Tools of Geometry

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Chapter 1 Overview

This chapter begins by addressing the building blocks of geometry which are the point, the line, and the plane. Students will construct line segments, midpoints, bisectors, angles, angle bisectors, perpendicular lines, parallel lines, polygons, and points of concurrency. A translation is a rigid motion that preserves the size and shape of segments, angles, and polygons. Students use the coordinate plane and algebra to determine the characteristics of lines, segments, and points of concurrency.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
1.1	Points, Lines, Planes, Rays, and Line Segments	G.CO.1	1	This lesson builds a foundation for common language in geometry. Students are introduced to construction, and the tools needed to get started. Questions then ask students to use undefined terms to define new terms.	X	X	X	X	
1.2	Translating and Constructing Line Segments	G.CO.1 G.CO.2 G.CO.4 G.CO.5 G.CO.6 G.CO.12 G.CO.13	2	This lesson uses the Pythagorean Theorem to derive the distance formula. Questions then ask students to translate a line segment into all four quadrants to conclude that translations preserve the length of a line segment. Students will then use construction tools to construct congruent radii, a regular hexagon, and to duplicate line segments.	X	X	X	X	
1.3	Midpoints and Bisectors	G.CO.12 G.GPE.6 G.MG.1	2	This lesson uses the coordinate plane to derive the Midpoint Formula. Students will use construction tools and patty paper to bisect a line segment and locate the midpoint.	X		X	X	

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
1.4	Translating and Constructing Angles and Angle Bisectors	G.CO.1 G.CO.2 G.CO.4 G.CO.5 G.CO.6 G.CO.12	1	This lesson connects the concept that translations of a line segment preserves the length of a line segment to translations of an angle preserves the measure of the angle. Students will use construction tools to copy an angle and bisect an angle.	X	X	X		
1.5	Constructing Perpendicular Lines, Parallel Lines, and Polygons	G.CO.12	1	This lesson provides opportunities for students to use construction tools to construct parallel lines, perpendicular lines, an equilateral triangle, an isosceles triangle, a square, and a rectangle.	X	X	X	X	
1.6	Points of Concurrency	G.CO.12 G.MG.3	3	The lesson explores the circumcenter, incenter, centroid, and orthocenter for acute, obtuse, and right angles. Students will use algebra to calculate points of concurrency for a triangle in the coordinate plane.	X		X	X	

Skills Practice Correlation for Chapter 1

Lesson		Problem Set	Objectives
1.1	Points, Lines, Planes, Rays, and Line Segments		Vocabulary
		1 – 4	Identify points, lines, and planes in figures
		5 – 8	Draw geometric figures given descriptions
		9 – 12	Identify examples of coplanar lines
		13 – 16	Identify skew lines in figures
		17 – 22	Draw geometric figures given symbols
		23 – 28	Use symbols to write names of geometric figures
		29 – 32	Measure line segments and express measurements using symbols
1.2	Translating and Constructing Line Segments		Vocabulary
		1 – 6	Calculate distances between points given coordinates
		7 – 12	Calculate distances between points on the coordinate plane
		13 – 18	Translate line segments on the coordinate plane
		19 – 24	Construct line segments
1.3	Midpoints and Bisectors		Vocabulary
		1 – 6	Determine midpoints of line segments given coordinates of endpoints
		7 – 12	Determine midpoints of line segments using the Midpoint Formula
		13 – 18	Locate midpoints of line segments using construction
1.4	Translating and Constructing Angles and Angle Bisectors		Vocabulary
		1 – 6	Translate angles on the coordinate plane
		7 – 12	Construct angles
		13 – 18	Construct angle bisectors

Lesson		Problem Set	Objectives
1.5	Constructing Perpendicular Lines, Parallel Lines, and Polygons	1 – 6	Construct lines perpendicular to given lines through given points
		7 – 12	Construct lines parallel to given lines through given points
		13 – 20	Construct polygons
1.6	Points of Concurrency		Vocabulary
		1 – 8	Draw the incenters of triangles
		9 – 16	Draw the circumcenters of triangles
		17 – 24	Draw the centroids of triangles
		25 – 32	Draw the orthocenters of triangles
		33 – 38	Answer questions about points of concurrency
		39 – 44	Classify triangles given the coordinates of the vertices

1.1

Let's Get This Started!

Points, Lines, Planes, Rays, and Line Segments

LEARNING GOALS

In this lesson, you will:

- Identify and name points, lines, planes, rays, and line segments.
- Use symbolic notation to describe points, lines, planes, rays, and line segments.
- Describe possible intersections of lines and planes.
- Identify construction tools.
- Distinguish between a sketch, a drawing, and a construction.

KEY TERMS

- point
- line
- collinear points
- plane
- compass
- straightedge
- sketch
- draw
- construct
- coplanar lines
- skew lines
- ray
- endpoint of a ray
- line segment
- endpoints of a line segment
- congruent line segments

ESSENTIAL IDEAS

- The three essential building blocks (undefined terms) of geometry are point, line and plane.
- A point is described as a location. It is something thought of as having a definite position in space but no size or shape.
- A line is described as a straight continuous arrangement of an infinite number of points. It is something thought of as having infinite length but no width.
- A plane is described as a flat surface. It is something thought of as having infinite length and width but no depth.
- A compass and a straightedge are basic tools used to create geometric constructions.
- Coplanar lines are lines located on the same plane and skew lines are lines that are not on the same plane.
- Congruent line segments are segments of equal measure.
- When "sketching" a geometric figure the results are not precise and it is done with a free-hand.
- When "drawing" a geometric figure the results are precise and it is done using measuring tools such as protractors and rulers.
- When "constructing" a geometric figure the results are exact and it is done using a compass and straightedge.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Experiment with transformations in the plane

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

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Overview

The undefined terms, point, line, and plane are described. The terms collinear points, compass, straightedge, coplanar lines, skew lines, ray, endpoint of a ray, line segment, endpoints of a line segment, and congruent line segments are introduced and defined. Symbols for the various geometric figures are used and students answer questions related to the geometric representations of each new term. The terms sketch, draw, and construct are distinguished. Students will need to use a compass and ruler or straightedge to complete several questions.

Warm Up

1. Which point is larger, point *A* or point *B*?

A point is a location, so both point *A* and point *B* are simply locations; they have no size. The dot representing point *A* appears to be larger than the dot representing point *B*, but the points themselves have no size are represent two different locations.

2. Which line is longer, line *AB* or line *CD*?

All lines have infinite length. The representation of line *AB* appears to be shorter than the representation of line *CD*, but the lines themselves as indicated by the arrowheads are infinite in length.

3. What are the similarities and differences between the two geometric figures shown.

The first figure is a line segment that begins at point *J*, includes all of the collinear points between point *J* and point *K* and ends at point *K*. The second figure is a line that travels through point *J* and point *K* and contains an infinite number of points collinear with point *J* and point *K*. Both geometric figures contain an infinite number of points.

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1.1

Let's Get This Started!

Points, Lines, Planes, Rays, and Line Segments

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- Use symbolic notation to describe points, lines, planes, rays, and line segments.
- Describe possible intersections of lines and planes.
- Identify construction tools.
- Distinguish between a sketch, a drawing, and a construction.

KEY TERMS

- point
- line
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- congruent line segments

Do you have techno-joy or techno-fear? For people who have techno-fear, learning about a new technology—whether it's a smart phone, new computer, or new TV—can be a nervous endeavor. For people in the techno-joy category, it's, "get out of the way, I can handle this new device! I don't need to read the directions!"

Technical writers are the kind of people who bridge the gap between techno-joy and techno-fear. Technical writers are people who write and edit the manuals for all kinds of devices—from cars to airplanes, from electronic tablets to blenders, from industrial ventilation fans to electronic medical devices, from vacuums to whatever device you can think of! Most technical writers take the technological language and specifications and convert them into language that is more comprehensible to the average user. Some technical writers will also translate manuals into different languages.

Have you ever had trouble building something by reading the instructions? It's a tough job to write instructions, especially when you don't really know the audience.

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Problem 1

The three essential building blocks of geometry, the point, the line and the plane, are introduced as undefined terms. Each undefined term is described and students are asked to respond to a series of questions related to these terms. Symbols and naming conventions are introduced for each undefined term. Collinear and non-collinear points are also defined.

Grouping

- Ask a student to read the paragraph written at the beginning of Problem 1. Discuss the context.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Discuss Phase

- Does a point have depth, width, or length?
- Are some points larger or smaller than other points?
- Does a line have depth, width, or length?
- Are some lines longer or shorter than other lines?

Guiding Questions for Share Phase, Questions 1 through 4

- How many different lines are in this diagram?
- Line s can be named how many different ways?

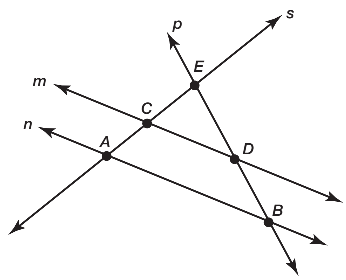
PROBLEM 1 Point, Line, Plane

There are three essential building blocks of geometry—the point, the line, and the plane. These three terms are called undefined terms; we can only describe and create mathematical models to represent them.

A **point** is described simply as a location. A point in geometry has no size or shape, but it is often represented using a dot. In a diagram, a point can be labeled using a capital letter.

A **line** is described as a straight, continuous arrangement of an infinite number of points. A line has an infinite length, but no width. Arrowheads are used to indicate that a line extends infinitely in opposite directions. In a diagram, a line can be labeled with a lowercase letter positioned next to the arrowhead.

A mathematical model of several points and lines is shown.



1. Does the name "line C" describe a unique line? Explain why or why not.
   No. The name "line C" does not describe a unique line because there is more than one line passing through point C.

2. Does the name "line CD" describe a unique line? Explain why or why not.
   Yes. The name "line CD" does describe a unique line because there is only one line passing through both point C and point D.

3. Does the name "line m" describe a unique line? Explain why or why not.
   Yes. The name "line m" does describe a unique line because there is only one line with that label.

4. How many points are needed to name a specific line?
   Two points are needed to name a line. A single italic letter can also be used to name a line.

5. What is another name for line AB?
   Another name for line AB is line BA or line n.

Line AB can be written using symbols as $\overleftrightarrow{AB}$ and is read as "line AB."

- Line p can be named how many different ways?
- Line m can be named how many different ways?
- Line n can be named how many different ways?
- Are there points on line s located between points A and C?
- How many points are on line p located between points B and D?

Grouping

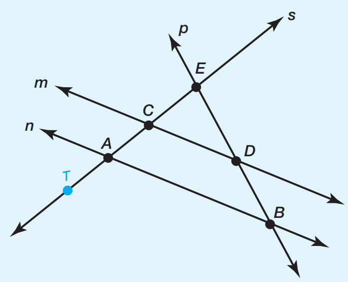
Have students complete Questions 5 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 8

- Is it possible to draw additional different lines in Brad's sketch through point *T*?
- How many additional lines can be drawn in Brad's sketch through point *T*?
- Is it possible to draw additional different lines through points *C* and *D*?
- How many additional lines can be drawn in Kara's sketch through points *C* and *D*?
- Are any two points collinear?
- Is it possible to draw two points that are not collinear?

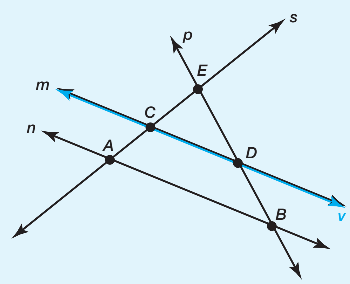
6. Analyze each model and explanation.

Brad



I drew point T on line s. Since only one line goes through T, point T describes just one line.

Kara



I drew line v through points C and D. So, line CD describes two lines.

Describe the inaccuracy in each students' reasoning.

It is not true that only one line goes through point *T* in Brad's diagram. Even though only one line is shown passing through point *T*, many different lines can be drawn through that point.

Even though Kara can draw and label line *v* through points *C* and *D*, line *v* is the same line as line *m*.

7. How many lines can be drawn through a single point?

An infinite number of lines can be drawn through a single point.

Collinear points are points that are located on the same line.

8. Use the diagram shown prior to Question 1.

a. Name three points that are collinear.

Answers will vary.

- Points *A*, *C*, and *E* are collinear.
- Points *E*, *D*, and *B* are collinear.

b. Name three points that are not collinear.

Answers will vary.

- Points *E*, *C*, and *D* are not collinear.
- Points *B*, *T*, and *A* are not collinear.
- Points *D*, *A*, and *C* are not collinear.

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Grouping

Have students complete Questions 9 through 11 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 9 through 11

- What is another model that could be used to represent a plane?
- Do planes have edges or boundaries?
- Are some planes larger or smaller than other planes?
- Is it possible for three planes to intersect at a single point? Why or why not?
- Is it possible for a line to intersect a plane at two distinct points? Why or why not?

Note

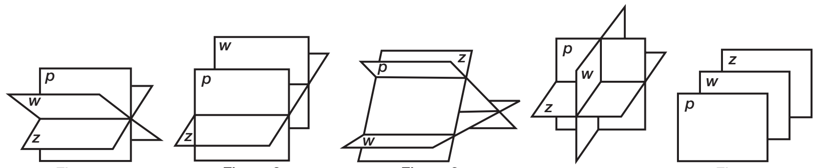
Each student will need access to a compass, and a straightedge or ruler for this lesson.

Misconceptions

Quite often students think of a point as a physical dot on a graph or in space rather than a geometric representation of a specified location. It is important to help students distinguish the difference between geometric representations of the undefined terms and the actual meaning of the undefined terms.

A **plane** is described as a flat surface. A plane has an infinite length and width, but no depth, and extends infinitely in all directions. One real-world model of a plane is the surface of a still body of water. Three non-collinear points describe a unique plane, but planes are usually named using one italic letter located near a corner of the plane as drawn.

Three planes can intersect in a variety of ways or may not intersect at all.



9. Describe the intersection of planes p, w, and z in each figure.

a. Figure 1

The intersection of planes p, w, and z is a single line.

b. Figure 2

Planes p and w do not intersect.

The intersection of planes p and z is a single line.

The intersection of planes w and z is a single line.

c. Figure 3

The intersection of planes w and z is a single line.

The intersection of planes w and p is a single line.

The intersection of planes p and z is a single line.

d. Figure 4

The intersection of planes w and z is a single line.

The intersection of planes w and p is a single line.

The intersection of planes p and z is a single line.

The intersection of planes p, w, and z is a single point.

e. Figure 5

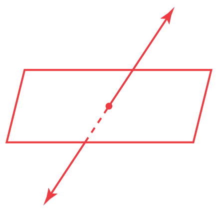
Planes p, w, and z do not intersect.

10. List all of the possible ways that three planes can intersect.

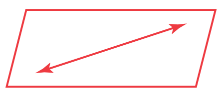
- The three planes do not intersect.
- The intersection of the three planes is a single line.
- Two planes do not intersect each other, but both intersect the third plane at different lines.
- Each plane intersects the other two planes at different lines.
- The three planes intersect at a single point.

11. Sketch and describe all possible ways that a line and a plane can intersect.

- The intersection of the line and the plane is a single point.



- The intersection of the line and the plane is the infinite number of points on the line itself.



- The line does not intersect the plane.

Problem 2

Students review the definitions of compass, and straightedge, and distinguish between the terms sketch, draw, and construct. Students are also given definitions for coplanar lines and skew lines.

Grouping

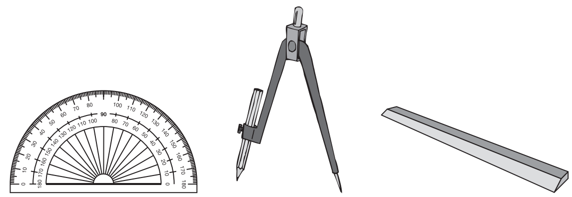
- Ask a student to read the definitions of compass, and straightedge. Discuss as a class.
- Have students review the definitions of sketch, draw, construct, coplanar lines, and skew lines as they are completing Questions 1 through 3 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What is the difference between a drawing and a sketch?
- What is the difference between a drawing and a construction?
- Are any two lines considered coplanar lines?
- Are any two intersecting lines considered coplanar lines?
- Are any two non-intersecting lines considered coplanar lines?
- Are any two lines considered skew lines?

PROBLEM 2 Creating Geometric Figures

You can use many tools to create geometric figures. Some tools, such as a ruler or a protractor, are classified as measuring tools. A **compass** is a tool used to create arcs and circles. A **straightedge** is a ruler with no numbers. It is important to know when to use each tool.



- When you **sketch** a geometric figure, the figure is created without the use of tools.
- When you **draw** a geometric figure, the figure is created with the use of tools such as a ruler, straightedge, compass, or protractor. A drawing is more accurate than a sketch.
- When you **construct** a geometric figure, the figure is created using only a compass and a straightedge.

1. Sketch and then draw each figure. Describe the steps that you performed to complete your sketch and your drawing.

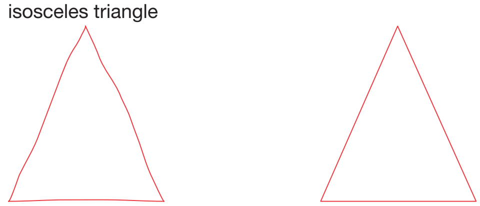
   a. square

   To complete the sketch, I drew what looked like 4 right angles and 4 equal sides using just a pencil.

   To complete the drawing, I used a ruler to draw a side that was 4 centimeters. Then I used a protractor to draw a right angle on each side of the segment. I then drew 2 more sides that were also 4 centimeters. Then I connected the points to form the last side.

   b. isosceles triangle

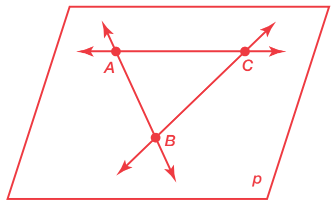
   To complete the sketch, I drew what looked like 2 equal sides using just a pencil.

   To complete the drawing, I used a ruler to draw 2 sides that were each 4 centimeters. Then I connected the points to form the last side.

- Are any two intersecting lines considered skew lines?
- Are any two non-intersecting lines considered skew lines?

Coplanar lines are two or more lines that are located in the same plane. **Skew lines** are two or more lines that do not intersect and are not parallel. Skew lines do not lie in the same plane.

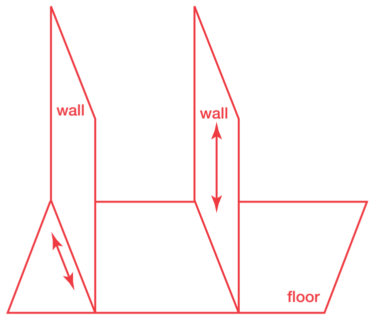
2. Draw and label three coplanar lines.



3. Look around your classroom. Describe the location of two skew lines.

Answers will vary.

One line would be a horizontal line on one wall and the other line would be a vertical line on the opposite wall. These two lines are skew lines.



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Problem 3

Students use the undefined terms to create new terms such as ray and line segment. Endpoints are defined related to both line segments and rays. Symbols and naming conventions are introduced for each term. Congruent line segments are defined as segments of equal measure. Several questions stress appropriateness of when to use an equal sign (=) versus when to use a congruent sign (≅).

Grouping

Have students review the definitions of ray, endpoint of a ray, line segment, and endpoints of a line segment as they are completing Questions 1 through 10 with a partner. Then share the responses as a class.

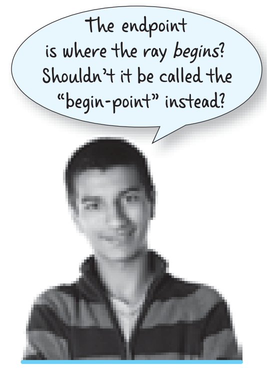
Guiding Questions for Share Phase, Questions 1 through 10

- Which point is the endpoint of ray *AB*?
- Which point is the endpoint of ray *BA*?
- What is the name of the ray that begins at point *A* and travels through point *B*?
- What is the name of the ray that begins at point *B* and travels through point *A*?
- What does ray *AB* have in common with ray *BA*?
- What is the difference between a line and a line segment?

PROBLEM 3 Using Undefined Terms to Define New Terms

A **ray** is a part of a line that begins with a single point and extends infinitely in one direction. The **endpoint of a ray** is the single point where the ray begins.

A ray is named using two capital letters, the first representing the endpoint and the second representing any other point on the ray. Ray *AB* can be written using symbols as $\overrightarrow{AB}$, which is read as "ray *AB*."



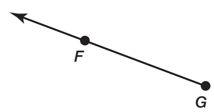
1. Sketch and label $\overrightarrow{AB}$.

2. Sketch and label $\overrightarrow{BA}$.

3. Are $\overrightarrow{AB}$ and $\overrightarrow{BA}$ names for the same ray? Explain why or why not.

   No. Ray *AB* begins at point *A* and travels through point *B*. Ray *BA* begins at point *B* and travels through point *A*.

4. Use symbols to name the geometric figure shown.



   $\overrightarrow{GF}$

- How many line segments are on a line?
- How are rays different than line segments?
- What is the difference between the distance from point *A* to point *B*, and the measure of line segment *AB*?
- How many rays are on line segment *FG*?
- How many rays are on line *FG*?

A **line segment** is a part of a line that includes two points and all of the collinear points between the two points. The **endpoints of a line segment** are the points where the line segment begins and ends.

A line segment is named using two capital letters representing the two endpoints of the line segment. Line segment *AB* can be written using symbols as $\overline{AB}$, which is read as "line segment *AB*."

5. Draw and label $\overline{AB}$.

6. Draw and label $\overline{BA}$.

7. Are $\overline{AB}$ and $\overline{BA}$ names for the same line segment? Explain why or why not.

Yes. They are names for the same line segment. A line segment that begins at point *A* and ends at point *B* and a line segment that begins at point *B* and ends at point *A* are the same line segment.

8. Use a ruler to measure $\overline{AB}$ in Question 5.

Answers will vary.

Line segment *AB* is 2.5 inches.

9. The measure of $\overline{AB}$ can be expressed in two different ways. Complete each statement:

a. "$AB =$ ___2.5___ inches" is read as "the distance from point *A* to point *B* is equal to ___2.5___ inches."

b. "$m\overline{AB} =$ ___2.5___ inches" is read as "the measure of line segment *AB* is equal to ___2.5___ inches."

c. How do you read "$m\overline{CF} = 3$ inches"?

The statement "$m\overline{CF} = 3$ inches" is read as "the measure of line segment *CF* is equal to three inches."

d. How do you read "$SP = 8$ inches"?

The statement "$SP = 8$ inches" is read as "the distance from point *S* to point *P* is equal to eight inches."

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10. Use symbols to name each geometric figure.

a. 

$\overline{FG}$ or $\overline{GF}$

b. 

$\overleftrightarrow{FG}$ or $\overleftrightarrow{GF}$

c. 

$\overrightarrow{FG}$

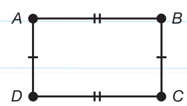
If two line segments have equal measure, then the line segments have the same length. **Congruent line segments** are two or more line segments of equal measure.

If $m\overline{AB} = m\overline{CD}$, then line segment AB is congruent to line segment CD by the definition of congruent line segments. This statement can be written using symbols as $\overline{AB} \cong \overline{CD}$ and is read as "line segment AB is congruent to line segment CD."

Use the congruence symbol, $\cong$, between references to congruent geometric figures; and the equal symbol, $=$, between references to equal lengths or distances.

Markers are used to indicate congruent segments in geometric figures. If a diagram has more than one set of congruent segments then sets of markers can be used.

The figure shows $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$.



11. Draw and label two congruent line segments. Then, use symbols to write a statement that describes their relationship.

$\overline{AB} \cong \overline{CD}$

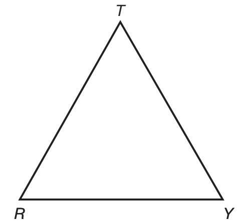
Grouping

Have students complete Questions 11 through 15 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 11 through 15

- What markers are used to denote congruent line segments?
- Are all lines congruent?
- Are all rays congruent?
- What is the definition of an equilateral triangle?
- What is the definition of an isosceles triangle?
- How is an isosceles triangle different than an equilateral triangle?
- How is an isosceles triangle similar to an equilateral triangle?
- What is the difference between a congruency statement and an equality statement?

12. Ms. Snyder drew the triangle shown and asked her students to classify it.



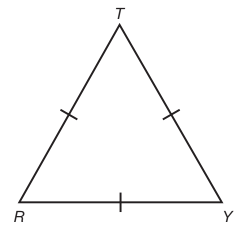
a. Mariah says the triangle is an equilateral triangle.

Is she correct?

Mariah is incorrect.

The triangle shown has no markers indicating sides of equal length. She assumed sides of equal length because they appeared equal in length. Assumptions are not valid reasons.

b. Ms. Snyder then drew markers and asked her students to classify the triangle.



Mariah says the triangle is an isosceles triangle. Justin says the triangle is an equilateral triangle.

Who is correct?

Both Mariah and Justin are correct.

The triangle shown has 3 sides of equal length. An isosceles triangle is defined as a triangle with at least 2 sides of equal length and an equilateral triangle is defined as a triangle with 3 sides of equal length.

c. Ms. Snyder also asked her students to write a statement that best describes the congruency of the line segments forming the triangle.

Mariah	Justin
$TR = RY = YR$	$\overline{TR} \cong \overline{RY} \cong \overline{YR}$

Who is correct?

Justin correctly wrote a congruency statement.

Mariah's statement is true, but it is not a congruency statement, it is an equality statement.

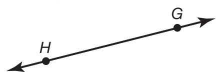
13. Use symbols to write three valid conclusions based on the figure shown. How do you read each conclusion?

$\overline{FG} \cong \overline{HI}$
Segment *FG* is congruent to segment *HI*.

$m\overline{FG} = m\overline{HI}$
The measure of segment *FG* is equal to the measure of segment *HI*.

$FG = HI$
The distance from point *F* to point *G* is equal to the distance from point *H* to point *I*.

14. Use symbols to name all lines, rays, or segments shown.



One line is shown, named $\overleftrightarrow{HG}$ *or* $\overleftrightarrow{GH}$.

There are an infinite number of line segments that can be defined. However only one line segment can be named using the points shown, $\overline{HG}$ *or* $\overline{GH}$.

There are an infinite number of rays that can be defined. However only two rays can be named using the points shown, $\overrightarrow{HG}$ and $\overrightarrow{GH}$.

15. Explain when it is appropriate to use the statement $JK = MN$ and when it is appropriate to use the statement $\overline{JK} \cong \overline{MN}$.
It is appropriate to use the statement $JK = MN$ when you want to state the distance from point *J* to point *K* is equal to the distance from point *M* to point *N*. It is appropriate to state that line segment *JK* is congruent to line segment *MN* when the line segments are drawn equal in length.

Talk the Talk

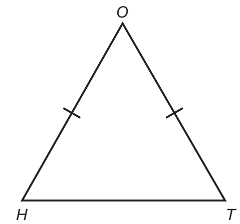
Students are given a diagram and several equality and congruency statements using various symbols. They are asked to determine which statements are correctly written.

Grouping

Have students complete the Question with a partner. Then have students share their responses as a class.

