More Probability, and Counting

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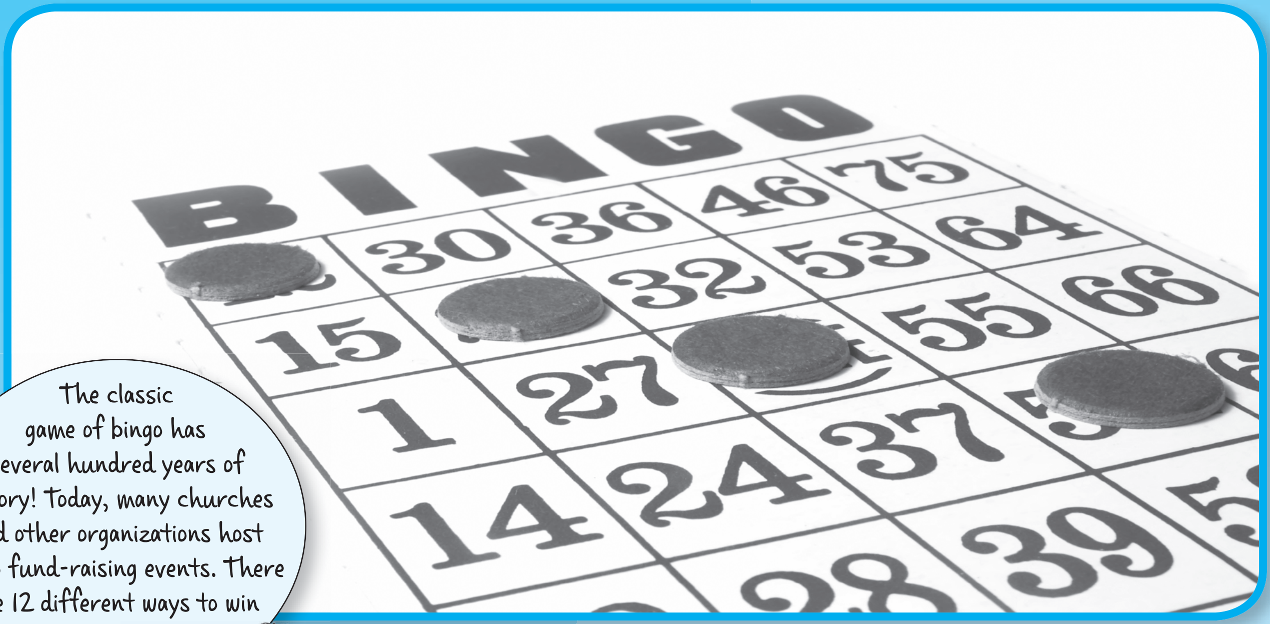

Chapter 20 Overview

This chapter addresses more compound probability concepts and more counting strategies. Compound probability concepts are presented using two-way frequency tables, conditional probability, and independent trials. The counting strategies include permutations, permutations with repetition, circular permutations, and combinations. The last lesson focuses on geometric probability and expected value.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
20.1	Compound Probability for Data Displayed in Two-Way Tables	S.CP.4	2	The lesson focuses on displaying data in two-way tables and using the data to determine compound probability. Questions ask students to complete two-way frequency tables, determine whether or not two events are independent, and determine the compound probability in a variety of scenarios.	X		X		
20.2	Conditional Probability	S.CP.3 S.CP.5 S.CP.6	2	The lesson focuses on determining conditional probability. Questions ask students to determine compound probability and conditional probability using two-way tables. Conditional probability is used to determine whether or not two events are independent. A formula for conditional probability is derived and used to calculate conditional probability in a variety of scenarios.	X	X	X		
20.3	Permutations and Combinations	S.CP.9	3	The lesson focuses on using permutations and combinations to calculate the size of sample spaces. Questions guide students to build conceptual understanding of factorials, permutations, permutations with repeated elements, circular permutations, and combinations. Students use factorials, permutations, permutations with repeated elements, circular permutations, and combinations to determine the size of sample spaces in variety of scenarios.	X	X	X	X	X

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
20.4	Independent Trials	S.CP.9	2	The lesson focuses on determining the probability of two or more trials of independent events. Questions guide students to build their conceptual knowledge using scenarios involving making or missing basketball free throws. The questions build on students' previous knowledge of probability and combinations to present Pascal's Triangle and derive the formula for calculating the probability of multiple trials of independent events.	X		X		
20.5	Expected Value	S.MD.6 S.MD.7	2	The lesson focuses on determining geometric probability and calculating the expected value of an event. Questions guide students to build their conceptual knowledge of geometric probability using various dart board scenarios. Students use their previous knowledge of probability to derive the calculations for expected value and determine the expected value in various scenarios.	X		X		

Skills Practice Correlation for Chapter 20

Lesson		Problem Set	Objectives
20.1	Compound Probability for Data Displayed in Two-Way Tables		Vocabulary
		1 – 8	Calculate relative frequencies
		9 – 22	Calculate relative frequencies in a two-way table
		23 – 34	Use a two-way table to calculate probabilities
		35 – 48	Use a table of relative frequencies to calculate probabilities
20.2	Conditional Probability		Vocabulary
		1 – 10	Use a table to determine probabilities
		11 – 14	Determine conditional probabilities
		15 – 20	Determine probabilities from a situation
		21 – 24	Determine probabilities from a two-way frequency table
20.3	Permutations and Combinations		Vocabulary
		1 – 6	Evaluate permutations expressions
		7 – 10	Calculate the number of possible outcomes in permutations situations
		11 – 16	Evaluate combinations expressions
		17 – 20	Calculate the number of possible outcomes in combinations situations
		21 – 24	Use permutations and combinations to solve problems
		25 – 28	Use permutations and combinations to solve problems
		29 – 34	Calculate permutations with repeated elements
		35 – 40	Calculate circular permutations
20.4	Independent Trials	1 – 6	Determine probabilities in problem situations
		7 – 12	Use formulas to determine probabilities
		13 – 18	Use formulas to determine probabilities
20.5	Expected Value		Vocabulary
		1 – 8	Determine geometric probabilities
		9 – 12	Determine expected values
		13 – 18	Determine expected values using spinners

20.1

Left, Left, Left, Right, Left

Compound Probability for Data Displayed in Two-Way Tables

LEARNING GOALS

In this lesson, you will:

- Determine probabilities of compound events for data displayed in two-way tables.
- Determine relative frequencies of events.

KEY TERMS

- two-way table
- frequency table
- two-way frequency table
- contingency table
- categorical data
- qualitative data
- relative frequency
- two-way relative frequency table

ESSENTIAL IDEAS

- A two-way table is a table that shows the relationship between two data sets, one organized in rows and one organized in columns.
- A frequency table is a table that shows the frequency of an item, number, or event appearing in a sample space.
- A two-way frequency table or contingency table shows the number of data points and their frequencies for two variables.
- A relative frequency is the ratio of occurrences within a category to the total number of occurrences.
- A two-way relative frequency table displays the relative frequencies for two categories of data.
- Two-way tables can be used to determine the probabilities of compound events.
- The converse of the multiplication rule for probability states that If the probability of two events *A* and *B* occurring together is $P(A) \cdot P(B)$, then the two events are independent.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

S-CP Conditional Probability and the Rules of Probability

Understand independence and conditional probability and use them to interpret data

4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities.

Overview

Two number cubes and the results of surveys are the contexts for creating sample spaces, organized lists, and tables. The converse of the multiplication rule is stated and used to determine when events are independent. The terms frequency, frequency table, two-way frequency table, relative frequency, and two-way relative frequency table are introduced. Students complete these types of tables and use the tables to answer questions related to the situations. Students convert ratios to percents.

Warm Up

A quiz in a magazine contains 5 true-false questions. The questions are written in a foreign language you do not recognize.

What is the probability of guessing the correct answers to all 5 questions?
Create a table or organized list to determine the probability.

	Question 1	Question 2	Question 3	Question 4	Question 5
1	Correct	Correct	Correct	Correct	Correct
2	Correct	Correct	Correct	Correct	Incorrect
3	Correct	Correct	Correct	Incorrect	Correct
4	Correct	Correct	Incorrect	Correct	Correct
5	Correct	Incorrect	Correct	Correct	Correct
6	Incorrect	Correct	Correct	Correct	Correct
7	Correct	Correct	Correct	Incorrect	Incorrect
8	Correct	Correct	Incorrect	Correct	Incorrect
9	Correct	Correct	Incorrect	Incorrect	Correct
10	Correct	Incorrect	Correct	Correct	Incorrect
11	Correct	Incorrect	Correct	Incorrect	Correct
12	Correct	Incorrect	Incorrect	Correct	Correct
13	Incorrect	Correct	Correct	Correct	Incorrect
14	Incorrect	Correct	Correct	Incorrect	Correct
15	Incorrect	Correct	Incorrect	Correct	Correct
16	Incorrect	Incorrect	Correct	Correct	Correct
17	Correct	Correct	Incorrect	Incorrect	Incorrect
18	Correct	Incorrect	Correct	Incorrect	Incorrect
19	Correct	Incorrect	Incorrect	Correct	Incorrect
20	Correct	Incorrect	Incorrect	Incorrect	Correct
21	Incorrect	Correct	Correct	Incorrect	Incorrect
22	Incorrect	Correct	Incorrect	Correct	Incorrect
23	Incorrect	Correct	Incorrect	Incorrect	Correct
24	Incorrect	Incorrect	Correct	Correct	Incorrect
25	Incorrect	Incorrect	Correct	Incorrect	Correct
26	Incorrect	Incorrect	Incorrect	Correct	Correct
27	Correct	Incorrect	Incorrect	Incorrect	Incorrect
28	Incorrect	Correct	Incorrect	Incorrect	Incorrect
29	Incorrect	Incorrect	Correct	Incorrect	Incorrect
30	Incorrect	Incorrect	Incorrect	Correct	Incorrect
31	Incorrect	Incorrect	Incorrect	Incorrect	Correct
32	Incorrect	Incorrect	Incorrect	Incorrect	Incorrect

The probability of guessing all 5 answers correctly is $\frac{1}{32}$.

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Left, Left, Left, Right, Left

20.1

Compound Probability for Data Displayed in Two-Way Tables

LEARNING GOALS

In this lesson, you will:

- Determine probabilities of compound events for data displayed in two-way tables.
- Determine relative frequencies of events.

KEY TERMS

- two-way table
- frequency table
- two-way frequency table
- contingency table
- categorical data
- qualitative data
- relative frequency
- two-way relative frequency table

One good thing about a multiple choice question is that you can always make an educated guess. This can be a good strategy, especially if you're not 100% sure about the solution.

On the flip side, relying too much on guessing means you may not have been prepared for the questions. Remember, a little extra preparation can go a long way!

Have you thought about the chances of guessing the correct answer to a multiple choice question? What about the chances of guessing the correct answers to a bunch of multiple choice questions?

Problem 1

Two number cubes are used to generate outcomes. Students write the sample space when two number cubes are rolled and the sum is calculated. The sample space is created in the form of an organized list, and then students represent all of the possible sums in a two-way table. The converse of the multiplication rule is stated and used to determine when events are independent. Next, students create a frequency table listing the frequencies of each possible sum and use the table to determine probabilities of specified events.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- How many possible outcomes are there for the first number cube?
- How many possible outcomes are there for the second number cube?
- How is the Counting Principle used to compute the number of possible outcomes in this situation?
- Did you use an organized list to write the sample space representing the possible outcomes?

PROBLEM 1 Rolling, Rolling, Rolling, Keep Those Number Cubes Rolling

When playing board games, it is common to roll two number cubes and calculate their sum.

1. Determine the number of possible outcomes when two number cubes are rolled once. Explain your conclusion.

   There are 36 outcomes for rolling two number cubes one time.

   I used the Counting Principle to calculate the answer. There are 6 possible outcomes for the first number cube and 6 possible outcomes for the second number cube.

   $6 \times 6 = 36$

2. Write the sample space for rolling two number cubes one time.

1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

A **two-way table** shows the relationship between two data sets, one data set is organized in rows and the other data set is organized in columns.

3. Complete the two-way table to represent all the possible sums that result from rolling two number cubes one time.

	2nd Number Cube					
1st Number Cube	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

4. What are the possible sums? Explain your reasoning.

   The possible sums are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12.
   I determined the possible sums from the two-way table. The least sum of 2 results from rolling two 1's. The greatest sum of 12 results from rolling two 6's.

- How many outcomes are possible?
- How did you decide what should be written in each cell of the two-way table?
- What kind of questions can be answered using this two-way table?
- Are all of the numbers in the two-way table possible sums of the two number cubes when rolled once?
- What is the least sum?
- What is the greatest sum?

Grouping

Have students complete Question 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- What is the purpose of the converse of the multiplication rule? In what situation is it helpful?
- If $P(A \text{ and } B) = P(A) \cdot P(B)$, are the events independent or dependent?
- What ratio represents $P(A)$ in this situation?
- What ratio represents $P(B)$ in this situation?
- What ratio represents $P(A \text{ and } B)$ in this situation?

You have learned that if two events *A* and *B* are independent, the probability of both events occurring, $P(A \text{ and } B)$, is $P(A) \cdot P(B)$.

The converse is also true: If the probability of two events *A* and *B*, $P(A \text{ and } B)$, occurring together is $P(A) \cdot P(B)$, then the two events are independent.

5. Use the converse of the multiplication rule to determine whether the events are independent. Explain your reasoning.

a. A result of 3 on the first number cube and a result of 3 on the second number cube

The events are independent because the probability of the compound event $P(A \text{ and } B)$ is equal to the product of the events' individual probabilities $P(A) \cdot P(B)$.

Let *A* represent the event resulting in a 3 on the first number cube.

Let *B* represent the event resulting in a 3 on the second number cube.

One of the 36 outcomes in the sample space results in a 3 on the first number cube and a 3 on the second number cube, which means $P(A \text{ and } B) = \frac{1}{36}$.

The individual probabilities of events *A* and *B* are $P(A) = \frac{1}{6}$ and $P(B) = \frac{1}{6}$, and $P(A) \cdot P(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.
Because the values of $P(A \text{ and } B)$ and $P(A) \cdot P(B)$ both equal $\frac{1}{36}$, events *A* and *B* are independent.

b. A result of two 3's and a sum of 6

The events are not independent because the probability of the compound event $P(A \text{ and } B)$ is not equal to the product of the events' individual probabilities $P(A) \cdot P(B)$.

Let *A* represent the event resulting in two 3's.

Let *B* represent the event resulting in a sum of 6.

One of the 36 outcomes in the sample space results in two 3's and a sum of 6, which means $P(A \text{ and } B) = \frac{1}{36}$.
The individual probabilities of events *A* and *B* are $P(A) = \frac{1}{36}$ and $P(B) = \frac{5}{36}$, and $P(A) \cdot P(B) = \frac{1}{36} \cdot \frac{5}{36} = \frac{5}{1296}$.
Because the value of $P(A \text{ and } B)$, $\frac{1}{36}$, is not equal to the value of $P(A) \cdot P(B)$, $\frac{5}{1296}$, events *A* and *B* are not independent.

c. A result of 6 on the first number cube and a sum less than 7

The events are not independent because the probability of the compound event $P(A \text{ and } B)$ is not equal to the product of the events' individual probabilities $P(A) \cdot P(B)$.

Let A represent the event resulting in a 6 on the first number cube.

Let B represent the event resulting in a sum less than 7.

None of the 36 outcomes in the sample space results in a 6 on the first number cube and a sum less than 7, which means $P(A \text{ and } B) = \frac{0}{36}$, or 0.

The individual probabilities of events A and B are $P(A) = \frac{1}{6}$ and $P(B) = \frac{15}{36}$, and $P(A) \cdot P(B) = \frac{1}{6} \cdot \frac{15}{36} = \frac{15}{216}$, or $\frac{5}{12}$.

Because the value of $P(A \text{ and } B)$, 0, is not equal to the value of $P(A) \cdot P(B)$, $\frac{5}{12}$, events A and B are not independent.

d. A result of any number on the first number cube and a sum less than 7

The events are independent because the probability of the compound event $P(A \text{ and } B)$ is equal to the product of the events' individual probabilities $P(A) \cdot P(B)$.

Let A represent the event resulting in any number on the first number cube.

Let B represent the event resulting in a sum less than 7.

Fifteen of the 36 outcomes in the sample space results in any number on the first number cube and a sum less than 7, which means $P(A \text{ and } B) = \frac{15}{36}$.

The individual probabilities of events A and B are $P(A) = \frac{6}{6}$ and $P(B) = \frac{15}{36}$, and $P(A) \cdot P(B) = \frac{6}{6} \cdot \frac{15}{36} = \frac{15}{36}$.

Because the values of $P(A \text{ and } B)$ and $P(A) \cdot P(B)$ both equal $\frac{15}{36}$, events A and B are independent.

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Grouping

Have students complete Question 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- What is the difference between a two-way table and a frequency table?
- How did you decide what should be written in each cell of the frequency table?
- What kind of questions can be answered using this frequency table?
- What does the word frequency mean?
- How is the frequency table used to determine probabilities of events occurring?
- What types of probabilities can be determined using this frequency table?
- What are the different ways the sum of 7 can be generated?
- Which sums are considered odd?
- Which sums are less than 7?

A **frequency table** shows the frequency of an item, number, or event appearing in a sample space.

6. Use the two-way table to answer each question.

a. Complete the frequency table shown.

Outcome (Sum of the Number Cubes)	Frequency
2	1
3	2
4	3
5	4
6	5
7	6
8	5
9	4
10	3
11	2
12	1

b. What is the sum of the frequencies? Why?

The sum of the frequencies is 36, which is equal to the size of the sample space.

c. Which sum appears most often in the frequency table? Why do you think this sum appears the most?

The sum of 7 appears the most within the frequency table.

The reason 7 appears the most within the frequency table is there are more ways to get a sum of 7 than any other number combination. The ways to get a sum of 7 are 1 + 6, 6 + 1, 5 + 2, 2 + 5, 3 + 4, and 4 + 3.

d. Using the frequency table, can you determine the probability of rolling a sum or 7? Why or why not?

Yes. The sum of the frequencies, 36, is the number of possible outcomes. The frequency for 7 tells me the number of outcomes for the event "sum of 7," which is 6. So the probability of rolling a sum of 7 is $\frac{6}{36}$, or $\frac{1}{6}$.

e. What is the probability of rolling an odd sum?

$P(\text{odd sum}) = \frac{18}{36} = \frac{1}{2}$

There are five odd sums, a sum of 3, a sum of 5, a sum of 7, a sum of 9, and a sum of 11. The frequencies for the five odd sums are 2, 4, 6, 4, and 2. The sum of the frequencies is $2 + 4 + 6 + 4 + 2 = 18$. So, $P(\text{odd sum}) = \frac{18}{36} = \frac{1}{2}$.

f. What is the probability of rolling a sum less than 7?

$P(\text{sum} < 7) = \frac{15}{36} = \frac{5}{12}$

There are five sums less than 7, a sum of 2, a sum of 3, a sum of 4, a sum of 5, and a sum of 6. The frequencies for the five odd sums are 1, 2, 3, 4, and 5. The sum of the frequencies is $1 + 2 + 3 + 4 + 5 = 15$. So, $P(\text{sum} < 7) = \frac{15}{36}$.

Problem 2

Students create a frequency table to determine the probability of passing a 4-question true-false test.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- How many outcomes are possible on this 4-question true-false test?
- Which outcome(s) is least probable on this 4-question true-false test?
- Which outcome(s) is most probable on this 4-question true-false test?
- What is the probability of answering 4 questions correctly?
- What is the probability of answering 3 questions correctly?
- What is the probability of answering 2 questions correctly?
- What is the probability of answering 1 question correctly?
- What is the probability of answering 0 questions correctly?

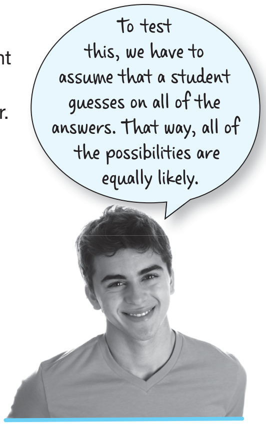
PROBLEM 2 True or False?

1. Rasheeda says that the probability of passing any true-false test is always 50%. Is Rasheeda correct?

Let's investigate Rasheeda's claim by first considering a true-false test with 4 questions. List all the ways a student could answer 4 test questions. Use a ✓ to represent a correct answer and an X to represent an incorrect answer.

✓✓✓✓
✓✓✓X
✓✓X✓
✓✓XX
✓X✓✓
✓X✓X
✓XX✓
✓XXX

XXXX
XXX✓
XX✓X
XX✓✓
X✓XX
X✓X✓
X✓✓X
X✓✓✓



2. Create a frequency table with the number of correct answers and their frequency.

Number of Correct Answers	Frequency
0	1
1	4
2	6
3	4
4	1

3. What is the probability that a student who guesses on this 4-question true-false test will pass the test? Assume a grade of 60% or higher is a passing grade.

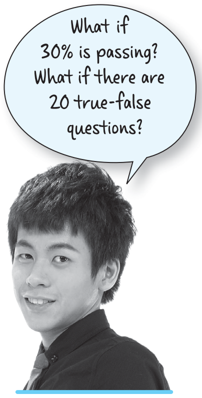
The probability of answering 3 or 4 questions correctly is $\frac{5}{16}$, or 31.25%.

To pass the test, a student must answer 3 questions correctly or 4 questions correctly. Answering 3 out of 4 questions correctly results in a grade of 75%. Answering 4 out of 4 questions correctly results in a grade of 100%.

Let A represent the event of answering 3 questions correctly.

Let B represent the event of answering 4 questions correctly.

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
$$= \frac{1}{16} + \frac{4}{16} - 0$$
$$= \frac{5}{16}$$



4. Is Rasheeda correct? Will a student who guesses on a 4-question true-false test have a 50% chance of passing?

No. I determined that a student who guesses on the 4-question test has a 31.25% chance of passing.

