Circles 9


That's no moon. It's a picture of a solar eclipse in the making. A solar eclipse occurs when the Moon passes between the Earth and the Sun. Scientists can predict when solar eclipses will happen years into the future.

Chapter 9 Overview

This chapter reviews information about circles, and then focuses on angles and arcs related to a circle, chords, and tangents. Several theorems related to circles are proven throughout the chapter.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
9.1	Introduction to Circles	G.CO.1 G.C.1 G.C.2 G.MG.1	1	The lesson focuses on reviewing information about circles. Questions ask students to explore, review, and identify segments, lines, points, angles, and arcs related to a circle. Rigid motion is used to prove all circles are similar.	X		X	X	
9.2	Central Angles, Inscribed Angles, and Intercepted Arcs	G.CO.1 G.C.2 G.MG.1	2	The lesson addresses the relationship between arcs, central angles, and inscribed angles of a circle. Questions ask students to explore and determine the measure of arcs, central angles, and inscribed angles. The Arc Addition Postulate is presented and students use it to prove the Inscribed Angles Theorem and Parallel Lines-Congruent Arcs Theorem.	X		X	X	
9.3	Measuring Angles Inside and Outside of Circles	G.C.2 G.MG.1	2	The lesson focuses on interior angles of a circle, exterior angles of a circle, and tangents to a circle. Questions ask students to explore and determine the measure of angles formed by chords, secants, and tangents. Students prove the Interior Angles of a Circle Theorem, the Exterior Angles of a Circle Theorem, and the Tangent to a Circle Theorem.	X	X			

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
9.4	Chords	G.C.2 G.MG.1	2	The lesson focuses on chords of a circle. Questions ask students to explore and identify the relationship between a chord and a diameter of a circle, and the relationship between congruent chords and their minor arcs. Students prove the Diameter-Chord Theorem, the Equidistant-Chord Theorem and its converse, the Congruent Chord-Congruent Arc Theorem and its converse, and the Segment-Chord Theorem.	X				
9.5	Tangents and Secants	G.C.4 G.MG.1	2	The lesson focuses on tangents and secants of a circle. Questions ask students to explore and identify the relationship between a tangent line and a radius of a circle, and the relationship between congruent tangent segments. Students construct a tangent line and prove the Tangent Segment Theorem, the Secant Segment Theorem, and the Secant Tangent Theorem.	X	X			

9

Skills Practice Correlation for Chapter 9

Lesson		Problem Set	Objectives
9.1	Introduction to Circles		Vocabulary
		1 – 8	Identify indicated parts of circles
		9 – 14	Identify inscribed angles and central angles
		15 – 20	Classify arcs as major arcs, minor arcs, or semicircles
		21 – 28	Draw described parts of circles
9.2	Central Angles, Inscribed Angles, and Intercepted Arcs		Vocabulary
		1 – 6	Determine the measure of minor arcs
		7 – 12	Determine the measure of central angles
		13 – 18	Determine the measure of inscribed angles
		19 – 24	Determine the measure of intercepted arcs
		25 – 30	Calculate the measures of angles using the Inscribed Angle Theorem
		31 – 36	Use information to answer questions about circles
9.3	Measuring Angles Inside and Outside of Circles		Vocabulary
		1 – 6	Write expressions for the measures of given angles
		7 – 12	List the intercepted arc for given angles
		13 – 18	Write expressions for the measures of given angles
		19 – 24	Create proofs using circle theorems
		25 – 30	Determine the measures of angles and arcs
9.4	Chords		Vocabulary
		1 – 6	Use information about circles to answer questions
		7 – 12	Determine measurements given information
		13 – 18	Compare measurements of arcs and segments in a circle
		19 – 24	Use the Segment Chord Theorem to write equations from diagrams
9.5	Tangents and Secants		Vocabulary
		1 – 6	Use tangents and tangent segments to calculate angle measures
		7 – 12	Use tangents to write congruence statements
		13 – 18	Use tangents and tangent segments to calculate angle measures
		19 – 24	Identify secant segments and external secant segments
		25 – 30	Use the Secant Segment Theorem to write equations from diagrams
		31 – 36	Identify tangent segments, secant segments, and external secant segments
		37 – 42	Use the Secant Tangent Theorem to write equations from diagrams

9.1

Riding a Ferris Wheel

Introduction to Circles

LEARNING GOALS

In this lesson, you will:

- Review the definition of line segments related to a circle such as chord, diameter, secant, and tangent.
- Review definitions of points related to a circle such as center and point of tangency.
- Review the definitions of angles related to a circle such as central angle and inscribed angle.
- Review the definitions of arcs related to a circle such as major arc, minor arc, and semicircle.
- Prove all circles are similar using rigid motion.