1

Talk the Talk



Circle each statement that is valid about triangle HOT.

$\overline{HO} = \overline{TO}$	$m\overline{HO} = m\overline{TO}$
$\overline{HO} \cong \overline{TO}$	$m\overline{HO} \cong m\overline{TO}$
$HO = TO$	$mHO = mTO$
$HO \cong TO$	$mHO \cong mTO$

Be prepared to share your solutions and methods.

Check for Students' Understanding

Sketch a geometric representation for each of the following.

1. $\overrightarrow{HP}$

2. $\overleftrightarrow{MW}$

3. $\overline{TX}$

4. Plane q

1.2

Attack of the Clones

Translating and Constructing Line Segments

LEARNING GOALS

In this lesson, you will:

- Determine the distance between two points.
- Use the Pythagorean Theorem to derive the Distance Formula.
- Apply the Distance Formula on the coordinate plane.
- Translate a line segment on the coordinate plane.
- Copy or duplicate a line segment by construction.

KEY TERMS

- Distance Formula
- transformation
- rigid motion
- translation
- pre-image
- image
- arc

CONSTRUCTIONS

- copying a line segment
- duplicating a line segment

ESSENTIAL IDEAS

- The Pythagorean Theorem is used to determine the distance between two points in a coordinate plane.
- Translations are described as functions that take points on the plane as inputs and give other points as outputs through the use of tables.
- Translations preserve the distance between two points.
- Translations preserve the length of a line segment.
- Construction tools can be used to duplicate a line segment.
- Radii of congruent circles are congruent.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Experiment with transformations in the plane

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not.

4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

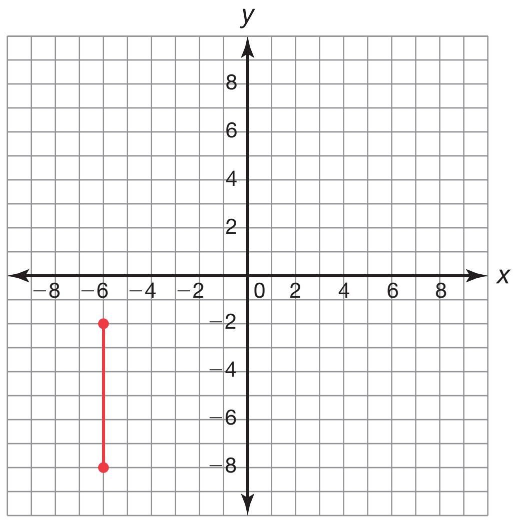
Overview

Students will apply the Pythagorean Theorem to a situation on a coordinate plane. They calculate various distances using coordinates of points aligned either horizontally or vertically using subtraction and diagonal distances using the Pythagorean Theorem. Using variables on the coordinate plane, students derive the Distance Formula. Students are given a problem in which two points are plotted on a grid and they are asked to use the Pythagorean Theorem to determine the distance between the two given points. Students then translate a line segment onto itself on a coordinate plane through a series of four translations to conclude that the pre-image and all other images remain congruent.

Using construction tools, students are guided through the steps necessary to construct circles with congruent radii, a regular hexagon, and duplicating a line segment. Other constructions will help students realize radii of congruent circles are congruent and an arc or the entire circle can be used in the process of duplicating a line segment.

Warm Up

Use the coordinate plane provided to answer each question.



1. Plot points A $(-6, -2)$ and B $(-6, -8)$.

   See coordinate plane.

2. Is the distance between points A and B considered a horizontal distance, a vertical distance, or a diagonal distance? Explain your reasoning.

   The distance between points A and B is a vertical distance because the points are directly above or below each other.

3. How do you calculate the distance between points A and B?

   To calculate the distance between points A and B, subtract 2 from 8.

4. What is the distance between points A and B?

   The distance between points A and B is equal to 6 units.

5. How do the negative coordinates affect the distance between points A and B?

   The negative coordinates have no affect on the distance between the two points. Distance is always a positive value.

1

1.2

Attack of the Clones

Translating and Constructing Line Segments

LEARNING GOALS

In this lesson, you will:

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- Apply the Distance Formula on the coordinate plane.
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- Copy or duplicate a line segment by construction.

KEY TERMS

- Distance Formula
- transformation
- rigid motion
- translation
- pre-image
- image
- arc

CONSTRUCTIONS

- copying a line segment
- duplicating a line segment

Are you better at geometry or algebra? Many students have a preference for one subject or the other; however, geometry and algebra are very closely related. While there are some branches of geometry that do not use much algebra, analytic geometry applies methods of algebra to geometric questions. Analytic geometry is also known as the study of geometry using a coordinate system. So anytime you are performing geometric calculations and it involves a coordinate system, you are studying analytic geometry. Be sure to thank Descartes and his discovery of the coordinate plane for this!

What might be the pros and cons of analytic geometry compared to other branches of geometry? Does knowing about analytic geometry change how you feel about your own abilities in geometry or algebra?

1

Problem 1

The locations of friends' houses are described using street names in a neighborhood laid out on a square grid. Students will determine the location of each on the coordinate plane and calculate distances between locations using either subtraction (for vertical or horizontal distances) or the Pythagorean Theorem (for diagonal distances).

Grouping

Have students complete Questions 1 through 10 with a partner. Then have students share their responses as a class.

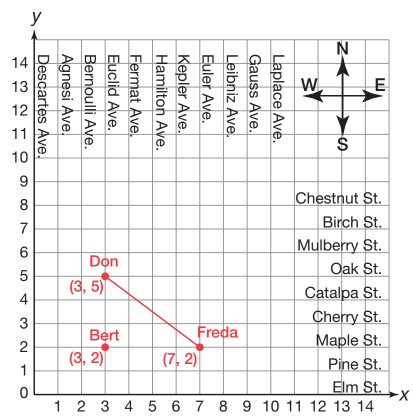
Guiding Questions for Share Phase, Questions 1 through 10

- What are the coordinates of Don's house?
- What are the coordinates of Freda's house?
- What are the coordinates of Bert's house?
- Whose houses are horizontally aligned?
- Whose houses are vertically aligned?
- Whose houses are diagonally aligned?
- What expression is used to represent the distance between Don's house and Bert's house?
- What expression is used to represent the distance between Bert's house and Freda's house?

PROBLEM 1 Where Do You Live?

Don, Freda, and Bert live in a town where the streets are laid out in a grid system.

1. Don lives 3 blocks east of Descartes Avenue and 5 blocks north of Elm Street. Freda lives 7 blocks east of Descartes Avenue and 2 blocks north of Elm Street. Plot points to show the locations of Don's house and Freda's house on the coordinate plane. Label each location with the student's name and the coordinates of the point.



a. Name the intersection of streets that Don lives on.

Don lives at the intersection of Euclid Avenue and Oak Street.

b. Name the intersection of streets that Freda lives on.

Freda lives at the intersection of Euler Avenue and Maple Street.

2. Bert lives at the intersection of the avenue that Don lives on, and the street that Freda lives on. Plot and label the location of Bert's house on the coordinate plane. Describe the location of Bert's house with respect to Descartes Avenue and Elm Street.

Bert lives 3 blocks east of Descartes Avenue and 2 blocks north of Elm Street.

3. How do the *x*- and *y*-coordinates of Bert's house compare to the *x*- and *y*-coordinates of Don's house and Freda's house?

The *x*-coordinate of Bert's house is the same as the *x*-coordinate of Don's house.
The *y*-coordinate of Bert's house is the same as the *y*-coordinate of Freda's house.

- What expression is used to represent the distance between Freda's house to Bert's house and the distance from Bert's house to Don's house?
- If Freda walks to Don's house, is she walking a horizontal, vertical, or diagonal distance?
- Do the distances between Freda's house, Bert's house, and Don's house form a right triangle? How do you know?
- The distance between which two houses is represented by the hypotenuse?

4. Use Don's and Bert's house coordinates to write and simplify an expression that represents the distance between their houses. Explain what this means in terms of the problem situation.

$5 - 2 = 3$

Don lives 3 blocks north of Bert.

5. Use Bert's and Freda's house coordinates to write and simplify an expression that represents the distance between their houses. Explain what this means in terms of the problem situation.

$7 - 3 = 4$

Freda lives 4 blocks east of Bert.

6. All three friends are planning to meet at Don's house to hang out. Freda walks to Bert's house, and then Freda and Bert walk together to Don's house.

a. Use the coordinates to write and simplify an expression that represents the total distance from Freda's house to Bert's house to Don's house.

$$(7 - 3) + (5 - 2) = 4 + 3$$
$$= 7$$

b. How far, in blocks, does Freda walk altogether?

Freda walks 7 blocks.

7. Draw the direct path from Don's house to Freda's house on the coordinate plane. If Freda walks to Don's house on this path, how far, in blocks, does she walk? Explain how you determined your answer.

Let d be the direct distance between Don's house and Freda's house.

$d^2 = 3^2 + 4^2$

$d^2 = 9 + 16$

$d^2 = 25$

$d = 5$

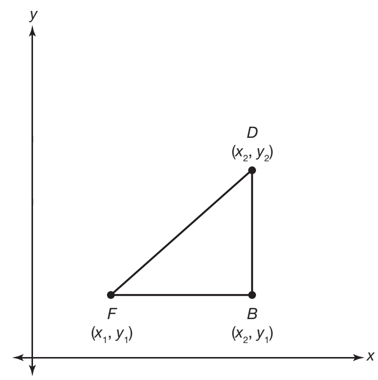
Freda walks 5 blocks to reach Don's house.

I used the Pythagorean Theorem to determine this distance.

8. Complete the summary of the steps that you took to determine the direct distance between Freda's house and Don's house. Let d be the direct distance between Don's house and Freda's house.

Distance between Bert's house and Freda's house		Distance between Don's house and Bert's house		Direct distance between Don's house and Freda's house
$(\boxed{7} - \boxed{3})^2$	+	$(\boxed{5} - \boxed{2})^2$	=	$\boxed{d}^2$
$\boxed{4}^2$	+	$\boxed{3}^2$	=	$\boxed{d}^2$
$\boxed{16}$	+	$\boxed{9}$	=	$\boxed{d}^2$
		$\boxed{25}$	=	$\boxed{d}^2$
		$\boxed{5}$	=	$\boxed{d}$

Suppose Freda's, Bert's, and Don's houses were at different locations but oriented in a similar manner. You can generalize their locations by using x_1, x_2, y_1, and y_2 and still solve for the distances between their houses using variables. Let point F represent Freda's house, point B represent Bert's house, and point D represent Don's house.



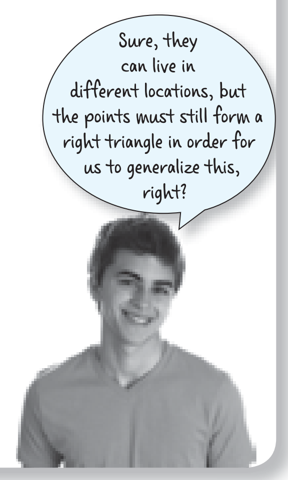
9. Use the graph to write an expression for each distance.

a. Don's house to Bert's house (DB)

$DB = y_2 - y_1$

b. Bert's house to Freda's house (BF)

$BF = x_2 - x_1$



10. Use the Pythagorean Theorem to determine the distance from Don's house to Freda's house (*DF*).

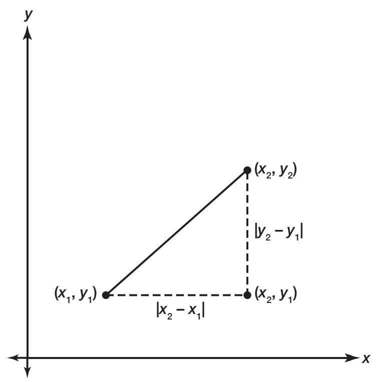
$DF^2 = BF^2 + DB^2$

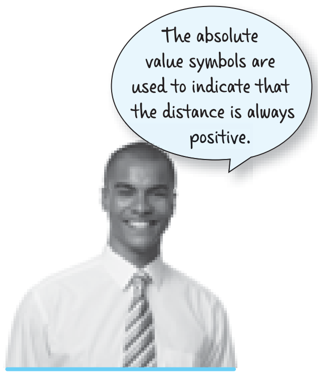
$DF^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$DF = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

You used the Pythagorean Theorem to calculate the distance between two points on the coordinate plane. Your method can be written as the *Distance Formula.*

The **Distance Formula** states that if (x_1, y_1) and (x_2, y_2) are two points on the coordinate plane, then the distance d between (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.





11. Do you think that it matters which point you identify as (x_1, y_1) and which point you identify as (x_2, y_2) when you use the Distance Formula? Use an example to justify your answer.

No. Once I square the differences, the values are the same so the order does not matter.

For example, $(3 - 5)^2 = (-2)^2 = 4$ and $(5 - 3)^2 = 2^2 = 4$.

1

Grouping

Have students complete Questions 12 through 14 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 12 through 14

- The value of the square root of 29 is between what two whole numbers?
- Is the square root of 29 closer to 5 or closer to 6?
- The value of the square root of 208 is between what two whole numbers?
- Is the square root of 208 closer to 14 or closer to 15?
- The value of the square root of 65 is between what two whole numbers?
- Is the square root of 65 closer to 8 or closer to 9?
- How is Question 13 similar to Question 12 part (a)?
- How would you describe this translation?
- In Question 14, will more than one value satisfy the Distance Formula?
- Why do two different values satisfy the Distance Formula?

Note

Encourage students to approximate the value of a square root without using a calculator by first identifying what two whole numbers are closest to the value of the square root, then estimating which whole number the square root is closer to by choosing a value for the tenth position. Also, students often overlook the possibility of a second solution when working with square roots. You may need to mention a possible negative and positive solution in Question 14.

12. Calculate the distance between each pair of points. Round your answer to the nearest tenth if necessary. Show all your work.

a. (1, 2) and (3, 7)

$x_1 = 1, y_1 = 2, x_2 = 3, y_2 = 7$

$d = \sqrt{(3 - 1)^2 + (7 - 2)^2}$

$= \sqrt{2^2 + 5^2}$

$= \sqrt{4 + 25}$

$= \sqrt{29}$

≈ 5.4

The distance is about 5.4 units.

b. (−6, 4) and (2, −8)

$x_1 = -6, y_1 = 4, x_2 = 2, y_2 = -8$

$d = \sqrt{[2 - (-6)]^2 + (-8 - 4)^2}$

$= \sqrt{8^2 + (-12)^2}$

$= \sqrt{64 + 144}$

$= \sqrt{208}$

≈ 14.4

The distance is about 14.4 units.

c. (−5, 2) and (−6, 10)

$x_1 = -5, y_1 = 2, x_2 = -6, y_2 = 10$

$d = \sqrt{[-6 - (-5)]^2 + (10 - 2)^2}$

$= \sqrt{(-1)^2 + 8^2}$

$= \sqrt{1 + 64}$

$= \sqrt{65}$

≈ 8.1

The distance is about 8.1 units.

d. (−1, −3) and (−5, −2)

$x_1 = -1, y_1 = -3, x_2 = -5, y_2 = -2$

$d = \sqrt{[-1 - (-5)]^2 + [-3 - (-2)]^2}$

$= \sqrt{[4^2 + (-1)^2]}$

$= \sqrt{16 + 1}$

$= \sqrt{17}$

≈ 4.1

The distance is about 4.1 units.

13. Carlos and Mandy just completed Question 12 parts (a) through (c). Now, they need to calculate the distance between the points (−4, 2) and (−2, 7). They notice the similarity between this problem and part (a).

Mandy

$d = \sqrt{(-4 - -2)^2 + (2-7)^2}$

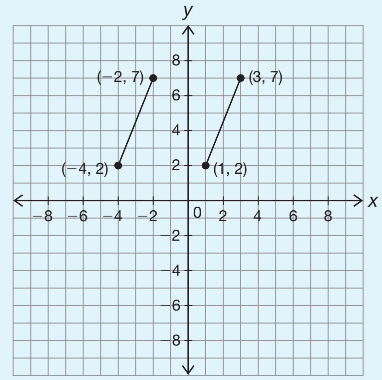
$d = \sqrt{(-2)^2 + (-5)^2}$

$d = \sqrt{4+25}$

$d = \sqrt{29}$

$d \approx 5.4$

Carlos



$(1, 2) \rightarrow (-4, 2)$ The point moved 5 units to the left

$(3, 7) \rightarrow (-2, 7)$ The point moved 5 units to the left

Since both points moved 5 units to the left, this did not alter the distance between the points, so the distance between points (−4, 2) and (−2, 7) is approximately 5.4.

Who used correct reasoning?

Both Mandy and Carlos used correct reasoning. Mandy's work shows an algebraic solution whereas Carlos's work shows geometric reasoning.

14. The distance between (x, 2) and (0, 6) is 5 units. Use the Distance Formula to determine the value of x. Show all your work.

$x_1 = x, y_1 = 2, x_2 = 0, y_2 = 6, d = 5$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$5 = \sqrt{(0 - x)^2 + (6 - 2)^2}$

$5 = \sqrt{-x^2 + 4^2}$

$5^2 = (\sqrt{-x^2 + 16})^2$

$25 = x^2 + 16$

$9 = x^2$

$-3, 3 = x$

The value of x can be −3 or 3.

1

Problem 2

The locations of friends' houses are used to form a line segment on a coordinate plane. The terms transformation, rigid motion, translation, image, and pre-image are introduced. Students will translate a line segment into all four quadrants of the graph and determine the location of each on the coordinate plane. They notice patterns used when writing the coordinates of the images and pre-image and conclude that translating a line segment does not alter the length of the line segment. Translations preserve the length of a line segment.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

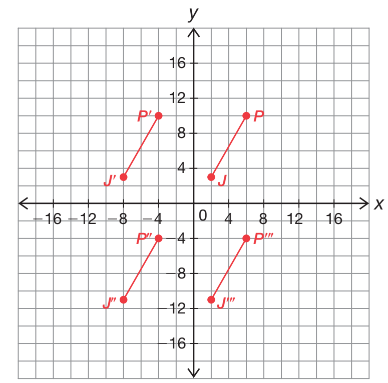
- Are Pedro's and Jethro's house positioned in a vertical or horizontal distance from each other?
- How would you best describe the slope of the line segment formed by connecting Pedro's and Jethro's house?
- The value of the square root of 65 is closest to what whole number?
- Is the the square root of 65 closer to 8 or closer to 9?

PROBLEM 2 Translating a Line Segment

1. Pedro's house is located at (6, 10). Graph this location on the coordinate plane and label the point *P*.
   See coordinate plane.

2. Jethro's house is located at (2, 3). Graph this location on the coordinate plane and label the point *J*.
   See coordinate plane.

3. Draw a line segment connecting the two houses to create line segment *PJ*.



4. Determine the length of line segment *PJ*.

$x_1 = 6; y_1 = 10; x_2 = 2; y_2 = 3$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(2 - 6)^2 + (3 - 10)^2}$

$= \sqrt{(-4)^2 + (-7)^2}$

$= \sqrt{16 + 49}$

$= \sqrt{65}$

≈ 8.1

The length of $\overline{PJ}$ is about 8.1 units.

- Which coordinates of the endpoints are the same if the line segment is described as vertical?
- Which coordinates of the endpoints are the same if the line segment is described as horizontal?

1

Grouping

Have students complete Questions 5 through 12 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 12

- If the location of Pedro's and Jethro's houses is translated to the left, is that considered a negative or positive direction? Why?
- If the location of Pedro's and Jethro's houses is translated down, is that considered a negative or positive direction? Why?
- If the location of Pedro's and Jethro's houses is translated to the right, is that considered a negative or positive direction? Why?
- If the location of Pedro's and Jethro's houses is translated up, is that considered a negative or positive direction? Why?
- It can be said that through translations line segment *PJ* maps onto itself, what does this mean?

A **transformation** is the mapping, or movement, of all the points of a figure in a plane according to a common operation.

A **rigid motion** is a transformation of points in space.

A **translation** is a rigid motion that "slides" each point of a figure the same distance and direction. Sliding a figure left or right is a horizontal translation, and sliding it up or down is a vertical translation.

The original figure is called the **pre-image**. The new figure created from the translation is called the **image**.

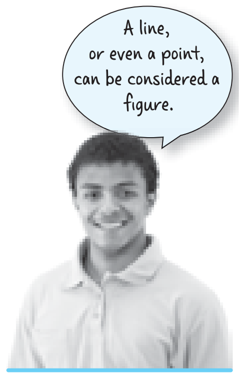
5. Line segment *PJ* is horizontally translated 10 units to the left.

a. Graph the image of pre-image $\overline{PJ}$. Label the new points P' and J'.

See graph.

b. Identify the coordinates of P' and J'.

The coordinates of P' are $(-4, 10)$, and the coordinates of J' are $(-8, 3)$.



6. Line segment $P'J'$ is vertically translated 14 units down.

a. Graph the image of pre-image $\overline{P'J'}$. Label the new points P'' and J''.

See graph.

b. Identify the coordinates of P'' and J''.

The coordinates of P'' are $(-4, -4)$, and the coordinates of J'' are $(-8, -11)$.

7. Line segment $P''J''$ is horizontally translated 10 units to the right.

a. Without graphing, predict the coordinates of P''' and J'''.

The coordinates of P''' will be $(6, -4)$ and J''' will be $(2, -11)$.

b. Graph the image of pre-image $\overline{P''J''}$. Label the new points P''' and J'''.

See graph.

8. Describe the translation necessary on $\overline{P'''J'''}$ so that it returns to the location of $\overline{PJ}$.
To return the line segment to the original location, I need to perform a vertical translation 14 units up.

9. How do the lengths of the images compare to the lengths of the pre-images? Explain how you could verify your answer.
The lengths of the images and the pre-images are the same. They are each approximately 8.1 units long.
I could use the Distance Formula to verify that each line segment is the same length.

10. Analyze the coordinates of the endpoints of each line segment.
  a. Identify the coordinates of each line segment in the table.

Line Segments	$\overline{PJ}$	$\overline{P'J'}$	$\overline{P''J''}$	$\overline{P'''J'''}$
Coordinates of Endpoints	(6, 10) (2, 3)	(−4, 10) (−8, 3)	(−4, −4) (−8, −11)	(6, −4) (2, −11)

  b. Describe how a horizontal translation changes the *x*- and *y*-coordinates of the endpoints.
  A horizontal translation changes the *x*-coordinates of the endpoints, but the *y*-coordinates remain the same.

  c. Describe how a vertical translation changes the *x*- and *y*-coordinates of the endpoints.
  A vertical translation changes the *y*-coordinate of both endpoints, but the *x*-coordinates remain the same.

11. Describe a sequence of two translations that will result in the image and the pre-image being the same.
The image and the pre-image will be the same if I translate up, down, left, or right and then translate in the opposite direction by the same number of units.

12. Describe a sequence of four translations that will result in the image and the pre-image being the same.
Answers will vary.
The image and the pre-image will be the same if I translate right by x units, down by y units, left by $-x$ units, and up by $-y$ units.

Problem 3

Students are instructed how to use a compass to draw a circle. They will construct a circle using a given point as the center and a given line segment as the radius of the circle. They then construct other radii in the same circle and conclude that all radii of the same circle are of equal length. In the next activity, the term arc is defined. Given a line segment, students will construct an arc using the line segment as a radius and one endpoint as the center of the circle. They conclude that all of the points on the arc are the same distance to the center of the circle, and all of the line segments formed by connecting a point on the arc to the center point of the circle are the same length. Students use congruent circles to create a hexagon. Students are given two congruent circles and use construction tools to duplicate congruent radii. They will conclude radii of congruent circles are congruent.

Grouping

- Ask a student to read the definitions. Discuss the context and complete Question 1 as a class.
- Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

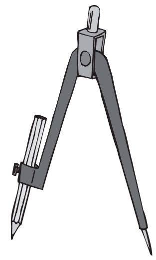
PROBLEM 3 Copying Line Segments

In the previous problem, you translated line segments on the coordinate plane. The lengths of the line segments on the coordinate plane are measurable.

In this problem, you will translate line segments when measuring is not possible. This basic geometric construction is called **copying a line segment** or **duplicating a line segment.** You will perform the construction using a compass and a straightedge.

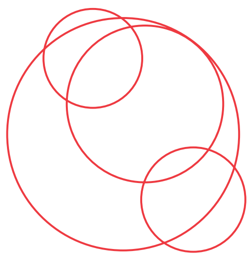
One method for copying a line segment is to use circles. But before you can get to that, let's review how to draw perfect circles with a compass.

Remember that a compass is an instrument used to draw circles and arcs. A compass can have two legs connected at one end.



One leg has a point, and the other holds a pencil. Some newer compasses may be different, but all of them are made to construct circles by placing a point firmly into the paper and then spinning the top of the compass around, with the pencil point just touching the paper.

1. Use your compass to construct a number of circles of different sizes.



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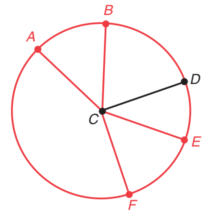
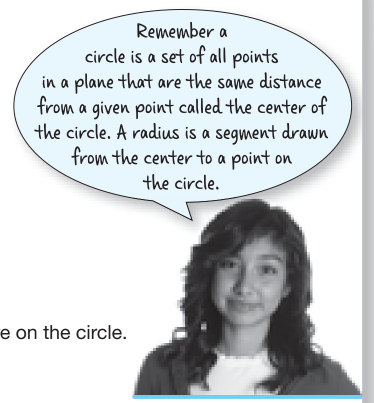
Guiding Questions for Share Phase, Questions 2 and 3

- What is a radius?
- What do you know about all radii of the same circle?
- What do you know about all radii of congruent circles?
- How is a circle related to an arc of the circle?
- How many radii are used to create an arc?
- How many arcs are in a circle?
- How many points determine an arc?
- How is the center of the circle used to create an arc?
- How are radii used to create an arc?

2. Point C is the center of a circle and $\overline{CD}$ is a radius.

a. Construct circle C.





b. Draw and label points A, B, E, and F anywhere on the circle.

c. Construct $\overline{AC}$, $\overline{BC}$, $\overline{EC}$, and $\overline{FC}$.

d. Shawna makes the following statement about radii of a circle.

> **Shawna**
>
> *All radii are the same length, because all of the points of a circle are equidistant from the circle's center.*

Explain how Shawna knows that all radii are the same length? Does this mean the line segments you constructed are also radii?

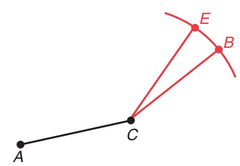
Shawna knows that the radii are the same length because the distance from the center to any point on the circle is the same distance. So, any segment drawn from the center to a point on the circle will have the same length.

Yes. The other line segments are radii of the circle because they are line segments that extend from the center of the circle to an endpoint on the circle.

An **arc** is a part of a circle. You can also think of an arc as the curve between two points on the circle.

3. Point C is the center of a circle and $\overline{AC}$ is the radius.

a. Construct an arc of circle C. Make your arc about one-half inch long. Construct the arc so that it does not pass through point A.



b. Draw and label two points B and E on the arc and construct $\overline{CE}$ and $\overline{CB}$.

c. What conclusion can you make about the constructed line segments?

The constructed segments must have the same length as $\overline{AC}$ because every point on the arc is the same distance from C as A is.

