. Explain how to calculate the following for a 45°–45°–90° triangle.

   a. The length of a leg given the length of the hypotenuse.

   To calculate the length of a leg, divide the length of the hypotenuse by the square root of 2.

   b. The length of the hypotenuse given the length of a leg.

   To calculate the length of the hypotenuse, multiply the length of a leg by the square root of 2.

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3. Explain how to calculate the following for a 30°–60°–90° triangle.

a. The length of the hypotenuse given the length of the shorter leg.

To calculate the length of the hypotenuse, multiply the length of the shorter leg by 2.

b. The length of the hypotenuse given the length of the longer leg.

To calculate the length of the hypotenuse, divide the length of the longer leg by the square root of 3 and then multiply by 2.

c. The length of the shorter leg given the length of the longer leg.

To calculate the length of the shorter leg, divide the length of the longer leg by the square root of 3.

d. The length of the shorter leg given the length of the hypotenuse.

To calculate the length of the shorter leg, divide the length of the hypotenuse by 2.

e. The length of the longer leg given the length of the shorter leg.

To calculate the length of the longer leg, multiply the length of the shorter leg by the square root of 3.

f. The length of the longer leg given the length of the hypotenuse.

To calculate the length of the longer leg, divide the length of the hypotenuse by 2 and then multiply by the square root of 3.

Be prepared to share your solutions and methods.

3

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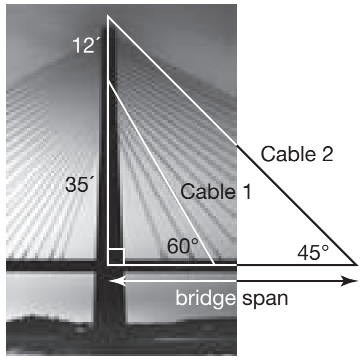
Check for Students' Understanding

A cable stayed bridge has one or more towers erected above piers in the middle of the span. From these towers, cables stretch down diagonally (usually to both sides).



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One cable forms a 60° angle with the bridge deck and is connected to the tower 35 feet above the bridge deck. A second cable forms a 45° angle with the bridge deck and is connected to the tower 12 feet above the point at which the first cable connects to the tower.

Calculate the length of both cables and the length of the bridge span from each cable to the tower.

Length of Cable 1:

$35 = \frac{1}{2}h\sqrt{3}$

$70 = h\sqrt{3}$

$h = \frac{70}{\sqrt{3}} = \frac{70\sqrt{3}}{3} \approx 40.41'$

Cable 1 bridge span:

$\frac{40.41}{2} \approx 20.205'$

Length of Cable 2:

$h = 47\sqrt{2}$

$h \approx 66.47'$

Cable 2 bridge span:

$l = 47'$

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Chapter 3 Summary

KEY TERMS

- remote interior angles of a triangle (3.1)

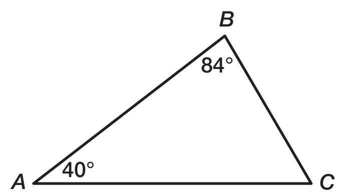
POSTULATES AND THEOREMS

- Triangle Sum Theorem (3.1)
- Exterior Angle Theorem (3.1)
- Exterior Angle Inequality Theorem (3.1)
- Triangle Inequality Theorem (3.2)
- 45°–45°–90° Triangle Theorem (3.3)
- 30°–60°–90° Triangle Theorem (3.4)

3.1 Using the Triangle Sum Theorem

The Triangle Sum Theorem states: "The sum of the measures of the interior angles of a triangle is 180°."