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Problem 3

A two-way frequency table or contingency table is introduced, and students complete the table within the context of a situation involving right-handed vs. left-handed people that may or may not participate in team or individual sports. The table is used to answer questions related to the situation. Next, the terms relative frequency and two-way relative frequency table are introduced. The information from the previous survey is used to calculate the relative frequency of each entry and students record the results in a two-way relative frequency table expressing each result as a fraction and then as a percent. The table is used to determine specified probabilities. Students also explain both correct and incorrect reasoning used in student work examples.

Grouping

- Ask a student to read aloud the information and definition. Complete the table as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- What makes a frequency table a two-way frequency table?
- How many people were surveyed?
- Do you think surveying 63 people is enough people to reach valid conclusions? Why or why not?
- Reading the table, how many right-handed people play team sports?
- What does ‘mixed’ mean in the table?
- How did you determine the totals in the two-way frequency table?
- What does the number 8 represent in the two-way frequency table?

PROBLEM 3 Left, Left, Left, Right, Left . . .

A **two-way frequency table**, also called a **contingency table**, shows the number of data points and their frequencies for two variables. One variable is divided into rows, and the other is divided into columns.

A recent study estimates that between 70% to 90% of the world’s population is right-handed. Another study suggests that almost 90% of athletes are right-handed. And yet another study shows that left-handed people have a higher percentage of participation in individual sports like wrestling or golf.

Favored hand and sports participation are examples of data sets that can be grouped into categories. These data are called **categorical data**, or **qualitative data**.

Mr. Harris’s math class decides to conduct a survey of 63 people to determine which hand is favored and whether the hand favored affects whether a person participates in certain types of sports or no sports at all. The results are shown in the two-way table.

Sports Participation

Favored Hand		Individual	Team	Does Not Play	Total
	Left	3	13	8	24
	Right	6	23	4	33
	Mixed	1	3	2	6
	Total	10	39	14	63

1. Name the two variables displayed in the table.
   The two variables are hand favored and sports participation.

2. Calculate the total for each row and each column in the table. Write each total next to the row or column it belongs to.

- What does the total of 14 represent in the two-way frequency table?
- Where are the names of the variables located on the two-way frequency table?
- Did Campbell calculate the percents for each category?

3. Use the two-way frequency table to answer each question. Describe how to use the rows and columns to determine each answer.

a. How many of those surveyed are left-handed and do not play sports?

Eight of the people surveyed are left-handed and do not play sports.

I determine the answer by identifying where the row labeled "Left" intersects with the column labeled "Does Not Play" column.

b. How many of those surveyed are right-handed or play team sports?

Forty-nine of the people surveyed are right-handed or play team sports.

First, I determined that 33 people are right-handed from the total of the middle row labeled "Right." Next, I determined that 39 people play team sports from the total of the middle column labeled "Team." Then I added 33 + 39 = 72. But I counted those who are both right-handed and play team sports twice, so I have to subtract the number of people who both are right-handed and play team sports, 23.

$72 - 23 = 49$

c. How many of those surveyed play any kind of sport?

Forty-nine of the people surveyed play some kind of sport.

I determined the answer by calculating the sum of the first two columns labeled "Individual" and "Team."

$39 + 10 = 49$

4. Campbell claims that the figures in the studies are correct because there were more right-handed people who participated in the survey than left-handed and mixed-handed people. Do you agree with Campbell? Explain your reasoning.

No. I do not agree with Campbell because she did not calculate the percents for each category.

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Grouping

Have students complete Questions 5 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 7

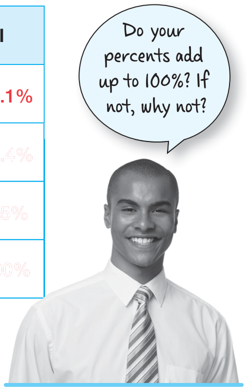
- What is the difference between frequency and relative frequency?
- What is the difference between a two-way frequency table and a two-way relative frequency table?
- What number is in the denominator of every fractional entry in the relative frequency table? Why?
- How did you convert each ratio into a percent?
- When is it appropriate to apply the Addition Rule for Probability?

Because there are 3 types of handed people who participated in the survey, Mr. Harris's students cannot claim that the studies' figures are correct by simply looking at the frequencies. Instead they must determine the *relative frequencies*. A **relative frequency** is the ratio of occurrences within a category to the total number of occurrences. To determine the ratio for each category, determine the part to the whole for each category.

A **two-way relative frequency table** displays the relative frequencies for two categories of data.

5. Use the survey results to calculate the relative frequency of each entry. Record the results in the two-way relative frequency table. Write each result as a fraction and as a percent rounded to the tenths place.

	Individual	Team	Does Not Play	Total
Left	$\frac{3}{63} \approx 4.8\%$	$\frac{13}{63} \approx 20.6\%$	$\frac{8}{63} \approx 12.7\%$	$\frac{24}{63} \approx 38.1\%$
Right	$\frac{6}{63} \approx 9.5\%$	$\frac{23}{63} \approx 36.5\%$	$\frac{4}{63} \approx 6.3\%$	$\frac{33}{63} \approx 52.4\%$
Mixed	$\frac{1}{63} \approx 1.6\%$	$\frac{3}{63} \approx 4.8\%$	$\frac{2}{63} \approx 3.2\%$	$\frac{6}{63} \approx 9.5\%$
Total	$\frac{10}{63} \approx 15.9\%$	$\frac{39}{63} \approx 61.9\%$	$\frac{14}{63} \approx 22.2\%$	$\frac{63}{63} = 100\%$



6. If you randomly select a student from the survey, what is the probability that they:

a. are left-handed?

P(left-handed) ≈ 38.1%

Twenty-four out of 63 people surveyed are left-handed.

$\frac{24}{63} \approx 0.381$

b. are right-handed?

P(right-handed) ≈ 52.4%

Thirty-three out of 63 people surveyed are right-handed.

$\frac{33}{63} \approx 0.524$

c. participate in some kind of sport?

P(participate in sport) ≈ 77.8%

Out of 63 people surveyed, 10 + 39, or 49, people participate in some kind of sport.

$\frac{49}{63} \approx 0.778$

7. If a student is randomly selected, what is the probability that they are:

a. left-handed participating in individual sports?

The probability of randomly selecting a left-handed person who participates in individual sports is $\frac{3}{63}$, or 4.8%.

I determined the answer by identifying where the row labeled “Left” intersects with the column labeled “Individual.”

b. right-handed participating in team sports?

The probability of randomly selecting a right-handed person who participates in team sports is $\frac{23}{63}$, or 36.5%.

I determined the answer by identifying where the row labeled “Right” intersects with the column labeled “Team.”

c. left-handed participating in individual sports or right-handed participating in team sports? Explain how you determined your answer.

The probability of randomly selecting a left-handed person who participates in individual sports or a right-handed person who participates in team sports is $\frac{23}{63}$, or 41.3%.

I added the number of left-handed people participating in individual sports to the number of right-handed people participating in team sports, $3 + 23 = 26$ or $4.8\% + 36.5\% \approx 41.3\%$.

20

Grouping

Have students complete Questions 8 through 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 8 through 10

- Which probability rule did you use to answer parts (a) and (b)?
- Why is Danielle's solution method incorrect? Explain.
- Why is Tyler's solution method incorrect? Explain.

Explain how you can use the probability rules you have learned to answer each question.

8. If a student is randomly selected, what is the probability they:

a. are left-handed or participate in team sports? Explain how you determined your answer.

$P(\text{left-handed or team sports}) \approx 79.4\%$

I used the Addition Rule for Probability which states $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

Let A represent the event of selecting a left-handed person.

Let B represent the event of selecting a person who participates in a team sport.

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$
$$\approx 38.1\% + 61.9\% - 20.6\%$$
$$\approx 79.4\%$$

b. do not play sports or are mixed-handed? Explain how you determined your answer.

$P(\text{no sports or mixed-handed}) \approx 28.5\%$

I used the Addition Rule for Probability which states $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

Let A represent the event of selecting a person who does not participate in sports.

Let B represent the event of selecting a mixed-handed person.

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$
$$\approx 22.2\% + 9.5\% - 3.2\%$$
$$\approx 28.5\%$$

9. How do the results of the survey compare with the estimate that between 70% and 90% of the world's population is right-handed?

The survey shows a lower percentage of right-handed students than what is estimated for the world's population.

I determine the answer by identifying the total percent for the row labeled "Right."

$52.4\% < 70\%$

10. How do the results of the survey compare with the estimate that almost 90% of athletes are right-handed?

Three students attempted to answer this question. Their work and reasoning are shown.

Danielle

Out of the 63 people surveyed, 6 + 23, or 29, are both right-handed AND play a sport.

$\frac{29}{63} \approx 46\%$

This is much less than the 90% estimate.

Tyler

Out of the 33 people in the survey who are right-handed, 6 + 23, or 29, play a sport.

$\frac{29}{33} \approx 87.9\%$

This is close to the 90% estimate.

Keisha

Out of the 10 + 39, or 49, people in the survey who play a sport, 6 + 23, or 29, are right-handed.

$\frac{29}{49} \approx 59.2\%$

This is much less than the 90% estimate.

Explain why Keisha's reasoning is correct, and describe what Danielle and Tyler did to get incorrect answers.

Keisha correctly determined the percent of athletes from the survey who are right-handed. She considered the total number of right-handed people who play sports, which includes the right-handers in individual sports and the right-handers in team sports. This is the correct comparison to the estimate that almost 90% of athletes are right-handed.

Danielle used the total number of people in the survey instead of the total number of people who play a sport to determine the answer. Her answer describes the percentage of all the people in the survey who are right-handed and play a sport.

Tyler used the total number of right-handed people in the survey instead of the total number of people who play a sport to determine the answer. His answer describes the percentage of right-handed people in the survey who are right-handed and play a sport.

Problem 4

The results of a survey concerning what students do when they are not attending school are listed in a two-way relative frequency table. The frequencies are provided but the students complete the table by computing the relative frequency ratios and equivalent percentages for each entry. The table is then used to answer several questions related to the situation.

Grouping

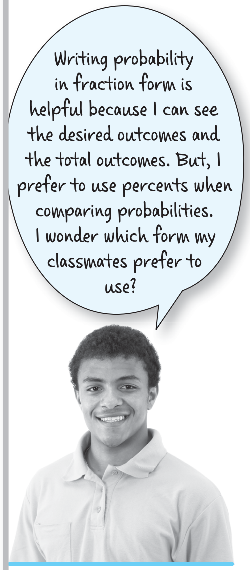
- Discuss the introduction and complete the table as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- How many students were surveyed?
- How many male students were surveyed?
- How many female students were surveyed?
- What number is in the denominator of every fractional entry in the relative frequency table? Why?
- How did you convert each ratio into a percent?

PROBLEM 4 More of My Favorite Things

A survey was conducted in one class about students' favorite after-school activities. The frequencies of male and female responses are shown in the table.



What is your favorite thing to do when you are not in school?

	Male		Female		Total	
Activity	Freq.	Rel. Freq.	Freq.	Rel. Freq.	Freq.	Rel. Freq.
Listen to music	5	$\frac{5}{30} \approx 16.7\%$	3	$\frac{3}{30} = 10\%$	8	$\frac{8}{30} \approx 26.7\%$
Watch TV	5	$\frac{5}{30} \approx 16.7\%$	4	$\frac{4}{30} \approx 13.3\%$	9	$\frac{9}{30} = 30\%$
Participate in sports	1	$\frac{1}{30} \approx 3.3\%$	2	$\frac{2}{30} \approx 6.7\%$	3	$\frac{3}{30} = 10\%$
Play video games	3	$\frac{3}{30} = 10\%$	0	0	3	$\frac{3}{30} = 10\%$
Surf the Internet	2	$\frac{2}{30} \approx 6.7\%$	2	$\frac{2}{30} \approx 6.7\%$	4	$\frac{4}{30} \approx 13.3\%$
Shop	0	0	1	$\frac{1}{30} \approx 3.3\%$	1	$\frac{1}{30} \approx 3.3\%$
Read	0	0	1	$\frac{1}{30} \approx 3.3\%$	1	$\frac{1}{30} \approx 3.3\%$
Other	1	$\frac{1}{30} \approx 3.3\%$	0	0	1	$\frac{1}{30} \approx 3.3\%$
Total	17	$\frac{17}{30} \approx 56.7\%$	13	$\frac{13}{30} \approx 43.3\%$	30	$\frac{30}{30} = 100\%$

1. Calculate the relative frequency of each entry. Record the results in the two-way relative frequency table. Write each result as a fraction and as a percent rounded to the tenths place.

2. What is the probability that a randomly chosen student from the class:

a. is male?

The probability of randomly selecting a male student is $\frac{17}{30}$, or 56.7%.

I determined the answer by locating the cell with the total number of males. It is located at the bottom of the column labeled "Male."

- What is $P(A) \cdot P(B)$? How does this compare to the survey results?
- If the ratios are not equal, are the events dependent or independent?

b. watches TV as their favorite thing to do when not in school?

The probability of randomly selecting a student whose favorite thing to do is watch TV is $\frac{9}{30}$, or 30%.

I determined the answer by locating the cell with the total number of people who said watching TV is their favorite thing to do. It is located in the last cell of the row labeled "Watch TV."

c. is a male who plays video games?

The probability of randomly selecting a male whose favorite thing to do is play video games is $\frac{3}{30}$, or 10%.

I determined the answer by identifying where the row labeled "Play video games" intersects with the column labeled "Male."

d. is a female who listens to music?

The probability of randomly selecting a female whose favorite thing to do is listen to music is $\frac{3}{30}$, or 10%.

I determined the answer by identifying where the row labeled "Listen to music" intersects with the column labeled "Female."

e. is female and watches TV for her favorite pastime?

The probability of randomly selecting a female whose favorite thing to do is watch TV is $\frac{4}{30}$, or 13.3%.

I determined the answer by identifying where the row labeled "Watch TV" intersects with the column labeled "Female."

f. is female or watches TV for his/her favorite pastime?

The probability of randomly selecting a female or a person whose favorite thing to do is watch TV is $\frac{18}{30}$, or 60%.

I solved the problem by adding the total number of females to the total number of people who selected watching TV as their favorite thing. Then, I subtracted the number counted twice, which is the number of females whose favorite thing to do is watch TV.

The cell with the total number of females is at the bottom of the column titled "Females."

The cell with the total number of people who selected watching TV as their favorite thing is at the end of the row titled "Watch TV."

The cell with the number of females whose favorite thing to do is watch TV is where the row labeled "Watch TV" intersects with the column labeled "Female."

$\frac{13}{30} + \frac{9}{30} - \frac{4}{30} = \frac{18}{30}$, or 60%

3. Is being male independent of listening to music as a favorite after-school activity? Explain your reasoning.

Being male is not independent of listening to music as a favorite after-school activity.

The events are not independent because the probability of the compound event $P(A \text{ and } B)$ is not equal to the product of the events' individual probabilities $P(A) \cdot P(B)$.

Let A represent the event of being male.

Let B represent the event of selecting music as your favorite after-school activity.

$P(A \text{ and } B) = \frac{5}{30}$, or 16.7%. I determined this number by identifying where the column labeled "Male" intersects with the row labeled "Listen to music."

The individual probabilities of events A and B are $P(A) = \frac{17}{30}$ and $P(B) = \frac{8}{30}$, and $P(A) \cdot P(B) = \frac{17}{30} \cdot \frac{8}{30} = \frac{136}{900}$, or 15.1%.

Because the value of $P(A \text{ and } B)$, 16.7%, is not equal to the value of $P(A) \cdot P(B)$, 15.1%, events A and B are not independent.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Suppose someone made the statement that most students in chess club are also on the honor roll or taking a foreign language.

Describe how you might go about showing the probabilities related to this statement.

I could begin by surveying all of the students that are members of the chess club and ask them if they are on the honor roll and if they are taking a foreign language.

I could enter the data from the survey into a table and compute the frequency of occurrence for each entry. Then I could calculate the percent of students that are members of the chess club and also on the honor roll, or also take a foreign language. If more than half of the students are in one or both groups, then the statement is truthful.

20.2

It All Depends

Conditional Probability

LEARNING GOALS

In this lesson, you will:

- Use conditional probability to determine the probability of an event given that another event has occurred.
- Use conditional probability to determine whether or not events are independent.

KEY TERM

- conditional probability

ESSENTIAL IDEAS

- Conditional probability is the probability of Event *B*, given that Event *A* has already occurred.
- The notation for conditional probability is $P(B|A)$, which reads, "the probability of Event *B*, given Event *A*."
- When $P(B|A) = P(B)$, the two events, *A* and *B*, are independent.
- When $P(B|A) \neq P(B)$, the two events, *A* and *B*, are dependent.
- The conditional probability formula is stated as $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

S-CP Conditional Probability and the Rules of Probability

Understand independence and conditional probability and use them to interpret data

3. Understand the conditional probability of *A* given *B* as $P(A \text{ and } B)/P(B)$, and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*.
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.

Use the rules of probability to compute probabilities of compound events in a uniform probability model

6. Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model.

Overview

Rolling two number cubes and calculating the sum is once again used to generate a two-way data table listing the possible outcomes. Different events are described and students calculate $P(A)$, $P(B)$, and $P(A \text{ and } B)$. The term conditional probability, $P(B|A)$ is defined. Students derive a formula for computing conditional probability, $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$. The conditional probability formula is applied to several different situations.

Warm Up

A middle school teacher remarked that it seems like younger and younger children have their own smart phones each year.

Suppose the teacher wants to determine the probabilities of a random male and female middle school student having their own smart phone, so she administers a random survey. She has quite a bit of data on several sheets of paper and these data need to be organized and recorded. She asks for your assistance.

Create a two-way frequency table that can be used to record the data.

	Male Student with cell	Male Student without cell	Female Student with cell	Female Student without cell	Total
Grade 6					
Grade 7					
Grade 8					
Total					

20.2

It All Depends

Conditional Probability

LEARNING GOALS

In this lesson, you will:

- Use conditional probability to determine the probability of an event given that another event has occurred.
- Use conditional probability to determine whether or not events are independent.

KEY TERM

- conditional probability

Don't you hate it when your sock drawer is unorganized and your socks are all over the place? A disorganized sock drawer means you'll probably spend more time looking for a matching pair of socks, which could make your head spin!

Believe it or not, one web site lists the top 10 cities with the messiest sock drawers and the top 10 cities with the most organized sock drawers.

Does your sock drawer belong to the messy category or organized category?

Problem 1

A two-way table listing the outcomes of rolling a number cube twice and computing the sum is provided. The table is used to determine compound probabilities. The term conditional probability is introduced, and notation for conditional probability is described. The two-way table is used to determine the probabilities of Event *A*, Event *B*, and the probability of Event *B*, given Event *A*. it is concluded that when two events are independent events, $P(B|A) = P(B)$, and when two events are dependent events, $P(B|A) \neq P(B)$.

Grouping

- Ask students to read the introduction. Discuss as a class.
- Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- How many possible outcomes are there for the first number cube roll?
- How many possible outcomes are there for the second number cube roll?
- How is the Counting Principle used to compute the number of possible outcomes in this situation?
- How many outcomes are possible?
- What kind of questions can be answered using this two-way table?
- What does $P(A)$ represent in this situation?
- When determining $P(A)$, how many cells were shaded in the two-way table?
- What does $P(B)$ represent in this situation?
- When determining $P(B)$, how many cells were shaded in the two-way table?
- What does $P(A \text{ and } B)$ represent in this situation?
- When determining $P(A \text{ and } B)$, how many cells were shaded in the two-way table?

PROBLEM 1 Rollin', Rollin', Rollin'

The two-way table represents a situation in which 2 number cubes are rolled, one at a time. The sums of the two rolls are shown.

Second Roll

First Roll		**1**	**2**	**3**	**4**	**5**	**6**
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

1. Suppose event *A* is rolling a 5 on the first roll, and event *B* is rolling a 4 or less on the second roll.

Determine the probability of rolling a 5 on the first roll and rolling a 4 or less on the second roll, $P(A \text{ and } B)$.

a. What is the probability of rolling a 5 on the first roll, $P(A)$? Shade the desired outcomes, and draw a border around the possible outcomes. Explain your reasoning.

Second Roll

First Roll		**1**	**2**	**3**	**4**	**5**	**6**
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

$P(A) = \frac{6}{36} = \frac{1}{6}$

I shaded all 6 cells in row 5 to represent all the desired outcomes of rolling a five on the first roll. The second roll can be any of the columns numbered 1 through 6, because its outcome was not specified. I drew a border around all 36 cells to represent all the possible outcomes.

b. What is the probability of rolling a 4 or less on the second roll, $P(B)$? Shade the desired outcomes, and draw a border around the possible outcomes. Explain your reasoning.

Second Roll

First Roll

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(B) = \frac{24}{36} = \frac{2}{3}$

I shaded all 6 cells in columns 1 through 4 to represent all the desired outcomes of rolling a four or less on the second roll. The first roll can be any row numbered 1 through 6, because its outcome was not specified. I drew a border around all 36 cells to represent all the possible outcomes.

c. What is the probability of rolling a 5 first AND a 4 or less second, $P(A \text{ and } B)$? Shade the desired outcomes, and draw a border around the possible outcomes. Explain your reasoning.

Second Roll

First Roll

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(A \text{ and } B) = \frac{4}{36} = \frac{1}{9}$

I determined the answer by identifying the overlapping cells in the table for rolling a 5 on the first roll and rolling a 4 or less on the second roll. I copied the shading for the two events from the previous questions and saw that the first four cells in row 5 are overlapping.

I can also calculate $P(A \text{ and } B)$ by multiplying $P(A)$ and $P(B)$, because A and B are independent events.

$$\begin{aligned} P(A \text{ and } B) &= P(A) \cdot P(B) \\ &= \frac{1}{6} \cdot \frac{2}{3} \\ &= \frac{2}{18} \\ &= \frac{1}{9} \end{aligned}$$

Grouping

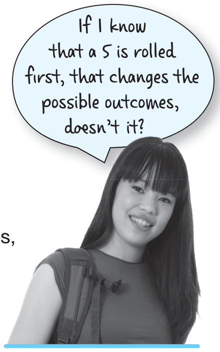
Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 2 and 3

- What is the difference between probability and conditional probability?
- If the occurrence of Event *A* doesn't change the probability of Event *B*, what information does this give you about the relationship between Event *A* and Event *B*?
- If the occurrence of Event *A* changes the probability of Event *B*, what information does this give you about the relationship between Event *A* and Event *B*?