Grouping

Have students complete Question 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- What does it mean to say to line segments are congruent?
- How could you show two line segments are congruent using a compass?
- How do you know when two line segments are congruent?
- What does it mean to say two arcs are congruent?
- How many arcs intersect at center point *A*?
- How many circles did it take to form the hexagon?
- What is a regular hexagon?
- How is a regular hexagon different from other hexagons?
- Is the hexagon formed a regular hexagon? How do you know?

Recall that congruent line segments are line segments that have the same length. The radii of a circle are congruent line segments because any line segment drawn from the center to a point on the circle has the same length.

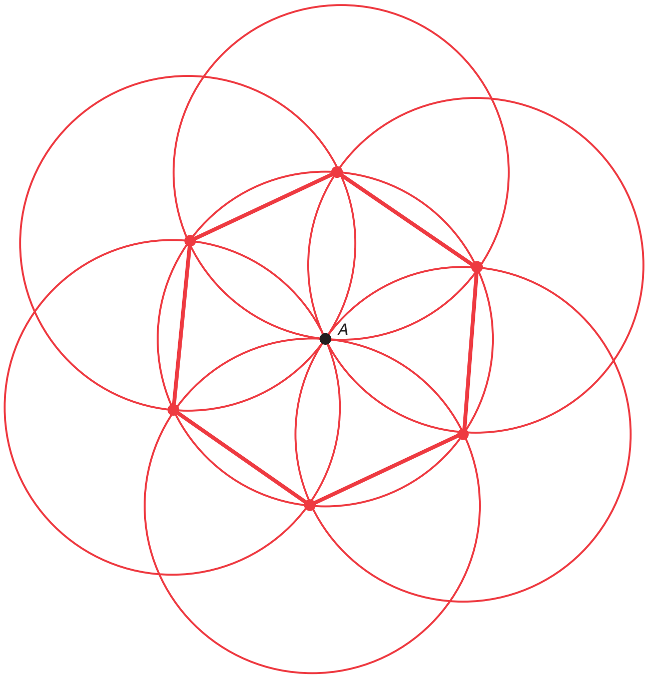
4. Construct a circle with the center *A* and a radius of about 1 inch.

a. Without changing the width of your compass, place the compass point on any point on the circle you constructed and then construct another circle.

b. Draw a dot on a point where the two circles intersect. Place the compass point on that point of intersection of the two circles, and then construct another circle.

c. Repeat this process until no new circles can be constructed.

d. Connect the points of the circles' intersections with each other.



e. Describe the figure formed by the line segments.

The figure formed by the line segments is a hexagon.

Grouping

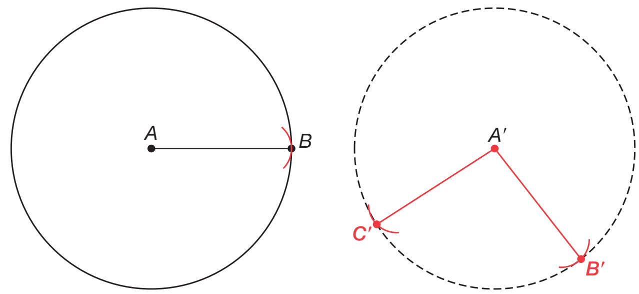
Have students complete Question 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- Is line segment *AB* considered a radius of circle *A*? Explain.
- How would you describe the location of point *B*’ on circle *A*’?
- Did your classmates locate point *B*’ in the same place? Explain.
- What do all of the different locations of point *B*’ have in common?
- Is line segment *A*’*B*’ considered a radius of circle *A*’? Explain.
- Are radii of congruent circles always congruent? Explain.
- How many radii can be drawn in one circle?

Now let’s use these circle-drawing skills to duplicate a line segment.

5. Circle *A* is congruent to Circle *A*′.



a. Duplicate $\overline{AB}$ in Circle *A*′. Use point *A*′ as the center of the circle, then label the endpoint of the duplicated segment as point *B*′.

See circle.

b. Describe the location of point *B*′.

Answers will vary.

Point *B*′ could be located anywhere on the circle.

c. If possible, construct a second line segment in Circle *A*′ that is a duplicate of $\overline{AB}$. Label the duplicate segment $\overline{A'C'}$. If it is not possible, explain why.

See circle.

It is possible to construct a second line segment that is a duplicate of $\overline{AB}$.

Problem 4

Students are instructed how to duplicate a line segment by construction. Using a compass and a straightedge they will copy a line segment. Step by step procedures are given for students to use as a guide.

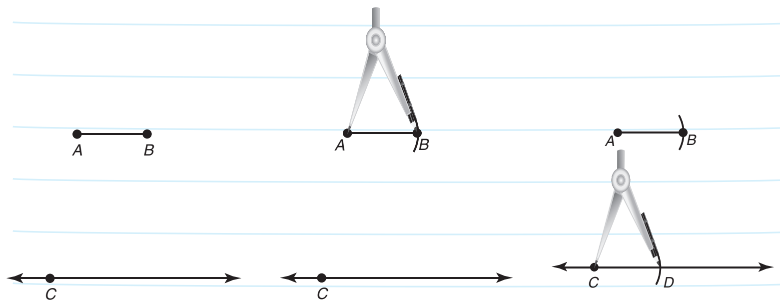
Grouping

Have students complete the Questions 1 through 4 with a partner. Then have students share their responses as a class.

PROBLEM 4 Another Method

To duplicate a line segment, you don't have to draw a full circle.

You can duplicate a line segment by constructing an exact copy of the original line segment.



Construct a Starter Line

Use a straightedge to construct a starter line longer than $\overline{AB}$. Label point C on the line.

Measure Length

Set your compass at the length AB.

Copy Length

Place the compass at C. Mark point D on the new segment.

Line segment CD is a duplicate of line segment AB.

1

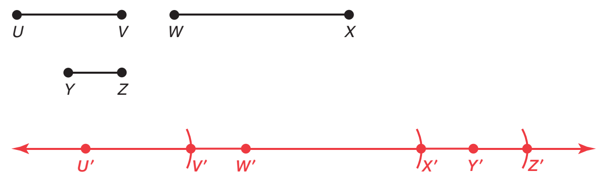
Guiding Questions for Share Phase, Questions 1 through 4

- What is a starter line?
- What is the purpose of a starter line?
- Why should every construction begin with a starter line?
- When it says to measure the length in Step 2, are you actually measuring the length?
- How is measuring the length in this construction different than measuring the length in other activities you have done?
- Could you easily construct a line segment that is three or four times the length of the given line segment? How would you do it?
- Did you duplicate the three line segments on the same starter line or did you use three different starter lines? Does it make a difference?
- Did Dave and Sandy use a starter line in their construction?
- What is the purpose of a starter line?
- Why should every construction begin with a starter line?
- If center point A were connected to any other point located on circle A, would the newly formed line segment be congruent to line segment AB? Why or why not?

1. Construct a line segment that is twice the length of $\overline{AB}$.

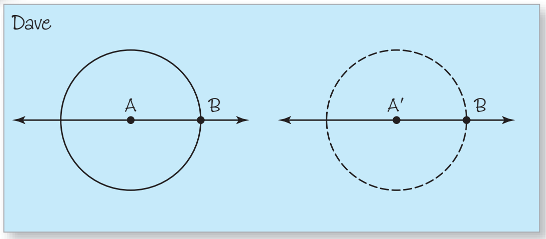
Line segment $A'C'$ is twice the length of $\overline{AB}$.

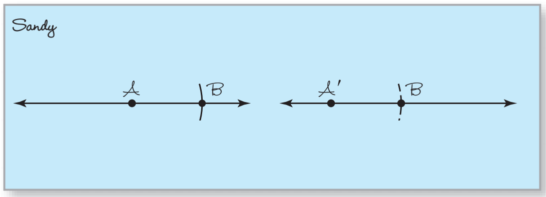
2. Duplicate each line segment using a compass and a straightedge.



- Is there an advantage to using Dave's method of duplicated a line segment? Explain.
- Is there an advantage to using Sandy's method of duplicate?

3. Dave and Sandy are duplicating $\overline{AB}$. Their methods are shown.





Which method is correct? Explain your reasoning.

Both methods are correct. Sandy and Dave are using the same method. The only difference is Dave used the compass to draw complete circles, whereas Sandy drew only the part of the circle, an arc, where it intersects the line segment.

4. Which method do you prefer? Why?

Answers may vary.

I prefer Sandy's method because I do not need to use the entire circle to duplicate a line segment.

1

Talk the Talk

In this lesson, students explored how line segments can be duplicated on a coordinate plane through translations, and how line segments can be duplicated by construction using construction tools. They also showed how vertically or horizontally translating a line segment resulted in preserving the length of the original segment. Students will state the similarities and differences of using the two methods to duplicate a line segment and conclude that segment lengths are preserved through translations.

Grouping

Have students complete the Questions 1 through 4 with a partner. Then have students share their responses as a class.

Talk the Talk

1. You translate a line segment vertically or horizontally. Is the length of the image the same as the length of the pre-image? Explain why or why not.
   Yes. Vertically or horizontally translating a line segment does not alter the length of the line segment. Because I am moving all points on the line segment the same distance and direction, the length is preserved.

2. If both endpoints of a line segment were not moved the same distance or direction, would the length of the line segment change? Would this still be considered a translation? Explain your reasoning.
   If both points on a line segment are not moved the same distance or direction, the length of the line segment would probably change. It would not be considered a translation because a translation slides each point of a figure the same distance and direction.

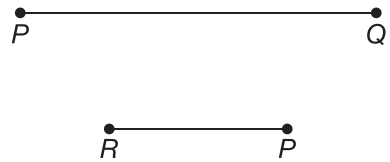
3. What can you conclude about the length of a line segment and the length of its translated image that results from moving points on a coordinate plane the same distance and the same direction?
   The length of the translated image of a line segment is congruent to the length of the pre-image line segment.

4. What can you conclude about the length of a line segment and the length of its translated image that results from construction?
   The length of the translated image of a line segment is congruent to the length of the pre-image line segment.

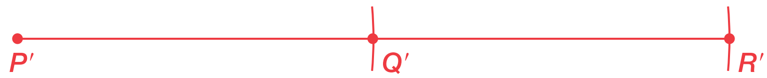
Be prepared to share your solutions and methods.

Check for Students' Understanding

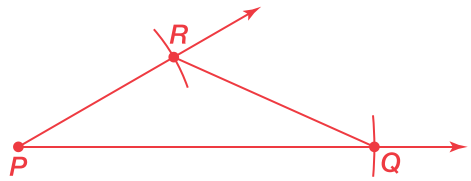
Use the given line segments in each of the following problems.



1. Construct a segment twice the length of line segment *PQ*.



2. Construct triangle *PQR*.



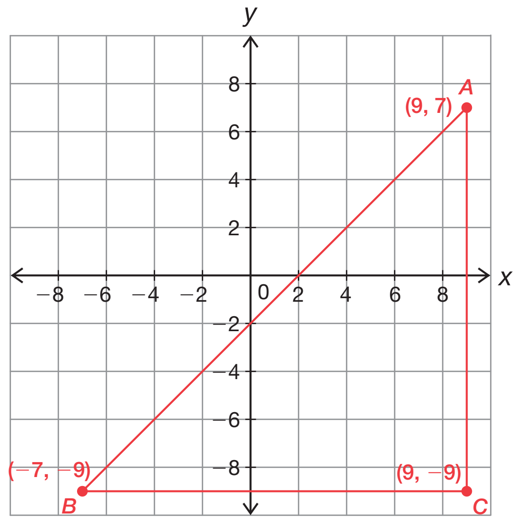
3. Do you suppose everyone in your class constructed the same triangle? Explain.

   Everyone could have constructed different triangles because the length of side *RQ* was not given and the measure of angle *P* was not given.

4. Your classmate was absent from school today and she is on the phone asking you how to duplicate a line segment. What will you tell her?

   I will tell her to draw a starter line, then locate and label endpoint *A* on the starter line. Point *A* will be one endpoint of the duplicated line segment. Using her compass, she needs to stretch it open so that the stylus is on one endpoint of the given line segment and the pencil is on the second endpoint. This captures the length of the given line segment. Using the same compass setting, place the stylus on point *A* and strike an arc through the starter line. Where the arc intersects the starter line is the second endpoint of the duplicated line segment.

5. Use the coordinate plane provided to answer the question. One unit represents one kilometer.



Vita's house is located at point *A* (9, 7). Her dog wandered away from home, but fortunately, the dog was wearing an identification tag which included Vita's phone number. Vita received a phone call that the dog was last seen at a location described by point *B* (−7, −9).

How far did the dog wander from its home?

$16^2 + 16^2 = c^2$

$256 + 256 = c^2$

$512 = c^2$

$c = \sqrt{512} \approx 22.6$

The dog wandered about 22.6 kilometers from its home.

1.3

Stuck in the Middle

Midpoints and Bisectors

LEARNING GOALS

In this lesson, you will:

- Determine the midpoint of a line segment on a coordinate plane.
- Use the Midpoint Formula.
- Apply the Midpoint Formula on the coordinate plane.
- Bisect a line segment using patty paper.
- Bisect a line segment by construction.
- Locate the midpoint of a line segment.

KEY TERMS

- midpoint
- Midpoint Formula
- segment bisector

CONSTRUCTIONS

- bisecting a line segment

ESSENTIAL IDEAS

- The midpoint of a line segment is the point on the segment that is equidistant from the endpoints of the line segment.
- The midpoint formula states that the midpoint between any two points (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- Construction tools can be used to bisect a line segment.
- Patty paper can be used to bisect a line segment.
- Bisecting a line segment locates the midpoint of the line segment.
- The length of the midsegment of a triangle is one half the length of the base of the triangle.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

G-GPE Expressing Geometric Properties with Equations

Use coordinates to prove simple geometric theorems algebraically

6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

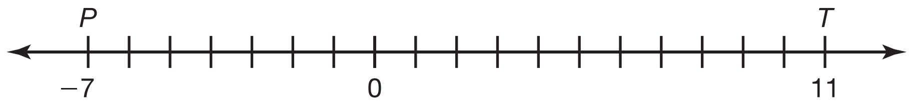
1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

Students will apply the Midpoint Formula to a situation on a coordinate plane. They then calculate the coordinates of various midpoints of line segments using the coordinates of points aligned horizontally or vertically using subtraction, and diagonally using the Pythagorean Theorem. Students derive the midpoint formula using generic coordinates such as (x_1, y_1) and (x_2, y_2), and then use the formula to solve problems. Using construction tools, students are guided through the steps necessary to bisect a line segment and locate the midpoint. Patty paper is also used to perform this construction. Students use the coordinate plane to show the length of the midsegment of a triangle is one half the length of the base of the triangle.

Warm Up

1. What is the value of the point half-way between points *P* and *T*?

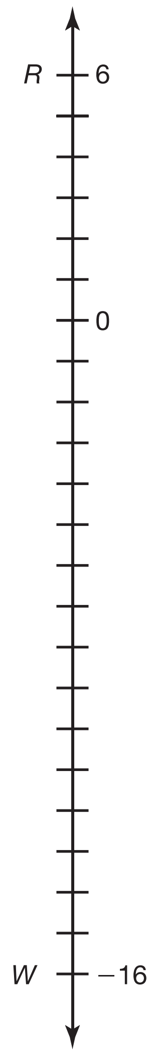


The value of the point half-way between points *P* and *T* is 2 units.

2. Bill insists the answer to Question 1 must be a positive number. Is Bill correct? Explain his reasoning.

Bill is correct. He knows the majority of distance between points *P* and *T* is on the positive side of the number line, therefore the middlemost point must also be on the positive side of the number line.

3. What is the value of the point half-way between points *R* and *W*?



The value of the point half-way between points *R* and *W* is −5 units.

1

4. Describe how you determined the value of the point half-way between two points in Question 3.

To determine the value of the point half-way between 6 and −16, I took the absolute value of each number and added them together to get the distance between the two points, or 22, then divided that distance in half, which is 11. Then I calculated a value that is 11 units away from both point R and W and got the value of −5.

5. What determines if the value of the point half-way between two points on a number line is positive number or a negative number?

The value of a point located half-way between two given points on a number line will have a positive value if more than half the distance between the two points is located on the positive side of the number line. If more than half the distance between the two points is on the negative side of a number line, the value located half-way between the points will also have a negative value.

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1.3

Stuck in the Middle

Midpoints and Bisectors

LEARNING GOALS

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- Determine the midpoint of a line segment on a coordinate plane.
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- Bisect a line segment by construction.
- Locate the midpoint of a line segment.

KEY TERMS

- midpoint
- Midpoint Formula
- segment bisector

CONSTRUCTIONS

- bisecting a line segment

When you hear the phrase "treasure hunt," you may think of pirates, buried treasure, and treasure maps. However, there are very few documented cases of pirates actually burying treasure, and there are no historical pirate treasure maps! So where did this idea come from?

Robert Louis Stevenson's book *Treasure Island* is a story all about pirates and their buried gold, and this book greatly influenced public knowledge of pirates. In fact, it is Stevenson who is often credited with coming up with the concept of the treasure map and using an X to mark where a treasure is located.

Have you ever used a map to determine your location or the location of another object? Did you find it difficult or easy to use? How does the idea of a treasure map relate to a familiar mathematical concept you are very familiar with?

1

Problem 1

The map of a playground situated in the first quadrant of the coordinate plane is used to locate various items. Students will answer questions by determining the coordinates of points describing vertical or horizontal distances and their respective midpoints.

Grouping

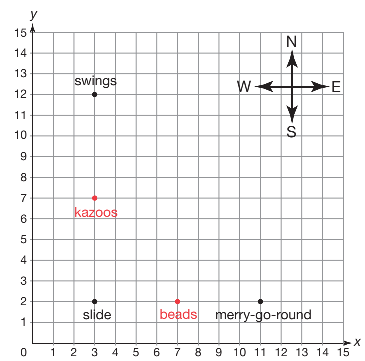
- Ask a student to read the introduction. Discuss the context and complete Question 1 as a class.
- Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

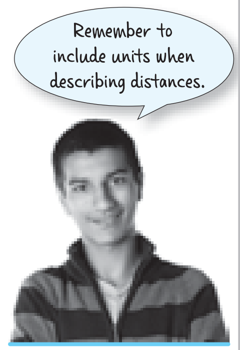
Guiding Questions for Discuss Phase

- Do the merry-go-round and the slide have a horizontal or vertical relationship?
- Do the slide and the swings have a horizontal or vertical relationship?
- Did you have to use the Distance Formula to determine the distance between the merry-go-round and the slide? Explain.

PROBLEM 1 Locating the Treasure

Ms. Lopez is planning a treasure hunt for her kindergarten students. She drew a model of the playground on a coordinate plane as shown. She used this model to decide where to place items for the treasure hunt, and to determine how to write the treasure hunt instructions. Each grid square represents one square yard on the playground.





1. Ms. Lopez wants to place some beads in the grass halfway between the merry-go-round and the slide.

   a. Determine the distance between the merry-go-round and the slide. Show all your work.

   $11 - 3 = 8$

   The merry-go-round and the slide are 8 yards apart.

   b. How far should the beads be placed from the merry-go-round and the slide?

   The beads should be placed 4 yards from the slide and 4 yards from the merry-go-round.

   c. Write the coordinates for the location exactly halfway between the merry-go-round and the slide. Graph a point representing the location of the beads on the coordinate plane.

   The coordinates for the location exactly halfway between the merry-go-round and the slide are (7, 2).

   d. How do the *x*- and *y*-coordinates of the point representing the location of the beads compare to the coordinates of the points representing the locations of the slide and the merry-go-round?

   The *y*-coordinates of all three points are the same.

   The *x*-coordinate of the point representing the beads is 4 units greater than the *x*-coordinate of the point representing the slide and is 4 units less than the *x*-coordinate of the point representing the merry-go-round.

Guiding Questions for Share Phase, Questions 2 and 3

How did you determine the half-way point between the merry-go-round and the slide?

2. Ms. Lopez wants to place some kazoos in the grass halfway between the slide and the swings.

   a. Write the coordinates for the location of the kazoos. Graph the location of the kazoos on the coordinate plane.

   The distance between the swings and the slide is 12 − 2, or 10 yards. So, the kazoos are 5 yards from the swings and 5 yards from the slide.

   The coordinates of the point representing the location of the kazoos are (3, 7).

   b. How do the *x*- and *y*-coordinates of the point representing the location of the kazoos compare to the coordinates of the points representing the locations of the slide and the swings?

   The *x*-coordinates for all three points are the same.

   The *y*-coordinate of the point representing the kazoos is 5 units greater than the *y*-coordinate of the point representing the slide and is 5 units less than the *y*-coordinate of the point representing the swings.

3. Ms. Lopez wants to place some buttons in the grass halfway between the swings and the merry-go-round.

   a. Determine the distance between the swings and the merry-go-round.

$$\begin{aligned} d &= \sqrt{(3-11)^2 + (12-2)^2} \\ &= \sqrt{(-8)^2 + 10^2} \\ &= \sqrt{64 + 100} \\ &= \sqrt{164} \\ &\approx 12.8 \end{aligned}$$

   The swings and the merry-go-round are about 12.8 yards apart.

   b. How far should the buttons be placed from the swings and the merry-go-round?

   The buttons should be placed about 6.4 yards from the swings and about 6.4 yards from the merry-go-round.

   c. How is determining the coordinates for the location of the buttons different than determining the coordinates for the locations of the beads or the kazoos?

   The beads were along a horizontal line with the slide and the merry-go-round so its *y*-coordinate was the same.

   The kazoos were along a vertical line with the slide and the swings so its *x*-coordinate was the same.

   The *x*-and *y*-coordinates of the buttons are both different than the *x*- and *y*-coordinates of the swings and the merry-go-round.

Grouping

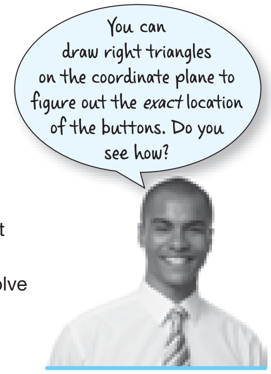
Have students complete Questions 4 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 7

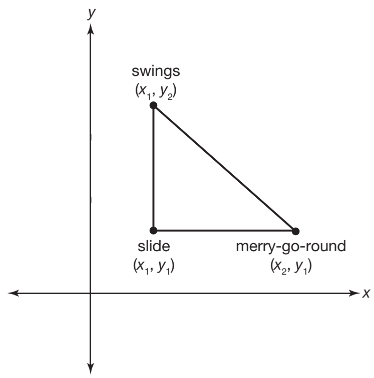
- How did you determine the half-way point between the slide and the swings?
- Why are the x-coordinates of the kazoos, the slide, and the swing the same value?
- Why aren't the y-coordinates of the kazoos, the slide, and the swing the same value?
- How is calculating the location of the pile of buttons different than calculating the location of the beads or kazoos in the previous questions?
- What geometric figure is formed when the locations of the swings, slide, and merry-go-round are connected?

d. Write the coordinates for the location of the kazoos. Graph the location of the buttons on the coordinate plane.

The coordinates of the point representing the location of the kazoos are (7, 7).



Suppose the slide, the swings, and the merry-go-round were at different locations but oriented in a similar manner. You can generalize their locations by using x_1, x_2, y_1, and y_2, and then solve for the distances between each using variables.



4. Use the diagram to describe each distance algebraically.

a. the vertical distance from the x-axis to the slide

The vertical distance to the slide is y_1.

b. the distance from the slide to the swings

The distance from the slide to the swings is $y_2 - y_1$.

c. half the distance from the slide to the swings

Half the distance from the slide to the swings is $\frac{y_2 - y_1}{2}$.

d. the vertical distance from the x-axis to the slide plus half the distance from the slide to the swings

The vertical distance from the x-axis to the slide plus half the distance from the slide to the swings is $y_1 + \frac{y_2 - y_1}{2}$.

5. Simplify your expression from Question 4, part (d).

$y_1 + \frac{y_2 - y_1}{2} = \frac{2y_1}{2} + \frac{y_2 - y_1}{2} = \frac{2y_1 + y_2 - y_1}{2} = \frac{y_1 + y_2}{2}$

6. Use the diagram to describe each distance algebraically.

   a. the horizontal distance from the *y*-axis to the slide

   The horizontal distance to the slide is x_1.

   b. the distance from the slide to the merry-go-round

   The distance from the slide to the merry-go-round is $x_2 - x_1$.

   c. half the distance from the slide to the merry-go-round

   Half the distance from the slide to the merry-go-round is $\frac{x_2 - x_1}{2}$.

   d. the horizontal distance from the *y*-axis to the slide plus half the distance from the slide to the merry-go-round

   The horizontal distance from the *y*-axis to the slide plus half the distance from the slide to the merry-go-round is $x_1 + \frac{x_2 - x_1}{2}$.

7. Simplify your expression from Question 6, part (d).

$x_1 + \frac{x_2 - x_1}{2} = \frac{2x_1}{2} + \frac{x_2 - x_1}{2} = \frac{2x_1 + x_2 - x_1}{2} = \frac{x_1 + x_2}{2}$

The coordinates of the points that you determined in Questions 5 and 7 are *midpoints*. A **midpoint** is a point that is exactly halfway between two given points. The calculations you performed can be summarized by the *Midpoint Formula*.

The **Midpoint Formula** states that if (x_1, y_1) and (x_2, y_2) are two points on the coordinate plane, then the midpoint of the line segment that joins these two points is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

8. Use the Midpoint Formula to determine the location of the buttons from Question 3.

The swings are at (3, 12) and the merry-go-round is at (11, 2). The buttons are halfway between the swings and the merry-go-round.

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$\left(\frac{3 + 11}{2}, \frac{12 + 2}{2}\right)$

$\left(\frac{14}{2}, \frac{14}{2}\right)$

(7, 7)

The buttons are located at (7, 7).

Guiding Questions for Discuss Phase

- Why must y_1 be added when writing the expression to determine the distance between the two points?
- Why must x_1 be added when writing the expression to determine the distance between the two points?
- What common denominator can be used to combine the terms in the expression?
- Can this midpoint formula be used with any two points on the coordinate plane? Explain.
- Why do you think division by two is used in the midpoint formula?
- What is a good way to remember the midpoint formula?

Grouping

Have students complete Question 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 10

- Looking only at the coordinates of the two points, how can you tell if you will be adding or subtracting the *x*-vales of the coordinates when using the midpoint formula?
- Looking only at the coordinates of the two points, how can you tell if you will be adding or subtracting the *y*-vales of the coordinates when using the midpoint formula?

9. Do you think it matters which point you identify as (x_1, y_1) and which point you identify as (x_2, y_2) when you use the Midpoint Formula? Explain why or why not.

No. It does not matter because addition is commutative.