Example



$m\angle A + m\angle B + m\angle C = 180°$

$40° + 84° + m\angle C = 180°$

$m\angle C = 180° - (40° + 84°)$

$m\angle C = 180° - 124°$

$m\angle C = 56°$

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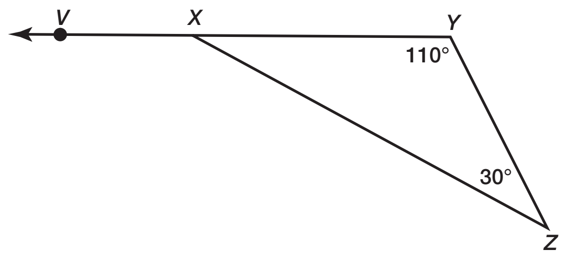
3.1 Using the Exterior Angle Theorem

The Exterior Angle Theorem states: "The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles of the triangle."

Example



$m\angle VXZ = m\angle Y + m\angle Z$

$m\angle VXZ = 110° + 30°$

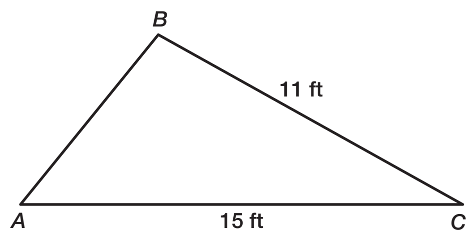
$m\angle VXZ = 140°$

3

3.2 Using the Triangle Inequality Theorem

The Triangle Inequality Theorem states: "The sum of the lengths of any two sides of a triangle is greater than the length of the third side."

Example



$AB < BC + AC$	$BC < AB + AC$	$AC < AB + BC$
$AB < 11 + 15$	$11 < AB + 15$	$15 < AB + 11$
$AB < 26$	$-4 < AB$	$4 < AB$

So, *AB* must be greater than 4 feet and less than 26 feet. (A length cannot be negative, so disregard the negative number.)

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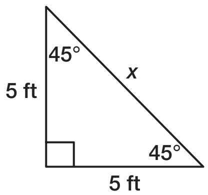
3.3 Using the 45°–45°–90° Triangle Theorem

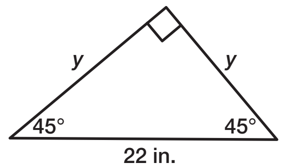
The 45°–45°–90° Triangle Theorem states: "The length of the hypotenuse in a 45°–45°–90° triangle is $\sqrt{2}$ times the length of a leg."

Examples



$x = 5\sqrt{2}$ ft

The length of the hypotenuse is $5\sqrt{2}$ feet.



$y\sqrt{2} = 22$

$$y = \frac{22}{\sqrt{2}} = \frac{22 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{22 \cdot \sqrt{2}}{2} = 11\sqrt{2} \text{ in.}$$

The length of each leg is $11\sqrt{2}$ inches.

3

3.4 Using the 30°–60°–90° Triangle Theorem

The 30°–60°–90° Triangle Theorem states: "The length of the hypotenuse in a 30°–60°–90° triangle is two times the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg."

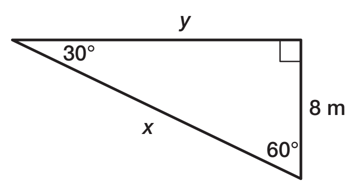
Examples

Hypotenuse:

$x = 2(8) = 16$ m

Longer leg:

$y = 8\sqrt{3}$ m



Shorter leg:

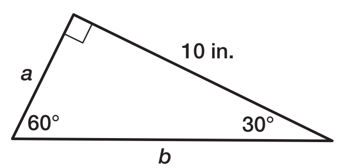
$a\sqrt{3} = 10$

$$a = \frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3} \text{ in.}$$

Hypotenuse:

$$b = 2a = 2\left(\frac{10\sqrt{3}}{3}\right)$$

$$= \frac{20\sqrt{3}}{3} \text{ in.}$$



Shorter leg:

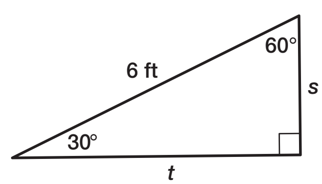
$2s = 6$

$s = 3$ ft

Longer leg:

$t = s\sqrt{3}$

$t = 3\sqrt{3}$ ft



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