A **conditional probability** is the probability of event *B*, given that event *A* has already occurred. The probability of rolling a 4 or less on the second roll of the number cube (*B*), given that a 5 is rolled first (*A*), is an example of a conditional probability.

The notation for conditional probability is $P(B|A)$, which reads, "the probability of event *B*, given event *A*."



2. What is the probability of rolling a 4 or less on the second roll, given that a 5 is rolled first, $P(B|A)$? Shade the desired outcomes, and draw a border around the possible outcomes. Explain your reasoning.

Second Roll

First Roll	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(B|A) = \frac{4}{6} = \frac{2}{3}$

The number of possible outcomes is 6, not 36, because of the given condition that a 5 is rolled first. So, I drew a border around all of the cells in row five. Also, I shaded the first 4 columns in row five to represent rolling a 4 or less on the second roll, given that a 5 is rolled first.

3. Compare $P(B|A)$ with $P(B)$.

a. What do you notice?

$P(B|A) = P(B)$

The probability of rolling a 4 or less on the second roll, given that a 5 is rolled first is $\frac{4}{6}$, or $\frac{2}{3}$.
The probability of rolling a 4 or less on the second roll is also $\frac{4}{6}$, or $\frac{2}{3}$.

b. Do you think event *A* and *B* are independent or dependent events? Explain your reasoning.

Events *A* and *B* are independent.

If $P(B|A) = P(B)$, that means the occurrence of event A does not affect the probability of event *B*. So, the two events are independent.

Grouping

Have students complete Question 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- Under what circumstance does $P(B|A) = P(B)$?
- Under what circumstance does $P(B|A) \neq P(B)$?

4. Suppose event *A* is rolling a 5 on the first roll, and event *B* is rolling a sum greater than or equal to 8.

a. What is the probability of rolling a 5 on the first roll, *P*(*A*)? Shade the desired outcomes, and draw a border around the possible outcomes.

Second Roll

First Roll

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(A) = \frac{6}{36} = \frac{1}{6}$

I shaded all 6 cells in row 5 to represent all the desired outcomes of rolling a five on the first roll. The second roll can be any column numbered 1 through 6, because its outcome was not specified. I drew a border around all 36 cells to represent all the possible outcomes.

b. What is the probability of rolling a sum greater than or equal to 8, *P*(*B*)? Shade in the table to show your answer. Shade the desired outcomes, and draw a border around the possible outcomes. Explain your reasoning.

Second Roll

First Roll

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(B) = \frac{15}{36} = \frac{5}{12}$

In rows 2 through 6, I shaded all the cells with value greater than or equal to 8. This represents all the desired outcomes for rolling a sum greater than or equal to 8. I drew a border around all 36 cells to represent all the possible outcomes.

c. What is the probability of rolling a 5 first AND a sum greater than or equal to 8, $P(A \text{ and } B)$? Shade the desired outcomes, and draw a border around the possible outcomes. Explain your reasoning.

Second Roll

First Roll

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(A \text{ and } B) = \frac{4}{36} = \frac{1}{9}$

I determined the answer by identifying the overlapping cells in the table for rolling a 5 on the first roll and rolling a sum greater than or equal to 8. I copied the shading from the previous questions and saw that the last four cells in row 5 are overlapping.

d. What is the probability of rolling a sum greater than or equal to 8, given that a 5 is rolled first, $P(B|A)$? Shade the desired outcomes, and draw a border around the possible outcomes. Explain your reasoning.

Second Roll

First Roll

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(B|A) = \frac{4}{6} = \frac{2}{3}$

The number of possible outcomes is 6, not 36, because of the given condition that a 5 is rolled first. So, I drew a border around all of the cells in row five. Also, I shaded the last 4 cells in row five to represent rolling a sum greater than or equal to 8.

e. Compare $P(B|A)$ with $P(B)$. What do you notice?

$P(B|A) \neq P(B)$

The probability of rolling a sum greater than or equal to 8, given that a 5 is rolled first is $\frac{4}{6}$, or $\frac{2}{3}$.

But, the probability of rolling a sum greater than or equal to eight is $\frac{15}{36}$, or $\frac{5}{12}$.

f. Do you think events *A* and *B* are independent or dependent events? Explain your reasoning.

Events *A* and *B* are dependent events.

If $P(B|A) \neq P(B)$, that means the occurrence of event *A* affects the probability of event *B*. So, the two events are dependent.

When two events are independent, $P(B|A) = P(B)$. When two events are dependent, $P(B|A) \neq P(B)$. Thus, two events are independent if and only if $P(B|A) = P(B)$.

PROBLEM 2 Building the Formula

Now you can derive the formula for the conditional probability of two independent events, $P(B|A)$.

1. Let *A* represent the event of rolling a 5 on the first roll. Let *B* represent the event of rolling a number less than or equal to 4 on the second roll.

Second Roll

First Roll

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

a. Complete the following.

	$P(A)$	$P(B)$	$P(A \text{ and } B)$	$P(B\|A)$
Desired Outcomes	6	24	4	4
Total Outcomes	36	36	36	6
Probability	$\frac{6}{36} = \frac{1}{6}$	$\frac{24}{36} = \frac{2}{3}$	$\frac{4}{36} = \frac{1}{9}$	$\frac{4}{6} = \frac{2}{3}$

Problem 2

Students use two-way tables from the previous problem and Event *A* and Event *B* to derive the formula for the conditional probability for two independent events, $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$.

Grouping

- Have students complete Question 1 with a partner. Then have students share their responses as a class.
- Discuss the worked example after Question 1 as a class.

Guiding Questions for Share Phase, Question 1

- If Event *A* is described as rolling a 5 on the first roll, which rows of the two-way table are not considered possible outcomes?
- If Event *B* is described as rolling 4 or less on the second roll, which columns of the two-way table are not considered possible outcomes?
- Which outcomes are no longer possible?
- What are the desired outcomes?
- When determining the conditional probability, what is the new sample space?
- When determining the conditional probability, what are the desired outcomes?
- Is $P(A \text{ and } B) = P(A) \cdot P(B)$?
- What is $\frac{P(A) \cdot P(B)}{P(A)}$ simplified?

b. Use the table to verify that event *A* and event *B* are independent.

Event *A* and event *B* are independent because $P(B) = P(B|A)$.
$P(B) = \frac{2}{3}$ and $P(B|A) = \frac{2}{3}$

c. What quantity in the table is equivalent to the number of desired outcomes for $P(B|A)$?

The number of desired outcomes for $P(A \text{ and } B)$ is equivalent to the number of desired outcomes for $P(B|A)$. The number of desired outcomes for $P(A \text{ and } B)$ is equal to 4 and the number of desired outcomes for $P(B|A)$ is also equal to 4.

d. What quantity in the table is equivalent to the number of total outcomes for $P(B|A)$?

The number of desired outcomes for $P(A)$ is equivalent to the number of total outcomes for $P(B|A)$. The desired outcomes for $P(A)$ is equal to 4 and the total outcomes for $P(B|A)$ is also equal to 4.

The conditional probability, $P(B|A)$, for independent events can be represented as

$\frac{\text{desired outcomes}}{\text{total outcomes}}$.

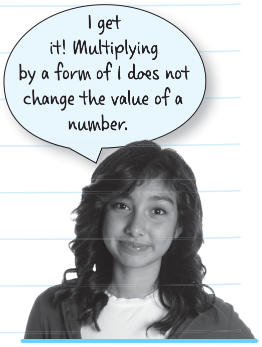
$$P(B|A) = \frac{\text{desired outcomes}}{\text{total outcomes}}$$

$$= \frac{A \text{ and } B}{A}$$

$$= \frac{A \text{ and } B}{A} \cdot \frac{\frac{1}{\text{total}}}{\frac{1}{\text{total}}}$$

$$= \frac{\frac{A \text{ and } B}{\text{total}}}{\frac{A}{\text{total}}}$$

$$= \frac{P(A \text{ and } B)}{P(A)}$$



Grouping

Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 2 and 3

- What does it mean for two events to be independent?
- Do you think $P(B|A) = P(A|B)$?
- Why is the denominator written in the formula for conditional probability?

2. Michael says the probability of rolling a number less than or equal to 4 on the second roll, given a 5 on the first roll, $P(B|A)$, is equal to the probability of rolling a number less than or equal to 4 on the second roll, $P(B)$, because Events *A* and *B* are independent.

Is Michael correct? Explain your reasoning.

Yes. Michael is correct.

The conditional probability $P(B|A)$ is equal to the probability of Event *B*, $P(B)$, because the outcomes of event *A* do not affect the outcomes of Event *B*.

3. How is the formula in the worked example related to Michael's statement in Question 2?

The conditional probability $P(B|A)$ for independent events *A* and *B* is equal to $\frac{P(A \text{ and } B)}{P(A)}$. Because the events are independent, $P(A \text{ and } B)$ is $P(A) \times P(B)$.

Thus, $\frac{P(A) \times P(B)}{P(A)} = P(B)$.

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Problem 3

A probability situation involving choosing socks from a drawer is presented.

Students determine that the conditional probability formula works for actions that affect the outcomes of other actions.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

- How many outcomes are in your organized list?
- How many outcomes contain at least one red sock?
- How many outcomes contain a red sock picked first?
- How many outcomes contain a red sock picked second?
- How many outcomes contain a red sock picked first and a red sock picked second?
- Is there replacement in this situation?
- Does the occurrence of Event *A* effect the number of possible outcomes for Event *B*?
- Before the occurrence of Event *A*, how many socks are in the drawer?
- After the occurrence of Event *A*, how many socks remain in the drawer?

PROBLEM 3 Can We Depend on Conditional Probability?

Let's see whether the formula works for actions that affect the outcomes of other actions!

Suppose you have 2 red socks, 1 blue sock, and 1 yellow sock in a drawer. You randomly choose a sock without replacing it, and then randomly choose a second sock.

1. Write an organized list to represent all of the possible outcomes.

R_1R_2	R_2R_1	BR_1	YR_1
R_1B	R_2B	BR_2	YR_2
R_1Y	R_2Y	BY	YB

2. What is the probability of randomly choosing a red sock first, $P(A)$?

$P(A) = \frac{6}{12} = \frac{1}{2}$

Six out of the 12 outcomes in the sample space result in choosing a red sock first, which means $P(A) = \frac{6}{12}$, or $\frac{1}{2}$.

Another way to determine the answer is to consider that 2 out of the 4 socks in the drawer are red, which means $P(A) = \frac{2}{4}$, or $\frac{1}{2}$.

3. What is the probability of randomly choosing a red sock second, given that a red sock is randomly chosen first, $P(B|A)$?

$P(B|A) = \frac{2}{6} = \frac{1}{3}$

There are 6 outcomes that result in choosing a red sock first and 2 of the 6 outcomes result in choosing a red sock second. That translates to $P(B|A) = \frac{2}{6}$, or $\frac{1}{3}$.

Another way to determine the answer is to consider that after picking a red sock first, only 1 of the 3 remaining socks is red. That translates to $P(B|A) = \frac{1}{3}$.

4. What is the probability of randomly choosing a red sock first and a red sock second, $P(A \text{ and } B)$?

$P(A \text{ and } B) = \frac{2}{12} = \frac{1}{6}$

Two out of the 12 outcomes in the sample space result in choosing a red sock first and a red second, which means $P(A \text{ and } B) = \frac{1}{12}$, or $\frac{1}{6}$.

Another way to determine the answer is to multiply $P(A)$ and $P(B|A)$.

$$\begin{aligned} P(A \text{ and } B) &= P(A) \cdot P(B|A) \\ &= \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{1}{6} \end{aligned}$$

5. Are events *A* and *B* independent or dependent events? Explain your reasoning.

Events *A* and *B* are dependent because the result of picking the first sock has an effect on the outcomes for picking the second sock: $P(B|A) \neq P(B)$.

6. Is the probability of randomly choosing a red sock second, given that a red sock is randomly chosen first, $P(B|A)$, equal to the probability of $\frac{P(A \text{ and } B)}{P(A)}$? Explain why or why not.

Yes. The two probabilities are equal.

$P(B|A) = \frac{1}{3}$

$$\begin{aligned} \frac{P(A \text{ and } B)}{P(A)} &= \frac{\frac{1}{6}}{\frac{1}{2}} \\ &= \frac{1}{6} \times 2 \\ &= \frac{2}{6} \\ &= \frac{1}{3} \end{aligned}$$

Problem 4

Students apply the conditional probability formula, $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$, to three different situations to determine conditional probability.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What is Event *A* in this situation?
- What is Event *B* in this situation?
- What is $P(A \text{ and } B)$ in this situation?
- What is $\frac{P(A \text{ and } B)}{P(A)}$ in this situation?
- Does $P(B|A) = P(B)$? What implication does this have on the relationship between Event *A* and Event *B*?
- Does $P(B|A) \neq P(B)$? What implication does this have on the relationship between Event *A* and Event *B*?

PROBLEM 4 Using the Formula for Conditional Probability

1. A biology teacher gave her students two tests. The probability that a student received a score of 90% or above on both tests is $\frac{1}{10}$. The probability that a student received a score of 90% or above on the first test is $\frac{1}{5}$, and the probability that a student received a score of 90% or above on the second test is $\frac{1}{2}$.

a. What is the probability that a student who received a score of 90% or above on the first test also received a score of 90% or above on the second test?

The probability that a student received a score of 90% or above on the second test if they already received a score of 90% or above on the first test is $\frac{1}{2}$.

Let *A* represent the event of scoring 90% or above on the first test.

Let *B* represent the event of scoring 90% or above on the second test.

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\frac{1}{10}}{\frac{1}{5}} = \frac{1}{2}$$

b. Are “scoring 90% or above on the first test” and “scoring 90% or above on the second test” independent or dependent events? Explain your reasoning.

The events are independent because the probability of scoring 90% or above on the second test is equal to the probability of scoring 90% or above on the second test, given that the student scored 90% or above on the first test.

Let *A* represent the event of scoring 90% or above on the first test.

Let *B* represent the event of scoring 90% or above on the second test.

$$P(B) = \frac{1}{2}$$

$$\begin{aligned} P(B|A) &= \frac{P(A \text{ and } B)}{P(A)} \\ &= \frac{\frac{1}{10}}{\frac{1}{5}} \\ &= \frac{1}{10} \times 5 \\ &= \frac{5}{10} \\ &= \frac{1}{2} \end{aligned}$$

$P(B) = P(B|A)$ because both probabilities equal $\frac{1}{2}$.

2. The probability of a positive test for a disease and actually having the disease is $\frac{1}{4000}$. The probability of a positive test is $\frac{1}{3000}$. Seventy out of 100 people have the disease.

a. What is the probability of actually having the disease given a positive test?

The probability of actually having the disease given a positive test is $\frac{3}{4}$.

Let A represent the event of a positive test.

Let B represent the event of actually having the disease.

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\frac{1}{4000}}{\frac{1}{3000}} = \frac{3}{4}$$

b. Are testing positive for the disease and having the disease independent events? Explain your reasoning.

The events are not independent because the probability of having the disease is not equal to the probability of having the disease, given a positive test.

Let A represent the event of a positive test.

Let B represent the event of having the disease.

$P(B) = \frac{70}{100}$, or $\frac{7}{10}$

$$\begin{aligned} P(B|A) &= \frac{P(A \text{ and } B)}{P(A)} \\ &= \frac{\frac{1}{4000}}{\frac{1}{3000}} \\ &= \frac{1}{4000} \times 3000 \\ &= \frac{3000}{4000} \\ &= \frac{3}{4} \end{aligned}$$

$P(B) \neq P(B|A)$ because $\frac{7}{10}$ is not equal to $\frac{3}{4}$.

3. A basketball player makes two out of two free throws 49% of the time. She makes 70% of her free throws.

a. What is the probability that she will make the second free throw after making the first?

The probability of making the second free throw after making the first is $\frac{7}{10}$.

Let A represent the event of making the first free throw.

Let B represent the event of making the second free throw.

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\frac{49}{100}}{\frac{70}{100}} = \frac{7}{10}$$

b. Are making the first free throw and making the second free throw independent or dependent events? Explain your reasoning.

The events are independent because the probability of making the second free throw is equal to the probability of making the second free throw, given that the first free throw was made.

Let A represent the event of making the first free throw.

Let B represent the event of making the second free throw.

$P(B) = 0.70$

$$\begin{aligned} P(B|A) &= \frac{P(A \text{ and } B)}{P(A)} \\ &= \frac{0.49}{0.70} \\ &= 0.70 \end{aligned}$$

$P(B) = P(B|A)$ because both probabilities equal 0.70.

Be prepared to share your solutions and methods.

Check for Students' Understanding

You and five friends decide to go out to dinner together, but everyone wants to go to a different restaurant.

One method for picking who gets to make the decision is to draw straws. Five straws of equal length and one straw slightly shorter are placed in a container. One at a time, each person draws one straw and whoever draws the shorter straw gets to make the decision.

1. Determine the probability of the first person drawing the shortest straw, $P(A)$.

The probability of the first person drawing the shortest straw is $\frac{1}{6}$.

2. Determine the probability of the first person not drawing the shortest straw, $P(\overline{A})$.

The probability of the first person not drawing the shortest straw is $\frac{5}{6}$.

3. Determine the probability of the second person drawing the shortest straw, $P(B)$.

The probability of the second person drawing the shortest straw is $\frac{1}{5}$.

4. Does $P(B|\overline{A}) = P(B)$? What does this imply about the relationship between Event *A* and Event *B*?

No. $P(B|\overline{A}) = \frac{5}{6}$ and $P(B) = \frac{1}{5}$. Event *A* and event *B* are dependent events.

5. Is this a fair method of selection? Why or why not?

The probability of the second person drawing the shortest straw is

$P(B) = P(B|\overline{A}) \cdot P(\overline{A})$

$P(B) = \frac{1}{5} \cdot \frac{5}{6} = \frac{1}{6}$

The probability that neither of the first two people drawing the shortest straw is $\frac{5}{6} \cdot \frac{4}{5} = \frac{4}{6}$, so the probability of the third person drawing the shortest straw is $\frac{1}{4} \cdot \frac{4}{6} = \frac{1}{6}$.

Yes, it is a fair selection method.

20.3

Counting

Permutations and Combinations

LEARNING GOALS

In this lesson, you will:

- Use permutations to calculate the size of sample spaces.
- Use combinations to calculate the size of sample spaces.
- Use permutations to calculate probabilities.
- Use combinations to calculate probabilities.
- Calculate permutations with repeated elements.
- Calculate circular permutations.

KEY TERMS

- factorial
- permutation
- circular permutation
- combination

ESSENTIAL IDEAS

- The factorial of n, which is written with an exclamation mark as $n!$ is the product of all non-negative integers less than or equal to n: $n(n - 1)(n - 2) \dots$
- A permutation is an ordered arrangement of items without repetition.
- The notation denoting a permutation or r elements taken from a collection of n items is: ${}_nP_r = P(n, r) = P^n_r$.
- The formula used to compute the number of permutations, P, of r elements chosen from n elements is: ${}_nP_r = \frac{n!}{(n - r)!}$.
- A combination is an unordered collection of items.
- The notation denoting a combination or r elements taken from a collection of n elements is: ${}_nC_r = C(n, r) = C^n_r$.
- The formula used to compute the number of combinations, C, of r elements chosen from n elements is: ${}_nC_r = \frac{n!}{(n - r)!r!}$.
- The formula for the number of permutations of n elements with k copies of an element is $\frac{n!}{k!}$.
- The formula for the number of permutations of n elements with k copies of one element and h copies of another element is $\frac{n!}{k!h!}$.
- The circular permutation of n objects is $(n - 1)!$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

S-CP Conditional Probability and the Rules of Probability

Use the rules of probability to compute probabilities of compound events in a uniform probability model

9. Use permutations and combinations to compute probabilities of compound events and solve problems.

Overview

The terms factorial, permutation, and combination are defined. Students derive the formulas to calculate permutations and combinations, then apply them in different situations. Situations involve permutations with and without repeated elements. Students answer questions, complete tables, and make the connections necessary to develop additional formulas related to combinations and permutations. Circular permutations are introduced. Students conclude that the formula for the circular permutation of n objects is $(n - 1)!$.

Warm Up

Consider the following outcomes.

Heads, Tails, Heads
Tails, Heads, Heads
Heads, Heads, Tails

1. How are the three outcomes similar?

   The three outcomes are similar because the three outcomes contain the same elements.

2. How are the three outcomes different?

   The three outcomes are different because the order of the elements is different in each outcome.

3. Which outcome might describe a coin flip that resulted in heads the first flip, heads the second flip, and tails the third flip?

   Heads, Heads, Tails, describes a coin flip that resulted in heads the first flip, heads the second flip, and tails the third flip.

4. Did the order matter when determining the answer to Question 3? Why or why not?

   Yes. Order mattered when answering Question 3 because the question specified the results in a particular order.

5. Which outcome might describe a coin flip that resulted in heads twice and tails once?

   All three outcomes describe a coin flip that resulted in heads twice and tails once.

6. Did the order matter when determining the answer to Question 5? Why or why not?

   No. Order did not matter when answering Question 5 because the question did not specify the results in any particular order.

7. If order matters, are the three outcomes considered names for the same outcome or are they considered three different outcomes?

   If order matters, the three outcomes are considered different because the order of the elements is different in each outcome.

8. If order doesn't matter, are the three outcomes considered names for the same outcome or are they considered three different outcomes?

   If order doesn't matter, the three outcomes are considered names for the same outcome because the elements are the same in each outcome.

20.3

Counting

Permutations and Combinations

LEARNING GOALS

In this lesson, you will:

- Use permutations to calculate the size of sample spaces.
- Use combinations to calculate the size of sample spaces.
- Use permutations to calculate probabilities.
- Use combinations to calculate probabilities.
- Calculate permutations with repeated elements.
- Calculate circular permutations.