10. Determine the midpoint of each line segment from the given endpoints. Show all of your work.

a. (0, 5) and (4, 3)

$x_1 = 0, y_1 = 5,$

$x_2 = 4, y_2 = 3$

$\left(\frac{0 + 4}{2}, \frac{5 + 3}{2}\right) = \left(\frac{4}{2}, \frac{8}{2}\right)$

$= (2, 4)$

b. (8, 2) and (6, 0)

$x_1 = 8, y_1 = 2,$

$x_2 = 6, y_2 = 0$

$\left(\frac{8 + 6}{2}, \frac{2 + 0}{2}\right) = \left(\frac{14}{2}, \frac{2}{2}\right)$

$= (7, 1)$

c. (−3, 1) and (9, −7)

$x_1 = -3, y_1 = 1,$

$x_2 = 9, y_2 = -7$

$\left(\frac{-3 + 9}{2}, \frac{1 - 7}{2}\right) = \left(\frac{6}{2}, \frac{-6}{2}\right)$

$= (3, -3)$

d. (−10, 7) and (−4, −7)

$x_1 = -10, y_1 = 7,$

$x_2 = -4, y_2 = -7$

$\left(\frac{-10 - 4}{2}, \frac{7 - 7}{2}\right) = \left(\frac{-14}{2}, \frac{0}{2}\right)$

$= (-7, 0)$

Note

Subtracting a negative number is often a difficult concept. Students are not always aware of why they are changing the signs, they just memorized the manipulation. An example of why this happens may be meaningful.

Problem 2

Scenarios are given asking students to determine a location $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of the distance between two specified points on the coordinate plane. Students use the Midpoint Formula and determine when it is useful to solve similar situations.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

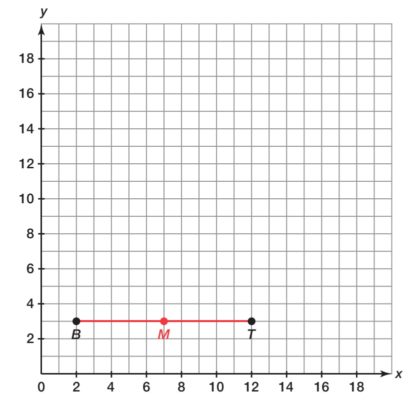
Guiding Questions for Share Phase, Questions 1 through 4

- How is this scenario different from the playground scenario?
- Is there more than one way to solve this problem?
- How would your solving this problem be different if the key was located $\frac{3}{4}$ of the way between the backdoor and the oak tree?
- How did you locate Jean's key when it was buried $\frac{1}{4}$ of the way between the front porch and the rose bush?
- Why couldn't you locate Jean's key when it was buried $\frac{1}{3}$ of the way between the front porch and the rose bush?
- In which situations was the midpoint formula helpful?
- In which situations was the midpoint formula not helpful?

PROBLEM 2 Jack's Spare Key

1. Jack buried a spare key to his house in the backyard in case of an emergency. He remembers burying the key halfway between the back door and an oak tree. The location of the back door is point $B(2, 3)$, and the location of the oak tree is point $T(12, 3)$.

   a. Determine the location of the key. Show all of your work. Then graph the location of the key as point M.



$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$\left(\frac{2 + 12}{2}, \frac{3 + 3}{2}\right) = \left(\frac{14}{2}, \frac{6}{2}\right) = (7, 3)$

The key is located halfway between point B and point T at point M (7, 3).

   b. Suppose Jack buried his spare key $\frac{1}{3}$ of the way between the back door and the oak tree. Determine the location of the key. Show all of your work.

$d = \sqrt{(12 - 2)^2 + (3 - 3)^2}$

$d = \sqrt{(10)^2 + (0)^2}$

$d = \sqrt{100 + 0}$

$d = \sqrt{100}$

$d = 10$

The distance between the back door and the oak tree is 10 units. The distance between the back door and the key is $\frac{10}{3}$ units. The x-coordinate of the key is $2 + \frac{10}{3}$, or $\frac{16}{3}$. The y-coordinate is 3.

The key is located $\frac{1}{3}$ of the way between point B and point T at $\left(\frac{16}{3}, 3\right)$.

2. Jean also buried her house key. She remembers burying the key between the front porch and a rose bush. The location of the front porch is point $P(1, 2)$ and the location of the rose bush is point $B(16, 14)$.

a. Suppose Jean buried her key $\frac{1}{2}$ of the way between the front porch and the rose bush. Determine the location of the key. Show all your work.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{1 + 16}{2}, \frac{2 + 14}{2}\right)$$
$$= \left(\frac{17}{2}, \frac{16}{2}\right)$$
$$= \left(8\frac{1}{2}, 8\right)$$

The point $\left(8\frac{1}{2}, 8\right)$ is halfway between the front porch and the rose bush.

b. Suppose Jean buried her key $\frac{1}{4}$ of the way between the front porch and the rose bush. Determine the location of the key. Show all your work.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{1 + 8.5}{2}, \frac{2 + 8}{2}\right)$$
$$= \left(\frac{9.5}{2}, \frac{10}{2}\right)$$
$$= \left(4\frac{1}{4}, 5\right)$$

The point $\left(4\frac{1}{4}, 5\right)$ is one-fourth of the way between the front porch and the rose bush.

c. Suppose Jean buried her key $\frac{1}{3}$ of the way between the front porch and the rose bush. Explain why the Midpoint Formula is not helpful in determining the location of Jean's spare key.

The Midpoint Formula is only helpful when I want to locate a point that is halfway between two other points. It will not locate a point that is one-third of the way between two points.

3. Rick and Courtney used different methods to determine the location that is $\frac{1}{3}$ of the way between the front porch and the rose bush.

Rick

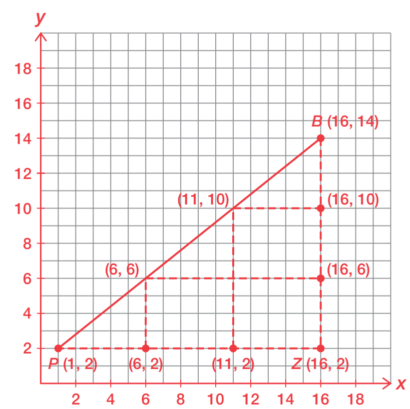
I drew a vertical line from point B and a horizontal line from point P and determined the coordinates of point Z where they intersected. I divided the horizontal line and the vertical line into 3 equal parts and wrote the coordinates. I used those coordinates to divide segment BP into three parts. Then, I calculated the coordinates along BP.

Courtney

I thought about the slope of segment BP. If I want to divide segment BP into three parts then I need to start at point P and perform three vertical and horizontal shifts to get to point B. So I can just divide the total horizontal and vertical shifts by three to calculate each vertical and horizontal shift. After that, determining the coordinates is a snap!

Calculate the location of the point that is $\frac{1}{3}$ of the way between the front porch and the rose bush using Rick's method and Courtney's method. Use a graph to support your work.

a. Rick's Method



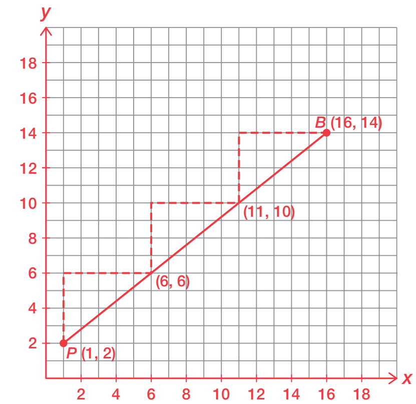
The coordinates of point Z are (16, 2).

Segment BZ is 14 −2, or 12 units. Dividing segment BZ into three equal segments means that each segment is 4 units long. So, the coordinates are (16, 6) and (16, 10).

Segment PZ is 16 − 1, or 15 units. Dividing segment PZ into three equal segments means that each segment is 5 units. So, the coordinates are (6, 2) and (11, 2).

The coordinates along BP are (6, 6) and (11, 10). The point (6, 6) is one-fourth of the way between the front porch and the rose bush.

b. Courtney's Method



The vertical distance between point *P* and point *B* is 14 − 2, or 12 units. So, I need three vertical shifts of 4 units to move from point *P* to point *B*.

The horizontal distance between point *P* and point *B* is 16 − 1, or 15 units. So, I need three horizontal shifts of 5 units to move from point *P* to point *B*.

Starting at point *P*, the coordinates along segment *BP* are (6, 6) and (11, 10).

4. Suppose Jean buried her key $\frac{1}{5}$ of the way between the front porch and the rose bush. Determine the location of the key. Show all your work.

The vertical distance between point *P* and point *B* is 14 − 2, or 12 units. So, I need five vertical shifts of 2.4 units to move from point *P* to point *B*.

The horizontal distance between point *P* and point *B* is 16 − 1, or 15 units. So, I need three horizontal shifts of 3 units to move from point *P* to point *B*.

Starting at point *P*, the coordinates along segment *BP* are (4, 4.4), (7, 6.8), (10, 9.2), and (13, 11.6).

Problem 3

Students are provided with the steps for locating a midpoint by bisecting a line segment using construction tools. Segment bisector is defined. Students first use patty paper to bisect a line segment then use a compass and straightedge. Students will then locate the midpoint of several line segments using construction tools. Each line segment is drawn in a different direction.

Grouping

- Ask a student to read the definitions and example. Discuss the context and complete Questions 1 and 2 as a class.
- Have students complete Questions 3 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Discuss Phase

- If the line segment was vertically drawn on the patty paper, how would you perform the construction?
- Why does this construction work?
- How wide should the compass be opened to strike the arcs?
- Why does this construction work?
- What is the significance of the location where two arcs intersect?

PROBLEM 3 Stuck in the Middle

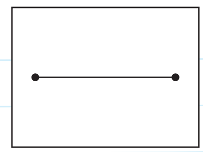
In the previous problem, you located the midpoint of a line segment on the coordinate plane. The lengths of the line segments on the plane are measurable.

In this problem, you will locate the midpoint of a line segment when measurement is not possible. This basic geometric construction used to locate a midpoint of a line segment is called **bisecting a line segment**. When bisecting a line segment, you create a *segment bisector*. A **segment bisector** is a line, line segment, or ray that divides a line segment into two line segments of equal measure, or two congruent line segments.

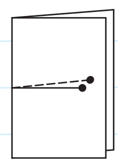
Just as with duplicating a line segment, there are a number of methods to bisect a line segment.

You can use tracing paper—also known as patty paper—to bisect a line.

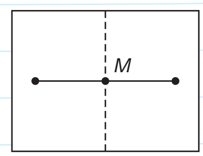
Draw a line segment on the paper.



Fold the paper so that the endpoints of the line segment lie on top of each other.



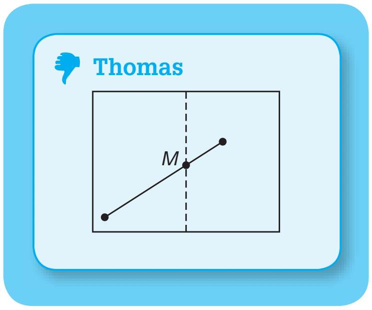
Open the paper. The crease represents the segment bisector, and the midpoint is located where the crease intersects the line segment.



1. Use tracing paper to duplicate a line segment. How do you know your bisector and midpoint are accurate?

   When I folded the paper, I made sure the endpoints were aligned. This means the crease must be exactly in the middle of the line.

2. Thomas determined the midpoint of $\overline{AB}$ incorrectly.

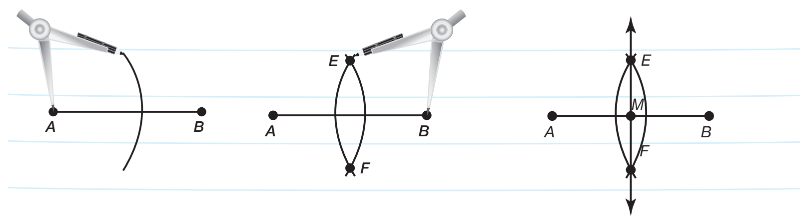


Explain what Thomas did incorrectly and how you can tell he is incorrect. Explain how he can correctly determine the midpoint of $\overline{AB}$.

Thomas just folded the paper in half which did not result in a correct midpoint. I can tell this is incorrect because the bisector did not divide the line segment into two equal parts.

To correctly determine the midpoint, Thomas must make sure that when he folds the paper the endpoints line up.

You can use a compass and straightedge to construct a segment bisector.



Construct an Arc

Open the radius of the compass to more than half the length of line segment *AB*. Use endpoint *A* as the center and construct an arc.

Construct Another Arc

Keep the compass radius and use point *B* as the center as you construct an arc. Label the points formed by the intersection of the arcs point *E* and point *F*.

Construct a Line

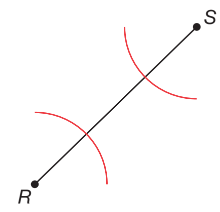
Connect points *E* and *F*. Line segment *EF* is the segment bisector of line segment *AB*. The point *M* represents the midpoint of $\overline{AB}$.

Line *EF* bisects line segment *AB*. Point *M* is the midpoint of line segment *AB*.

Guiding Questions for Share Phase, Questions 3 through 6

- How does the direction the line segments are drawn (vertical, horizontal, slanted) affect the construction of the midpoint?
- Do all lines have midpoints? How can this be proven?
- Do you prefer drawing the first arc using the left or the right endpoint of the line segment? Why?
- How can a compass be used to verify a line segment has been bisected?

3. Aaron is determining the midpoint of line segment *RS*. His work is shown.

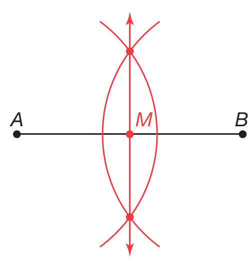


He states that because the arcs do not intersect, this line segment does not have a midpoint. Kate disagrees and tells him he drew his arcs incorrectly and that he must redraw his arcs to determine the midpoint. Who is correct? Explain your reasoning.

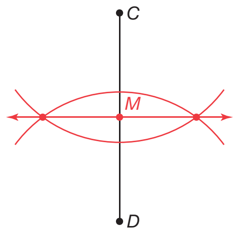
Kate is correct. All line segments have midpoints. If the arcs do not intersect, Aaron should open his compass further and redraw the arcs.

4. Use construction tools to locate the midpoint of each given line segment. Label each midpoint as *M*.

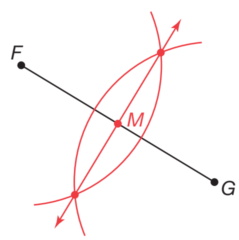
a.



b.

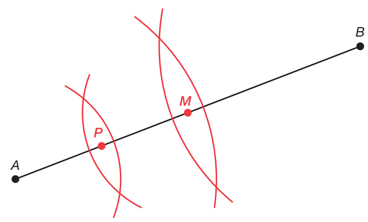


c.



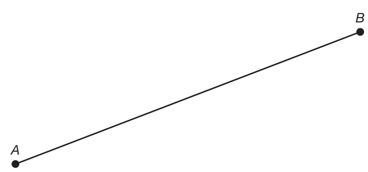
5. Perform each construction shown. Then explain how you performed each construction.

a. Locate a point one-fourth the distance between point A and point B.



Point P is $\frac{1}{4}$ the way between point A and point B. First, I constructed the midpoint M of $\overline{AB}$. Point M is one-half the distance between points A and B. Then I constructed the midpoint P of $\overline{AM}$. Point P is one-fourth the distance between points A and B.

b. Locate a point one-third the distance between point A and point B.



This construction cannot be done based on what students know at this point in the course.

6. Explain how you can duplicate a line segment to verify that the midpoint resulting from bisecting the line segment is truly the midpoint of the segment.

I can use my compass to copy each line segment formed by bisecting the line segment to see if they map onto themselves.

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Talk the Talk

In this lesson, students explored how to locate the midpoint of a line segment using the Midpoint Formula and by construction using construction tools. Students will state the similarities and differences of using the two methods to locate the midpoint of a line segment.

Grouping

Have students complete the Questions 1 through 3 with a partner. Then have students share their responses as a class.

Talk the Talk

1. When bisecting a line segment using construction tools, does it make a difference which endpoint you use to draw the first arc?

   No. It does not make a difference which endpoint I use to draw the first arc. The results are the same.

2. When locating the midpoint of a line segment on a coordinate plane using the Midpoint Formula, does it make a difference which endpoint you use as x_1 and y_1?

   No. It does not make a difference which endpoint I use as x_1 and y_1. The results are the same.

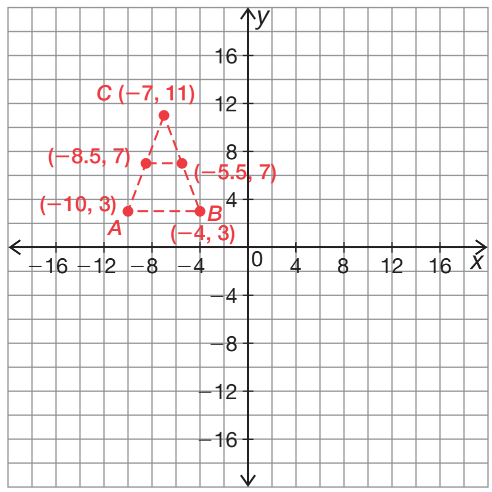
3. How will you decide if you should use the Midpoint Formula or construction tools to locate a midpoint?

   If I know the coordinates of the endpoints of the line segment, then I can use the Midpoint Formula. If I do not know the coordinates of the endpoints, I must use construction tools.

Be prepared to share your solutions and methods.

Check for Students' Understanding

1. Graph the three points on the coordinate plane.
   $A\ (-10, 3)$, $B\ (-4, 3)$, $C\ (-7, 11)$



2. Connect the three points to form triangle *ABC*.

3. Solve for the coordinates of M_1, the midpoint of side *AC*.
   $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-7 - 10}{2}, \frac{11 + 3}{2}\right) = \left(\frac{-17}{2}, \frac{14}{2}\right) = (-8.5, 7)$$

4. Solve for the coordinates of M_2, the midpoint of side *BC*.
   $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-7 - 4}{2}, \frac{11 + 3}{2}\right) = \left(\frac{-11}{2}, \frac{14}{2}\right) = (-5.5, 7)$$

5. Connect the two midpoints M_1 and M_2.

6. Calculate the distance between points M_1 and M_2.
   $$M_1M_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
   $$M_1M_2 = \sqrt{(-5.5 - -8.5)^2 + (7 - 7)^2}$$
   $$M_1M_2 = \sqrt{(3)^2 + (0)^2}$$
   $$M_1M_2 = \sqrt{9 + 0}$$
   $$M_1M_2 = \sqrt{9} = 3$$

   The distance between points M_1 and M_2 is 3 units.

1

7. Calculate the distance between points A and B.

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$AB = \sqrt{(-4 - -10)^2 + (3 - 3)^2}$

$AB = \sqrt{(6)^2 + (0)^2}$

$AB = \sqrt{36 + 0}$

$AB = \sqrt{36} = 6$

The distance between points A and B is 6 units.

8. Compare the length of the midsegment (line segment M_1M_2) of the triangle to the length of the base of the triangle (line segment AB).

The length of the midsegment of a triangle is equal to one-half the length of the base of the triangle.

1.4

What's Your Angle?

Translating and Constructing Angles and Angle Bisectors

LEARNING GOALS

In this lesson, you will:

- Translate an angle on the coordinate plane.
- Copy or duplicate an angle by construction.
- Bisect an angle by construction.

KEY TERMS

- angle
- angle bisector

CONSTRUCTIONS

- copying an angle
- duplicating an angle
- bisecting an angle

ESSENTIAL IDEAS

- Translations preserve the measure of an angle.
- Construction tools can be used to duplicate or copy an angle.
- Construction tools can be used to bisect an angle.
- Patty paper can be used to bisect an angle.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Experiment with transformations in the plane

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not.

4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

Make geometric constructions

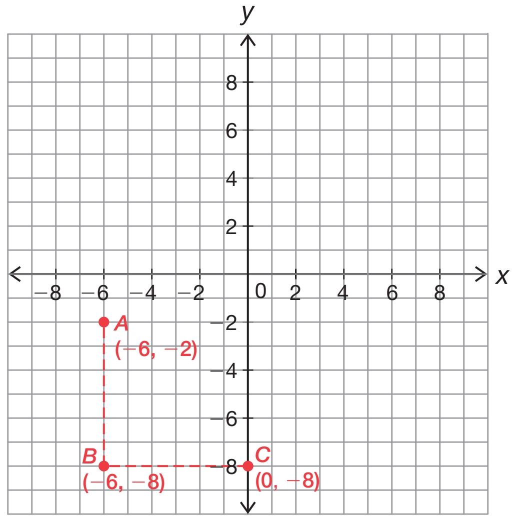
12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

Overview

Students will translate an angle onto itself on a coordinate plane through a series of four translations to conclude that the pre-image and all other images remain congruent. Using construction tools, students are guided through the steps necessary to bisect an angle. Patty paper is also used to perform this construction.

Warm Up

Use the coordinate plane provided to answer each question.



1. Plot points A (−6, −2), B (−6, −8) and C (0, −8).

2. If angle B appears to be a right angle, is this enough to conclude the measure of angle B is equal to 90 degrees? Explain your reasoning.

   No, it is not possible to conclude that the measure of angle *B* is equal to 90 degrees based only upon appearance. It must be proven or shown first that the line segments forming the angle are perpendicular to each other before concluding the angle is a right angle and the measure of the angle is equal to 90 degrees.

   It can be shown that line segment *AB* is vertical and line segment *BC* is horizontal, and they intersect at point *B*, then the angle formed at the intersection of the two segments must be a right angle, therefore the measure of the angle is equal to 90 degrees.

3. How can the Distance Formula and the Pythagorean Theorem be helpful in determine that the measure of angle B is equal to 90 degrees?

   The Distance Formula could be used to determine the length of line segment *AC*, and then the lengths of line segments *AB*, *BC*, and *AC,* used in conjunction with the Pythagorean Theorem to show that the theorem is satisfied. Only the lengths of the sides of a right triangle will satisfy the Pythagorean Theorem.

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1.4

What's Your Angle?

Translating and Constructing Angles and Angle Bisectors

LEARNING GOALS

In this lesson, you will:

- Translate an angle on the coordinate plane.
- Copy or duplicate an angle by construction.
- Bisect an angle by construction.

KEY TERMS

- angle
- angle bisector

CONSTRUCTIONS

- copying an angle
- duplicating an angle
- bisecting an angle

You may have never thought of it this way, but drawing and geometry are closely linked. Drawing is the process of deliberately arranging lines and curves to create an image. Most drawings have a number of different angles that are created through the intersection of these lines and curves. However, an art movement known as De Stijl (pronounced duh SHTEEL) limits drawings to using only horizontal and vertical lines. They also limit the colors used to the primary colors. While you may think this sounds restricting, many artists have created many works of art in this style. In fact, an architect even designed a house adhering to the De Stijl principles!

If De Stijl limits the artists to only using horizontal and vertical lines, what types of angles can be created in their art work? What types of angles cannot be created? What might be some challenges with drawing or painting in this style?

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Problem 1

Since angles are formed at the intersection of two line segments, then angles can be translated by translating the line segments forming the angle. Students will translate an angle into all four quadrants of the graph and determine the location of each on the coordinate plane. They then notice patterns used when writing the coordinates of the images and pre-image and conclude that translating an angle does not alter the measure of the angle. Translations preserve the measure of an angle.

Grouping

- Ask a student to read the introduction and definition. Discuss the context and complete Questions 1 and 2 as a class.
- Have students complete Questions 3 through 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Discuss Phase

- How is translating an angle similar to translating a line segment?
- When a line segment is translated, does this alter the direction or orientation of the line segment? Explain.

PROBLEM 1 Translating an Angle

Previously, you practiced translating a line segment on the coordinate plane horizontally or vertically.

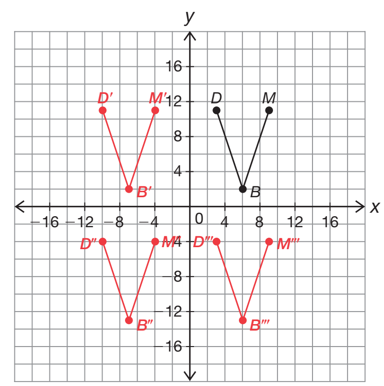
1. Describe how to translate a line segment on a coordinate plane.

   To translate a line segment, I must move all points of the line segment the distance and direction specified.

An **angle** is formed by two rays or line segments that share a common endpoint. The sides of the angle are represented by the two rays or segments. Each ray of an angle contains an infinite number of line segments. The $\angle$ symbol represents "angle." Angle *DMB* can be written as $\angle DMB$.

2. Analyze $\angle DBM$ shown on the coordinate plane. Describe how you would translate this angle on the coordinate plane.

   To translate this angle, I would translate points *D*, *B*, and *M* the specific distance and direction, and then I would connect the endpoints to form the sides of the angle, $\overline{DB}$ and $\overline{BM}$.



- It can be said that through translations angle *B* maps onto itself, what does this mean?
- When angle *DBM* was translated to the left, down, to the right, or up, does it appear to alter the measurement of the angle? Why or why not?

Guiding Questions for Share Phase, Questions 3 through 9

- Why must four translations take place for the angle to return to the original location?
- If only the vertex point of the angle were moved in a vertical direction, would the angle appear to be the same measurement?
- If only the vertex point of the angle were moved in a horizontal direction, would the angle appear to be the same measurement?

3. Complete each translation.

a. Horizontally translate $\angle DBM$ 13 units left. Label the image $\angle D'B'M'$.

b. Vertically translate $\angle D'B'M'$ 15 units down. Label the image $\angle D''B''M''$.

c. Horizontally translate $\angle D''B''M''$ 13 units right. Label the image $\angle D'''B'''M'''$.

d. Use the graph to complete the tables by determining the endpoints of each line segment.

Line Segments	$\overline{MB}$	$\overline{M'B'}$	$\overline{M''B''}$	$\overline{M'''B'''}$
Coordinates of Endpoints	(9, 11) (6, 2)	(−4, 11) (−7, 2)	(−4, −4) (−7, −13)	(9, −4) (6, −13)

Line Segments	$\overline{DB}$	$\overline{D'B'}$	$\overline{D''B''}$	$\overline{D'''B'''}$
Coordinates of Endpoints	(3, 11) (6, 2)	(−10, 11) (−7, 2)	(−10, −4) (−7, −13)	(3, −4) (6, −13)

4. Describe how a horizontal translation changes the x- and y-coordinates of the endpoints of each side of an angle.

A horizontal translation changes the x-coordinates of all the endpoints, but the y-coordinates remain the same.

5. Describe how a vertical translation changes the x- and y-coordinates of the angle endpoints of each side of an angle.

A vertical translation changes the y-coordinates of all the endpoints, but the x-coordinates remain the same.

6. Describe a sequence of two translations that will result in the image and the pre-image of an angle being the same.

The image and the pre-image of an angle will be the same if I translate up, down, left, or right and then translate in the opposite direction by the same number of units.

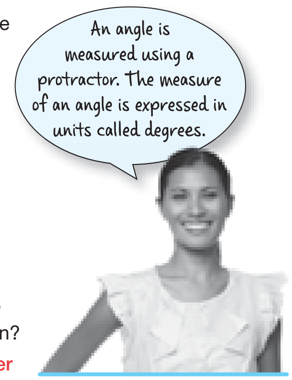
7. Describe a sequence of four translations that will result in the image and the pre-image of an angle being the same.

Answers will vary.

The image and the pre-image of an angle will be the same if I translate right by x units, down by y units, left by $-x$ units, and up by $-y$ units.

8. Measure each angle on the coordinate plane. How do the measures of each image compare to its corresponding pre-image?

The measures of each image and its corresponding pre-image is the same.



9. What is the result of moving only one angle endpoint a specified distance or direction? How does this affect the measure of the angle? Is this still considered a translation?

If only one endpoint of an angle is moved, and the other two are not, the measure of the angle would probably change.

This movement is no longer a translation because a translation consists of sliding each point of a figure the same distance and direction.

Problem 2

Students are provided with the steps for duplicating or copying an angle using construction tools.

Grouping

- Have a student read the definitions, and as a class, practice the construction.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

PROBLEM 2 Constructing an Angle

Previously, you translated an angle on the coordinate plane using line segments that were associated with units of measure. You can also translate an angle not associated with units of measure.

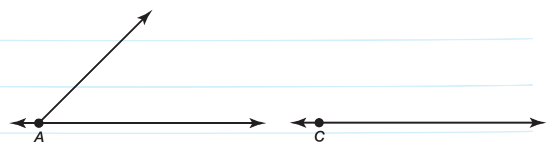
Guiding Questions for Discuss Phase

- How would you describe the positioning of the compass in the first step?
- How would you describe the positioning of the compass in the second step?
- Why does this construction work?
- What is the significance of the location the where two arcs intersect?

This basic geometric construction to translate an angle not associated with units of measure is called **copying an angle** or **duplicating an angle**. The construction is performed using a compass and a straightedge.

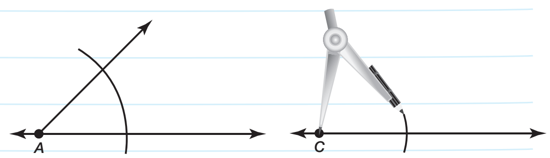
Construct a Starter Line

Use a straightedge to construct a starter line. Label point *C* on the new segment.



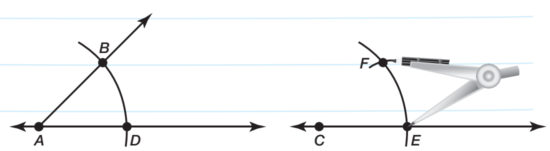
Construct an Arc

Construct an arc with center *A*. Using the same radius, construct an arc with center *C*.



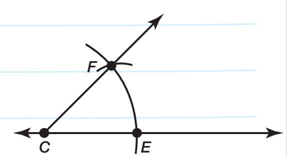
Construct Another Arc

Label points *B*, *D*, and *E*. Construct an arc with radius *BD* and center *E*. Label the intersection *F*.



Construct a Ray

Construct ray *CF*.
$\angle BAD \cong \angle FCE$.



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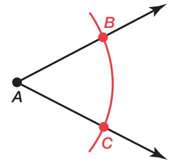
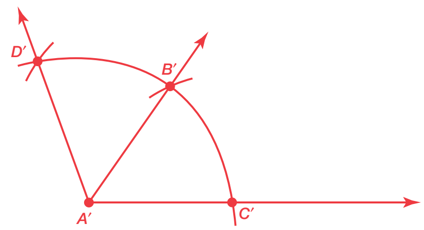
Guiding Questions for Share Phase, Questions 1 and 2

- How could you show the angle you created is twice the measure of angle *A*?
- Is there more than one way to do this construction? How so?

1. Construct an angle that is twice the measure of $\angle A$. Then explain how you performed the construction.





First, I duplicated $\angle BAC$ and labeled it as $\angle B'A'C'$. Then I used $\overline{A'B'}$ as my starter line, duplicated $\angle BAC$ again, and labeled it as $\angle D'A'C'$. Angle $D'A'C'$ is twice the measure of $\angle ABC$.

2. How is duplicating an angle similar to duplicating a line segment? How is it different?
When duplicating a line segment and an angle, the same tools are used. However, when duplicating a line segment, I only have to make one mark. When duplicating an angle, I must make multiple marks because I must duplicate both rays of the angle.

Problem 3

Students are provided with the steps bisecting an angle using patty paper and using construction tools.

Grouping

- Have a student read the definitions and as a class, practice the construction, and complete Question 1.
- Have students complete Questions 2 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Discuss Phase

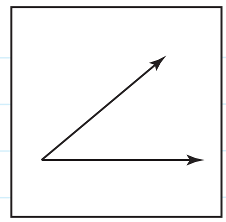
- Does the orientation of the original angle drawn on the patty paper change the result?
- Can patty paper be used to construct an angle one-fourth the measure of the original angle?
- Can patty paper be used to construct an angle one-the measure of the original angle
- Why does this construction work?

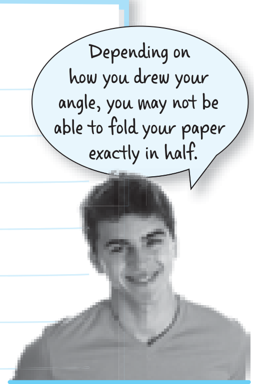
PROBLEM 3 Bisecting an Angle

Just as line segments can be bisected, angles can be bisected too. If a ray is drawn through the vertex of an angle and divides the angle into two angles of equal measure, or two congruent angles, this ray is called an **angle bisector**. The construction used to create an angle bisector is called **bisecting an angle**.

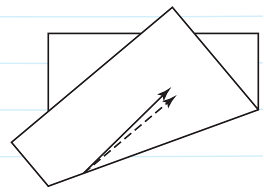
One way to bisect an angle is using tracing paper.

Draw an angle on the paper.

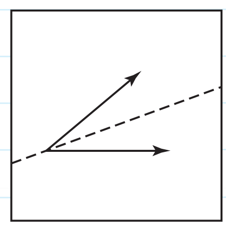




Fold the paper so that the rays lie on top of each other. The fold should pass through the vertex of the angle.



Open the paper. The crease represents the angle bisector.



1. Angela states that as long as the crease goes through the vertex, it is an angle bisector. Is she correct? Why or why not?

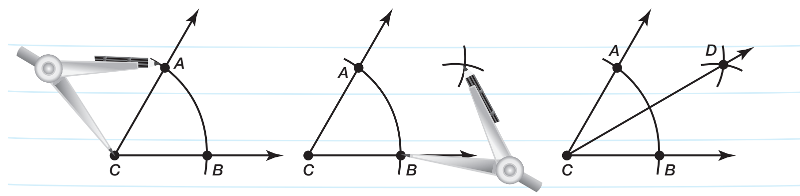
   No. Angela is not correct. Just because the crease goes through the vertex does not mean it bisects the angle. To bisect the angle, the bisector must divide the angle into two equal parts.

1

Guiding Questions for Share Phase, Questions 2 through 5

- How many bisectors were needed to construct an angle one-fourth the measure of the original angle?
- How many bisectors were needed to construct an angle one-eighth the measure of the original angle?
- Can all angles be bisected? Why or why not?
- Can an angle have more than one bisector? Why or why not?
- If an angle is bisected, what can you conclude?

You can also bisect an angle using a compass and a straightedge.



Construct an Arc

Place the compass at *C*. Construct an arc that intersects both sides of the angle. Label the intersections *A* and *B*.

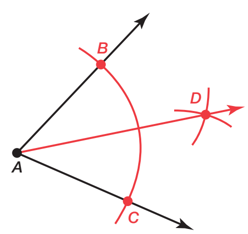
Construct Another Arc

Place the compass at *A*. Construct an arc. Then, place the compass point at *B*. Using the same radius, construct another arc.

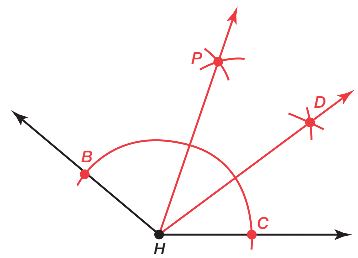
Construct a Ray

Label the intersection of the two arcs *D*. Use a straightedge to construct a ray through *C* and *D*. Ray *CD* bisects $\angle C$.

2. Construct the bisector of $\angle A$.



3. Construct an angle that is one-fourth the measure of $\angle H$. Explain how you performed the construction.

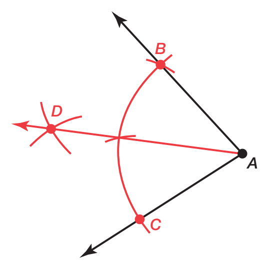


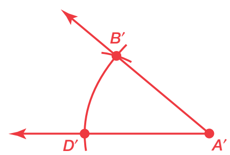
First I bisected $\angle H$ which formed two angles that are each one-half the measure of $\angle H$. Then I bisected one of those angles that are each one-fourth the measure of $\angle H$.

4. Describe how to construct an angle that is one-eighth the measure of $\angle H$ from Question 3.

I must bisect angle *DHC* or angle *DHP* that I constructed in Question 3. The resulting pair of angles will both be one-eighth the measure of angle *H*.

5. Use a compass and straightedge to show that the two angles formed by the angle bisector of angle *A* are congruent. Explain how you performed the construction.





Bisect $\angle A$ and label the two angles that are formed as $\angle BAD$ and $\angle CAD$. Duplicate $\angle BAD$ and label the new angle $\angle B'A'D'$. Duplicate $\angle CAD$ on top of $\angle B'A'D'$. They are the same angle, so $\angle BAD$ and $\angle CAD$ are congruent.

1

Talk the Talk

In this lesson, students explored how to translate an angle on the coordinate plane by translating the line segments forming the angle and by construction using construction tools. Students will state the similarities and differences of using the two methods to translate an angle.

Grouping

Have students complete the Questions 1 through 3 with a partner. Then have students share their responses as a class.

Talk the Talk

Translating an angle on the coordinate plane using coordinates and translating an angle by construction using construction tools both preserve the measure of the angle.

1. How are the two methods of translation similar?

   Both methods result in an angle that is congruent to the original angle.

2. How are the two methods of translation different?

   Translating angles on a coordinate plane enables me to know the coordinates of the angle pre-image and image. I also know that the angle measure is preserved, but I will still need a protractor to determine the angle measure.

   Copying an angle by construction does not involve coordinates. However, I know the measure of the angle is preserved without measuring the angle because the tools are accurate.

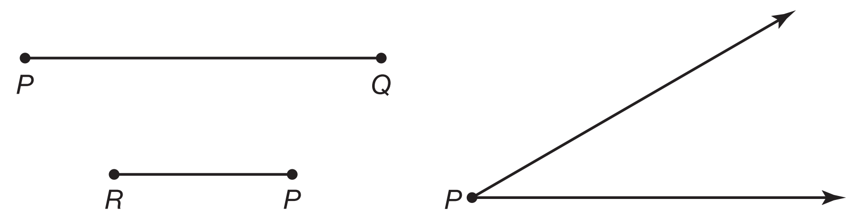
3. Does either a vertical or a horizontal translation of an angle alter the measure of the angle? Explain why or why not.

   Vertically or horizontally translating an angle does not alter the angle measure because the points that form the angle move the same distance and in the same direction when translating.

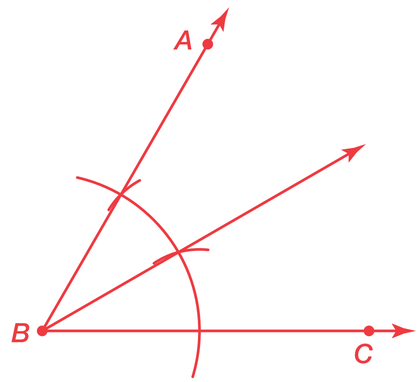
Be prepared to share your solutions and methods.

Check for Students' Understanding

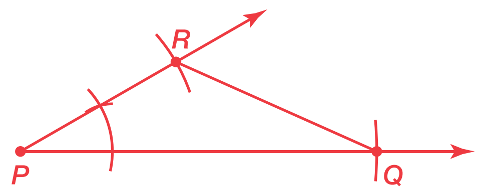
Use the given line segments and angle in each question.



1. Construct an angle twice the measure of angle *P*.



2. Construct triangle *PQR*.



3. Do you suppose everyone in your class constructed the same triangle? Explain your reasoning.

Everyone constructed the same triangle because the length of the third side (*RQ*) was determined by the measures of the given parts of the triangle.

1

4. A classmate was absent from school today and she is on the phone asking you how to duplicate an angle. What will you tell her?

I will tell her to draw a starter line, then locate and label a point *A* on the starter line. Using her compass, place the stylus on the vertex of the given angle and draw an arc that intersects both sides of the angle. Then place the stylus on point *A* and draw the same arc intersecting the starter line. Label the point at which the arc intersects the starter line point *B*. Using the given angle place the stylus on the point corresponding to point *B* and stretch the compass open so that the pencil strikes an arc on the second side of the given angle exactly where the first arc intersected the side. Using that compass setting, place the stylus on point *B* and strike an arc that intersects the arc you previously drew. Where the two arcs intersect, label the point *C*. Connect points *C*, *A* and *B* to form the duplicated angle *CAB*.

If You Build It . . .

1.5

Constructing Perpendicular Lines, Parallel Lines, and Polygons

LEARNING GOALS

In this lesson, you will:

- Construct a perpendicular line to a given line.
- Construct a parallel line to a given line through a point not on the line.
- Construct an equilateral triangle given the length of one side of the triangle.
- Construct an isosceles triangle given the length of one side of the triangle.
- Construct a square given the perimeter (as the length of a given line segment).
- Construct a rectangle that is not a square given the perimeter (as the length of a given line segment).

KEY TERM

- perpendicular bisector

CONSTRUCTIONS

- a perpendicular line to a given line through a point on the line
- a perpendicular line to a given line through a point not on the line

ESSENTIAL IDEAS

- Perpendicular lines can be constructed using construction tools given a line and a point on the line.
- Perpendicular lines can be constructed using construction tools given a line and a point not on the line.
- Parallel lines can be constructed using construction tools given a line and a point not on the line.
- An equilateral triangle can be constructed using construction tools given the length of one side of the triangle.
- An isosceles triangle can be constructed using construction tools given the length of one side of the triangle.
- A square can be constructed using construction tools given the perimeter.
- A rectangle that is not a square can be constructed using construction tools given the perimeter.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

1

Overview

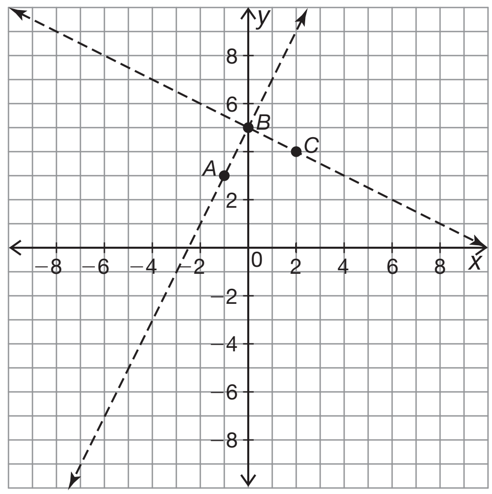
Students will use a compass and straightedge to construct a perpendicular line to a given line through a point on the line, a perpendicular line to a given line through a point not on the line, and a parallel line to a given line through a point not on the line. Students are shown that two lines perpendicular to the same line are parallel to each other and use this knowledge to construct two parallel lines.

Students then use a compass and straightedge to construct an equilateral triangle, an isosceles triangle, a square, and a rectangle that is not a square. A given perimeter is provided for the construction of the square and the construction of the rectangle that is not a square.

Warm Up

Graph each equation and describe the slope of each line.

1. Determine if line *AB* is perpendicular to line *BC*. State your reasoning.



Line *AB* contains points *A* (−1, 3) and *B* (0, 5).

Line *BC* contains points *B* (0, 5) and *C* (2, 4).

$$m_{AB} = \frac{y_1 - y_2}{x_1 - x_2}$$

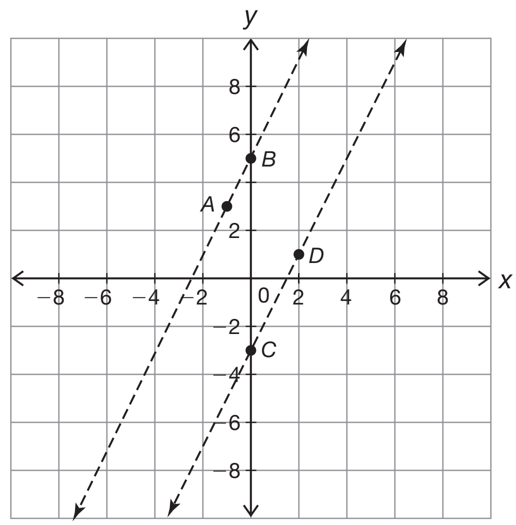
$$m_{AB} = \frac{5 - 3}{0 - (-1)} = \frac{2}{1}$$

$$m_{BC} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$m_{BC} = \frac{5 - 4}{0 - 2} = \frac{1}{-2} = -\frac{1}{2}$$

Line *AB* is perpendicular to line *BC* because their slopes have a negative reciprocal relationship.

2. Determine if line AB is parallel to line CD. State your reasoning.



Line AB contains points A $(-1, 3)$ and B $(0, 5)$.

Line CD contains points C $(0, -3)$ and D $(2, 1)$.

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

$$m = \frac{5 - 3}{0 - -1} = \frac{2}{1}$$

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

$$m = \frac{-3 - 1}{0 - 2} = \frac{-4}{-2} = \frac{1}{2}$$

Line AB is parallel to line CD because their slopes are equal to each other.

1.5

If You Build It . . .

Constructing Perpendicular Lines, Parallel Lines, and Polygons

LEARNING GOALS

In this lesson, you will:

- Construct a perpendicular line to a given line.
- Construct a parallel line to a given line through a point not on the line.
- Construct an equilateral triangle given the length of one side of the triangle.
- Construct an isosceles triangle given the length of one side of the triangle.
- Construct a square given the perimeter (as the length of a given line segment).
- Construct a rectangle that is not a square given the perimeter (as the length of a given line segment).

KEY TERM

- perpendicular bisector

CONSTRUCTIONS

- a perpendicular line to a given line through a point on the line
- a perpendicular line to a given line through a point not on the line

There's an old saying that you might have heard before: "They broke the mold when they made me!" A person says this to imply that they are unique. Of course, humans do not come from molds, but there are plenty of things that do.

For example, take a look at a dime if you have one handy. Besides some tarnish on the coin and the year the coin was produced, it is identical to just about every other dime out there. Creating and duplicating a coin a few billion times is quite a process involving designing the coin, creating multiple molds (and negatives of the molds), cutting the design onto metal, and on and on.

Can you think of any times when the "original" might be more important than a duplicate? Can you think of any examples where the "original" product might be more expensive than a generic brand of the same product?

1

Problem 1

Students are guided through the process of constructing a line perpendicular to a given line through a point on the given line. Students are then guided through the process of constructing a line perpendicular to a given line through a point not on the given line.

Grouping

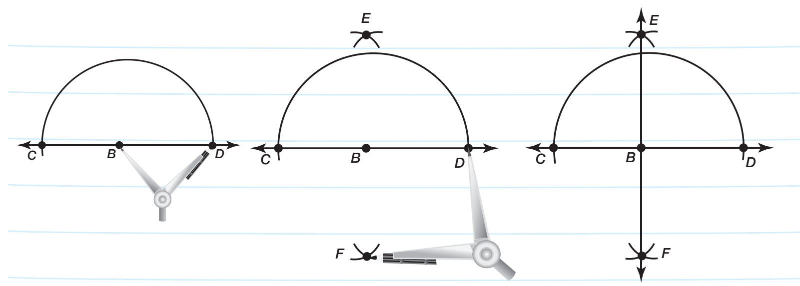
- As a class, have students practice the construction.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

PROBLEM 1 Constructing Perpendicular Lines

Previously, you practiced bisecting a line segment and locating the midpoint of a line segment by construction. In fact, you were also constructing a line segment perpendicular to the original line segment.

A **perpendicular bisector** is a line, line segment, or ray that bisects a line segment and is also perpendicular to the line segment.

Follow the steps to construct a perpendicular line through a point on the line.



Construct an Arc

Use *B* as the center and construct an arc. Label the intersections points *C* and *D*.

Construct Other Arcs

Open the compass larger than the radius. Use *C* and *D* as centers and construct arcs above and below the line. Label the intersection points *E* and *F*.

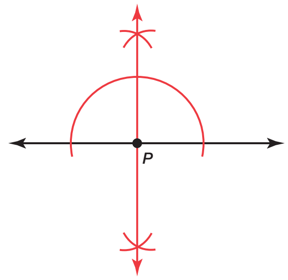
Construct a Line

Use a straightedge to connect points *E* and *F*. Line *EF* is perpendicular to line *CD*.

Guiding Questions for Share Phase, Questions 1 through 3

- Why does this construction seem familiar to you?
- Does it make a difference how much you open the compass radius to draw the intersecting arcs? Why or why not?
- If the point on the starter line was moved to the extreme left or the extreme right, could you still perform this construction? Explain.
- Why does this construction work?

1. Construct a line perpendicular to the given line through point *P*.



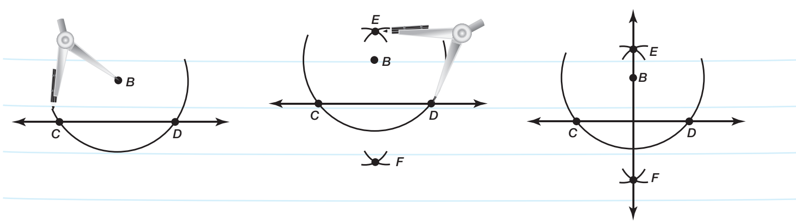
2. How is constructing a segment bisector and constructing a perpendicular line through a point on a line different?

A segment bisector bisects a line segment, so the segment bisector is exactly in the middle of the given line segment. A perpendicular line is perpendicular to a given line, but I cannot bisect a line because each line extends in opposite directions infinitely.

3. Do you think that you can only construct a perpendicular line through a point that is on a line? Why or why not?

I can construct a point not on a line because the point is in the same plane as the line. Therefore, it is possible to construct a perpendicular line through a point not on a line.

Follow these steps to construct a perpendicular line through a point not on a line.



Construct an Arc

Use *B* as the center and construct an arc. Label the intersection points *C* and *D*.

Construct Other Arcs

Open the compass larger than the radius. Use *C* and *D* as centers and construct arcs above and below the line. Label the intersection points *E* and *F*.

Construct a Line

Use a straightedge to connect points *E* and *F*. Line *EF* is perpendicular to line *CD*.

4. Amos claims that it is only possible to construct a perpendicular line through horizontal and vertical lines because the intersection of the points must be right angles. Loren claims that a perpendicular line can be constructed through any line and any point on or not on the line. Who is correct? Correct the rationale of the student who is *not* correct.

Loren is correct.

Amos must realize that right angles can occur through any orientation of a line as long as the constructed line is perpendicular. By definition, perpendicular lines intersect at right angles.

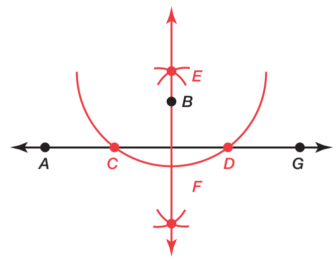
Grouping

Have students complete Questions 4 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 8

- How is this construction similar to the previous construction?
- Does it make a difference how much you open the compass radius to draw the intersecting arcs? Why or why not?
- If the point on the starter line was moved to the extreme left or the extreme right, could you still perform this construction? Explain.
- Why does this construction work?

5. Construct a line perpendicular to $\overleftrightarrow{AG}$ through point B.



6. How is the construction of a perpendicular line through a point on a line different from the construction of a perpendicular line through a point not on the line?
The steps are identical. The only difference is the location of the point.

7. Choose a point on the perpendicular bisector of $\overline{AG}$ and measure the distance from your point to point A and point G. Choose another point on the perpendicular bisector and measure the distance from this point to point A and point G. What do you notice.
The distances from the points that I chose to the endpoints of line segment AG are equal.

8. Make a conjecture about the distance from any point on a perpendicular bisector to the endpoints of the original segment.
Any point on the perpendicular bisector is equidistant to the endpoints of the line segment.

1

Problem 2

Students are shown that two lines perpendicular to the same line are parallel to each other. Using this information as a foundation, they will explore how to construct parallel lines using perpendicular line constructions and describe their construction.

Grouping

- Have students complete Question 1 independently. Discuss as a class.
- Have students complete Question 2 with a partner. Then have students share their responses as a class.

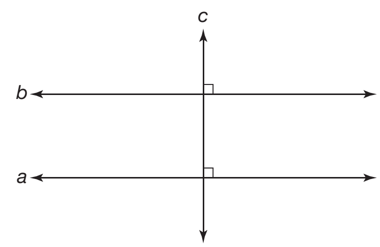
Guiding Questions for Discuss Phase

- What kind of angle is formed where line *a* intersects line *c*?
- What kind of angle is formed where line *b* intersects line *c*?
- What special pair of angles is formed by the intersection of lines *b* and *c*, and the intersection of lines *a* and *c*?
- If a pair of corresponding angles is congruent, are the lines forming the angles parallel to each other?
- Are two lines perpendicular to the same line always parallel to each other? Why or why not?

PROBLEM 2 Constructing Parallel Lines

To construct a line parallel to a given line, you must use a perpendicular line.

1. Analyze the figure shown.



Describe the relationship between the lines given.

a. *a* and *c*

Lines *a* and *c* are perpendicular.

b. *b* and *c*

Lines *b* and *c* are perpendicular.

c. *a* and *b*

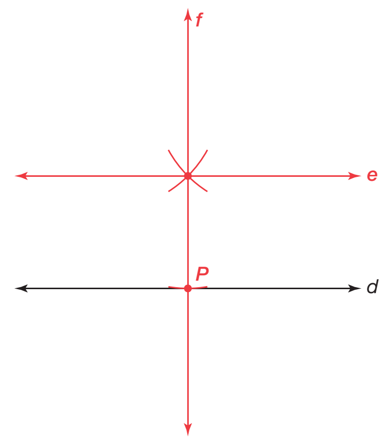
Lines *a* and *b* are parallel.

Guiding Questions for Share Phase, Question 2

- How do you begin this construction?
- How many perpendicular line constructions are needed to construct parallel lines?
- Are corresponding pairs of angles congruent? How do you know?
- Why does this construction work?

1

2. Construct line *e* parallel to line *d*. Then, describe the steps you performed for the construction.



To construct a line parallel to line *d*, I performed the following steps:

- First I chose a point on line *d* and labeled it point *P*.
- Then, I constructed a line perpendicular to line *d* and labeled this perpendicular line *f*.
- Next, I chose a point on line *f* and constructed another line that was perpendicular to line *f*. I labeled that line *e*.
- Therefore, by construction, lines *d* and e are parallel.

1

Problem 3

Students are given a starter line and the length of one side of a triangle. They will construct an equilateral triangle using the given length and describe the process.

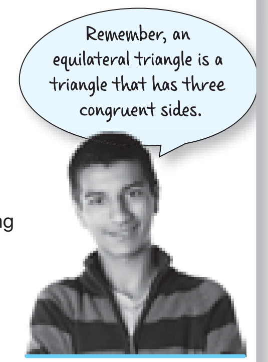
Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- How many arcs or circles are needed to complete this construction?
- Where is a good place to place the stylus when drawing the first arc or circle?
- What do all of the points on this arc or circle have in common?
- Where is a good place to place the stylus when drawing the second arc or circle?
- What is the significance of the intersecting arcs or circles?
- Why does this construction work?