KEY TERMS

- factorial
- permutation
- circular permutation
- combination

An estimated 50 million Americans do crossword puzzles each day!

Have you tried a crossword puzzle?

Let's say you were trying to figure out the letters for a 4-letter word by randomly choosing letters. It would take you a long time to randomly try all of the different arrangements of letters. In fact, for a 4-letter word there are 456,976 possible arrangements! Good thing crossword puzzles provide hints for each word.

Problem 1

When sample spaces are large and too cumbersome to compose organized lists, students explore patterns when calculating the number of possible outcomes. They generalize by concluding that if there are n letters, and they are taken three at a time without repetition, there can only be n choices for the first letter, $n - 1$ choices for the second letter, and $n - 2$ choices for the third letter, for a total of $n(n - 1)(n - 2)$. And if there are n letters, chosen three at a time with repetition, there can only be n choices for the first letter, n choices for the second letter, and n choices for the third letter, for a total of $n(n)(n)$ or n^3.

Grouping

Have students complete Questions 1 through 11 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 11

- What are the first four letters of the alphabet
- What is a three-letter string?
- Is ACA a possible three-letter string? Why not?
- If the first letter is chosen, how many choices remain for the second letter in the three-letter string?
- If the first and second letters are chosen, how many choices remain for the third letter in the three-letter string?
- What happens when you multiply the number of choices for each position? Does this product look familiar? Why?
- How many choices are there for the first letter in a four-letter string?
- If the first letter is chosen, how many choices remain for the second letter in the four-letter string?

PROBLEM 1 Strings and Factorials

Calculating large sample spaces can present several challenges because it can be too time consuming or impractical to list all of the possible outcomes. Even for relatively small numbers of options, listing the sample space can be challenging.

1. Using the first four letters of the alphabet, list all of the three-letter strings, such as DBA, that can be formed without using the same letter twice in one string.

ABC, ABD, ACB, ACD, ADB, ADC
BAC, BAD, BCA, BCD, BDA, BDC
CAB, CAD, CBA, CBD, CDA, CDB
DAB, DAC, DBA, DBC, DCA, DCB

2. How many different strings are possible?

There are 24 different three-letter strings.

3. For each string, how many possible letters could be first? Second? Third?

There are four choices for the first letter, three for the second, and two for the third.

4. How can your answer to Question 3 help you to calculate the number of possible three-letter strings?

You can multiply the number of choices for each letter to determine the total number of possibilities.

$4 \times 3 \times 2 = 24$

5. How many different four-letter strings can be made using the first four letters of the alphabet? Explain your reasoning.

There would still only be 24 because, in each case, the number of choices for each letter would be 4, then 3, then 2, then 1.

$4 \times 3 \times 2 \times 1 = 24$

6. If you were able to use the letters more than once, how many three-letter strings could you list? How many four-letter strings? Explain your reasoning.

There are 64 three-letter strings. I calculated the answer by multiplying the number of choices for each letter. The number of choices for each letter is 4 because the letters can be repeated.

$4 \times 4 \times 4 = 64$

There are 256 four-letter strings. I calculated the answer by multiplying the number of choices for each letter. The number of choices for each letter is 4 because the letters can be repeated.

$4 \times 4 \times 4 \times 4 = 256$

- If the first and second letters are chosen, how many choices remain for the third letter in the four-letter string?
- If the first, second and third letters are chosen, how many choices remain for the fourth letter in the four-letter string?
- How does introducing replacement change the situation?
- If the situation involves replacement, how many choices are there for each of the four letters in the three-letter string?
- If the situation involves replacement, how many choices are there for each of the four letters in the four-letter string?
- How many letters are in the alphabet?
- How many choices for the first letter? The second letter? The third letter?
- If this situation involved replacement, what effect would that have on the possible number of three-letter strings?
- If there are n choices for the first letter, what algebraic phrase represents one less choice?
- If there are n choices for the first letter, what algebraic phrase represents two less choices?

7. If each letter could only be used once, how many 10-letter strings could you list using the first 10 letters of the alphabet? Explain your reasoning.

There are 3,628,800 ten-letter strings.

I calculated the answer by multiplying the number of choices for each letter. The number of possibilities for each successive letter decrease by 1 because letters cannot be repeated.

$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3{,}628{,}800$

8. If you use the entire alphabet, how many three-letter strings can be made if each letter was only used once in each string? Explain your reasoning.

There are 15,600 three-letter strings.

I calculated the answer by multiplying the number of choices for each letter. The number of possibilities for each successive letter decrease by 1 because letters cannot be repeated.

$26 \times 25 \times 24 = 15{,}600$

9. Calculate the number of 26-letter strings possible without replacement.

There are approximately 4.03×10^{26} 26-letter strings.

I calculated the answer by multiplying the number of choices for each letter. The number of possibilities for each successive letter decrease by 1 because letters cannot be repeated.

$26 \times 25 \times 24 \ldots \times 3 \times 2 \times 1 = 4.03 \times 10^{26}$

10. In general, if there are n letters, how many three-letter lists are possible without repetition?

There can only be n choices for the first letter, $n - 1$ for the second, and $n - 2$ for the third, for a total of $n(n - 1)(n - 2)$.

11. In general, if there are n letters, how many three-letter lists are possible with repetition?

The number of three-letter outcomes using n letters, with repeating letters, is n^3 because there are n choices for each place in the list.

Problem 2

The term factorial is defined. Students calculate several factorials and write expressions as ratios of two factorials. They simplify fractions involving factorials by dividing out common factors. Answers to questions in Problem 1 are rewritten as factorials or factorial fractions.

Grouping

Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- What are integers?
- What are non-negative integers?
- Does your calculator have a factorial function? Where is it located? How do you use it?
- Is Josh ignoring the factorial signs?
- Why can't you ignore the factorial signs?
- What is the equivalent of 10!?
- What is the equivalent of 5!?
- Is 10! one half of 5!? Why not?
- When expressing the product as a ratio of 2 factorials, how do you know which factorial to place in the numerator?

PROBLEM 2 Give Me 5!

In 1808, Christian Kramp introduced the *factorial*, which could help you to perform some of the calculations in the previous questions. The **factorial** of n, which is written with an exclamation mark as $n!$, is the product of all non-negative integers less than or equal to n. For example, $3! = 3 \times 2 \times 1 = 6$.

1. Calculate each factorial.

a. $5! = 5(4)(3)(2)(1) = 120$

b. $7! = 7(6)(5)(4)(3)(2)(1) = 5040$

c. $10! = 10(9)(8)7! = 720(5040) = 3,628,800$

2. Josh says that $\frac{10!}{5!}$ is 2! because $10 \div 5 = 2$. Is Josh correct? If so, explain Josh's reasoning. If not, show how to calculate the correct quotient.

Josh is not correct because $\frac{10!}{5!} = 30,240$ and $2! = 2$.

$$\frac{10!}{5!} = \frac{(10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot \cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1})}{\cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = 30,240$$

$2! = 2 \times 1 = 2$

3. Write each expression as a ratio of 2 factorials.

a. $(5)(4)(3) = \frac{5!}{2!}$

b. $7 \cdot 6 \cdot 5 \cdot 4 = \frac{7!}{3!}$

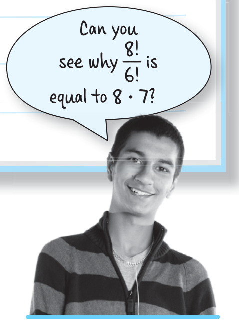
- When expressing the product as a ratio of 2 factorials, what feature of the product helps to determine the factorial that is placed in the denominator?
- If 6! appears as a factor in both the numerator and the denominator, can it be crossed out and divided into itself 1 time?

You can simplify fractions involving factorials by dividing out common factors. For example:

$$\frac{8!}{6!} = \frac{8 \times 7 \times \overset{1}{\cancel{6!}}}{\underset{1}{\cancel{6!}}}$$

$$= 8 \times 7$$

$$= 56$$



4. Simplify the following fractions.

a. $\frac{8!}{6!} = \frac{8 \cdot 7 \cdot \overset{1}{\cancel{6!}}}{\underset{1}{\cancel{6!}}} = 56$

b. $\frac{11!}{9!} = \frac{11 \cdot 10 \cdot \overset{1}{\cancel{9!}}}{\underset{1}{\cancel{9!}}} = 110$

c. $\frac{7!}{4!} = \frac{7(6)(5)\overset{1}{\cancel{(4)(3)(2)(1)}}}{\underset{1}{\cancel{4(3)(2)(1)}}} = 210$

d. $\frac{7!5!}{8!3!} = \frac{\cancel{7(6)(5)(4)(3)(2)(1)} \cdot (5)(4)\overset{1}{\cancel{(3)(2)(1)}}}{8\cancel{(7)(6)(5)(4)(3)(2)(1)} \cdot \underset{1}{\cancel{(3)(2)(1)}}} = \frac{20}{8} = \frac{5}{2}$

5. Using factorials, rewrite your answers to the following questions in Problem 1.

a. How many different four-letter strings can be made using the first four letters of the alphabet?

$4!$ or $\frac{4!}{1!}$

b. If each letter could only be used once, how many 10-letter strings could you list using the first 10 letters of the alphabet?

$10!$

c. If you use the entire alphabet, how many three-letter strings can be made if each letter was only used once in each string?

$\frac{26!}{23!}$

d. In general, if there are *n* letters, how many three-letter lists are possible without repetition?

$\frac{n!}{(n-3)!}$

Problem 3

The term permutation is defined and permutation notation is introduced. Students practice writing permutations as ratios of factorials. Students write a formula to describe the number of permutations, P, of r letters chosen from n letters; ${}_nP_r = \frac{n!}{(n-r)!}$. Students apply the formula to different situations to answer questions related to permutations.

Grouping

- Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- If you are forming a two-letter string from the first 4 letters of the alphabet, are you replacing the letter after you chose it?
- How is ${}_4P_2$ read?
- How do you write the number of permutations as a ratio of factorials?
- How is ${}_5P_3$ read?
- How do you express this situation using permutation notation?
- What is placed in the numerator of the formula?
- What is placed in the denominator of the formula?
- What does n represent in the formula with respect to the problem situation?

PROBLEM 3 Permutations

An ordered arrangement of items without repetition is called a **permutation**. In Questions 1 through 4 of Problem 1, the different 3-letter strings formed from the first 4 letters of the alphabet are permutations. There are different notations that are used for the permutations of r elements taken from a collection of n items:

$${}_nP_r = P(n, r) = P_r^n$$

1. Suppose you form 2-letter strings from the first 4 letters of the alphabet.

a. How many choices are there for the first letter? The second letter?

There are 4 choices for the first letter and 3 choices for the second letter.

b. How many different permutations are there of 2 letters from the first 4 letters of the alphabet, ${}_4P_2$?

There are 4 × 3, or 12, different permutations of 2 letters.

c. Write the number of permutations as a ratio of factorials.

$${}_4P_2 = 4 \times 3 = \frac{4 \times 3 \times \cancel{2} \times \cancel{1}}{\cancel{2} \times \cancel{1}} = \frac{4!}{2!}$$

2. Suppose you form 3-letter strings from any 5 letters of the alphabet. Write the number of permutations as a ratio of factorials.

$${}_5P_3 = 5 \times 4 \times 3 = \frac{5 \times 4 \times 3 \times \cancel{2} \times \cancel{1}}{\cancel{2} \times \cancel{1}} = \frac{5!}{2!}$$

3. How many 4-letter strings can be formed from the first 10 letters of the alphabet? Write the number of permutations as a ratio of factorials.

$${}_{10}P_4 = 10 \times 9 \times 8 \times 7 = \frac{10 \times 9 \times 8 \times 7 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}{\cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}} = \frac{10!}{6!}$$

4. Study the ratios of factorials you wrote in Questions 1–3. Let r be the number of letters in a permutation. Let n be the number of letters to choose from.

Write a formula to describe the number of permutations, P, of r letters chosen from n letters.

$${}_nP_r = \frac{n!}{(n-r)!}$$

- What does r represent in the formula with respect to the problem situation?
- How did you determine what the value of n is in this situation?
- How did you determine what the value of r is in this situation?
- Can the denominator of a fraction be equal to zero? Why or why not?
- Does 0! equal zero or one?
- Will the formula work if 0! is equal to zero?
- Will the formula work if 0! is equal to one?

Grouping

Have students complete Questions 6 and 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 6 and 7

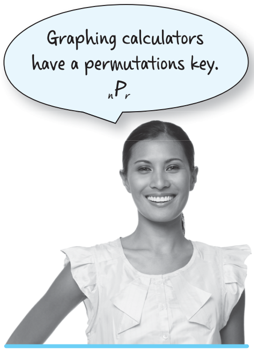
- How do you express this situation using permutation notation?
- What does *n* represent in the formula with respect to the problem situation?
- What does *r* represent in the formula with respect to the problem situation?

5. Calculate the following permutations using the formula ${}_nP_r = \frac{n!}{(n-r)!}$.

a. ${}_6P_3 = \frac{6!}{(6-3)!} = \frac{6(5)(4)\overset{1}{\cancel{(3)(2)(1)}}}{\underset{1}{\cancel{3(2)(1)}}} = 120$

b. ${}_{10}P_1 = \frac{10!}{(10-1)!} = \frac{(10)\overset{1}{\cancel{(9)(8)(7)(6)(5)(4)(3)(2)(1)}}}{\underset{1}{\cancel{(9)(8)(7)(6)(5)(4)(3)(2)(1)}}} = 10$

c. ${}_5P_2 = \frac{5!}{(5-2)!} = \frac{5(4)\overset{1}{\cancel{(3)(2)(1)}}}{\underset{1}{\cancel{(3)(2)(1)}}} = 20$



6. A sundae shop sells yogurt sundaes with 2 layers, 4 layers, or 5 layers of yogurt flavors. The shop carries 65 flavors in all.

a. How many different 4-layer sundaes could you create without repeating flavors?

I could create 16,248,960 different 4-layer sundaes.

${}_{65}P_4 = \frac{65!}{(65-4)!} = 65 \times 64 \times 63 \times 62 = 16{,}248{,}960$

b. Which type of sundae would give you the greatest number of choices if you do not repeat flavors?

2-layer sundae choices: ${}_{65}P_2 = \frac{65!}{(65-2)!} = 65 \times 64 = 4160$

4-layer sundae choices: ${}_{65}P_4 = \frac{64!}{(64-4)!} = 65 \times 64 \times 63 \times 62 = 16{,}248{,}960$

5-layer sundae choices: ${}_{65}P_5 = \frac{65!}{(65-5)!} = 65 \times 64 \times 63 \times 62 \times 61 = 991{,}186{,}560$

The 5-layer sundae would give me the greatest number of choices.

20

7. A briefcase lock has 3 dials, each with the digits 0–9. The 3-digit code that unlocks the briefcase has no digits repeating.

a. If you tried one code every 5 seconds, what is the greatest amount of time it could take you to open the briefcase?

The greatest amount of time it could take is 3600 seconds, or 1 hour. This is the amount of time it would take if I tried the correct code last.

I need to determine the number of 3-digit codes, or 3-digit permutations, there are from a collection of 10 elements (10 digits from 0–9).

$$ {}_{10}P_3 = \frac{10!}{(10-3)!} = 10 \times 9 \times 8 = 720 $$

There are 720 possible codes to try. At one code every 5 seconds, it would take at most 5 × 720, or 3600, seconds to open the briefcase. This is 60 minutes, or 1 hour.

b. Suppose the briefcase has 10 dials, each with the digits 0–9. How many different 10-digit codes are possible with no repeated digits?

$$ {}_{10}P_{10} = \frac{10!}{(10-10)!} = \frac{10!}{0!} = 3{,}628{,}800 $$

c. What conclusion can you make about the value of 0!? Explain your reasoning.

In order for the formula to work, 0! must be equal to 1.

Problem 4

Students use a formula to determine the number of permutations in a situation and create an organized list to verify their answer. Next, order doesn't matter enters the situation which effects the number of permutations. Permutations of other situations are compared when order matters and then when order doesn't matter. Students use answers from previous questions to complete a table which helps them write a formula for the number of permutations of *n* elements with k copies of an element: $\frac{n!}{k!}$. Next, the write a formula for the number of permutations of *n* elements with *k* copies of one element and *h* copies of another element: $\frac{n!}{k!h!}$.

Grouping

- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.
- Discuss the paragraph below Question 4 as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- Is the order of the books important in this situation?
- Are N_1N_2RS and N_1N_2SR both considered different permutations in this situation?

PROBLEM 4 Permutations with Repeated Elements

Sofie has 4 books on her shelf—2 novels (N_1 and N_2), 1 science book (S), and 1 reference book (R).

1. How many permutations of all 4 books are there? Write an organized list to verify your answer.

There are 24 permutations.

N_1N_2RS	~~N_2N_1RS~~	RN_1N_2S	SN_1N_2R
N_1N_2SR	~~N_2N_1SR~~	RN_1SN_2	SN_1RN_2
N_1RN_2S	~~N_2RN_1S~~	~~RN_2N_1S~~	~~SN_2N_1R~~
N_1RSN_2	~~N_2RSN_1~~	~~RN_2SN_1~~	~~SN_2RN_1~~
N_1SN_2R	~~N_2SN_1R~~	RSN_1N_2	SRN_1N_2
N_1SRN_2	~~N_2SRN_2~~	~~RSN_2N_1~~	~~SRN_2N_1~~

2. Suppose that Sofie wants to arrange her books by book type. The order of the 2 novels doesn't matter to her.

a. Does this change the number of permutations? Explain your reasoning.

Yes. If the order of the novels doesn't matter, then N_2N_1 is the same as N_1N_2. That means that any arrangement which repeats the order of the novels from any other arrangement is not a permutation.

b. If necessary, modify the organized list in Question 1 to show the new number of permutations.

See organized list.

I crossed off 12 repeated arrangements.

So, the number of permutations by book type is 12.

- If Sophie wants to arrange her books by book type, does this mean order is important when determining the permutations?
- Is N_1N_2 considered the same permutation as N_2N_1?
- Is $RN_1N_2N_3$ considered the same permutation as $RN_2N_1N_3$?
- Is $RN_1N_2N_3$ considered the same permutation as $N_2N_1RN_3$?
- Is $N_4N_1N_2N_3$ considered the same permutation as $N_2N_1N_3N_4$?

3. Suppose Sofie has 3 novels (N_1, N_2, N_3) and 1 reference book (R) on her shelf.

a. How many permutations of all 4 books are there?

If each novel is considered separately, there are 24 permutations.

$${}_4P_4 = \frac{4!}{(4-4)!} = 4! = 24$$

b. How many permutations are there if the order of the novels doesn't matter? Write an organized list to determine your answer.

There would be 4 permutations if the order of the novels doesn't matter.

NNNR
NNRN
NRNN
RNNN

4. Suppose all 4 of Sophie's books are novels (N_1, N_2, N_3, N_4).

a. How many permutations of all 4 books are there?

If each novel is considered separately, there would be 24 permutations.

$${}_4P_4 = \frac{4!}{(4-4)!} = 4! = 24$$

b. How many permutations are there if the order of the novels doesn't matter? Write an organized list to determine your answer.

There would be only 1 permutation if the order of the novels doesn't matter.

NNNN

When the order of all of the elements in an arrangement matters, the arrangement contains 1 copy of each element. For example, the arrangement of Sofie's books N_1N_2RS contains 1 copy of each element.

When the order of some or all of the elements in an arrangement does not matter, the arrangement contains 2 or more copies of one or more of the elements. For example, the arrangement NNRS of Sofie's 4 books contains 2 copies of the element N (2 novels).

Grouping

Have students complete Questions 5 through 12 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 12

- What do you notice about the relationship between the middle two columns of the table?
- How does the number of permutations when order matters compare to the number of permutations when order doesn't matter?
- Under what condition can we think of N_1N_2 as NN?
- Under what condition can we not think of N_1N_2 as NN?
- What did you divide by? Is it the same as the product of the number of copies of each element?
- Is the number of permutations always a whole number? Why or why not?

5. Use your answers from Questions 1–4 to complete the first three rows of the table.

List of Elements	Number of Permutations (order of novels matters)	Repeated Elements	Factorial of Number of Repeated Elements	Number of Permutations (order of novels doesn't matter)
N_1, N_2, S, R	24	2	$2! = 2$	12
N_1, N_2, N_3, R	24	3	$3! = 6$	4
N_1, N_2, N_3, N_4	24	4	$4! = 24$	1

6. Refer to the table in Question 5 to describe how to calculate the number of permutations when the order of the novels doesn't matter.

I can determine the number of permutations when the order of novels doesn't matter by dividing the number of permutations when the order of the novels matters by the factorial of the number of repeated elements.

7. Write a formula to describe the number of permutations of *n* elements with *k* copies of an element.

$\frac{n!}{k!}$

What if there were more than 1 repeated element?

8. Suppose Sofie has 2 novels and 2 science books. The order of each type of book doesn't matter. Write an organized list of all the different permutations of Sofie's books.

NNSS NSNS NSSN
SSNN SNSN SNNS

9. Suppose Sofie has 3 reference books and 2 science books. The order of each type of book doesn't matter. Write an organized list of all the different permutations of Sofie's books.

RRRSS RRSRS RRSSR RSRSR RSRRS RSSRR
SSRRR SRSRR SRRSR SRRRS

10. Use your answers from Questions 8 and 9 to complete the table.

List of Elements	Number of Permutations (order or repeated elements matters)	First Repeated Element	Factorial of First Number of Repeated Elements	Second Repeated Element	Factorial of Second Number of Repeated Elements	Product of Factorials	Number of Permutations (order of repeated elements does not matter)
N_1, N_2, S_1, S_2	24	2	$2! = 2$	2	$2! = 2$	$2 \cdot 2 = 4$	6
R_1, R_2, R_3, S_1, S_2	120	3	$3! = 6$	2	$2! = 2$	$6 \cdot 2 = 12$	10

11. Compare the value that you divided by each time to the number of copies. What do you notice?

If one item had multiple copies, then I divided by the factorial of that number of copies. For example, in Question 4 there are 4 novels, so I divided by 4!, or 12.