PROBLEM 3 Constructing an Equilateral Triangle



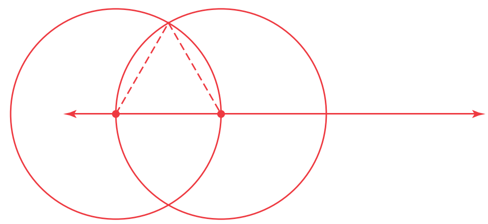
In the rest of this lesson, you will construct an equilateral triangle, an isosceles triangle, a square, and a rectangle that is not a square. To perform the constructions, use only a compass and straightedge and rely on the basic geometric constructions you have learned such as duplicating a line segment, duplicating an angle, bisecting a line segment, bisecting an angle, constructing perpendicular lines, and constructing parallel lines.

1. The length of one side of an equilateral triangle is shown.

a. What do you know about the other two sides of the equilateral triangle you will construct given the line segment shown?

The other two sides of the equilateral triangle must be the same length as the given line segment.

b. Construct an equilateral triangle using the given side length. Then, describe the steps you performed for the construction.



First, I duplicated the line segment on a starter line to serve as one side of the equilateral triangle. Next, I used each endpoint of the given line segment as the center of a circle and the length of the line segment as the length of the radii of both circles. Then, I connected the endpoints of the line segments to the point at which the circles intersect to form the equilateral triangle.

2. Sophie claims that she can construct an equilateral triangle by duplicating the line segment three times and having the endpoints of all three line segments intersect. Roberto thinks that Sophie's method will not result in an equilateral triangle. Who is correct? Explain why the incorrect student's rationale is not correct.

Roberto is correct.

While Sophie's method of constructing an equilateral triangle seems logical, it would also involve measurement. She would need to duplicate the line segment at an exact angle measure of 60 degrees. However, because construction does not involve units of measure, Sophie's method will not work because it involves an angle measure for each of the duplicated line segments.

Problem 4

Students are given a starter line and the length of one side of a triangle. They will construct an isosceles triangle using the given length and describe the process. They may choose to use the given side as the base or as one of the two congruent legs.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

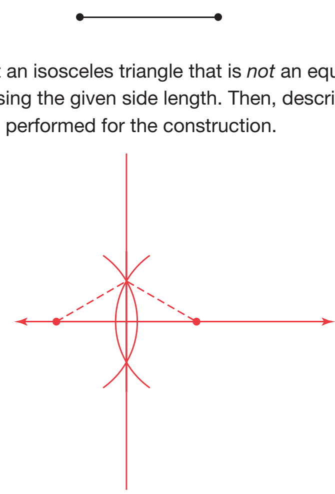
Guiding Questions for Share Phase, Question 1

- Are you using the given side as the base of the isosceles triangle or as one of the two congruent sides of the isosceles triangle?
- How many arcs or circles are needed to complete this construction?
- Where is a good place to place the stylus when drawing the first arc or circle?
- What do all of the points on this arc or circle have in common?
- Where is a good place to place the stylus when drawing the second arc or circle?
- What is the significance of the intersecting arcs or circles?
- Why does this construction work?

PROBLEM 4 Constructing an Isosceles Triangle

1. The length of one side of an isosceles triangle that is not an equilateral triangle is shown.

a. Construct an isosceles triangle that is *not* an equilateral triangle using the given side length. Then, describe the steps you performed for the construction.



First, I duplicated the line segment on a starter line to serve as one side of the isosceles triangle. Then, I constructed a perpendicular bisector of the given line segment and connected the endpoints of the line segment to any point on the perpendicular bisector to form an isosceles triangle.

b. Explain how you know your construction resulted in an isosceles triangle that is not an equilateral triangle.

Because I must use the line segment given, I know that I can bisect the line segment. By performing the segment bisector construction, I have divided the segment into two congruent segments. Therefore, when I connect the endpoints of the original segment to the segment bisector, I know that the other two sides are equivalent lengths; and therefore the construction is an isosceles triangle with two sides that are equal.

c. How does your construction compare to your classmates' constructions?

Answers will vary.

Some students used the given line segment as one of the two congruent sides of an isosceles triangle and some students used the given line segment as the base of the isosceles triangle, therefore resulting in different triangles.

Problem 5

Students are given a starter line and a line segment which is equivalent to the perimeter of a square. They will construct the square and describe the process.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

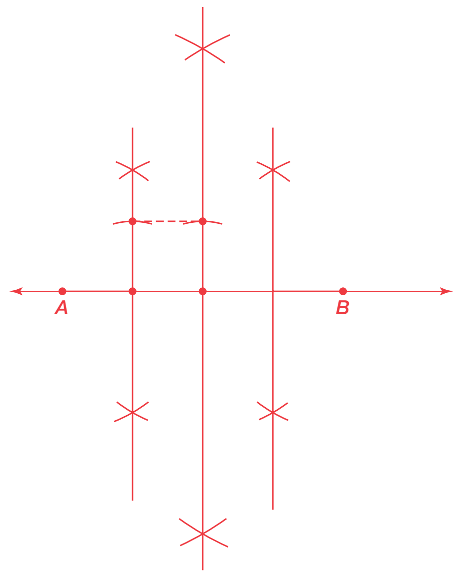
- How is the perimeter of a square determined?
- What do you know about the lengths of the sides of a square?
- How many different squares are determined from this given perimeter?
- How do you determine the length of one side of the square if you are only given the perimeter of the square?
- What construction must be done first? Why?
- How many parts are in this construction?
- Why does this construction work?

PROBLEM 5 Constructing a Square Given the Perimeter

Now you will construct a square using a given perimeter.

1. The perimeter of a square is shown by $\overline{AB}$.

a. Construct the square. Then, describe the steps that you performed for the construction.



First, I constructed a starter line and duplicated the given perimeter. Then, I bisected the duplicated line segment using a perpendicular bisector. Then I bisected the two resulting line segments to create 4 equal line segments. Since a square has 4 equal sides, I duplicated one of the line segments along two of the perpendicular bisectors to create the height of the square. Finally, I connected the two endpoints of the line segments representing the height to complete the square.

b. How does your construction compare to your classmates' constructions?

The constructions are similar.

Problem 6

Students are given a starter line and a line segment which is equivalent to the perimeter of a rectangle that is not a square. They will construct the rectangle and describe the process.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

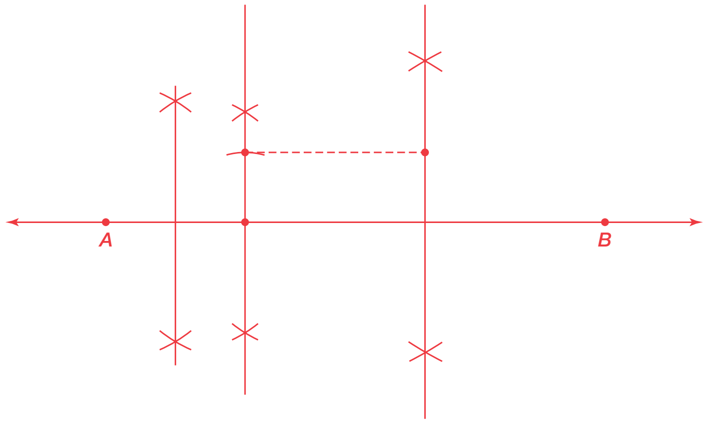
Guiding Questions for Share Phase, Question 1

- How is the perimeter of a rectangle determined?
- What do you know about the lengths of the sides of a rectangle that is not a square?
- How many different rectangles are determined from this given perimeter?
- How do you determine the length of one side of the rectangle if you are only given the perimeter of the rectangle?
- What construction must be done first? Why?
- How is this construction different than the previous construction?
- How can you be sure you are not constructing another square?

PROBLEM 6 Constructing a Rectangle Given the Perimeter

1. The perimeter of a rectangle is shown by $\overline{AB}$.

a. Construct the rectangle that is not a square. Then, describe the steps you performed for the construction.



First, I constructed a starter line and duplicated the given perimeter. Because the rectangle cannot have equal sides, I placed a point anywhere on the line segment except in the middle so the line segment is divided into two unequal line segments. Next, I drew perpendicular bisectors through each of the line segments to create four line segments. One of the line segments is now the base of the rectangle. I duplicated another line segment that is not the same size as the base on two of the perpendicular bisectors to use as the height of the rectangle. Finally, I connected the endpoints of the line segments representing the height to create a rectangle.

b. How does this construction compare to your classmates' constructions?

The constructions are similar. However, the orientation of the rectangle can be different.

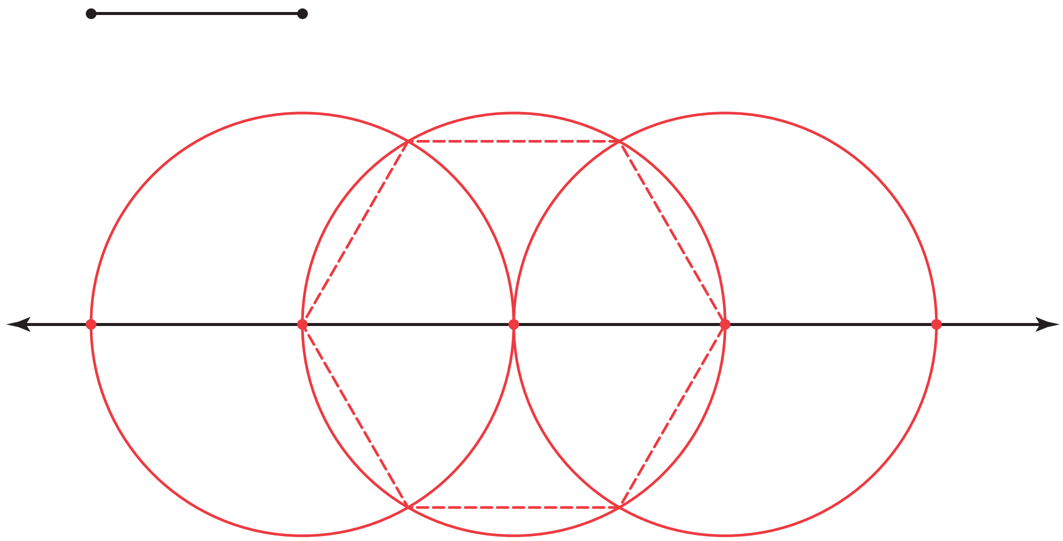
Be prepared to share your solutions and methods.

- How many parts are in this construction?
- Why does this construction work?

1

Check for Students' Understanding

Given the radius of a circle and a starter line, construct a regular hexagon using only **three** arcs or circles.



1.6

What's the Point?

Points of Concurrency

LEARNING GOALS

In this lesson, you will:

- Construct the incenter, circumcenter, centroid, and orthocenter.
- Locate points of concurrency using algebra.

KEY TERMS

- concurrent
- point of concurrency
- circumcenter
- incenter
- median
- centroid
- altitude
- orthocenter

ESSENTIAL IDEAS

- A point of concurrency is a point at which three lines, rays, or line segments intersect.
- The circumcenter is the point at which the three perpendicular bisectors of the sides of a triangle are concurrent.
- The incenter is the point at which the three angle bisectors of a triangle are concurrent.
- The median of a triangle is a line segment formed by connecting a vertex of a triangle to the midpoint of the opposite side of the triangle.
- The centroid is the point at which the three medians of a triangle are concurrent.
- The distance from the centroid to the vertex is twice the distance from the centroid to the midpoint of the opposite side.
- The orthocenter is the point at which the three altitudes or lines containing the altitudes of a triangle are concurrent.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

3. Apply geometric methods to solve design problems.

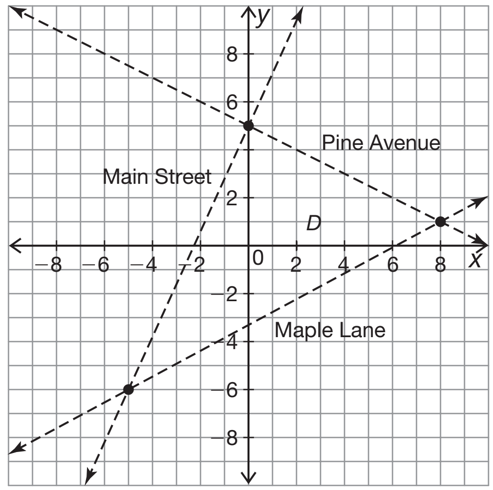
1

Overview

Students will use a compass and straightedge to construct the four points of concurrency in a triangle; the incenter, the circumcenter, the centroid, and the orthocenter. They also locate the points of concurrency using algebraic methods on the coordinate plane. Students explore triangles where the points of concurrency are collinear and also where the points of concurrency converge at one point. They determine which points of concurrency are most helpful in various situations.

Warm Up

Describe how you could locate a point that is equidistant from all three roads as shown.



Answers will vary.

I could randomly choose a point that appeared to be equidistant from each road, use a ruler to measure the distances from the point to each road along a perpendicular segment, and readjust the location of the point until all three distances were equal.

1

1.6

What's the Point?

Points of Concurrency

LEARNING GOALS

In this lesson, you will:

- Construct the incenter, circumcenter, centroid, and orthocenter.
- Locate points of concurrency using algebra.

KEY TERMS

- concurrent
- point of concurrency
- circumcenter
- incenter
- median
- centroid
- altitude
- orthocenter

Imagine playing a game of darts with a lot of people. You're all aiming for the bullseye, but of course it is the most difficult spot to hit on the dartboard. Therefore, it is probably the least hit spot on the board. Maybe you play a few hundred rounds, with each person having three throws on each turn, and each hit—no matter where it is—is recorded.

Now imagine that the dartboard is taken away, but all of the hits are shown as little dots on an empty wall. Could you determine the bullseye location using just the locations of those dots? How might you do it?

1

Problem 1

The concept of currency is introduced. Concurrent lines, rays, or line segments are three or more lines, rays, or line segments intersecting at a single point. The students will construct perpendicular bisectors. Then, they will use a ruler to determine that the distances from each point to the point of concurrency are equal.

Grouping

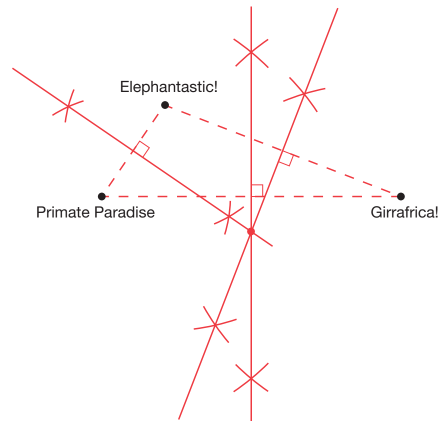
Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- Why are you constructing perpendicular bisectors?
- Are the perpendicular bisectors of the three sides of a triangle always concurrent?
- Is there another way to determine that the distances from each kiosk to the exhibit are equal?

PROBLEM 1 The New Zoo Review

Josh and Lezlee have been given the task of building an information kiosk for the zoo. They want the kiosk to be easily accessible from each of the various exhibits. Three of the exhibits at the zoo are shown.



1. Josh suggests building the kiosk so that it is equal distance from Giraffrica! and Primate Paradise.

   a. Lezlee replies that the only location of the kiosk can be determined by determining the midpoint of the line segment between the two exhibits.

   Is Lezlee correct? Explain your reasoning.

   No. Lezlee is not correct. The midpoint of the line segment between the two exhibits is one possible location for the kiosk but it is not the only possible location.

   b. Use a construction to show all possible locations of the kiosk so that it is equal distance from Giraffrica! and Primate Paradise. Explain your reasoning.

   See diagram.

   I constructed a perpendicular bisector to the line segment between the two exhibits. Any point along this bisector is a possible location of the kiosk, because any point on the perpendicular bisector is equidistant from the endpoints of the segment.

2. Lezlee suggests building the kiosk so that it is equal distance from Elephantastic! and Primate Paradise. Use a construction to show all possible locations of the kiosk so that it is equal distance from Elephantastic! and Primate Paradise. Explain your reasoning.

I drew a line segment from Primate Paradise to Elephantastic!. Then, I constructed a perpendicular bisector of this line segment. Any point along this perpendicular bisector is a possible location of the kiosk, because any point on the perpendicular bisector is equidistant from the endpoints of the segment.

3. Josh then wonders where the kiosk could be built so that it is equal distance from Elephantastic! and Giraffrica!. Use a construction to show all possible locations of the kiosk so that it is equal distance from Elephantastic! and Giraffrica!. Explain your reasoning.

I drew a line segment from Giraffrica!to Elephantastic!. Then, I constructed a perpendicular bisector of this line segment. Any point along this perpendicular bisector is a possible location of the kiosk, because any point on the perpendicular bisector is equidistant from the endpoints of the segment.

4. Describe how to determine a location that is the same distance from all three exhibits. Is there more than one possible location that is equidistant from all three exhibits? Explain your reasoning.

Each of the perpendicular bisectors represents possible locations for the kiosk equidistant from a pair of exhibits. So, the intersection of all the perpendicular bisectors represents the location that is equidistant from all three exhibits.

There is only one possible location that is equidistant from all three exhibits because the three perpendicular bisectors only intersect at one point.

5. Verify that the location you described in Question 4 is equidistant from each exhibit.

I used a ruler to determine the distance from the kiosk to each exhibit. All three distances are equal.

1

Problem 2

Students will draw and label lines, rays, and line segments that are concurrent.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

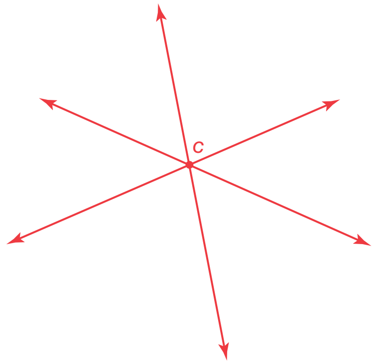
- What is an example in real-life where you would see three concurrent lines?
- What is an example in real-life where you would see three concurrent rays?
- What is an example in real-life where you would see three concurrent line segments?

PROBLEM 2 Concurrence

Concurrent lines, rays, or line segments are three or more lines, rays, or line segments intersecting at a single point. The **point of concurrency** is the point at which concurrent lines, rays, or segments intersect.

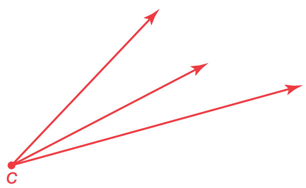
1. Draw three concurrent lines and label *C* as the point of concurrency.

Answers will vary.

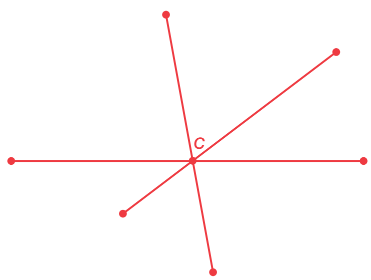


2. Draw three concurrent rays and label *C* as the point of concurrency.

Answers will vary.



3. Draw three concurrent line segments and label *C* as the point of concurrency.



Problem 3

Students will plot points to form acute, obtuse, right, and equilateral triangles. They will explore the circumcenter by constructing the perpendicular bisector of each side of the triangle and identifying the coordinates of the point at which the perpendicular bisectors are concurrent.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

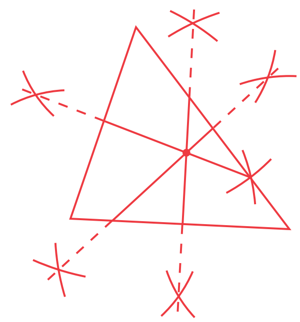
- Could you have determined the point of concurrency by constructing only two perpendicular bisectors in the acute triangle? Explain.
- Is the circumcenter equidistant from each of the sides of the acute triangle?
- Could you have determined the point of concurrency by constructing only two perpendicular bisectors in the obtuse triangle? Explain.
- Is the circumcenter equidistant from each of the sides of the obtuse triangle?
- Why do you suppose this point of concurrency is called the circumcenter?
- What significance would the circumcenter have in terms of a real-life example?

PROBLEM 3 Investigating the Circumcenter

1. Make a conjecture about the perpendicular bisectors of an acute triangle by performing the following steps.

   a. Draw an acute triangle that is not an equilateral triangle.

   Answers will vary.



   b. Construct the three perpendicular bisectors of your acute triangle.

   See drawing.

   c. Make a conjecture about the intersection of the three perpendicular bisectors of an acute triangle.

   The three perpendicular bisectors of an acute triangle intersect at a single point.

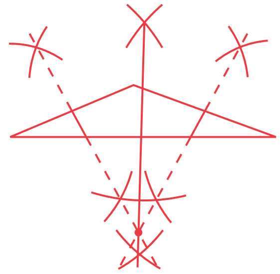
   d. Compare your conjecture to the conjectures of your classmates. What do you notice?

   My conjecture is the same as the conjectures made by my classmates.

2. Make a conjecture about the perpendicular bisectors of an obtuse triangle by performing the following steps.

   a. Draw an obtuse triangle.

   Answers will vary.



   b. Construct the three perpendicular bisectors of your obtuse triangle.

   See drawing.

   c. Make a conjecture about the intersection of the perpendicular angle bisectors of an obtuse triangle.

   The three perpendicular bisectors of an obtuse triangle intersect at a single point.

   d. Compare your conjecture to the conjectures of your classmates. What do you notice?

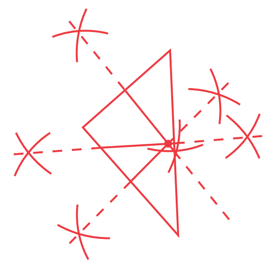
   My conjecture is the same as the conjectures made by my classmates.

1

3. Make a conjecture about the perpendicular bisectors of a right triangle by performing the following steps.

   a. Draw a right triangle.

      Answers will vary.



   b. Construct the three perpendicular bisectors of your right triangle.

      See drawing.

   c. Make a conjecture about the intersection of the three perpendicular bisectors of a right triangle.

      The three perpendicular bisectors of a right triangle intersect at a single point.

   d. Compare your conjecture to the conjectures of your classmates. What do you notice?

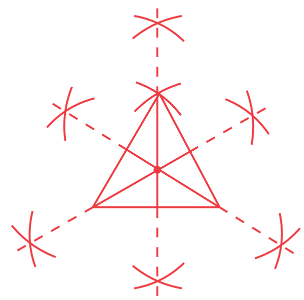
      My conjecture is the same as the conjectures made by my classmates.

1

4. Make a conjecture about the perpendicular bisectors of an equilateral triangle by performing the following steps.

   a. Construct an equilateral triangle.

   Answers will vary.



   b. Construct the three perpendicular bisectors of your equilateral triangle.

   See drawing.

   c. Make a conjecture about the intersection of the three perpendicular bisectors of an equilateral triangle.

   The three perpendicular bisectors of an equilateral triangle intersect at a single point.

   d. Compare your conjecture to the conjectures of your classmates. What do you notice?

   My conjecture is the same as the conjectures made by my classmates.

5. Make a conjecture about the intersection of the three perpendicular bisectors of any triangle. Is the intersection on the interior, exterior, or on the triangle?

   The three perpendicular bisectors of any triangle will intersect at a single point. For an acute triangle, this point will be in the interior of the triangle, for an obtuse triangle this point will be on the exterior of the triangle, and for a right triangle this point will be on the hypotenuse of the triangle.

1

The **circumcenter** is the point of concurrency of the three perpendicular bisectors of a triangle.

6. Consider the four triangles that you drew in Questions 1 through 4.

a. Measure the distance from the circumcenter to each vertex of the triangle.

Answers will vary.

b. Is the circumcenter always, sometimes, or never equidistant from each vertex of the triangle? Explain your reasoning.

The circumcenter is always equidistant from each vertex of the triangle.

c. Measure the distance from the circumcenter to each side of the triangle.

Answers will vary.

d. Is the circumcenter always, sometimes, or never equidistant from each side of the triangle? Explain your reasoning.

The circumcenter is only equidistant from each side of the triangle if the triangle is an equilateral triangle.

1

Problem 4

Students will plot points to form acute, obtuse, right, and equilateral triangles. They will explore the incenter by constructing the three angle bisectors of the triangle and identifying the coordinates of the point at which the angle bisectors are concurrent.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

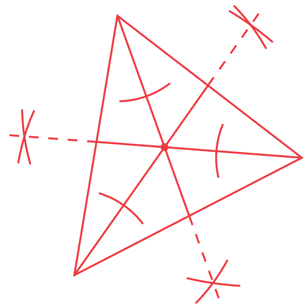
- Could you have determined the point of concurrency by constructing only two angle bisectors in the acute triangle? Explain.
- Is the incenter equidistant from each of the sides of the acute triangle?
- Could you have determined the point of concurrency by constructing only two angle bisectors in the obtuse triangle? Explain.
- Is the incenter equidistant from each of the sides of the obtuse triangle?
- Why do you suppose this point of concurrency is called the incenter?
- What significance would the incenter have in terms of a real-life example?

PROBLEM 4 Investigating the Incenter

1. Make a conjecture about the angle bisectors of an acute triangle by performing the following steps.

   a. Draw an acute triangle that is not an equilateral triangle.

   Answers will vary.

   b. Construct the three angle bisectors of your acute triangle.

   See drawing.

   c. Make a conjecture about the intersection of the three angle bisectors of an acute triangle.

   The three angle bisectors of an acute triangle intersect at a single point.

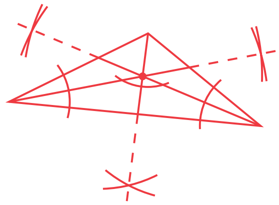
   d. Compare your conjecture to the conjectures of your classmates. What do you notice?

   My conjecture is the same as the conjectures made by my classmates.

2. Make a conjecture about the angle bisectors of an obtuse triangle by performing the following steps.

a. Draw an obtuse triangle.

Answers will vary.



b. Construct the three angle bisectors of your obtuse triangle.

See drawing.

c. Make a conjecture about the intersection of the three angle bisectors of an obtuse triangle.

The three angle bisectors of an obtuse triangle intersect at a single point.

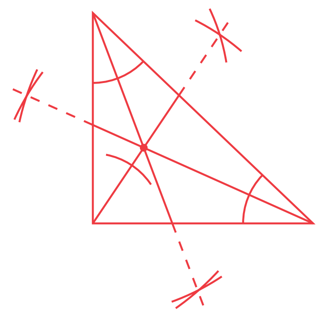
d. Compare your conjecture to the conjectures of your classmates. What do you notice?

My conjecture is the same as the conjectures made by my classmates.

3. Make a conjecture about the angle bisectors of a right triangle by performing the following steps.

   a. Draw a right triangle.

   Answers will vary.



   b. Construct the three angle bisectors of your right triangle.

   See drawing.

   c. Make a conjecture about the intersection of the three angle bisectors of a right triangle.

   The three angle bisectors of a right triangle intersect at a single point.

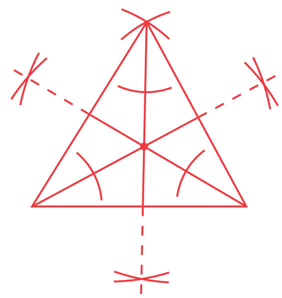
   d. Compare your conjecture to the conjectures of your classmates. What do you notice?

   My conjecture is the same as the conjectures made by my classmates.

4. Make a conjecture about the angle bisectors of an equilateral triangle by performing the following steps.

a. Construct an equilateral triangle.

Answers will vary.



b. Construct the three angle bisectors of your equilateral triangle.

See drawing.

c. Make a conjecture about the intersection of the three angle bisectors of an equilateral triangle.

The three angle bisectors of an equilateral triangle intersect at a single point.

d. Compare your conjecture to the conjectures of your classmates. What do you notice?

My conjecture is the same as the conjectures made by my classmates.

5. Make a conjecture about the intersection of the three angle bisectors of any triangle. Is the intersection on the interior, exterior, or on the triangle?

The three angle bisectors of any triangle will intersect at a single point on the interior of the triangle.

The **incenter** is the point of concurrency of the three angle bisectors of a triangle.

6. Consider the four triangles that you drew in Questions 1 through 4.

a. Measure the distance from the incenter to each vertex of the triangle.

Answers will vary.

b. Is the incenter always, sometimes, or never equidistant from each vertex of the triangle? Explain your reasoning.

The incenter is only equidistant from each vertex of the triangle if the triangle is an equilateral triangle.

c. Measure the distance from the incenter to each side of the triangle.

Answers will vary.

d. Is the incenter always, sometimes, or never equidistant from each side of the triangle? Explain your reasoning.

The incenter is always equidistant from each side of the triangle.

Problem 5

Students will plot points to form acute, obtuse, right, and equilateral triangles. They will explore the centroid by constructing the three medians of the triangle and identifying the coordinates of the point at which the medians are concurrent.

Grouping

Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 7

- Could you have determined the point of concurrency by constructing only two medians in the acute triangle? Explain.
- Is the centroid equidistant from each of the sides of the acute triangle?
- Could you have determined the point of concurrency by constructing only two medians in the obtuse triangle? Explain.
- Is the centroid equidistant from each of the sides of the obtuse triangle?
- Why do you suppose the centroid is the center of balance in the triangle?
- Why do you suppose this point of concurrency is called the centroid?

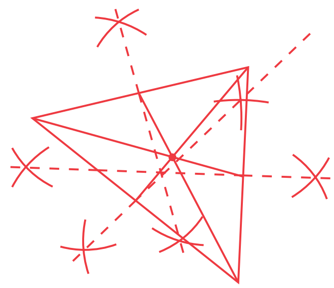
PROBLEM 5 Investigating the Centroid

A **median** of a triangle is a line segment that connects a vertex to the midpoint of the opposite side.

1. Make a conjecture about the medians of an acute triangle by performing the following steps.

   a. Draw an acute triangle that is not an equilateral triangle.

   Answers will vary.



   b. Construct the three medians of your acute triangle.

   See drawing.

   c. Make a conjecture about the intersection of the three medians of an acute triangle.

   The three medians of an acute triangle intersect at a single point.

   d. Compare your conjecture to the conjectures of your classmates. What do you notice?

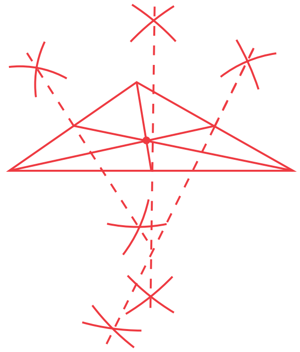
   My conjecture is the same as the conjectures made by my classmates.

- Which segment on each median is twice the length of the remaining segment?
- What is the importance of this ratio?
- What significance would the centroid have in terms of a real-life example?

2. Make a conjecture about the medians of an obtuse triangle by performing the following steps.

   a. Draw an obtuse triangle.

   Answers will vary.



   b. Construct the three medians of your obtuse triangle.

   See drawing.

   c. Make a conjecture about the intersection of the three medians of an obtuse triangle.

   The three medians of an obtuse triangle intersect at a single point.

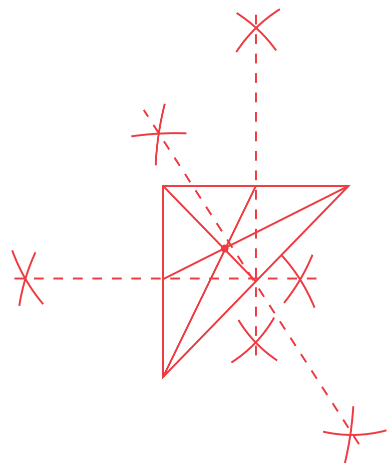
   d. Compare your conjecture to the conjectures of your classmates. What do you notice?

   My conjecture is the same as the conjectures made by my classmates.

3. Make a conjecture about the medians of a right triangle by performing the following steps.

   a. Draw a right triangle.

   Answers will vary.



   b. Construct the three medians of your right triangle.

   See drawing.

   c. Make a conjecture about the intersection of the three medians of a right triangle.

   The three medians of a right triangle intersect at a single point.

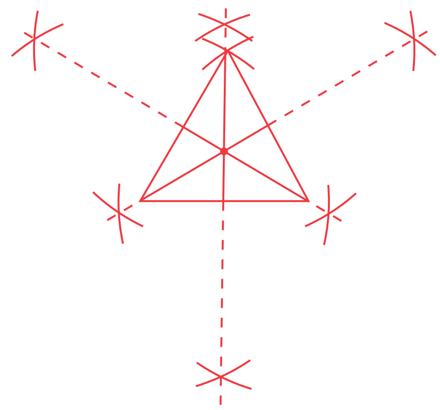
   d. Compare your conjecture to the conjectures of your classmates. What do you notice?

   My conjecture is the same as the conjectures made by my classmates.

4. Make a conjecture about the medians of an equilateral triangle by performing the following steps.

   a. Construct an equilateral triangle.

   Answers will vary.



   b. Construct the three medians of your equilateral triangle.

   See drawing.

   c. Make a conjecture about the intersection of the three medians of an equilateral triangle.

   The three medians of an equilateral triangle intersect at a single point.

   d. Compare your conjecture to the conjectures of your classmates. What do you notice?

   My conjecture is the same as the conjectures made by my classmates.

5. Make a conjecture about the intersection of the three medians of any triangle. Is the intersection on the interior, exterior, or on the triangle?

   The three medians of any triangle will intersect at a single point on the interior of the triangle.

The **centroid** is the point of concurrency of the three medians of a triangle.

6. Consider the four triangles that you drew in Questions 1 through 4.

a. Measure the distance from the centroid to each vertex of the triangle.

Answers will vary.

b. Is the centroid always, sometimes, or never equidistant from each vertex of the triangle? Explain your reasoning.

The centroid is only equidistant from each vertex of the triangle if the triangle is an equilateral triangle.

c. Measure the distance from the centroid to each side of the triangle.

Answers will vary.

d. Is the centroid always, sometimes, or never equidistant from each side of the triangle? Explain your reasoning.

The centroid is only equidistant from each side of the triangle if the triangle is an equilateral triangle.

7. The centroid divides each median into two segments. Compare the distance from the centroid to the vertex and the distance from the centroid to the midpoint of the opposite side in each of the triangles you created. What is the ratio?

The distance from the centroid to the vertex is twice the distance from the centroid to the midpoint of the opposite side.

1

Problem 6

Students will plot points to form acute, obtuse, right, and equilateral triangles. They will explore the orthocenter by constructing the three altitudes of the triangle and identifying the coordinates of the point at which the altitudes are concurrent.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

- Could you have determined the point of concurrency by constructing only two altitudes in the acute triangle? Explain.
- Is the orthocenter equidistant from each of the sides of the acute triangle?
- Could you have determined the point of concurrency by constructing only two altitudes in the obtuse triangle? Explain.
- Is the orthocenter equidistant from each of the sides of the obtuse triangle?
- Why do you suppose this point of concurrency is called the orthocenter?
- What significance would the orthocenter have in terms of a real-life example?

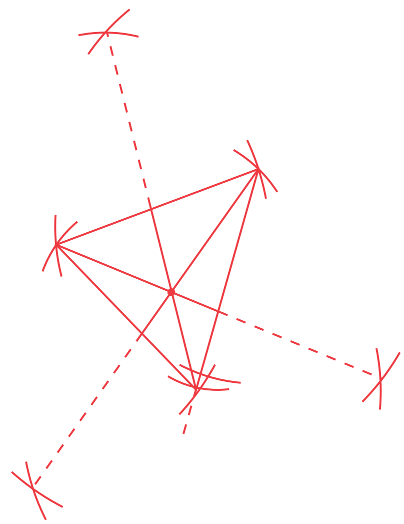
PROBLEM 6 Investigating the Orthocenter

An **altitude** of a triangle is a line segment that is perpendicular to a side of the triangle and has one endpoint at the opposite vertex.

1. Make a conjecture about the altitudes of an acute triangle by performing the following steps.

   a. Draw an acute triangle that is not an equilateral triangle.

   Answers will vary.



   b. Construct the three altitudes of your acute triangle.

   See drawing.

   c. Make a conjecture about the intersection of the three altitudes of an acute triangle.

   The three altitudes of an acute triangle intersect at a single point.

   d. Compare your conjecture to the conjectures of your classmates. What do you notice?

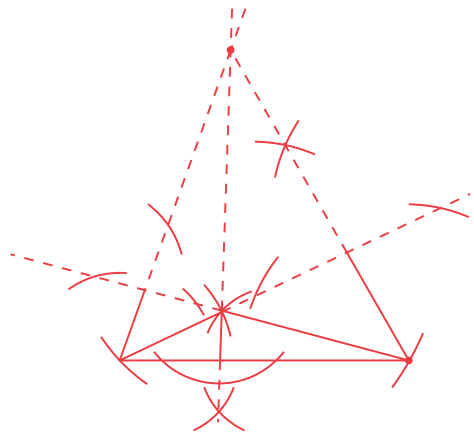
   My conjecture is the same as the conjectures made by my classmates.

1

2. Make a conjecture about the altitudes of an obtuse triangle by performing the following steps.

a. Draw an obtuse triangle.

Answers will vary.



b. Construct the three altitudes of your obtuse triangle.

See drawing.

c. Make a conjecture about the intersection of the three altitudes of an obtuse triangle.

The three altitudes of an obtuse triangle intersect at a single point.

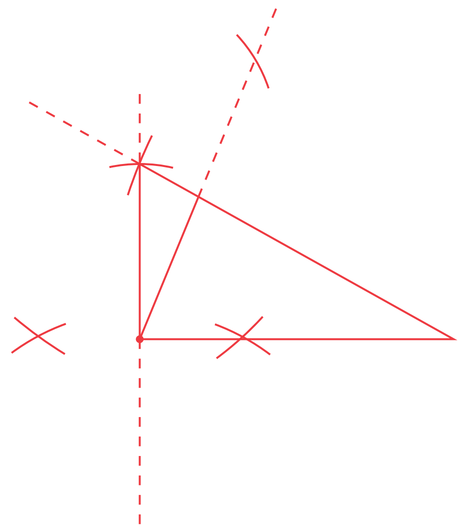
d. Compare your conjecture to the conjectures of your classmates. What do you notice?

My conjecture is the same as the conjectures made by my classmates.

3. Make a conjecture about the altitudes of a right triangle by performing the following steps.

   a. Draw a right triangle.

   Answers will vary.



   b. Construct the three altitudes of your right triangle.

   See drawing.

   c. Make a conjecture about the intersection of the three altitudes of a right triangle.

   The three altitudes of a right triangle intersect at a single point.

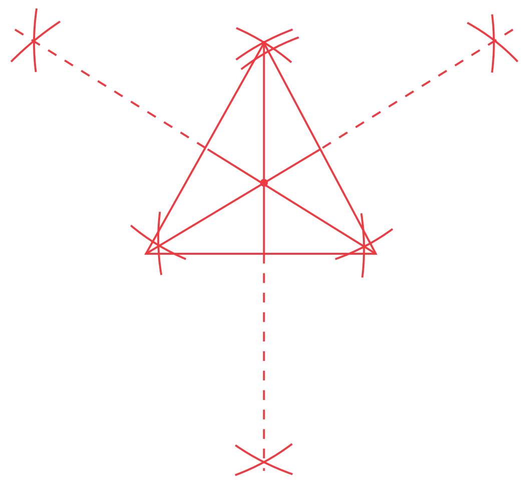
   d. Compare your conjecture to the conjectures of your classmates. What do you notice?

   My conjecture is the same as the conjectures made by my classmates.

4. Make a conjecture about the altitudes of an equilateral triangle by performing the following steps.

   a. Construct an equilateral triangle.

   Answers will vary.



   b. Construct the three altitudes of your equilateral triangle.

   See drawing.

   c. Make a conjecture about the intersection of the three altitudes of an equilateral triangle.

   The three altitudes of an equilateral triangle intersect at a single point.

   d. Compare your conjecture to the conjectures of your classmates. What do you notice?

   My conjecture is the same as the conjectures made by my classmates.

5. Make a conjecture about the intersection of the three altitudes of any triangle. Is the intersection on the interior, exterior, or on the triangle?

   The three altitudes of any triangle will intersect at a single point. For an acute triangle, this point will be in the interior of the triangle, for an obtuse triangle this point will be on the exterior of the triangle, and for a right triangle this point will be at the vertex of the right angle.

The **orthocenter** is the point of concurrency of the three altitudes of a triangle.

6. Consider the four triangles that you drew in Questions 1 through 4.

a. Measure the distance from the orthocenter to each vertex of the triangle.

Answers will vary.

b. Is the orthocenter always, sometimes, or never equidistant from each vertex of the triangle? Explain your reasoning.

The orthocenter is only equidistant from each vertex of the triangle if the triangle is an equilateral triangle.

c. Measure the distance from the orthocenter to each side of the triangle.

Answers will vary.

d. Is the orthocenter always, sometimes, or never equidistant from each side of the triangle? Explain your reasoning.

The orthocenter is only equidistant from each side of the triangle if the triangle is an equilateral triangle.

Problem 7

Students summarize their findings after constructing the incenter, circumcenter, centroid, and orthocenter for acute, obtuse, right, and equilateral triangles.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Discuss Phase

- What is an equilateral triangle?
- How is an equilateral triangle different from an isosceles triangle?
- Are there any similarities or differences in the locations of the incenter for acute, obtuse, and right triangles?
- Are there any similarities or differences in the locations of the circumcenter for acute, obtuse, and right triangles?
- Are there any similarities or differences in the locations of the centroid for acute, obtuse, and right triangles?
- Are there any similarities or differences in the locations of the orthocenter for acute, obtuse, and right triangles?

PROBLEM 7 Points of Concurrency

1. Examine the incenter, circumcenter, centroid, and orthocenter for the equilateral triangles you created. What do you notice?
The incenter, circumcenter, centroid, and orthocenter for an equilateral triangle are the same point.

2. Complete the table to describe the location of each point of currency for acute, obtuse, and right triangles.

	Acute Triangle	Obtuse Triangle	Right Triangle
Incenter	Interior	Interior	Interior
Circumcenter	Interior	Exterior	On Hypotenuse
Centroid	Interior	Interior	Interior
Orthocenter	Interior	Exterior	At Vertex of Right Angle

1

Problem 8

Students will draw an isosceles triangle on the coordinate plane. They show the triangle is isosceles using both an algebraic method and a geometric method and they conclude the incenter, centroid, circumcenter, and orthocenter are collinear points.

Grouping

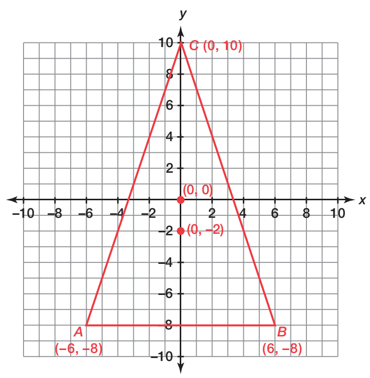
Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What is an isosceles triangle?
- What formula is needed to show the triangle is isosceles?
- Are the centroid, circumcenter, orthocenter, and incenter names for the same point?
- What is the centroid?
- How is algebra used to locate the centroid?
- Is there more than one algebraic method that can be used to locate the coordinates of the centroid in the isosceles triangle?
- Is it necessary to determine the equation for the medians? Why or why not?
- Given the equations of two lines, how do you determine the point of intersection?

PROBLEM 8 Using Algebra with Points of Concurrency

1. Form a triangle on the grid by connecting the points $A(-6, -8)$, $B(6, -8)$, and $C(0, 10)$.



2. Classify triangle *ABC* by performing the following steps.

   a. Calculate the length of each side of the triangle using the Distance Formula.

$$AC = \sqrt{(-6-0)^2 + (-8-10)^2}$$
$$= \sqrt{(-6)^2 + (-18)^2}$$
$$= \sqrt{36 + 324}$$
$$= \sqrt{360}$$
$$\approx 18.97$$

$$BC = \sqrt{(6-0)^2 + (-8-10)^2}$$
$$= \sqrt{(6)^2 + (-18)^2}$$
$$= \sqrt{36 + 324}$$
$$= \sqrt{360}$$
$$\approx 18.97$$

Side *AB* is a horizontal segment, so I can subtract the *x*-coordinates to calculate its length.

$AB = 6 - (-6) = 12$

   b. Use the side lengths to classify the triangle. Explain your reasoning.

Two side lengths are equal so the triangle is an isosceles triangle.

- What is the circumcenter?
- How is algebra used to locate the circumcenter?
- Is there more than one algebraic method that can be used to locate the coordinates of the circumcenter in the isosceles triangle?
- Is it necessary to determine the equation for the perpendicular bisectors? Why or why not?
- What is the orthocenter?

- How is algebra used to locate the orthocenter?
- Is there more than one algebraic method that can be used to locate the coordinates of the orthocenter in the isosceles triangle?
- Is it necessary to determine the equation for the altitudes? Why or why not?
- What is the incenter?
- Can algebra be used to locate the incenter?

3. Describe how you could use algebra to calculate the coordinates of each point of concurrency.

   a. centroid

   The centroid is the point of concurrency of the medians.

   First, I can calculate the midpoint of each side of the triangle using the Midpoint Formula. Then, I can use each midpoint and the opposite vertex to write an equation for each median. Finally, I can calculate the point of intersection of the medians.

   b. circumcenter

   The circumcenter is the point of concurrency of the perpendicular bisectors.

   First, I can calculate the midpoint of each side of the triangle using the Midpoint Formula. Then, I can calculate the slope of each side of the triangle. The slope of a perpendicular bisector is the negative reciprocal of the slope of the side. So, I can use the slope and the median to write an equation for each perpendicular bisector. Finally, I can calculate the intersection of the perpendicular bisectors.

   c. orthocenter

   The orthocenter is the point of concurrency of the altitudes.

   First, I can calculate the slope of each side of the triangle. The slope of an altitude is the negative reciprocal of the slope of the side. So, I can use the slope and the coordinates of the opposite vertex to write an equation for each altitude. Finally, I can calculate the intersection of the altitudes.

   d. incenter

   The incenter is the point of concurrency of the angle bisectors.

   I don't have any way of calculating the equations of the lines representing the angle bisectors. So, I cannot calculate the incenter algebraically.

1

Grouping

Have students complete Questions 4 through 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 9

- What are the steps involved in calculating the centroid?
- What are the steps involved in calculating the circumcenter?
- What are the steps involved in calculating the orthocenter?
- Are the centroid, circumcenter, orthocenter, and incenter collinear points?
- Do the four points of concurrency appear to be collinear?
- Do you think this is true for all isosceles triangles? Why or why not?
- Do you think this is true for any other triangles? If so, which ones?

4. Calculate the centroid of triangle *ABC*.

a. Calculate the midpoint of each side of the triangle.

Midpoint of $\overline{AB} = \left(\frac{-6+6}{2}, \frac{-8+(-8)}{2}\right) = \left(\frac{0}{2}, \frac{-16}{2}\right) = (0, -8)$

Midpoint of $\overline{AC} = \left(\frac{-6+0}{2}, \frac{-8+10}{2}\right) = \left(\frac{-6}{2}, \frac{2}{2}\right) = (-3, 1)$

Midpoint of $\overline{BC} = \left(\frac{0+6}{2}, \frac{10+(-8)}{2}\right) = \left(\frac{6}{2}, \frac{2}{2}\right) = (3, 1)$

b. Write an equation for each median.

Median connecting midpoint of side *AB* and point *C*

Use points (0, −8) and (0, 10).

The equation of a vertical line is of the form $x = a$.

$x = 0$

Median connecting midpoint of side *AC* and point *B*

Use points (−3, 1) and (6, −8).

Slope $= \frac{1-(-8)}{-3-6} = \frac{9}{-9} = -1$

Use the point-slope form.

$y - (-8) = -1(x - 6)$

$y + 8 = -x + 6$

$y = -x - 2$

Median connecting midpoint of side *BC* and point *A*

Use points (3, 1) and (−6, −8).

Slope $= \frac{1-(-8)}{3-(-6)} = \frac{9}{9} = 1$

Use the point-slope form.

$y - 1 = 1(x - 3)$

$y - 1 = x - 3$

$y = x - 2$

c. Calculate the coordinates of the centroid.

Determine the intersection of $x = 0$ and $y = x - 2$.

$y = x - 2$

$y = 0 - 2$

$y = -2$

The coordinates of the centroid are (0, −2).

d. Plot the centroid on the grid in Question 1.

See graph.

5. Calculate the circumcenter of triangle *ABC*.

a. Calculate the midpoint of each side of the triangle.

Midpoint of $\overline{AB} = \left(\frac{-6+6}{2}, \frac{-8+(-8)}{2}\right) = \left(\frac{0}{2}, \frac{-16}{2}\right) = (0, -8)$

Midpoint of $\overline{AC} = \left(\frac{-6+0}{2}, \frac{-8+10}{2}\right) = \left(\frac{-6}{2}, \frac{2}{2}\right) = (-3, 1)$

Midpoint of $\overline{BC} = \left(\frac{0+6}{2}, \frac{10+(-8)}{2}\right) = \left(\frac{6}{2}, \frac{2}{2}\right) = (3, 1)$

b. Calculate the slope of each side of the triangle.

Slope of $\overline{AB} = \frac{-8-(-8)}{6-(-6)} = \frac{0}{12} = 0$

Slope of $\overline{AC} = \frac{10-(-8)}{0-(-6)} = \frac{18}{6} = 3$

Slope of $\overline{BC} = \frac{-8-10}{6-0} = \frac{-18}{6} = -3$

c. Write an equation for each perpendicular bisector.

<u>Perpendicular bisector of side *AB*</u>

The slope of side *AB* is 0, so the slope of the perpendicular bisector is undefined.
A vertical line has an undefined slope.

The equation of a vertical line is of the form $x = a$.

$x = 0$

<u>Perpendicular bisector of side *AC*</u>

The slope of side *AC* is 3 so the slope of the perpendicular bisector is $-\frac{1}{3}$.

Use the point-slope form with the slope of $-\frac{1}{3}$ and the midpoint of side *AC* $(-3, 1)$.

$y - 1 = -\frac{1}{3}(x - (-3))$

$y - 1 = -\frac{1}{3}(x + 3)$

$y - 1 = -\frac{1}{3}x - 1$

$y = -\frac{1}{3}x$

<u>Perpendicular bisector of side *BC*</u>

The slope of side *BC* is -3 so the slope of the perpendicular bisector is $\frac{1}{3}$.

Use the point-slope form with the slope of $\frac{1}{3}$ and the midpoint of side *BC* $(3, 1)$.

$y - 1 = \frac{1}{3}(x - 3)$

$y - 1 = \frac{1}{3}x - 1$

$y = \frac{1}{3}x$

d. Calculate the coordinates of the circumcenter.

Determine the intersection of $x = 0$ and $y = \frac{1}{3}x$.

$y = \frac{1}{3}(0)$

$y = 0$

The coordinates of the circumcenter are (0, 0).

e. Plot the circumcenter on the grid in Question 1.

See graph.

6. Calculate the orthocenter of triangle *ABC*.

a. Calculate the slope of each side of the triangle.

Slope of $\overline{AB} = \frac{-8 - (-8)}{6 - (-6)} = \frac{0}{12} = 0$

Slope of $\overline{AC} = \frac{10 - (-8)}{0 - (-6)} = \frac{18}{6} = 3$

Slope of $\overline{BC} = \frac{-8 - 10}{6 - 0} = \frac{-18}{6} = -3$

b. Write an equation for each altitude.

Altitude passing through point *C*

The slope of side *AB* is 0 so the slope of the altitude is undefined. A vertical line has an undefined slope.

The equation of a vertical line is of the form $x = a$.

$x = 0$

Altitude passing through point *A*

The slope of side *BC* is -3 so the slope of the altitude is $\frac{1}{3}$.

Use the point-slope form with the slope of $\frac{1}{3}$ and point *A* $(-6, -8)$.

$y - (-8) = \frac{1}{3}(x - (-6))$

$y + 8 = \frac{1}{3}(x + 6)$

$y + 8 = \frac{1}{3}x + 2$

$y = \frac{1}{3}x - 6$

Altitude passing through point *B*

The slope of side *AC* is 3, so the slope of the altitude is $-\frac{1}{3}$.

Use the point-slope form with the slope of $-\frac{1}{3}$ and point *A* $(6, -8)$.

$y - (-8) = -\frac{1}{3}(x - 6)$

$y + 8 = -\frac{1}{3}x + 2$

$y = -\frac{1}{3}x - 6$

c. Calculate the coordinates of the orthocenter.

Determine the intersection of $x = 0$ and $y = \frac{1}{3}x - 6$.

$y = \frac{1}{3}(0) - 6$

$y = 0 - 6$

$y = -6$

The coordinates of the orthocenter are $(0, -6)$.

d. Plot the orthocenter on the grid in Question 1.

See graph.

7. What do you notice about the location of the centroid, circumcenter, and orthocenter for isosceles triangle *ABC*?

The centroid, circumcenter, and orthocenter are collinear.

8. Do you think the behavior that you described in Question 8 will also be true for the incenter? Explain your reasoning.

Yes. The incenter should also be collinear with the centroid, circumcenter, and orthocenter.

The angle bisector of angle *C* is the line $x = 0$ which is the line that all of the other points of concurrency lie on.

9. Do you think the behavior that you described in Questions 7 through 8 will be true for a triangle that is not isosceles? Explain how you could test your theory.

I could draw a triangle that is not isosceles and calculate the location of each point of concurrency to determine if they are collinear.

1

Problem 9

Students are given the location of three small towns on the coordinate plane. They determine the location of a new mall that is equidistant from each of the three towns and determine the cost of building the necessary access roads.

Grouping

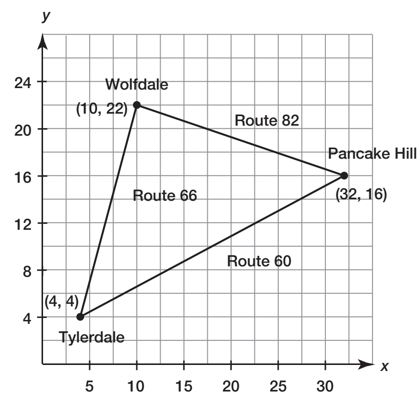
- Ask students to read introduction. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- Why is the circumcenter most useful in this situation?
- What steps are necessary to determine the coordinates of the circumcenter?
- Why isn't the centroid useful in this situation?
- How is the distance from the circumcenter to each town determined?
- How are the costs of the new access roads determined?