If more than one item has multiple copies, then I divide by the product of the factorial of each number of copies. For example, in Question 9 there are 3 reference books and 2 science books, so I divided by $3! \cdot 2!$, or 12.

12. Write a formula to describe the number of permutations of *n* elements with *k* copies of one element and *h* copies of another element.

$$\frac{n!}{k!h!}$$

Grouping

Have students complete Questions 13 through 16 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 13 through 16

- Is the formula $\frac{n!}{k!}$ used when order matters or when order doesn't matter?
- When is it appropriate to use the formula $\frac{n!}{k!}$?
- When determining the number of 6-letter strings formed by the word CANNON, does order matter? How do you know?
- When determining the number of 7-letter strings formed by the word Alabama, does order matter? How do you know?

13. Suppose Sofie has 3 novels, a science book, and a reference book. She chooses 2 books. The order of the novels doesn't matter. How many different permutations of 2 books are possible?

Gary and Denise tried to solve this problem.

Denise

There are 5 elements in the collection, and Sofie chooses 2 with 3 elements repeating, so I think the number of permutations is

$$ {}_nP_r = \frac{n!}{(n-r)!(r!)} = \frac{5!}{(5-2)!(3!)} = \frac{120}{36} \approx 3.3\ldots $$

There are approximately 3.3 permutations.

Gary

The organized list looks like this:

NN NS NR
SN SR
RN RS

There are 7 permutations.

Explain why Gary is correct and Denise is incorrect.

Gary's list shows all of the possibilities for choosing a novel first, all the possibilities for choosing the science book first, and all the possibilities for choosing the reference book first.

I know Denise's answer is incorrect because the number of permutations cannot be a decimal. It has to be a whole number.

14. Does $\frac{n!}{k!}$ describe the correct number of permutations in the situation in Question 13? Explain your reasoning.

No.

$\frac{5!}{3!}$ is equal to 20, but there are only 7 permutations.

15. Calculate the number of 6-letter strings that can be formed from the word CANNON.

One-hundred twenty 6-letter strings can be formed from the word CANNON.

$\frac{n!}{k!} = \frac{6!}{3!} = 6 \cdot 5 \cdot 4 = 120$

16. Calculate the number of seven-letter strings that can be formed from the word *Alabama*.

Two-hundred ten 7-letter strings can be formed from the word *Alabama*.

$\frac{n!}{k!} = \frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 210$

Problem 5

The term circular permutation is described. A round table seating arrangement is used to list examples of circular permutations. Students list elements that are equivalent to specified orders. The circular permutation of *n* objects is $(n - 1)!$. Students use this formula to answer questions related to the situation.

Grouping

Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

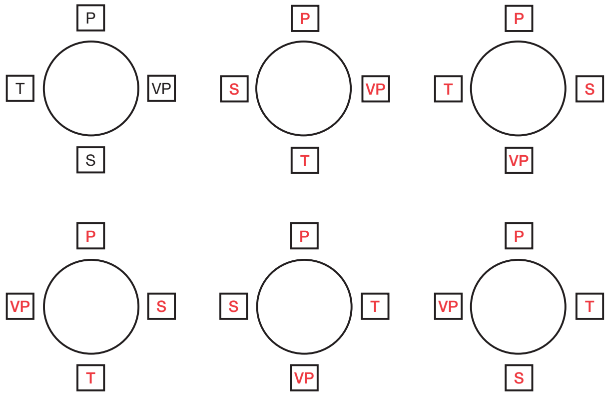
Guiding Questions for Share Phase, Questions 1 through 7

- How many different ways can the officers be arranged in a line?
- Does order matter when arranging the officers in a line?
- How did you determine which elements were equivalent to the specified table seating?
- Why do you suppose the circular permutation of *n* objects is $(n - 1)!$?

PROBLEM 5 Circular Permutations

A club consists of four officers: a president (P), a vice-president (VP), a secretary (S), and a treasurer (T).

1. List the different ways that the four officers could be seated at a round table



2. List the different ways that the officers could be arranged in a line.

P, VP, S, T
P, VP, T, S
P, S, VP, T
P, S, T, VP
P, T, VP, S
P, T, S, VP
VP, P, S, T
VP, P, T, S
VP, S, P, T
VP, S, T, P
VP, T, P, S
VP, T, S, P
S, P, T, VP
S, P, VP, T
S, VP, T, P
S, VP, P, T
S, T, P, VP
S, T, VP, P
T, P, VP, S
T, P, S, VP
T, VP, P, S
T, VP, S, P
T, S, P, VP
T, S, VP, P

3. Which elements from Question 2 are equivalent to the table seating of President, Vice-President, Secretary, Treasurer?

P, VP, S, T

T, P, VP, S

S, T, P, VP

VP, S, T, P

4. Which elements from Question 2 are equivalent to the table seating of President, Secretary, Vice-President, Treasurer?

P, S, VP, T

T, P, S, VP

VP, T, P, S

S, VP, T, P

5. Which elements from Question 2 are equivalent to the table seating of President, Treasurer, Secretary, Vice-President?

P, T, S, VP

VP, P, T, S

S, VP, P, T

T, S, VP, P

6. Based on Questions 3 through 5, how many equivalent elements are included for each seating arrangement? How does this number appear in the original problem?

Each seating arrangement consists of 4 equivalent elements. There are 4 officers in the original problem and 4 equivalent elements for each seating arrangement.

The seating arrangement problem is a *circular permutation*. A **circular permutation** is a permutation in which there is no starting point and no ending point. The circular permutation of n objects is $(n - 1)!$.

7. Calculate the number of table arrangements for each number of officers.

a. Five officers

There are 24 seating arrangements for 5 officers. I calculated the answer using the formula for a circular permutation.

$(5 - 1)! = 24$

b. Six officers

There are 120 seating arrangements for 6 officers. I calculated the answer using the formula for a circular permutation.

$(6 - 1)! = 120$

c. Ten officers

There are 362,880 seating arrangements for 10 officers. I calculated the answer using the formula for a circular permutation.

$(10 - 1)! = 362{,}880$

Problem 6

The term combination is defined and combination notation is introduced. Students first determine combinations by determining the permutations and then removing all permutations containing the same elements. Students write a formula to describe the number of combinations, C, of r elements taken from a collection of n elements; $_nC_r = \frac{n!}{(n-r)!r!}$. The connection between this formula and $_nC_r = \frac{_nP_r}{r!}$ is made. Students apply the formula to different situations to answer questions related to combinations.

Grouping

- Have students complete Question 1, parts (a) and (b) with a partner. Then have students share their responses as a class.
- Ask students to read the narrative about combinations. Discuss as a class.

Guiding Questions for Share Phase, Question 1, parts (a) and (b)

- When determining the 3-letter permutations of DEFG, does order matter?
- What is the total number of permutations?
- How many outcomes contain the exact same elements as DEF?

PROBLEM 6 Combinations

1. The 3-letter permutations of DEFG are shown:

DEF	~~EDF~~	~~FDE~~	~~GDE~~
DEG	~~EDG~~	~~FDG~~	~~GDF~~
~~DFE~~	~~EFD~~	~~FED~~	~~GED~~
DFG	EFG	~~FEG~~	~~GEF~~
~~DGE~~	~~EGD~~	~~FGD~~	~~GFD~~
~~DGF~~	~~EGF~~	~~FGE~~	~~GFE~~

$_4P_3 = \frac{4!}{(4-3)!} = 24$ permutations

a. Start at the top left (DEF). Cross out all the outcomes that contain the exact same elements as DEF, but in a different order. One example is DFE.

See list.

b. Move to the next outcome in the first column that is not crossed out, DEG, and repeat the process. Continue until you cannot cross out any outcomes.

See list.

The arrangements that you did not cross off are all the *combinations* of 3 letters from the collection of 4 letters. A **combination** is an unordered collection of items. Different notations can be used for the combinations of r elements taken from a collection of n elements:

$$_nC_r = C(n, r) = C^n_r$$

c. How many combinations of 3-letter strings are there, $_4C_3$?

There are 4 combinations of 3-letter strings.
I determined the answer by counting the number of outcomes in the sample space.

d. The notation $_4C_3$ represents the number of combinations of 3-letter strings formed from the 4 letters DEFG. Write a value in factorial notation to make the equation true.

$$_4C_3 = \frac{_4P_3}{\boxed{3!}}$$

- How many outcomes did you end up removing? How many combinations are in this situation?

Grouping

Have students complete Question 1, part (c) through Question 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1, parts (c) through Question 7

- How is $_4C_3$ read?
- 24 divided by what number equals 4?
- How can 6 be rewritten as a factorial?
- Will 3! satisfy the equation?
- How is 3! related to $_4C_3$?
- How is 2! related to $_4C_2$?
- How is 3! related to $_3C_3$?
- How is 2! related to $_5C_2$?

2. The 2-letter permutations of DEFG are shown:

DE	~~ED~~	~~FD~~	~~GD~~
DF	EF	~~FE~~	~~GE~~
DG	EG	FG	~~GF~~

$_4P_2 = \frac{4!}{(4-2)!} = \frac{24}{2} = 12$ permutations

a. How many combinations of 2-letter strings are there, $_4C_2$?

There are 6 combinations of 2-letter strings.

b. Write a value in factorial notation to make the equation true.

$$_4C_2 = \frac{_4P_2}{\boxed{2!}}$$

3. The 3-letter permutations of WXY are shown:

WXY	~~XWY~~	~~YWX~~
~~WYX~~	~~XYW~~	~~YXW~~

$_3P_3 = \frac{3!}{(3-3)!} = \frac{6}{1} = 6$ permutations

a. How many combinations of 3-letter strings are there, $_3C_3$?

There is 1 combination of 3-letter strings.

b. Write a value in factorial notation to make the equation true.

$$_3C_3 = \frac{_2P_3}{\boxed{3!}}$$

4. The 2-letter permutations of RSTVX are shown:

RS	~~SR~~	~~TR~~	~~VR~~	~~XR~~
RT	ST	~~TS~~	~~VS~~	~~XS~~
RV	SV	TV	~~VT~~	~~XT~~
RX	SX	TX	VX	~~XV~~

$_5P_2 = \frac{5!}{(5-2)!} = \frac{120}{6} = 20$ permutations

a. How many combinations of 2-letter strings are there, $_5C_2$?

There are 10 combinations of 2-letter strings.

b. Write a value in factorial notation to make the equation true.

$$_5C_2 = \frac{_5P_2}{\boxed{2!}}$$

5. Use your answers in Questions 1–4 to complete the table.

	n	r	$_nP_r$	Divided by	$_nC_r$
Question 1	4	3	24	6 or (3!)	4
Question 2	4	2	12	2 or (2!)	6
Question 3	3	3	6	6 or (3!)	1
Question 4	5	2	20	2 or (2!)	10

6. Compare the value that you divided by each time to the value of r. What do you notice?

The value that I divided by is equivalent to $r!$.

7. Study the table. Write a value in factorial notation to make this general statement true.

$$_nC_r = \frac{_nP_r}{\boxed{r!}}$$

8. Calculate the following combinations using the formula $_nC_r = {_nP_r}$, or $_nC_r = \frac{n!}{(n-r)!r!}$.

a. $_6C_3 = \frac{6!}{(6-3)!3!} = \frac{(6)(5)(4)(3)(2)(1)}{(3)(2)(1)(3)(2)(1)} = 20$

b. $_{10}C_1 = \frac{10!}{(10-1)!1!} = \frac{(10)(9)(8)(7)(6)(5)(4)(3)(2)(1)}{(9)(8)(7)(6)(5)(4)(3)(2)(1)(1)} = 10$

c. $\binom{7}{3} = \frac{7!}{(7-3)!3!} = \frac{(7)(6)(5)(4)(3)(2)(1)}{4(3)(2)(1)(3)(2)(1)} = 35$

Grouping

Have students complete Questions 8 and 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 8 and 9

- Are there any shortcuts when calculating the combinations using the formula or must you multiply all of the factors in both the numerator and denominator?
- Can like factorials be divided into each other?
- How is $_{10}C_4$ read?
- How is $_{10}C_6$ read?
- How is $_{10}C_1$ read?
- What is the value of 0!? Does that make sense with respect to this situation?
- What happens when 0! is all that is left in the denominator?

9. Using an organization of 10 members and the formula for combinations, answer the following.

a. How many four-member committees can be chosen?

$${}_{10}C_4 = \frac{10!}{(10-4)!4!} = \frac{(10)(\overset{3}{\cancel{9}})(\cancel{8})(7)(\cancel{6})(\cancel{5})\overset{1}{\cancel{(4)(3)(2)(1)}}}{(\cancel{6})(\cancel{5})(\cancel{4})(\cancel{3})(\cancel{2})(1)\underset{1}{\cancel{(4)(3)(2)(1)}}} = 210$$

A total of 210 committees can be chosen.

b. How many six-member committees can be chosen?

$$\binom{10}{6} = \frac{10!}{(10-6)6!} = \frac{(10)(\overset{3}{\cancel{9}})(\cancel{8})(7)\overset{1}{\cancel{(6)(5)(4)(3)(2)(1)}}}{(\cancel{4})(\cancel{3})(\cancel{2})(1)\underset{1}{\cancel{(6)(5)(4)(3)(2)(1)}}} = 210$$

A total of 210 committees can be chosen.

c. How many one-member committees can be chosen?

$${}_{10}C_1 = \frac{10!}{(10-1)!1!} = \frac{(10)\overset{1}{\cancel{(9)(8)(7)(6)(5)(4)(3)(2)(1)}}}{\underset{1}{\cancel{(9)(8)(7)(6)(5)(4)(3)(2)(1)}}(1)} = 10$$

A total of 10 committees can be chosen.

d. How many 10-member committees can be chosen?

$$\binom{10}{10} = \frac{10!}{(10-10)!10!} = \frac{10!}{0!} = \frac{\overset{1}{\cancel{(10)(9)(8)(7)(6)(5)(4)(3)(2)(1)}}}{0!\underset{1}{\cancel{(10)(9)(8)(7)(6)(5)(4)(3)(2)(1)}}} = 1$$

One committee can be chosen.

Talk the Talk

Different situations are described and students decide whether it is associated with a permutation or a combination. Then they calculate the answer.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Talk the Talk

1. State whether each question uses permutations or combinations. Then calculate the answer.

   a. Using a standard deck of playing cards, how many different five-card hands can be dealt without replacement?

   Combinations; $\binom{52}{5} = \frac{52!}{(52-5)!5!} = 2,598,960$

   b. How many different numbers can be made using any three digits of 12,378?

   Permutations; ${}_5P_3 = 5(4)(3) = 60$

   c. How many different ways can you arrange 10 CDs on a shelf?

   Permutations; ${}_{10}P_{10} = 10! = 3,628,800$

   d. A professional basketball team has 12 members, but only five can play at any one time. How many different groups of players can be on the court at one time?

   Combinations; ${}_{12}C_5 = \frac{12!}{(12-5)!5!} = 792$

20

2. Calculate the following probabilities.

a. Using a standard deck of playing cards, what is the probability that a person is randomly dealt a five-card hand containing an ace, a king, a queen, a jack, and a ten?

The probability a person is dealt a five-card hand containing an ace, a King, a Queen, a Jack, and a ten is $\frac{64}{162{,}435}$. I determined the number of desired outcomes by using the Counting Principle and the total number of possible outcomes by using combinations.

$\frac{64}{162{,}435}$.

$P(\text{ace}, K, Q, J, \text{and ten}) = \frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{{}_{52}C_5} = \frac{1024}{2{,}598{,}960} = \frac{64}{162{,}435}$

b. Consider the number 12,378. Using any three digits, what is the probability of making a three-digit number whose value is greater than 700?

The probability of making a three-digit number whose value is greater than 700 is $\frac{2}{5}$. I determined the number of desired outcomes by using the Counting Principle and the number of total possible outcomes by using permutations.

In order for the three-digit number to be greater than 700, the first digit must be 7 or 8.

$P(\text{number} > 700) = \frac{2 \cdot 4 \cdot 3}{{}_5P_3} = \frac{24}{60} = \frac{2}{5}$

Be prepared to share your solutions and methods.

Check for Students' Understanding

Explain the usefulness of each formula.

1. ${}_nC_r = \frac{n!}{(n-r)!r!}$

This is the formula used to compute the number of combinations, *C*, of *r* letters chosen from *n* letters.

2. $\frac{n!}{k!h!}$

This is the formula for the number of permutations of *n* elements with *k* copies of one element and *h* copies of another element.

3. ${}_nP_r = \frac{n!}{(n-r)!r!}$

This is the formula used to compute the number of permutations, *P*, of *r* letters chosen from *n* letters.

4. $\frac{n!}{k!}$

This is the formula for the number of permutations of *n* elements with *k* copies of an element.

5. $(n-1)!$

This is the formula used to compute circular permutation of objects of *n* objects.

6. $n(n-1)(n-2)\ldots$

This is the formula used to calculate *n*! (factorial).

20.4

Trials

Independent Trials

LEARNING GOALS

In this lesson, you will:

- Calculate the probability of two trials of two independent events.
- Calculate the probability of multiple trials of two independent events.
- Determine the formula for calculating the probability of multiple trials of independent events.

ESSENTIAL IDEA

If the probability of Event A is p and the probability of Event B is $1 - p$, then the probability of Event A occurring r times and Event B occurring $n - r$ times in n trials is: P(A occurring r times and B occurring $n - r$ times) or ${}_nC_r\,(p)^r\,(1 - p)^{n-r}$

COMMON CORE STATE STANDARDS FOR MATHEMATICS

S-CP Conditional Probability and the Rules of Probability

Use the rules of probability to compute probabilities of compound events in a uniform probability model

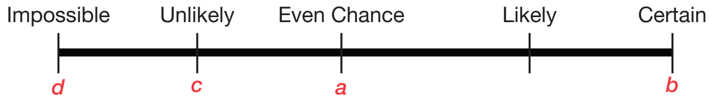
9. Use permutations and combinations to compute probabilities of compound events and solve problems.

Overview

Situations in this lesson focus on multiple trials for two independent events. Making free throw shots, rolling number cubes, and rolling a tetrahedron are used to generate the probabilities of two independent events. Outcomes are organized in a table and the table is connected to Pascal's Triangle. Students use Pascal's Triangle to compute the probability of an occurrence. A formula using combinations is applied to different situations to calculate probabilities for two independent events over multiple trials.

Warm Up

Consider the following illustration of possible probabilities of an event occurring.



Place each situation in an appropriate place on the line and explain your reasoning.

a. The probability of flipping a coin that results in heads.

The probability of flipping a coin and getting heads is 1 out of 2. I think this outcome has a 50% chance of occurring, so it has an even chance that it will occur.

b. The probability of flipping a coin that results in heads or tails.

The probability of flipping a coin and getting heads or tails is a certainty.

c. The probability of rolling a 3 on a six-sided number cube.

The probability of rolling a 3 on a number cube is 1 out of 6. I think this outcome has a less than 50% chance of occurring, so it is unlikely to occur.

d. The probability of rolling a 7 on a six-sided number cube.

The probability of rolling a 7 on a number cube is impossible. The sides are numbered 1 through 6.

e. Describe an event that is likely.

Answers will vary.

The probability of rolling a 1, 2, 3, or 4 on a number cube is likely.

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20.4

Trials

Independent Trials

LEARNING GOALS

In this lesson, you will:

- Calculate the probability of two trials of two independent events.
- Calculate the probability of multiple trials of two independent events.
- Determine the formula for calculating the probability of multiple trials of independent events.

What, do you think, is a typical free-throw percentage for an NBA player? What is the greatest free-throw percentage?

In May of 2013, the all-time leader was Steve Nash, who by that time had attempted 3360 free throws and made 3038 of them!

If Nash steps up to the free-throw line, what's the probability he will miss?

Problem 1

Improving free throw shooting is used to determine various probabilities associated with two or more trials of independent events. Students create a formula for calculating the probability of multiple trials of independent events. Outcomes are written in an arrangement that forms Pascal's Triangle and this connection is discussed. Pascal's Triangle is used to calculate the number of ways that an outcome can occur.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Is Walter's probability of making or missing an example of experimental or theoretical probability?
- If you didn't know about Walter's past performance, would Michael's answer be correct?
- What are the different ways Walter can make one free throw and miss one?

PROBLEM 1 Make or Miss?

Walter is a basketball player who is constantly working at improving his free throw shooting. He's worked his way up to consistently making 2 out of 3 free throws.

He is about to shoot 3 free throws.

1. Walter shoots 1 free throw. What's the probability of Walter making the free throw? Missing the free throw?

   The probability of Walter making the free throw, *P*(make), is $\frac{2}{3}$.

   The probability of Walter missing the free throw, *P*(miss), is $\frac{1}{3}$.

2. Walter shoots 2 free throws. What is the probability of Walter making 1 out of 2 free throws?

   Michael, Julie, and Erica shared their responses.

👎 Michael

The probability of Walter making 1 out of 2 free throws is $\frac{2}{4}$, or $\frac{1}{2}$ because 2 out of the 4 outcomes in the sample space show 1 made free throw.

make-make   miss-miss

make-miss   miss-make

👎 Julie

The probability of Walter making 1 out of 2 free throws is $\frac{2}{9}$.

I calculated the probability by multiplying the probability of a make and the probability of a miss.

$$\begin{aligned} P(1 \text{ make and } 1 \text{ miss}) &= P(1 \text{ make}) \cdot P(1 \text{ miss}) \\ &= \frac{2}{3} \cdot \frac{1}{3} \\ &= \frac{2}{9} \end{aligned}$$

👍 Erica

The probability of Walter making 1 out of 2 free throws is $\frac{4}{9}$.

I answered the question by multiplying the probability of a make and the probability of a miss. Then, I multiplied that result by 2 because Walter can make 1 out of 2 in two different ways. He could make the first and miss the second, or miss the first and make the second.

$$\begin{aligned} P(1 \text{ make and } 1 \text{ miss}) &= 2[P(1 \text{ make}) \cdot P(1 \text{ miss})] \\ &= 2[\frac{2}{3} \cdot \frac{1}{3}] \\ &= 2[\frac{2}{9}] \\ &= \frac{4}{9} \end{aligned}$$

a. Why is Michael's response incorrect? Explain.

Michael's response is incorrect because it does not take into account that Walter making a free throw is not equally likely to Walter missing a free throw. Walter makes 2 out of 3 free throws, which means he misses 1 out of 3.

If Walter had an equal probability of making or missing a free throw, $\frac{1}{2}$, then his response would have been correct.

b. Why is Julie's response incorrect? Explain.

Julie's response is incorrect because it does not take into account that Walter can make 1 out of 2 free throws in two different ways. He could make the first free throw and miss the second, or miss the first free throw and make the second.

Julie's calculation shows the probability of Walter making the first free throw and missing the second. But, her calculation does not include the probability of Walter missing the first free throw and making the second.

c. What's different about Erica's calculation and Julie's calculation?

Erica multiplied by 2, but Julie did not

d. Explain why Erica's method is correct and Julie's is incorrect.

Erica's method is correct because she multiplied the probability of making 1 out of 2 free throws by 2. Julie did not multiply by 2.

Multiplying by 2 represents Walter making 1 out of 2 free throws in two different ways. He could make the first free throw and miss the second, or miss the first free throw and make the second.

Grouping

Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- Is there a way to figure out the number of ways Walter can make a certain number and miss a certain number of free throws?
- Could you use exponents in your work?
- What would the exponents mean?

3. Calculate each probability.

a. How many different outcomes result in Walter making 2 out of 2 free throws? What is the probability of Walter making 2 out of 2 free throws?

One outcome results in Walter making 2 out of 2 free throws: make-make. In all the other outcomes, Walter makes less than 2 free throws: make-miss, miss-make, and miss-miss.

The probability of Walter making 2 out of 2 free throws, $P(\text{make and make})$, is $\frac{4}{9}$.

I calculated the probability by multiplying the probability of a make and the probability of a make.

$$\begin{aligned} P(\text{2 makes}) &= P(\text{make}) \cdot P(\text{make}) \\ &= \frac{2}{3} \cdot \frac{2}{3} \\ &= \frac{4}{9} \end{aligned}$$

b. How many different outcomes result in Walter making 1 out of 2 free throws? What is the probability of Walter making 1 out of 2 free throws?

Two outcomes result in Walter making 1 out of 2 free throws: make-miss and miss-make. The probability of Walter making 1 out of 2 free throws, $P(\text{make and make})$, is $\frac{4}{9}$.

First, I multiplied the probability of 1 make and 2 misses. Then, I multiplied that result by 2 because there are 2 different ways Walter could make 1 out of 2 free throws: make-miss and miss-make.

$$\begin{aligned} P(\text{1 make and 1 miss}) &= 2 \cdot [P(\text{make}) \cdot P(\text{miss})] \\ &= 2 \cdot \left[\frac{2}{3} \cdot \frac{1}{3}\right] \\ &= 2 \cdot \left[\frac{2}{9}\right] \\ &= \frac{4}{9} \end{aligned}$$

c. How many different outcomes result in Walter making 0 out of 2 free throws? What is the probability of Walter making 0 out of 2 free throws?

One outcome results in Walter making 0 out of 2 free throws: miss-miss.

The probability of Walter making 0 out of 2 free throws, P(miss and miss), is $\frac{1}{9}$.

I calculated the probability by multiplying the probability of a miss and the probability of a miss.

$$\begin{aligned} P(\text{0 makes}) &= P(\text{miss}) \cdot P(\text{miss}) \\ &= \frac{1}{3} \cdot \frac{1}{3} \\ &= \frac{1}{9} \end{aligned}$$

4. Walter shoots 3 free throws. Determine the probability of the following events.

   a. What is the probability of Walter making 0 out of 3 free throws?

   The probability of Walter making 0 out of 3 free throws, $P(0 \text{ makes})$, is $\frac{1}{27}$.

   I calculated the probability by multiplying the probability of 3 misses.

$$\begin{aligned} P(0 \text{ makes}) &= P(\text{miss}) \cdot P(\text{miss}) \cdot P(\text{miss}) \\ &= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \\ &= \frac{1}{27} \end{aligned}$$

   b. What is the probability of Walter making 1 out of 3 free throws?

   The probability of Walter making 1 out of 3 free throws, $P(1 \text{ make and 2 misses})$, is $\frac{6}{27}$.

   First, I multiplied the probability of 1 make and 2 misses. Then, I multiplied that result by 3 because there are 3 different ways Walter could make 1 out of 3 free throws: make-miss-miss, miss-make-miss, and miss-miss-make.

$$\begin{aligned} P(1 \text{ make and 2 misses}) &= 3 \cdot [P(\text{make}) \cdot P(\text{miss}) \cdot P(\text{miss})] \\ &= 3 \cdot \left[\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}\right] \\ &= 3 \cdot \frac{2}{27} \\ &= \frac{6}{27} \\ &= \frac{2}{9} \end{aligned}$$

c. What is the probability of Walter making 2 out of 3 free throws?

The probability of Walter making 2 out of 3 free throws, P(2 makes and 1 miss), is $\frac{12}{27}$.

First, I multiplied the probability of 2 makes and 1 miss. Then, I multiplied that result by 3 because there are 3 different ways Walter could make 2 out of 3 free throws: make-make-miss, make-miss-make, and miss-make-make.

$$\begin{aligned} P(\text{2 makes and 1 miss}) &= 3 \cdot [P(\text{make}) \cdot P(\text{make}) \cdot P(\text{miss})] \\ &= 3 \cdot \left[\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3}\right] \\ &= 3 \cdot \left[\frac{4}{27}\right] \\ &= \frac{12}{27} \\ &= \frac{4}{9} \end{aligned}$$

d. What is the probability of Walter making 3 out of 3 free throws?

The probability of Walter making 3 out of 3 free throws, P(3 makes), is $\frac{8}{27}$.

I calculated the probability by multiplying the probability of 3 makes.

$$\begin{aligned} P(\text{3 makes}) &= P(\text{make}) \cdot P(\text{make}) \cdot P(\text{make}) \\ &= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \\ &= \frac{8}{27} \end{aligned}$$

Grouping

Have students complete Question 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- How does Susan use combinations to improve the efficiency of calculations?
- What does it mean when the numerator and denominator are equal factorials?
- What does 0! in the denominator mean?

5. Susan claims that she can calculate the probability of Walter making free throws more efficiently, using combinations.

a. For determining the "3 ways" Walter can make 1 out of 3 free throws, she multiplied ${}_3C_1$ and $P(1 \text{ make})$.

$$\begin{aligned} P(1 \text{ make and } 2 \text{ misses}) &= {}_3C_1 \cdot [P(\text{make}) \cdot P(\text{miss}) \cdot P(\text{miss})] \\ &= 3 \cdot [P(\text{make}) \cdot P(\text{miss}) \cdot P(\text{miss})] \\ &= 3 \cdot \left[\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}\right] \\ &= 3 \cdot \left[\frac{2}{27}\right] \\ &= \frac{6}{27} \\ &= \frac{2}{9} \end{aligned}$$

Verify that Susan's method works for the calculations in Questions 3 and 4 by using combinations to calculate each probability.

b. Walter making 2 out of 2 free throws

Using combinations, I verified that the probability of Walter making 2 out of 2 free throws is $\frac{4}{9}$.

$$\begin{aligned} P(2 \text{ makes}) &= {}_2C_2 \cdot [P(\text{make}) \cdot P(\text{make})] \\ &= \frac{2!}{(2-2)!2!} \cdot \left[\frac{2}{3} \cdot \frac{2}{3}\right] \\ &= \frac{2!}{0!2!} \cdot \left[\frac{4}{9}\right] \\ &= \frac{2!}{2!} \cdot \left[\frac{4}{9}\right] \\ &= 1 \cdot \left[\frac{4}{9}\right] \\ &= \frac{4}{9} \end{aligned}$$

c. Walter making 1 out of 2 free throws

Using combinations, I verified that the probability of Walter making 1 out of 2 free throws is $\frac{4}{9}$.

$$\begin{aligned} P(1 \text{ make}) &= {}_2C_1 \cdot [P(\text{make}) \cdot P(\text{miss})] \\ &= \frac{2!}{(2-1)!1!} \cdot \left[\frac{2}{3} \cdot \frac{1}{3}\right] \\ &= \frac{2!}{1!1!} \cdot \left[\frac{2}{9}\right] \\ &= 2! \cdot \left[\frac{2}{9}\right] \\ &= 2 \cdot 1 \cdot \left[\frac{2}{9}\right] \\ &= 2 \cdot \left[\frac{2}{9}\right] \\ &= \frac{4}{9} \end{aligned}$$

d. Walter making 0 out of 2 free throws

Using combinations, I verified that the probability of Walter making 0 out of 2 free throws is $\frac{1}{9}$.

$$\begin{aligned} P(0\text{ makes}) &= {}_2C_0 \cdot [P(\text{miss}) \cdot P(\text{miss})] \\ &= \frac{2!}{(2-0)!0!} \cdot \left[\frac{1}{3} \cdot \frac{1}{3}\right] \\ &= \frac{2!}{2!0!} \cdot \left[\frac{1}{9}\right] \\ &= \frac{2!}{2!} \cdot \left[\frac{1}{9}\right] \\ &= 1 \cdot \left[\frac{1}{9}\right] \\ &= \frac{1}{9} \end{aligned}$$

e. Walter making 3 out of 3 free throws

Using combinations, I verified that the probability of Walter making 3 out of 3 free throws is $\frac{8}{27}$.

$$\begin{aligned} P(3\text{ makes}) &= {}_3C_3 \cdot [P(\text{make}) \cdot P(\text{make}) \cdot P(\text{make})] \\ &= \frac{3!}{(3-3)!3!} \cdot \left[\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}\right] \\ &= \frac{3!}{0!3!} \cdot \left[\frac{8}{27}\right] \\ &= \frac{3!}{3!} \cdot \left[\frac{8}{27}\right] \\ &= 1 \cdot \left[\frac{8}{27}\right] \\ &= \frac{8}{27} \end{aligned}$$

f. Walter making 2 out of 3 free throws

Using combinations, I verified that the probability of Walter making 2 out of 3 free throws is $\frac{4}{9}$.

$$\begin{aligned} P(2\text{ makes}) &= {}_3C_2 \cdot [P(\text{make}) \cdot P(\text{make}) \cdot P(\text{miss})] \\ &= \frac{3!}{(3-2)!2!} \cdot \left[\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3}\right] \\ &= \frac{3!}{1!2!} \cdot \left[\frac{4}{27}\right] \\ &= \frac{3!}{2!} \cdot \left[\frac{4}{27}\right] \\ &= \frac{3 \cdot 2 \cdot 1}{2 \cdot 1} \cdot \left[\frac{4}{27}\right] \\ &= 3 \cdot \left[\frac{4}{27}\right] \\ &= \frac{12}{27} \\ &= \frac{4}{9} \end{aligned}$$

$${}_3C_1 = \frac{3!}{(3-1)!1!} = \frac{3!}{2!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1} = 3$$

g. Walter making 1 out of 3 free throws

Using combinations, I verified that the probability of Walter making 1 out of 3 free throws is $\frac{2}{9}$.

$$\begin{aligned}
P(1\text{ make}) &= {}_3C_1 \cdot [P(\text{make}) \cdot P(\text{miss}) \cdot P(\text{miss})] \\
&= \frac{3!}{(3-1)!1!} \cdot \left[\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}\right] \\
&= \frac{3!}{2!} \cdot \left[\frac{2}{27}\right] \\
&= \frac{3 \cdot 2 \cdot 1}{2 \cdot 1} \cdot \left[\frac{2}{27}\right] \\
&= 3 \cdot \left[\frac{2}{27}\right] \\
&= \frac{6}{27} \\
&= \frac{2}{9}
\end{aligned}$$

$${}_3C_1 = \frac{3!}{(3-1)!1!} = \frac{3!}{2!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1} = 3$$

h. Walter making 0 out of 3 free throws

Using combinations, I verified that the probability of Walter making 0 out of 3 free throws is $\frac{1}{27}$.

$$\begin{aligned}
P(0\text{ makes}) &= {}_3C_0 \cdot [P(\text{miss}) \cdot P(\text{miss}) \cdot P(\text{miss})] \\
&= \frac{3!}{(3-0)!0!} \cdot \left[\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}\right] \\
&= \frac{3!}{3!} \cdot \left[\frac{1}{27}\right] \\
&= 1 \cdot \left[\frac{1}{27}\right] \\
&= \frac{1}{27}
\end{aligned}$$

$${}_3C_1 = \frac{3!}{(3-1)!1!} = \frac{3!}{2!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1} = 3$$

Grouping

Have students complete Question 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- How many made free throws is most likely?
- How many made free throws is least likely?
- Is Walter more likely to make all 4 free throws or miss all 4 free throws?
- What do your solution methods for parts (a) through (e) have in common?
- What is different about your solution methods for parts (a) through (e)?
- In general, describe the solution method you used for parts (a) through (e).

6. Walter shoots 4 free throws. Use combinations to determine the probability of each event.

a. Walter making 0 out of 4 free throws

The probability of Walter making 0 out of 4 free throws is $\frac{1}{81}$.

$$P(0\text{ makes}) = {}_4C_0 \cdot [P(\text{miss}) \cdot P(\text{miss}) \cdot P(\text{miss}) \cdot P(\text{miss})]$$

$$= \frac{4!}{(4-0)!0!} \cdot \left[\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}\right]$$

$$= \frac{4!}{4!0!} \cdot \left[\frac{1}{81}\right]$$

$$= \frac{4!}{4!} \cdot \left[\frac{1}{81}\right]$$

$$= 1 \cdot \left[\frac{1}{81}\right]$$

$$= \frac{1}{81}$$

b. Walter making 1 out of 4 free throws

The probability of Walter making 1 out of 4 free throws is $\frac{8}{81}$.

$$P(1\text{ makes}) = {}_4C_1 \cdot [P(\text{make}) \cdot P(\text{miss}) \cdot P(\text{miss}) \cdot P(\text{miss})]$$

$$= \frac{4!}{(4-1)!1!} \cdot \left[\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}\right]$$

$$= \frac{4!}{3!1!} \cdot \left[\frac{2}{81}\right]$$

$$= \frac{4!}{3!} \cdot \left[\frac{2}{81}\right]$$

$$= \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} \cdot \left[\frac{2}{81}\right]$$

$$= 4 \cdot \left[\frac{2}{81}\right]$$

$$= \frac{8}{81}$$

c. Walter making 2 out of 4 free throws

The probability of Walter making 2 out of 4 free throws is $\frac{8}{27}$.

$$P(2\text{ makes}) = {}_4C_2 \cdot [P(\text{make}) \cdot P(\text{make}) \cdot P(\text{miss}) \cdot P(\text{miss})]$$

$$= \frac{4!}{(4-2)!2!} \cdot \left[\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}\right]$$

$$= \frac{4!}{2!2!} \cdot \left[\frac{4}{81}\right]$$

$$= \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} \cdot \left[\frac{4}{81}\right]$$

$$= 6 \cdot \left[\frac{4}{81}\right]$$

$$= \frac{24}{81}$$

$$= \frac{8}{27}$$

d. Walter making 3 out of 4 free throws

The probability of Walter making 3 out of 4 free throws is $\frac{32}{81}$.

$$
\begin{aligned}
P(\text{3 makes}) &= {}_4C_3 \cdot [P(\text{make}) \cdot P(\text{make}) \cdot P(\text{make}) \cdot P(\text{miss})] \\
&= \frac{4!}{(4-3)!3!} \cdot \left[\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3}\right] \\
&= \frac{4!}{1!3!} \cdot \left[\frac{8}{81}\right] \\
&= \frac{4!}{3!} \cdot \left[\frac{8}{81}\right] \\
&= \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} \cdot \left[\frac{8}{81}\right] \\
&= 4 \cdot \left[\frac{8}{81}\right] \\
&= \frac{32}{81}
\end{aligned}
$$

e. Walter making 4 out of 4 free throws

The probability of Walter making 4 out of 4 free throws is $\frac{16}{81}$.

$$
\begin{aligned}
P(\text{4 makes}) &= {}_4C_4 \cdot [P(\text{make}) \cdot P(\text{make}) \cdot P(\text{make}) \cdot P(\text{make})] \\
&= \frac{4!}{(4-4)!4!} \cdot \left[\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}\right] \\
&= \frac{4!}{0!4!} \cdot \left[\frac{16}{81}\right] \\
&= \frac{4!}{4!} \cdot \left[\frac{16}{81}\right] \\
&= 1 \cdot \left[\frac{16}{81}\right] \\
&= \frac{16}{81}
\end{aligned}
$$

Grouping

Have students complete Questions 7 and 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 7 and 8

- What patterns can you find in the table?
- What patterns are there in the combinations notation used to show the results in the table?
- How can you continue Pascal's Triangle?

7. Complete the table to summarize the combination calculation that determines the number of ways each outcome can occur.

Number of Free Throws Attempted	Outcome 1	Outcome 2	Outcome 3	Outcome 4	Outcome 5	Outcome 6
1	0 makes, 1 miss occurs 1 way $_1C_0 = 1$	1 make, 1 miss occur 1 way $_1C_1 = 1$				
2	0 makes, 2 misses occur 1 way $_2C_0 = 1$	1 make, 1 miss occur 2 ways $_2C_1 = 2$	2 makes, 0 misses occur 1 way $_2C_1 = 1$			
3	0 makes, 3 misses occur 1 way $_3C_0 = 1$	1 make, 2 misses occur 3 ways $_3C_1 = 3$	2 makes, 1 miss occur 3 ways $_3C_2 = 3$	3 makes, 0 misses occur 1 way $_2C_3 = 1$		
4	0 makes, 4 misses occur 1 way $_4C_0 = 1$	1 make, 3 misses occur 4 ways $_4C_1 = 4$	2 makes, 2 misses occur 6 ways $_4C_2 = 6$	3 makes, 1 miss occur 4 ways $_4C_3 = 4$	4 makes, 0 misses occur 1 way $_4C_4 = 1$	
5	0 makes, 5 misses occur 1 way $_5C_0 = 1$	1 make, 4 misses occur 5 ways $_5C_1 = 5$	2 makes, 3 misses occur 10 ways $_5C_2 = 10$	3 makes, 2 misses occur 10 ways $_5C_3 = 10$	4 makes, 1 miss occur 5 ways $_5C_4 = 5$	5 makes, 0 misses occur 1 way $_5C_5 = 1$

8. The following is a different way to organize the number of ways for each outcome. Analyze any patterns you see. Describe these patterns and use them to write the next two rows.

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

The triangle of numbers is **Pascals' Triangle** named after Blaise Pascal, a 17th century mathematician. In the free throw shooting scenario, the numbers in Pascal's triangle represent the number of ways an outcome can occur.

Answers will vary.

Student responses could include, but are not limited to the following.

- Across each row, start with the number 1 and then add the number above and to the left and the number above and to the right to get the next number.
- Across the first diagonal the numbers are all 1s, across the next diagonal are natural numbers, across the next diagonal are triangular numbers, etc.
- All of the numbers are positive.
- The numbers are symmetric about a vertical line through the apex of the triangle.

Grouping

Have students complete Questions 9 through 11 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 9 through 11

- What pattern can you see in the exponents?
- What other situations can you think of where you could use this formula?

9. Summarize the probability of Walter making each number of free throws.

Number of Free Throws Walter Shoots	Number of Free Throws Walter Makes	Probability of Walter Making Each Fee Throw	Probability of Walter Missing Each Free Throw	Probability of Walter Making *r* out of *n* Free Throws
n	r	$P(A)$	$P(\sim A)$	${}_nC_r[P(A)]^r\,[P(\sim A)]^{n-r}$
1	0	$\frac{2}{3}$	$\frac{1}{3}$	${}_1C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^1$
	1	$\frac{2}{3}$	$\frac{1}{3}$	${}_1C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^0$
2	0	$\frac{2}{3}$	$\frac{1}{3}$	${}_2C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^2$
	1	$\frac{2}{3}$	$\frac{1}{3}$	${}_2C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^1$
	2	$\frac{2}{3}$	$\frac{1}{3}$	${}_2C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^0$
3	0	$\frac{2}{3}$	$\frac{1}{3}$	${}_3C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^3$
	1	$\frac{2}{3}$	$\frac{1}{3}$	${}_3C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2$
	2	$\frac{2}{3}$	$\frac{1}{3}$	${}_3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1$
	3	$\frac{2}{3}$	$\frac{1}{3}$	${}_3C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0$

a. The probability of Walter making 0 out of 1 free throw is provided. Complete the table to show the additional probabilities.

See table

b. Describe how to calculate the probability of Walter making *r* free throws out of *n* total free throws.

For the combination calculation, I used the number of free throws Walter shoots and the number of free throws he made. For example, the 3 and 2 in ${}_3C_2$ represent Walter making 2 out of 3 free throws. The exponent for $\frac{2}{3}$ represents the number of makes and the exponent for $\frac{1}{3}$ represents the number of misses.

c. Write an expression to calculate the probability of Walter making *r* free throws out of *n* total free throws.

See table.

10. Walter's free throw shooting was not as good in the past. He used to make 2 out of every 5 free throws. What was the probability of Walter making 8 out of 10 free throws?

The probability of Walter making 8 out of 10 free throws was about 1.1%.

$$
\begin{aligned}
P(\text{8 makes}) &= {}_{10}C_8 \cdot [P(\text{make})]^8 \cdot [P(\text{miss})]^2 \\
&= \frac{10!}{(10-2)!2!} \cdot \left[\frac{2}{5}\right]^8 \cdot \left[\frac{3}{5}\right]^2 \\
&= \frac{10!}{8!2!} \cdot \left[\frac{256}{390,625}\right] \cdot \left[\frac{9}{25}\right] \\
&= \frac{10 \cdot 9}{2} \cdot \left[\frac{2304}{9,765,625}\right] \\
&= 45 \cdot \left[\frac{2304}{9,765,625}\right] \\
&= \frac{103,680}{9,765,625} \\
&\approx 0.011
\end{aligned}
$$

11. Walter is working to improve his free throw shooting. His goal is to consistently make 9 out of every 10. What is the probability of Walter making 8 out of 10 free throws?