PROBLEM 9 The Mall

A construction company plans to build a large mall to serve three small towns as shown on the map below. They are considering a location that is equidistant from each of the three towns and agree to pay for building new roads connecting the mall to the three towns. Cost for building new roads is $150,000 per mile. Each unit on the graph represents one mile.



1. What point of concurrency is most useful in this situation? Explain your reasoning.

   The circumcenter is most useful because it is equidistant from each of the vertices.

2. Determine the approximate coordinates of the location of the mall.

Calculate the midpoint of each side of the triangle.

Midpoint of $\overline{WP} = \left(\frac{10+32}{2}, \frac{22+16}{2}\right) = \left(\frac{42}{2}, \frac{38}{2}\right) = (21, 19)$

Midpoint of $\overline{PT} = \left(\frac{32+4}{2}, \frac{16+4}{2}\right) = \left(\frac{36}{2}, \frac{20}{2}\right) = (18, 10)$

Midpoint of $\overline{TW} = \left(\frac{4+10}{2}, \frac{4+22}{2}\right) = \left(\frac{14}{2}, \frac{26}{2}\right) = (7, 13)$

Calculate the slope of each side of the triangle.

Slope of $\overline{WP} = \frac{22-16}{10-32} = \frac{6}{-22} = -\frac{3}{11}$

Slope of $\overline{PT} = \frac{16-4}{32-4} = \frac{12}{28} = \frac{3}{7}$

Slope of $\overline{TW} = \frac{22-4}{10-4} = \frac{18}{6} = 3$

Perpendicular bisector of side *WP*

The slope of side *WP* is $-\frac{3}{11}$ so the slope of the perpendicular bisector is $\frac{11}{3}$.

Use the point-slope form with the slope of $\frac{11}{3}$ and the midpoint of side *WP* (21, 19).

$$y - 19 = \frac{11}{3}(x - 21)$$
$$y - 19 = \frac{11}{3}x - 77$$
$$y = \frac{11}{3}x - 58$$

Perpendicular bisector of side *PT*

The slope of side *PT* is $\frac{3}{7}$ so the slope of the perpendicular bisector is $-\frac{7}{3}$.

Use the point-slope form with the slope of $-\frac{7}{3}$ and the midpoint of side *PT* (18, 10).

$$y - 10 = -\frac{7}{3}(x - 18)$$
$$y - 10 = -\frac{7}{3}x + 42$$
$$y = -\frac{7}{3}x + 52$$

To calculate the coordinates of the circumcenter, determine the intersection of $y = \frac{11}{3}x - 58$ and $y = -\frac{7}{3}x + 52$.

$$\frac{11}{3}x - 58 = -\frac{7}{3}x + 52$$
$$\frac{18}{3}x = 110$$
$$6x = 110$$
$$x = \frac{110}{6} = 18\frac{1}{3}$$
$$y = \frac{11}{3}x - 58$$
$$y = \frac{11}{3}\left(\frac{110}{6}\right) - 58$$
$$y = \frac{1210}{18} - 58$$
$$y = \frac{1210}{18} - \frac{1044}{18}$$
$$y = \frac{166}{18} = \frac{83}{9} = 9\frac{2}{9}$$

The coordinates of the circumcenter are $\left(18\frac{1}{3}, 9\frac{2}{9}\right)$.

3. Determine the approximate distance from the mall location to each town. Round to the nearest mile.

The distance from the mall to each town is approximately 15.26 miles.

$$d = \sqrt{\left(18\tfrac{1}{3} - 4\right)^2 + \left(9\tfrac{2}{9} - 4\right)^2}$$

$$\approx 15.26$$

4. How much money should be budgeted for road construction in this project?

Each new road will cost approximately 15.26 × 150,000, or $2,289,000.

The total budget should be at least 2,289,000 × 3, or $6,867,000.

Talk the Talk

Students are given different situations and determine which point of concurrency is most helpful.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- Why would the circumcenter be most helpful in Question 1 part (a)?
- Why would the centroid be most helpful in Question 1 part (b)?
- Why would the incenter be most helpful in Question 1 part (c)?
- Why would the orthocenter be most helpful in Question 2?
- Why would the centroid be most helpful in Question 3?

Talk the Talk

1. Determine which point of concurrency would be most helpful in each situation.

   a. A flea market is situated on a triangular piece of land. Each entrance is located at one of the three vertices of triangle. Joanie wants to set up her merchandise at a location that is equidistant from all three entrances.

   The circumcenter would be most helpful for Joanie to determine where to set up her merchandise.

   b. An artist is building a mobile with several metal triangles of various sizes. The triangles are connected to each other using steel rods and the rods are welded onto each triangle at a point which would allow the triangle to balance horizontally.

   The centroid would be most helpful for the creation of this mobile.

   c. Jim's backyard is a triangular plot of land. He is using fencing to build a circular dog pen. He wants the dog pen to be as large as possible and needs to determine the location of the center of the circular dog pen.

   The incenter would be most helpful to determine the center of the dog pen.

2. The intersection of the three altitudes of any triangle is best described by this point.

   The orthocenter would be most helpful.

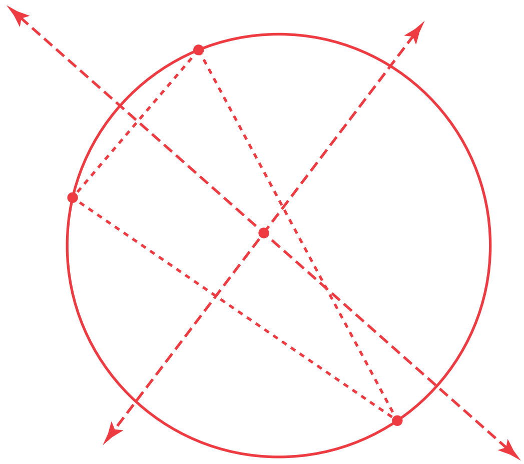
3. The point of concurrency located two-thirds the way from the vertex to the midpoint of the opposite side and otherwise known as the center of gravity.

   The centroid would be most helpful.

Be prepared to share your solutions and methods.

Check for Students' Understanding

1. Plot three non-collinear points.



2. Jane wants to construct a circle that passes through each of the three non-collinear points. What point of concurrency is most helpful?

   The circumcenter would be most helpful to construct a circle that passes through the three non-collinear points because it is equidistant from each vertex of a triangle.

3. Construct the circle using this point of concurrency.

Chapter 1 Summary

KEY TERMS

- point (1.1)
- line (1.1)
- collinear points (1.1)
- plane (1.1)
- compass (1.1)
- straightedge (1.1)
- sketch (1.1)
- draw (1.1)
- construct (1.1)
- coplanar lines (1.1)
- skew lines (1.1)
- ray (1.1)
- endpoint of a ray (1.1)
- line segment (1.1)
- endpoints of a line segment (1.1)
- congruent line segments (1.1)
- Distance Formula (1.2)
- transformation (1.2)
- rigid motion (1.2)
- translation (1.2)
- pre-image (1.2)
- image (1.2)
- arc (1.2)
- midpoint (1.3)
- Midpoint Formula (1.3)
- segment bisector (1.3)
- angle (1.4)
- angle bisector (1.4)
- perpendicular bisector (1.5)
- concurrent (1.6)
- point of concurrency (1.6)
- circumcenter (1.6)
- incenter (1.6)
- median (1.6)
- centroid (1.6)
- altitude (1.6)
- orthocenter (1.6)

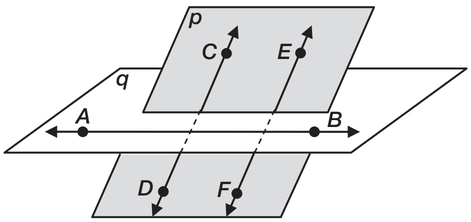
CONSTRUCTIONS

- copying a line segment (1.2)
- duplicating a line segment (1.2)
- bisecting a line segment (1.3)
- copying an angle (1.4)
- duplicating an angle (1.4)
- bisecting an angle (1.4)
- perpendicular line to a given line through a point on the line (1.5)
- perpendicular line to a given line through a point not on the line (1.5)

1.1 Identifying Points, Lines, and Planes

A point is a location in space that has no size or shape. A line is a straight continuous arrangement of an infinite number of points. A plane is a flat surface that has an infinite length and width, but no depth. Collinear points are points that are located on the same line. Coplanar lines are two or more lines that are located in the same plane. Skew lines are two or more lines that are not in the same plane.

Example



Points A and B lie on $\overline{AB}$, points C and D lie on $\overline{CD}$, and points E and F lie on $\overline{EF}$.

Line AB lies in plane q. Lines CD and EF lie in plane p.

Points A and B are collinear. Points C and D are collinear. Points E and F are collinear.

Lines CD and EF are coplanar.

Lines AB and CD are skew. Lines AB and EF are skew.

Planes p and q intersect.

1.2 Applying the Distance Formula

The Distance Formula can be used to calculate the distance between two points on the coordinate plane. The Distance Formula states that if (x_1, y_1) and (x_2, y_2) are two points on the coordinate plane, then the distance d between (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example

Calculate the distance between the points $(3, -2)$ and $(-5, 1)$.

$x_1 = 3, y_1 = -2, x_2 = -5, y_2 = 1$

$$\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(-5 - 3)^2 + [1 - (-2)]^2} \\
&= \sqrt{(-8)^2 + (3)^2} \\
&= \sqrt{64 + 9} \\
&= \sqrt{73} \\
&\approx 8.5
\end{aligned}$$

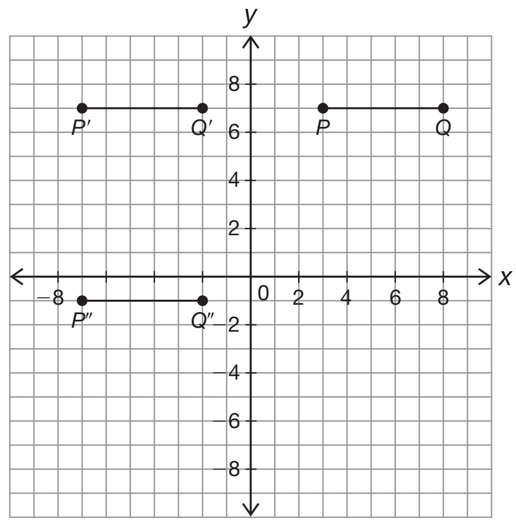
The distance between the points $(3, -2)$ and $(-5, 1)$ is $\sqrt{73}$ units, or approximately 8.5 units.

1.2 Translating Line Segments on the Coordinate Plane

A translation is a rigid motion that slides each point of a figure the same distance and direction. A horizontal translation of a line segment on the coordinate plane changes the *x*-coordinates of both endpoints while leaving the *y*-coordinates the same. A vertical translation changes the *y*-coordinates of both endpoints while leaving the *x*-coordinates the same.

Example

Line segment *PQ* is translated horizontally 10 units to the left to create $\overline{P'Q'}$. Line segment *P′Q′* is translated vertically 8 units down to create line segment $\overline{P''Q''}$.



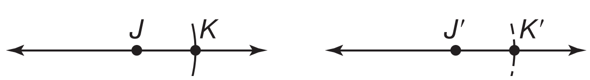
Line Segment	$\overline{PQ}$	$\overline{P'Q'}$	$\overline{P''Q''}$
Coordinates of Endpoints	(3, 7) (8, 7)	(−7, 7) (−2, 7)	(−7, −1) (−2, −1)

The lengths of the images and the pre-images remain the same after each translation.

1.2 Duplicating a Line Using Construction Tools

A straightedge and compass can be used to duplicate a line.

Example



Line segment *JK* can be duplicated using a straightedge and compass by drawing a starter line and then duplicating a line segment that is the same length as $\overline{JK}$.

1

1.3 Applying the Midpoint Formula

A midpoint is a point that is exactly halfway between two given points. The Midpoint Formula can be used to calculate the coordinates of a midpoint. The Midpoint Formula states that if (x_1, y_1) and (x_2, y_2) are two points on the coordinate plane, then the midpoint of the line segment that joins these two points is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example

Calculate the midpoint of a line segment with the endpoints $(-8, -3)$ and $(4, 6)$.

$x_1 = -8, y_1 = -3, x_2 = 4, y_2 = 6$

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-8 + 4}{2}, \frac{-3 + 6}{2}\right)$$

$$= \left(\frac{-4}{2}, \frac{3}{2}\right)$$

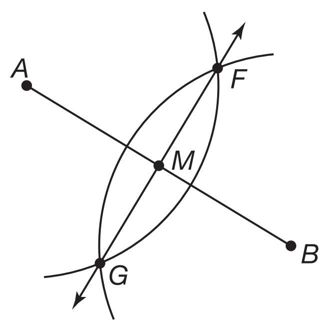
$$= \left(-2, \frac{3}{2}\right)$$

The midpoint of the line segment is $\left(-2, \frac{3}{2}\right)$.

1.3 Bisecting a Line Segment Using Construction Tools

Construction tools can be used to bisect a line segment.

Example



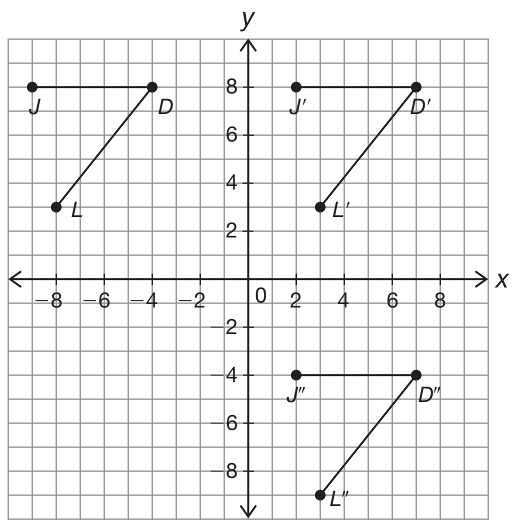
Open the radius of the compass to more than half the length of the original line segment. Construct an arc using one endpoint as the center. Keeping the compass at the same radius, construct an arc using the other endpoint as center. Label and connect the points created by the intersection of the arcs. Line segment FG bisects $\overline{AB}$.

1.4 Translating an Angle on the Coordinate Plane

Translating an angle on the coordinate plane is a rigid motion that slides the angle, either horizontally or vertically, on the coordinate plane. Because it is a rigid motion, the angle measures of the image and the pre-image are the same. Horizontal translations only impact the *x*-coordinates of the endpoints; vertical translations only impact the *y*-coordinates of the endpoints.

Example

Angle *JDL* is translated horizontally 11 units right to form $\angle J'D'L'$. Angle $J'D'L'$ is translated vertically 12 units down to create $\angle J''D''L''$.



Line Segment	$\overline{JD}$	$\overline{J'D'}$	$\overline{J''D''}$
Coordinates of Endpoints	(−9, 8) (−4, 8)	(2, 8) (7, 8)	(2, −4) (7, −4)

Line Segment	$\overline{DL}$	$\overline{D'L'}$	$\overline{D''L''}$
Coordinates of Endpoints	(−4, 8) (−8, 3)	(7, 8) (3, 3)	(7, −4) (3, −9)

The measure of the angle images and pre-images remain the same after each translation.

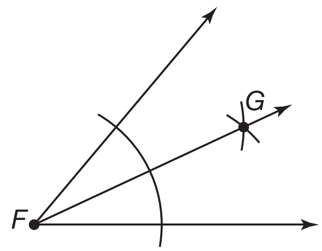
1

1.4 Bisecting an Angle Using Construction Tools

An angle bisector is a ray drawn through the vertex of an angle that divides the angle into two angles of equal measure.

Example

Angle *F* can be bisected using construction tools.

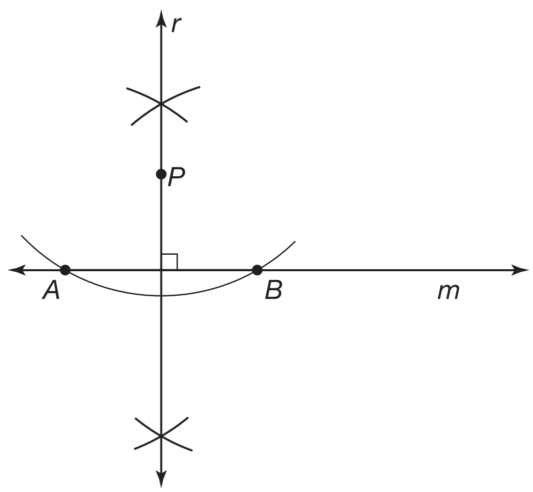


Place the compass on the vertex of the angle. Construct an arc that intersects both sides of the angle. Place the compass at one of the intersection points and construct an arc, then using the same radius of the compass construct an arc using the other intersection point. Construct a ray connecting the vertex to the intersection of the arcs. Ray *FG* bisects $\angle F$.

1.5 Constructing Perpendicular Lines

Perpendicular lines can be constructed through a given point using construction tools.

Example



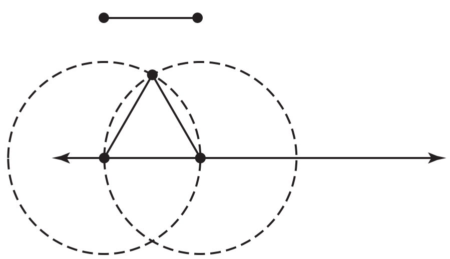
Use the given point *P* as the center and construct an arc that passes through the given line. Open the compass radius. Construct an arc above and below the given line using one of the intersection points just created. Keeping the radius the same, construct an arc above and below the given line using the other intersection point. Connect the intersection points of the arcs which should also pass through the given point. Line *r* is perpendicular to line *m*.

1.5 Constructing Equilateral Triangles

Equilateral triangles have 3 congruent sides. Construction tools can be used to construct an equilateral triangle given the length of one side.

Example

Construct an equilateral triangle with the side length shown.



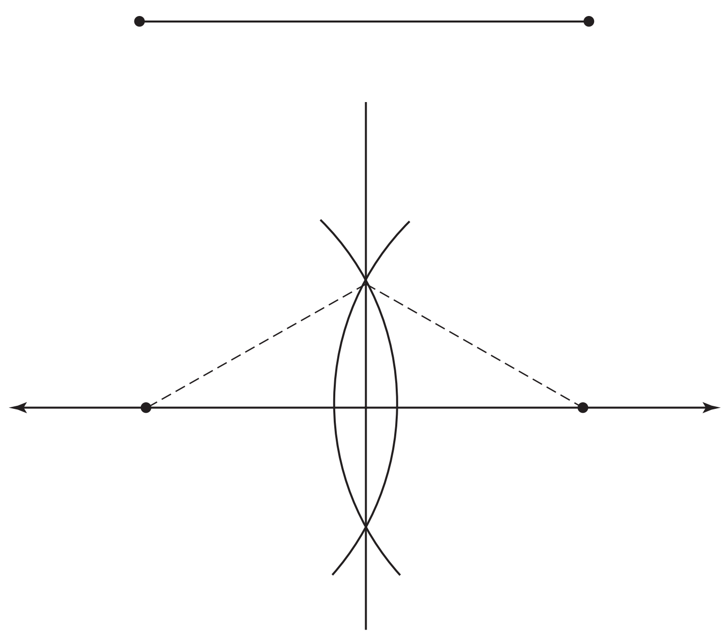
Construct a starter line and duplicate the given segment onto the starter line. Construct a circle using an endpoint of the line segment as the center. Then construct another circle using the other endpoint as the center. Connect the point of intersection of the circles to each endpoint using line segments.

1.5 Constructing Isosceles Triangles

An isosceles triangle is a triangle that has at least two sides of equal length.

Example

Construct an isosceles triangle with the side length shown.



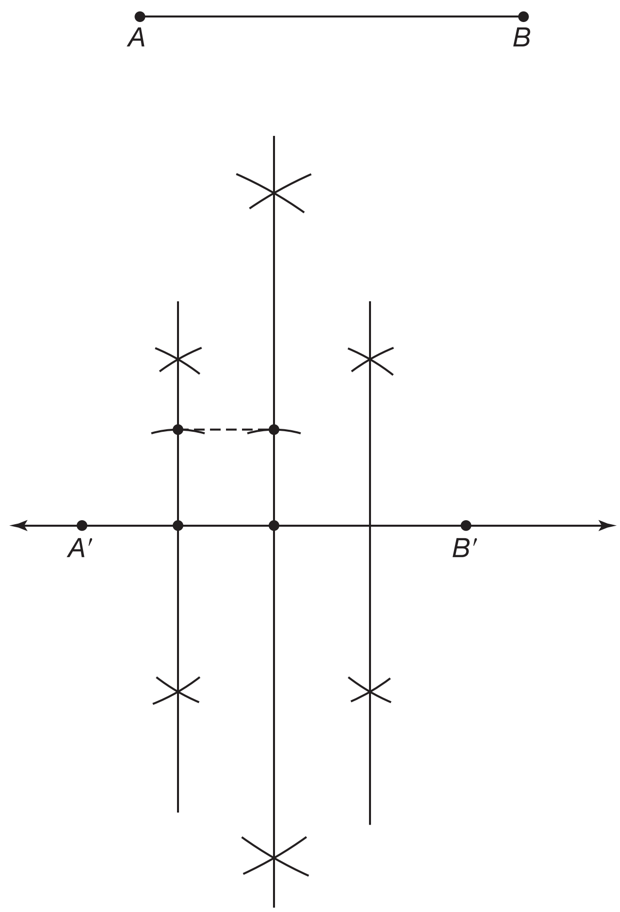
Construct a starter line and duplicate the given line segment. Then construct a perpendicular bisector through the line segment. Connect the endpoints of each line segment to a point on the bisector.

1.5 Constructing Squares

A square can be constructed using construction tools.

Example

Construct a square using the perimeter given.



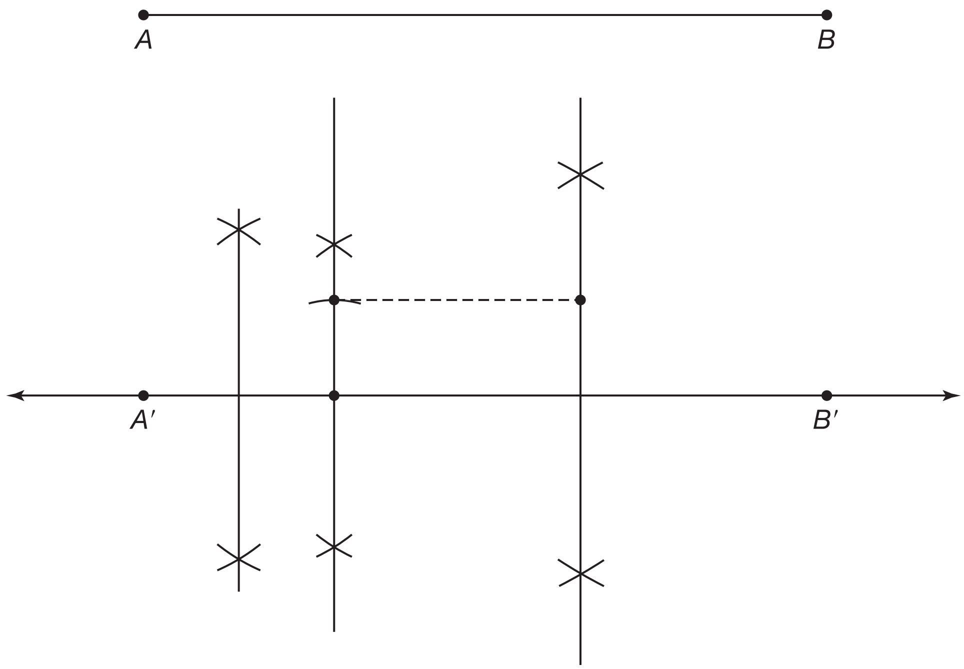
Construct a starter line and duplicate the given perimeter. Bisect the line segment using a perpendicular bisector. Then, bisect each of the created line segments to create 4 line segments of equal length. Duplicate one of the line segments along two perpendicular bisectors to create the height of the square. Connect the two endpoints of the line segments representing the height to complete the square.

1.5 Constructing Rectangles That Are Not Squares

A rectangle can be constructed in a similar method to constructing a square using a given perimeter of the rectangle.

Example

Construct a rectangle using the perimeter given.



Construct a starter line and duplicate the given perimeter. Place a point anywhere on the line segment except in the middle dividing the line segment into two unequal line segments. Then, draw perpendicular bisectors through each of the line segments to create four line segments. Choose one of the line segments to use as the base of the rectangle. Duplicate another line segment that is not the same size as the base on two of the perpendicular bisectors to use as the height of the rectangle. Finally, connect the endpoints of the line segments representing the height to create a rectangle.

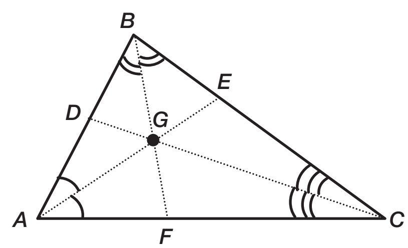
1.6 Identifying Points of Concurrency

When three or more lines intersect at the same point, the lines are called concurrent lines. The point at which the concurrent lines intersect is called the point of concurrency.

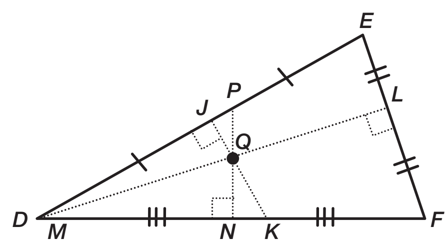
There are special types of points of concurrency in triangles. The incenter of a triangle is the point at which the three angle bisectors of a triangle are concurrent. The circumcenter of a triangle is the point at which the three perpendicular bisectors of a triangle are concurrent. The centroid is the point at which the three medians of a triangle are concurrent. The orthocenter is the point at which the three altitudes of a triangle are concurrent.

Examples

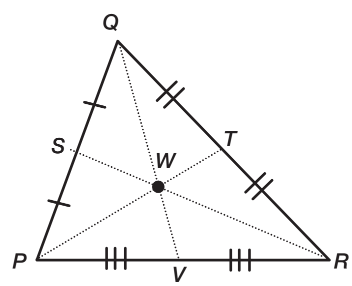
In $\triangle ABC$, $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$ are angle bisectors. So, point G is the incenter, and $DG = EG = FG$.



In $\triangle DEF$, $\overline{JK}$, $\overline{LM}$, and $\overline{NP}$ are perpendicular bisectors. So, point Q is the circumcenter, and the distances from the circumcenter to each vertex are the same.



In $\triangle PQR$, $\overline{PT}$, $\overline{QV}$, and $\overline{RS}$ are medians. So, point W is the centroid, and $PW = 2TW$, $QW = 2VW$, and $RW = 2SW$.



In $\triangle XYZ$, $\overline{XB}$, $\overline{YC}$, and $\overline{ZA}$ are altitudes. So, point D is the orthocenter.