The probability of Walter making 8 out of 10 free throws is about 19.4%.

$$
\begin{aligned}
P(\text{8 makes}) &= {}_{10}C_8 \cdot [P(\text{make})]^8 \cdot [P(\text{miss})]^2 \\
&= \frac{10!}{(10-2)!2!} \cdot \left[\frac{9}{10}\right]^8 \cdot \left[\frac{1}{10}\right]^2 \\
&= \frac{10!}{8!2!} \cdot \left[\frac{43,046,721}{100,000,000}\right] \cdot \left[\frac{1}{100}\right] \\
&= \frac{10 \cdot 9}{2} \cdot \left[\frac{43,046,721}{10,000,000,000}\right] \\
&= 45 \cdot \left[\frac{43,046,721}{10,000,000,000}\right] \\
&= \frac{1937102445}{10,000,000,000} \\
&\approx 0.194
\end{aligned}
$$

Problem 2

Rolling a color-coded number cube is used. It involves multiple trials for two independent events. Combinations are used to solve these problems due to the large number of trials. Students apply the formula for determining the probability resulting from multiple trials of two independent events to a problem situation involving a color coded tetrahedron to compute probabilities.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- How is rolling 4 reds and 1 blue the same as 5 objects taken 4 at a time?
- How is rolling 3 reds and 3 blues the same as 6 objects taken 3 at a time?
- What other problem situations can you create to use this formula?

PROBLEM 2 Using a Formula for Multiple Trials

1. Suppose you roll a number cube with 4 red faces and 2 blue faces. Calculate each probability.

a. Rolling 4 reds and 1 blue when the number cube is rolled five times.

The probability of rolling 4 reds and 1 blue is $\frac{80}{243}$.

$$\begin{aligned} P(4R \text{ and } 1B) &= {}_5C_4\left(\frac{4}{6}\right)^4\left(\frac{2}{6}\right) \\ &= 5\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) \\ &= 5\left(\frac{16}{81}\right)\left(\frac{1}{3}\right) \\ &= \frac{80}{243} \end{aligned}$$

b. Rolling 3 reds and 3 blues when the number cube is rolled six times.

The probability of rolling 3 reds and 3 blues is $\frac{160}{729}$.

The number of ways to get 3 reds is the 4th number in the row, 20.

$$\begin{aligned} P(3R \text{ and } 3B) &= {}_6C_3\left(\frac{4}{6}\right)^3\left(\frac{2}{6}\right)^3 \\ &= 20\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3 \\ &= 20\left(\frac{8}{27}\right)\left(\frac{1}{27}\right) \\ &= \frac{160}{729} \end{aligned}$$

c. Rolling 1 red and 6 blues when the number cube is rolled seven times.

The probability of rolling 1 red and 6 blues is $\frac{14}{2187}$.

$$\begin{aligned} P(1R \text{ and } 6B) &= {}_7C_1\left(\frac{4}{6}\right)^1\left(\frac{2}{6}\right)^6 \\ &= 7\left(\frac{2}{3}\right)^1\frac{1^6}{3} \\ &= 7\left(\frac{2}{3}\right)\left(\frac{1}{729}\right) \\ &= \frac{14}{2187} \end{aligned}$$

2. Calculate each probability.

a. Rolling 3 reds and 3 blues when the number cube is rolled six times.

The probability of rolling 3 reds and 3 blues is $\frac{160}{729}$.

$$P(3R \text{ and } 3B) = {}_6C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3$$
$$= 20\left(\frac{8}{27}\right)\left(\frac{1}{27}\right)$$
$$= \frac{160}{729}$$

b. Rolling 1 red and 6 blues when the number cube is rolled seven times.

The probability of rolling 3 reds and 7 blues is $\frac{14}{2187}$.

$$P(1R \text{ and } 6B) = {}_7C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^6$$
$$= 7\left(\frac{2}{3}\right)\left(\frac{1}{729}\right)$$
$$= \frac{14}{2187}$$

c. Rolling 3 reds and 7 blues when the number cube is rolled ten times.

The probability of rolling 3 reds and 7 blues is $\frac{960}{59{,}049}$.

$$P(3R \text{ and } 7B) = {}_{10}C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^7$$
$$= 120\left(\frac{8}{27}\right)\left(\frac{1}{2187}\right)$$
$$= \frac{960}{59{,}049}$$

d. Rolling 3 reds and 4 blues when the number cube is rolled seven times.

The probability of rolling 3 reds and 4 blues is $\frac{280}{2187}$.

$$P(3R \text{ and } 4B) = {}_7C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^4$$
$$= 35\left(\frac{8}{27}\right)\left(\frac{1}{81}\right)$$
$$= \frac{280}{2187}$$

3. A regular tetrahedron is a four-sided solid with each face an equilateral triangle. A regular tetrahedron has three sides painted blue (B) and one side painted red (R).

a. What is $P(R)$, the probability of rolling a tetrahedron that lands on a red face?

The probability of rolling a tetrahedron that lands on a red face, $P(R)$, is $\frac{1}{4}$.

b. What is $P(B)$, the probability of rolling a tetrahedron that lands on a blue face?

The probability of rolling a tetrahedron that lands on a blue face, $P(B)$, is $\frac{3}{4}$.

4. Calculate each probability for the regular tetrahedron from Question 3.

a. Landing on five blues and two reds in seven rolls.

The probability of rolling a tetrahedron that lands on blue 5 times and red 2 times is $\frac{5103}{16,384}$.

$$\begin{aligned} P(5B \text{ and } 2R) &= {}_7C_5 \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)^2 \\ &= 21\left(\frac{243}{1024}\right)\left(\frac{1}{16}\right) \\ &= \frac{5103}{16,384} \end{aligned}$$

b. Landing on four blues and one red in five rolls.

The probability of rolling a tetrahedron that lands on blue 4 times and red 1 time is $\frac{405}{1024}$.

$$\begin{aligned} P(4B \text{ and } 1R) &= {}_5C_4 \left(\frac{3}{4}\right)^4 \left(\frac{1}{4}\right)^1 \\ &= 5\left(\frac{81}{256}\right)\left(\frac{1}{4}\right) \\ &= \frac{405}{1024} \end{aligned}$$

c. Landing on five blues and five reds in ten rolls.

The probability of rolling a tetrahedron that lands on blue 5 times and red 5 times is $\frac{61,236}{1,048,576}$.

$$\begin{aligned} P(5B \text{ and } 5R) &= {}_{10}C_5 \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)^5 \\ &= 252\left(\frac{243}{1024}\right)\left(\frac{1}{1024}\right) \\ &= \frac{61,236}{1,048,576} \end{aligned}$$

Be prepared to share your solutions and methods.

Check for Students' Understanding

A regular octahedron is an eight-sided solid with each face an equilateral triangle. A regular octahedron has six sides painted yellow and two sides painted green.

1. What is the probability of rolling a black?

   $P(Y) = \frac{6}{8} = \frac{3}{4}$

2. What is the probability of rolling a green?

   $P(G) = \frac{2}{8} = \frac{1}{4}$

3. What is the probability of rolling seven yellows and one green in eight rolls?

   $P(7Y \text{ and } 1G) = {}_8C_7\left(\frac{3}{4}\right)^7\left(\frac{1}{4}\right)^1 = 8\left(\frac{243}{1024}\right)\left(\frac{1}{4}\right) = \frac{243}{4,096}$

4. What is the probability of rolling six yellows and two greens in eight rolls?

   $P(6Y \text{ and } 2G) = {}_8C_6\left(\frac{3}{4}\right)^6\left(\frac{1}{4}\right)^2 = 28\left(\frac{729}{4096}\right)\left(\frac{1}{16}\right) = \frac{20,412}{65,536} = \frac{5,103}{16,384}$

5. What is the probability of rolling five yellows and two greens in seven rolls?

   $P(5Y \text{ and } 2G) = {}_7C_5\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)^2 = 21\left(\frac{243}{1024}\right)\left(\frac{1}{16}\right) = \frac{5,103}{16,384}$

20.5

To Spin or Not to Spin

Expected Value

LEARNING GOALS

In this lesson, you will:

- Determine geometric probability.
- Calculate the expected value of an event.

KEY TERMS

- geometric probability
- expected value

ESSENTIAL IDEAS

- Geometric probability is the likelihood of an event occurring based on geometric relationships such as area, surface area, volume, and so on.
- Expected value is the sum of the values of a random variable with each value multiplied by its probability of occurrence.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

S-MD Using Probability to Make Decisions

Use probability to evaluate outcomes of decisions

6. Use probabilities to make fair decisions.

7. Analyze decisions and strategies using probability concepts.

Overview

The terms geometric probability and expected value are introduced in this lesson. Dartboards containing geometric shapes are used to determine geometric probabilities. Money wheels divided into eight equal regions are used to determine expected values.

Warm Up

Suppose you are a contestant on a game show. The host shows you 10 doors and tells you there is a car behind one of the doors and the remaining 9 doors have zonks. You choose door number 3 because it's your lucky number. The host decides to open 8 doors that he knows will all reveal zonks.

The host asks you one more time, do you want to keep your door or switch with the remaining closed door.

What should you do? Use probabilities to defend your decision.

SWITCH DOORS!!

The probability of choosing the door with the car was originally $\frac{1}{10}$, therefore the probability of winning the car by changing selection in this instance is $\frac{9}{10}$.

To Spin or Not to Spin

Expected Value

20.5

LEARNING GOALS

In this lesson, you will:

- Determine geometric probability.
- Calculate the expected value of an event.

KEY TERMS

- geometric probability
- expected value

We were rooting for the game show contestant to go for a bigger prize by spinning the wheel, but the contestant decided not take the risk and keep the current winnings.

Sometimes it pays off for the contestant to take the risk. Other times it doesn't.

If you were in the contestant's shoes, what would you do?

Problem 1

Geometric shapes such as triangles and circles are drawn/shaded on a dartboard to represent desired targets. Students determine the probabilities of randomly throwing a dart and the dart landing in a desired location. The areas of the geometric shapes are determined using given information and area formulas. Then students compare the area of the desired outcome to the area of the entire dartboard to calculate probability percents. The term geometric probability is introduced.

Grouping

- Discuss the information and the diagrams above Question 1 as a class.
- Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

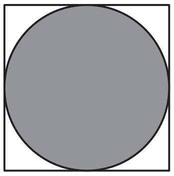
Guiding Questions for Share Phase, Questions 1 through 5

- Does the entire area of the dart board contain all possible outcomes?
- Did you choose the shaded area or the unshaded area to represent the desirable outcomes?
- What is the area of the entire dartboard?

PROBLEM 1 Bullseye!

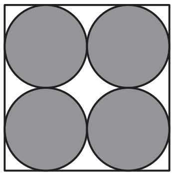
A class is hosting a dart competition to raise money. Three different designs under consideration are shown. All of the dartboards are squares that measure 50 cm by 50 cm. For this situation, you have to assume that a dart is thrown randomly at a dart board and that the dart will land somewhere on the dart board.

Big Bullseye



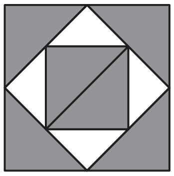
1 shaded circle

Four Circles



4 congruent shaded circles

Tricky Triangles



6 congruent shaded triangles

1. Which dart board will give the player the greatest chance of hitting a shaded section? Why?

   Answers will vary.

   Example Response: I think the player has the best chance of hitting the Big Bullseye because it seems to have the least white space.

2. You can use what you know about probability to determine the chance a person has of hitting a shaded section on a dart board.

   a. What value would you use to represent the number of possible outcomes?

   Answers may vary.

   I can use the total area of the dart board as the value that represents the number of possible outcomes.

   b. What value would you use to represent the number of desired outcomes?

   Answers may vary.

   I can use the total area of the shaded sections in each dart board as the value that represents the number of desired outcomes.

- What is the area of the large circle on the Bullseye dartboard?
- What operation is used to determine the probability of the dart landing in the shaded area?
- What is the area of one of the four circles on the Four Circles dartboard?
- What is the area of all four circles on the Four Circles dartboard?
- What is the area of one triangle on the Tricky Triangles dartboard?
- What is the area of all six triangles on the Tricky Triangles dartboard?

- Do any of the dartboards have equal probabilities? Which ones?
- Which dartboard would you like to use? Why?

3. Devin says that the number of desired outcomes for Big Bullseye is 1 because there is 1 shaded section. That means that Four Circles has 4 desired outcomes, and Tricky Triangles has 6 desired outcomes.

Is Devin's reasoning correct? Explain why or why not.

No. Devin is not correct. There are two outcomes for each dart board. The desired outcome results in the dart landing on a shaded section on the dart board and the undesired outcome results in the dart landing on an unshaded section.

4. Determine the probability that a dart lands in a shaded section of each dartboard.

a. Big Bullseye

The probability of the dart landing in the shaded area is 0.785, or 78.5%.

I determined the answer by calculating the area of the circle and the area of the square. The area of the circle represents the desired outcomes and the area of the square represents the total possible outcomes.

$$\begin{aligned} \text{Area of the Square} &= \text{length} \cdot \text{width} \\ &= 50 \cdot 50 \\ &= 2500 \end{aligned}$$

The area of the square is 2500 square centimeters.

$$\begin{aligned} \text{Area of the Circle} &= \pi \cdot \text{radius}^2 \\ &\approx 3.14 \cdot 25^2 \\ &\approx 3.14 \cdot 625 \\ &\approx 1962.5 \end{aligned}$$

The area of the circle is approximately 1962.5 square centimeters.

$$\begin{aligned} P(\text{shaded section}) &= \frac{\text{desired outcomes}}{\text{possible outcomes}} \\ &\approx \frac{1962.5}{2500} \\ &\approx 0.785 \end{aligned}$$

The probability of the dart landing on the shaded section of the Big Bullseye dartboard is approximately 0.785, or 78.5%.

b. Four Circles

Area of the square = 2500 cm^2

Area of one circle = $\pi(12.5)^2 = 156.25\pi \approx 490.625$ cm^2

Area of four circles ≈ 4(490.625) = 1962.5 cm^2

The probability of the dart landing in the shaded area is approximately 0.785, or 78.5%.

I determined the answer by calculating the area of all 4 circles and the area of the square. The area of the circles represents the desired outcomes and the area of the square represents the total possible outcomes.

$$\begin{aligned}\text{Area of the Square} &= \text{length} \cdot \text{width}\\ &= 50 \cdot 50\\ &= 2500\end{aligned}$$

The area of the square is 2500 square centimeters.

$$\begin{aligned}\text{Area of the 4 Circles} &= 4 \cdot \pi \cdot \text{radius}^2\\ &\approx 4 \cdot 3.14 \cdot 12.5^2\\ &\approx 4 \cdot 3.14 \cdot 156.25\\ &\approx 1962.5\end{aligned}$$

The area of the 4 circles is approximately 1962.5 square centimeters.

$$\begin{aligned}P(\text{shaded section}) &= \frac{\text{desired outcomes}}{\text{possible outcomes}}\\ &\approx \frac{1962.5}{2500}\\ &\approx 0.785\end{aligned}$$

The probability of the dart landing on a shaded section of the Four Circles dartboard is approximately 0.785, or 78.5%.

c. Tricky Triangles

Area of the square = 2500 cm^2

Area of one triangle = 0.5(25)(25) = 312.5 cm^2

Area of six triangles = 1875 cm^2

The probability of the dart landing in the shaded area is 0.75, or 75%.

I determined the answer by calculating the area of all 6 triangles and the area of the square. The area of the triangles represents the desired outcomes and the area of the square represents the total possible outcomes.

$$\begin{aligned}\text{Area of the Square} &= \text{length} \cdot \text{width} \\ &= 50 \cdot 50 \\ &= 2500\end{aligned}$$

The area of the square is 2500 square centimeters.

$$\begin{aligned}\text{Area of the 6 Triangles} &= 4 \cdot \frac{1}{2} \cdot \text{base} \cdot \text{height} \\ &= 6 \cdot \frac{1}{2} \cdot 25 \cdot 25 \\ &= 1875\end{aligned}$$

The area of the 4 circles is approximately 1875 square centimeters.

$$\begin{aligned}P(\text{shaded section}) &= \frac{\text{desired outcomes}}{\text{possible outcomes}} \\ &= \frac{1875}{2500} \\ &= 0.75\end{aligned}$$

The probability of the dart landing on a shaded section of the Tricky Triangles dartboard is 0.75, or 75%.

5. Which dart board should the class use to make the competition the most difficult? Explain your reasoning.

The Tricky Triangles dartboard would make the competition most difficult because it has the smallest shaded area.

Grouping

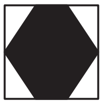
Have students complete Question 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- How are geometric probabilities different from other probabilities?
- What determines if a probability is a geometric probability?
- What strategy can be used to determine the area of the hexagon on the dartboard?
- Is there more than one strategy to determine the area of the hexagon on the dartboard?
- How did you represent the length of a side of the square dartboard?
- How many right triangles are on the dartboard?
- Do all of the right triangles on the dartboard have equal area? How do you know?
- Are all of the right triangles on the dartboard congruent to each other? How do you know?
- Do you have enough information to determine the area of the right triangles on the dartboard?

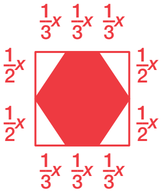
To determine the probability of hitting a shaded section on a dart board, you used *geometric probability*. **Geometric probability** is probability that involves a geometric measure, such as length, areas, volumes, and so on.

6. The dart board shown is a hexagon inscribed in a square. Two of the vertices of the hexagon bisect two of the sides of the square. The remaining 4 vertices trisect the other two sides of the square. What is the probability that a dart will hit the shaded area of the dart board? Show your work.



The probability that a dart will hit the shaded area of the dartboard is $\frac{2}{3}$.

Let x represent the side length of the square. Then the area of the square is x^2. For the sides that are bisected, each part is $\frac{1}{2}x$. For the sides that are trisected, each part is $\frac{1}{3}x$.



The area of each of the 4 right triangles is $\frac{1}{2}\left(\frac{1}{2}x\right)\left(\frac{1}{3}x\right) = \frac{1}{12}x^2$.

The total area of the 4 right triangles is $4\left(\frac{1}{12}x^2\right) = \frac{1}{3}x^2$.

The area of the shaded region is $x^2 - \frac{1}{3}x^2 = \frac{2}{3}x^2$.

The probability a dart lands in the shaded region is $\frac{\frac{2}{3}x^2}{x^2}$, or $\frac{2}{3}$.

- Is it easier to determine the area of the hexagon, or the area of the unshaded region of the dartboard?

Problem 2

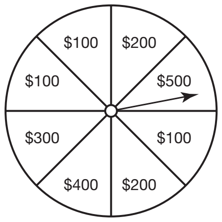
As contestants on a game show, students decide if they should spin a wheel divided into eight equal regions to win more money or keep the money they were given. Students first determine the probability of winning different amounts on the wheel if it is spun once. Next, they describe how often they would expect to win each amount. They then multiply each amount on the wheel by the probability of winning the amount, add all of the values together, and arrive at the expected value. The expected value in this situation represents the amount of money you would expect to win if you spun the wheel once. Knowing the expected value is helpful in deciding if they should keep the money they were given or spin the wheel. The term expected value is introduced.

Grouping

- Have students complete Questions 1 through 4, part (b) with a partner. Then have students share their responses as a class.
- Discuss the information above Question 4, part (c) and complete part (c) as a class.

PROBLEM 2 First Round

You are a contestant on a game show. In the first round, the host gives you $200. You can choose to keep the money or give the money back and spin the wheel to determine your winnings.



1. Would you choose to keep the $200 or spin the wheel? Why?

   Answers will vary.

   Example Response: I would choose to keep the $200 because there is a chance I could walk away with only $100 by spinning the wheel.

2. If you spin the wheel, what is the probability that you will win each amount?

   a. $100

   $P(\$100) = \frac{3}{8}$

   b. $200

   $P(\$200) = \frac{2}{8} = \frac{1}{4}$

   c. $300

   $P(\$300) = \frac{1}{8}$

   d. $400

   $P(\$400) = \frac{1}{8}$

   e. $500

   $P(\$500) = \frac{1}{8}$

3. If you spin the wheel, how often would you expect to win each amount?

   a. $100

   I would expect to win $100 three eighths of the time.

   b. $200

   I would expect to win $200 one fourth of the time.

   c. $300

   I would expect to win $300 one eighth of the time.

   d. $400

   I would expect to win $400 one eighth of the time.

   e. $500

   I would expect to win $500 one eighth of the time.

Guiding Questions for Share Phase, Questions 1 through 4, part (b)

- How many regions are on the wheel?
- Is each region the same area?
- How many different outcomes are on the wheel?
- Are any of the outcomes the same? Which ones?
- Do all outcomes have an equal probability of occurring each time the wheel is spun?
- Which outcome would you expect to get if you spin the wheel? Why?
- Which outcome would you least expect to get if you spin the wheel? Why?

Guiding Questions for Discuss Phase, Question 4, part (c)

- How can the expected value be $237.50 when that outcome is not on the wheel?
- Is it possible to get $237.50 when you spin the wheel?
- Is the expected value a realistic outcome? Why not?
- How does knowing the expected value help to decide whether you should spin the wheel or take the money?

4. Let's calculate the expected value for the game.

a. Multiply each amount by the probability of winning that amount.

$\$100 \cdot P(\$100) = \$100 \cdot \frac{3}{8} = \$37.50$

$\$200 \cdot P(\$200) = \$200 \cdot \frac{1}{4} = \$50$

$\$300 \cdot P(\$300) = \$300 \cdot \frac{1}{8} = \$37.50$

$\$200 \cdot P(\$400) = \$400 \cdot \frac{1}{8} = \$50$

$\$200 \cdot P(\$500) = \$500 \cdot \frac{1}{8} = \$62.50$

b. Calculate the sum of the values from Question 4.

$\$37.50 + \$50 + \$37.50 + \$50 + \$62.50 = \$237.50$

The sum in Question 5 is the **expected value**. The expected value is the average value when the number of trials is large. In this problem, the expected value represents the amount that you could expect to receive from a single spin.

c. Based on the expected value, should you keep the $200 or spin the wheel? Explain.

For a large number of trials, there is a better chance of winning additional money by spinning the wheel. I can expect to receive an average of $237.50 for each spin, which is more than the $200 I started with. But, that does not guarantee I will win on the next spin.

Problem 3

A money wheel similar to the last problem is used again. But this time different amounts appear on the wheel and the amount of money the contestant is given changes. Students decide if the contestant should spin the wheel by calculating the expected value if the wheel is spun once.

Grouping

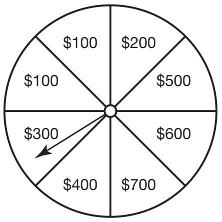
Have students complete Questions 1 through 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- How many regions are on the wheel?
- Is each region the same area?
- How many different outcomes are on the wheel?
- Are any of the outcomes the same? Which ones?
- Do all outcomes have an equal probability of occurring each time the wheel is spun?
- Which outcome would you expect to get if you spin the wheel? Why?
- Which outcome would you least expect to get if you spin the wheel? Why?
- How can the expected value be $350 when that outcome is not on the wheel?
- Is it possible to get $350 when you spin the wheel?
- Is the expected value a realistic outcome? Why not?
- How does knowing the expected value help to decide whether you should spin the wheel or take the money?

PROBLEM 3 Second Round

In the second round of the game show, you are offered another choice. The host gives you $300. You can choose to keep the money or give the money back and spin the wheel shown to determine your winnings. All the sections on the wheel are equal in size.



1. Calculate the expected value for one spin of the wheel.

   The expected value for one spin of the wheel is $350.

   I calculated the answer by multiplying the probability of each outcome by its dollar value. Then, I added all the products.

   $\$0 \cdot P(\$0) = \$0 \cdot \frac{1}{8} = \$0$

   $\$100 \cdot P(\$100) = \$100 \cdot \frac{1}{8} = \$12.50$

   $\$200 \cdot P(\$200) = \$200 \cdot \frac{1}{8} = \$25.00$

   $\$300 \cdot P(\$300) = \$300 \cdot \frac{1}{8} = \$37.50$

   $\$400 \cdot P(\$400) = \$400 \cdot \frac{1}{8} = \$50.00$

   $\$500 \cdot P(\$500) = \$500 \cdot \frac{1}{8} = \$62.50$

   $\$600 \cdot P(\$600) = \$600 \cdot \frac{1}{8} = \$75.00$

   $\$700 \cdot P(\$700) = \$700 \cdot \frac{1}{8} = \$87.50$

   $\$0 + \$12.50 + \$25 + \$37.50 + \$50 + \$62.50 + \$75 + \$87.50 = \$350$

2. Should you keep the $300 or spin the wheel? Explain your reasoning.

   For a large number of trials, there is a better chance of winning additional money by spinning the wheel. I can expect to receive an average of $350 for each spin, which is more than the $300 I started with. But, that does not guarantee I will win on the next spin.

Problem 4

A dartboard is used to generate probabilities. The probability of a randomly thrown dart landing in specified regions on the board is directly related to the point value of the score. Students use area formulas to determine the area of different regions on the dartboard. Inner and outer circles involve determining the area of an annulus. Knowing the area of each region on the dartboard, students are able to calculate the probability of a dart landing in each region and compute the expected value.

Grouping

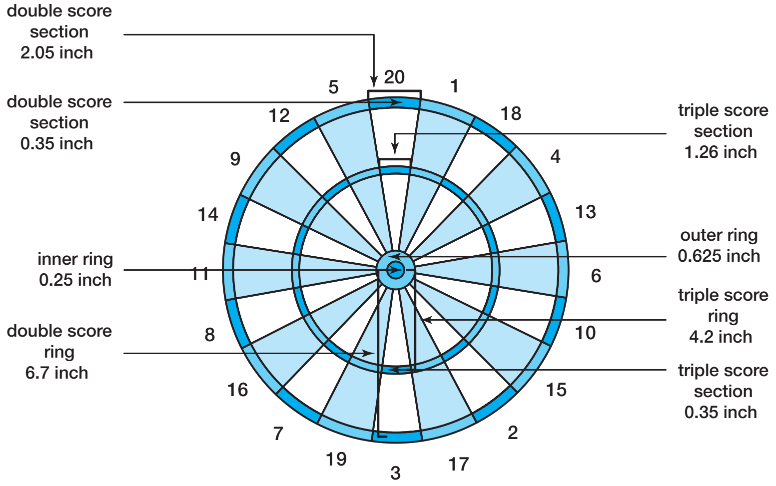
- Ask students to read the information. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- How can you use division to determine the areas of individual sections?
- What part of a circle does the measurement of 6.7 inches indicate?
- What does the measurement 0.25 inch indicate?
- About how many points should a person expect to score in three dart throws?

PROBLEM 4 Bullseye!

The dart board shown has a diameter of 17.75 inches and 20 congruent sections in a circular scoring area. Players can score points by throwing darts inside the scoring area. A dart that lands outside the scoring area scores no points.



1. Determine each of the approximate probabilities, assuming that 1 dart is thrown at random at the board. Explain your reasoning and show your work. Round values to the nearest ten-thousandth.

   a. probability of getting any score

   The probability of getting any score is approximately 57%.

   The probability of scoring is the ratio of the circular scoring area to the area of the entire dart board.

   Scoring area $= \pi r^2$

   $= \pi(6.7)^2 \approx 141.0261$ in.^{2}

   Area of dart board $= \pi r^2$

   $= \pi\left(\frac{17.75}{2}\right)^2 = \pi(8.875)^2 \approx 247.4495$ in.^{2}

   $\frac{141.0261}{247.4495} \approx 0.5699$ ($\approx$ 57%)

b. probability of landing anywhere on the section labeled 20

The probability of landing anywhere on the section labeled 20 is approximately 2.85%.

The probability of landing anywhere on the section labeled 20 is the ratio of the area of the section labeled 20 to the area of the entire dart board.

Area of "20" section $= \frac{\pi r^2}{20}$, where r is the radius of the circular scoring area

$= \frac{\pi(6.7)^2}{20} \approx 7.0513$ in.^{2}

Area of dart board $= \pi r^2$

$= \pi(8.875)^2 \approx 247.4495$ in.^{2}

$\frac{7.0513}{247.4495} \approx 0.0285$ (2.85%)

c. probability of landing within the outer ring but not the inner ring of the bullseye

The probability of landing anywhere within the outer ring but not the inner ring of the bullseye is approximately 41%.

The probability of landing within the outer ring but not in the inner ring is the ratio of the area of the outer ring minus the area of the inner ring over the area of the dart board.

Area of outer ring $= \pi\left(\frac{1.25}{2}\right)^2$

$= \pi(0.625)^2 \approx 1.2272$ in.^{2}

Area of inner ring $= \pi\left(\frac{0.5}{2}\right)^2$

$= \pi(0.25)^2 \approx 0.1963$ in.^{2}

Area of outer ring $-$ area of inner ring $\approx 1.2272 - 0.1963 = 1.0309$

Area of dart board $= \pi r^2$

$= \pi(8.875)^2 \approx 247.4495$ in.^{2}

$\frac{1.0309}{247.4495} \approx 0.0041$ (0.41%)

d. probability of landing within the inner ring of the bullseye

The probability of landing within the inner ring of the bullseye is approximately 8%.

The probability of landing within the inner ring of the bullseye is the ratio of the area of the inner ring to the area of the entire dart board.

Area of inner ring $= \pi\left(\frac{0.5}{2}\right)^2$

$= \pi(0.25)^2 \approx 0.1963$ in.^{2}

Area of dart board $= \pi r^2$

$= \pi(8.875)^2 \approx 247.4495$ in.^{2}

$\frac{0.1963}{247.4495} \approx 0.0008$ (0.08%)

e. probability of getting a triple 20

The probability of getting a triple 20 is approximately 0.18%.

The probability of getting a triple 20 is the ratio of the area of 1 rectangular triple-score section to the area of the dart board.

Area of triple-score section $\approx (0.35)(1.26)$

≈ 0.441 in.^{2}

Area of dart board $= \pi r^2$

$= \pi(8.875)^2 \approx 247.4495$ in.^{2}

$\frac{0.441}{247.4495} \approx 0.0018$ (0.18%)

f. probability of getting a double 20

The probability of getting a double 20 is approximately 2.38%.

The probability of getting a double 20 is the ratio of the area of 1 rectangular double-score section to the area of the dart board.

Area of double-score section $\approx (0.35)(2.05)$

≈ 0.7175 in.^{2}

Area of dart board $= \pi r^2$

$= \pi(8.875)^2 \approx 247.4495$ in.^{2}

$\frac{0.7175}{247.4495} \approx 0.0029$ (0.29%)

g. probability of landing on 20 without a double or triple score

I can subtract the sum of my answers in parts (e) and (f) from my answer in part (b).

$0.0285 - (0.0018 + 0.0029) = 0.0238$ (2.38%)

2. Suppose the inner bullseye is worth 50 points and the outer bullseye is worth 25 points. What is the expected value for 1 dart throw? Show your work.

The expected value for 1 dart throw is 7.4925.

The probability of landing on any 1 section without getting a double or triple score is 0.0238. Multiply each score from 1 to 20 by this probability and add the results:

$0.0238(20 + 19 + 18 + 17 \ldots) = 0.0238(210) = 4.998$

The probability of landing on a double-score section is 0.0029. Multiply the double of each score from 1 to 20 by this probability and add the results:

$0.0029(2)(20 + 19 + 18 + 17 \ldots) = 0.0029(420) = 1.218$

The probability of landing on a triple-score section is 0.0018. Multiply the triple of each score from 1 to 20 by this probability and add the results:

$0.0018(3)(20 + 19 + 18 + 17 \ldots) = 0.0018(630) = 1.134$

The probability of landing in the outer ring without landing in the inner ring is 0.0041. Multiply the score of 25 by this probability:

$0.0041 \times 25 = 0.1025$

The probability of landing in the inner ring of the bullseye is 0.0008. Multiply the score of 50 by this probability:

$0.0008 \times 50 = 0.04$

Add all of the values. The sum is the expected value for 1 dart throw:

$4.998 + 1.218 + 1.134 + 0.1025 + 0.04 = 7.4925$

Be prepared to share your solutions and methods.

Check for Students' Understanding

The Birthday Problem

How many randomly chosen people are needed before the probability is greater than 50% that at least two of them have the same birthday?

P(no match within a group of 2) $= \frac{364}{365}$

P(no match within a group of 3) $= \frac{364}{365} \cdot \frac{363}{365}$

P(no match within a group of 4) $= \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365}$

P(no match within a group of 5) $= \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365}$

P(no match within a group of 6) $= \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \cdot \frac{360}{365}$

P(no match within a group of 23) $= \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \cdot \frac{360}{365} \cdots \frac{343}{365} = 0.49$

P(match within a group of 23) $= 1 - 0.49 = 0.51$

If 23 people are randomly chosen, the probability of two people sharing the same birthday is greater than 50%, it is 51%.

Chapter 20 Summary

KEY TERMS

- two-way table (20.1)
- frequency table (20.1)
- two-way frequency table (20.1)
- contingency table (20.1)
- categorical data (20.1)
- relative frequency (20.1)
- two-way relative frequency table (20.1)
- conditional probability (20.2)
- factorial (20.3)
- permutation (20.3)
- circular permutation (20.3)
- combination (20.3)
- geometric probability (20.5)
- expected value (20.5)

20.1 Determine Probabilities of Compound Events Using Two-Way Frequency Tables

A two-way frequency table shows the number of data points and their frequencies for two variables. One variable is divided into rows, and the other is divided into columns. Two-way frequency tables can be used to calculate probabilities of compound events.

Example

The two-way frequency table shows how many 9th and 10th graders are studying Spanish, French, or German. What is the probability that a randomly selected student is a 10th grader or is studying Spanish?

What Foreign Language Are You Studying?

		Spanish	French	German	Total
Grade	9th Grade	41	19	25	85
	10th Grade	36	21	18	75
	Total	77	40	43	160

Apply the Addition Rule for Probability because the events are not mutually exclusive.

There are 75 students who are 10th graders, 77 students who are studying Spanish, and 36 students who are 10th graders and studying Spanish.

So, there are $75 + 77 - 36 = 116$ students who are 10th graders or studying Spanish.

$P(10^{\text{th}}\text{ grader or Spanish}) = \frac{116}{160} = 0.725 = 72.5\%$

20.1 Determine Relative Frequencies of Events

A relative frequency is the ratio of occurrences with a category to the total number of occurrences. To determine the ratio for each category, determine the part to the whole for each category.

Example

Refer to the two-way frequency table in the previous example. What is the relative frequency of students who are 9th graders and studying German? Round to the nearest thousandth.

There are 160 students in all. Of those, 25 students are 9th graders and studying German. Divide to determine the relative frequency.

$$\frac{25}{160} \approx 0.156$$

The relative frequency of 9th graders who are studying German is about 0.156.

20.2 Determine Conditional Probability

Conditional probability is the probability of Event B, given that Event A has already occurred. For example, the probability of rolling a 6 with a number cube, given that the number is greater than 3, is a conditional probability. The notation for conditional probability is $P(B|A)$, which reads, "the probability of Event B, given Event A." The formula $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$ can be used to calculate conditional probability.

Example

Employees at a company have the opportunity to earn additional certifications by taking optional training classes. The probability that a randomly selected employee has been promoted recently is 0.56. The probability that a randomly selected employee has earned additional certifications is 0.6. The probability that an employee has earned additional certifications and been promoted recently is 0.48. Calculate the probability that an employee has been promoted recently given that the employee has earned additional certifications.

$$\begin{aligned} P(\text{promotion} \mid \text{additional certifications}) &= \frac{P(\text{promotion and additional certifications})}{P(\text{additional certifications})} \\ &= \frac{0.48}{0.6} = 0.8 \end{aligned}$$

The probability that an employee has been promoted recently given that the employee has earned additional certifications is 0.8 or 80%.

20

20.3 Use Permutations and Combinations to Determine the Size of a Sample Space

The factorial of a positive integer n, written $n!$, is the product of all non-negative integers less than or equal to n: $n(n - 1)(n - 2) \ldots$ A permutation is an ordered arrangement of items without repetition. The number of permutations of n items taken r at a time is given by the formula ${}_nP_r = \frac{n!}{(n - r)!}$. A combination is an unordered collection of items. The number of combinations of n items taken r at a time is given by the formula ${}_nC_r = \frac{n!}{(n - r)!r!}$.

Example

Mr. Rivas will randomly select 4 students from a group of 20 students to lead a class activity. How many groups are possible if the order in which students are chosen is important? How many groups are possible if the order in which students are chosen is not important?

If order is important, then the situation represents a permutation.

$${}_{20}P_4 = \frac{20!}{(20 - 4)!} = \frac{20!}{16!} = 20 \times 19 \times 18 \times 17 = 116{,}280$$

If order is not important, then the situation represents a combination.

$${}_{20}C_4 = \frac{20!}{(20 - 4)!4!} = \frac{20!}{16!4!} = \frac{20 \times 19 \times 18 \times 17}{4 \times 3 \times 2 \times 1} = 4845$$

There are 116,280 possible permutations if order is important. There are 4845 possible combinations if order is not important.

20.3 Use Permutations to Calculate Probability

Permutations can be used to calculate probabilities. For situations that involve ordered groups of items, use permutations to calculate the number of possible outcomes in the sample space. Then use permutations to calculate the number of successful outcomes and compare to the number of possible outcomes to calculate the probability.

Example

There are 12 students competing in the finals of an academic competition. First, second, and third places earn a gold, silver, and bronze medal, respectively. Marissa, Clyde, and Logan are 3 friends who prepared for the competition together. What is the probability that the 3 friends will win first, second, and third places? Assume that each competitor is equally likely to win a medal.

Because the order in which competitors finish in the competition is important, this situation represents permutations.

There are ${}_{12}P_3 = 1320$ ways the medals can be awarded.

There are ${}_3P_3 = 6$ ways Marissa, Clyde, and Logan can win all of the medals.

The probability that the 3 friends win all of the medals is $\frac{6}{1320} = \frac{1}{220}$.

20.3 Use Combinations to Calculate Probability

Combinations can also be used to calculate probabilities. For situations that involve unordered groups of items, use combinations to calculate the number of possible outcomes in the sample space. Then use combinations to calculate the number of successful outcomes and compare to the number of possible outcomes to calculate the probability.

Example

There are 12 office workers, 3 managers, and 2 vice presidents participating in team building exercises. Suppose 5 people are chosen at random from the group to demonstrate an activity. What is the probability that the group will consist of 2 office workers, 2 managers, and 1 vice president?

Because the order in which the participants are selected is not important, this situation represents combinations.

There are ${}_{17}C_5 = 6188$ ways the members of the group can be selected.

There are $({}_{12}C_2)({}_{3}C_2)({}_{2}C_1) = 396$ ways that 2 office workers, 2 managers, and 1 vice president can be selected.

The probability that 2 office workers, 2 managers, and 1 vice president will be selected is

$\frac{396}{6188} = \frac{99}{1547}$.

20.3 Calculate Permutations with Repeated Elements

Some sets of objects may contain identical elements. For example, a jar may contain 3 identical red marbles, 2 identical blue marbles, and a green marble. To calculate the number of permutations of n objects with a copies of one identical object, b copies of another identical object, c copies of another identical object, and so on, use the formula $\frac{n!}{a!b!c!\ldots}$.

Example

How many different ways can the letters of the word STATISTICS be arranged?

There are 10 letters with 3 copies of the letter S, 3 copies of the letter T, and 2 copies of the letter I.

$\frac{10!}{3!3!2} = 50{,}400$

There are 50,400 ways the letters of the word STATISTICS can be arranged.

20.3 Calculate Circular Permutations

A circular permutation is an ordered arrangement of objects around a circle. The number of circular permutations of n objects is given by the expression $(n - 1)!$.

Example

A family of 2 parents and 3 children is sitting around the dining room table for dinner. In how many ways can the members of the family be seated?

There are $2 + 3 = 5$ family members.

The number of ways the family members can be seated around the table is $(5 - 1)! = 4! = 24$.

20.4 Calculate the Probability of Two Trials of Independent Events

Two events are called independent events if the outcome of one event does not have any effect on the outcome of the second event. To calculate the probability of two trials of independent events, multiply the probability of the event by itself.

Example

Lars rolls a number cube two times. What is the probability that he will roll an odd number on both rolls?

The outcome of the first number cube does not affect the outcome of the second number cube, so the situation represents independent events. Multiply the probability of rolling an odd number by itself.

$P(\text{both odd}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

The probability of rolling an odd number on both rolls is $\frac{1}{4}$.

20.4 Calculate the Probability of Multiple Trials of Independent Events

When multiple trials of independent events are conducted, you can use combinations to calculate probabilities. If an event has two possible outcomes, success with probability s or failure with probability f, then the probability of exactly b successes in a trials is given by the expression ${}_aC_bS^bf^{a-b}$.

Example

According to a recent survey, 75% of high school students plan to attend a 4-year college immediately after graduating from high school. Suppose 8 high school students are selected at random. What is the probability that 5 of the students plan to attend a 4-year college immediately after graduation?

The probability that a randomly selected student plans to attend a 4-year college is 0.75, and the probability that the student does not plan to attend a 4-year college is $1 - 0.75 = 0.25$.

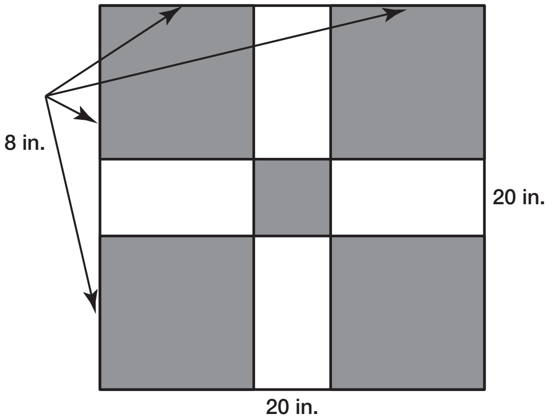
$P(\text{5 of 8 students plan to attend a 4-year college}) = {}_8C_5(0.75)^5(0.25)^3 \approx 0.208 = 20.8\%$.

20.5 Determine Geometric Probability

Geometry is a ratio of lengths, areas, volumes, and so on. To calculate a geometric probability using a figure, compare the length, area, or volume of the figure that represents a success to the entire length, area, or volume.

Example

A dartboard has the size and shape shown. The gray shaded area represents a scoring section of the dartboard. Calculate the probability that a dart that lands on a random part of the target will land in a gray scoring section.



Calculate the area of the dartboard: $20(20) = 400$ in.^{2}

There are 4 gray scoring squares with 8-in. sides and a gray scoring square with $20 - 8 - 8 = 4$-in. sides. Calculate the area of the gray scoring sections: $4(8)(8) + 4(4) = 272$ in.^{2}

Calculate the probability that a dart will hit a gray scoring section: $\frac{272}{400} = 0.68 = 68\%$.

20.5 Calculate Expected Value

The expected value of an experiment is the average value when the number of trials of the experiment is large. To calculate the expected value of an experiment, multiply the probability of each outcome by the value associated with that outcome. Add all of these products to calculate the expected value.

Example

Marina's change purse contains 4 pennies, 3 nickels, 7 dimes, and 6 quarters. Suppose she selects a coin from the purse at random. What is the expected value, in cents, of the coin?

There are $4 + 3 + 7 + 6 = 20$ coins.

$P(\text{penny}) = \frac{4}{20} = \frac{1}{5}$, $P(\text{nickel}) = \frac{3}{20}$, $P(\text{dime}) = \frac{7}{20}$, $P(\text{quarter}) = \frac{6}{20} = \frac{3}{10}$

Expected value: $(1¢)\frac{1}{5} + (5¢)\frac{3}{20} + (10¢)\frac{7}{20} + (25¢)\frac{3}{10} = 11.95¢$

The expected value of the coin is 11.95¢.