KEY TERMS

- center of a circle
- radius
- chord
- diameter
- secant of a circle
- tangent of a circle
- point of tangency
- central angle
- inscribed angle
- arc
- major arc
- minor arc
- semicircle

ESSENTIAL IDEAS

- A circle is a set of points on a plane that are equidistant from a fixed point. The fixed point is called the center of the circle.
- A radius of a circle is a line segment drawn from the center of the circle to a point on the circle.
- A diameter of a circle is a line segment drawn from a point on the circle to a second point on the circle passing through the center of the circle.
- A secant is a line that intersects a circle at exactly two points.
- A chord is a line segment that intersects the circle at exactly two points.
- A tangent is a line that intersects a circle at exactly one point. The point of intersection is called the point of tangency.
- A central angle is an angle of a circle whose vertex is the center point of the circle.
- An inscribed angle is an angle of a circle whose vertex is on the circle.
- A major arc of a circle is the largest arc formed by a secant and a circle.
- A minor arc of a circle is the smallest arc formed by a secant and a circle.
- A semicircle is half of a circle.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Experiment with transformations in the plane

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

G-C Circles

Understand and apply theorems about circles

1. Prove that all circles are similar.

2. Identify and describe relationships among inscribed angles, radii, and chords.

G-MG Modeling with Geometry

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Apply geometric concepts in modeling situations

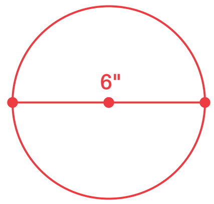
1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

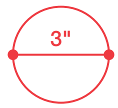
Students identify and analyze parts of a circle, including arcs, inscribed angles, secants, and tangents. Lines and line segments associated with a circle are classified and organized. Students use dilation to show that all circles are similar.

Warm Up

1. Draw a circle with the length of a diameter equal to 6 inches.



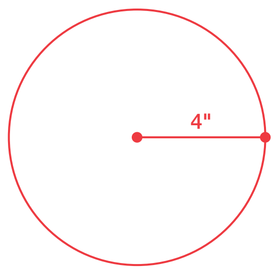
2. Draw a circle with the length of a diameter equal to 3 inches.



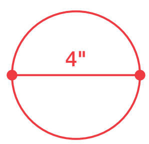
3. How does the area of the circle in Question 1 compare to the area of the circle in Question 2?

   The area of the circle drawn in Question 1 is equal to 9π in.^{2} and the area of the circle drawn in Question 2 is equal to 2.25π in.^{2}. The larger circle's area is 4 times greater than the smaller circle's area.

4. Draw a circle with the length of a radius equal to 4 inches.



5. Draw a circle with the length of a diameter equal to 4 inches.



6. How does the circumference of the circle in Question 1 compare to the circumference of the circle in Question 2?

   The circumference of the circle drawn in Question 1 is equal to 8π inches and the circumference of the circle drawn in Question 2 is equal to 4π inches. The larger circle's circumference is double the smaller circle's circumference.

9

9.1

Riding a Ferris Wheel

Introduction to Circles

9

LEARNING GOALS

In this lesson, you will:

- Review the definition of line segments related to a circle such as chord, diameter, secant, and tangent.
- Review definitions of points related to a circle such as center and point of tangency.
- Review the definitions of angles related to a circle such as central angle and inscribed angle.
- Review the definitions of arcs related to a circle such as major arc, minor arc, and semicircle.
- Prove all circles are similar using rigid motion.

KEY TERMS

- center of a circle
- radius
- chord
- diameter
- secant of a circle
- tangent of a circle
- point of tangency
- central angle
- inscribed angle
- arc
- major arc
- minor arc
- semicircle

Amusement parks are a very popular destination. Many people like rides that go fast, like roller coasters. Others prefer more relaxing rides. One of the most popular rides is the Ferris wheel.

The invention of the Ferris wheel is credited to George Washington Gale Ferris, Jr., who debuted his new ride at the World's Columbian Exposition in Chicago, Illinois in 1893. It was 264 feet tall, had a capacity of 2160 people, took 10 minutes to complete a revolution, and cost 50 cents to ride. Of course 50 cents was quite a bit of money at the time.

The well-known London Eye in England is the tallest Ferris wheel in the Western Hemisphere. The Singapore Flyer, located near the Singapore River, is currently the tallest in the world. It is more than a third of a mile high!

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Problem 1

The scenario is about a Ferris wheel. The Ferris wheel represents a circle and points are labeled around the wheel denoting seat locations. Students identify the location of points with respect to the circle. Points are located on the circle, at the center of the circle, inside the circle, or outside the circle. The definitions of circle, radius, diameter, center, and chord are reviewed. The terms secant, tangent, and point of tangency are defined.

Grouping

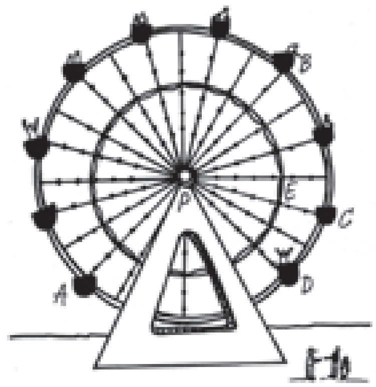
- Ask a student to read aloud the information and definition. Discuss as a class.
- Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- How did you determine the name of the circle?
- How many endpoints are associated with a radius?
- Which endpoint do all radii of a circle have in common?
- How many points are located on a circle?
- What do all of the points on a circle have in common?
- Which segment passes through the center point of the circle?
- How many diameters can be drawn in a circle?

PROBLEM 1 Going Around and Around

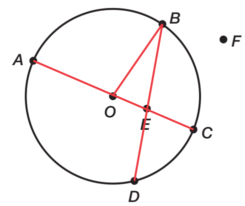
A Ferris wheel is in the shape of a circle.



Recall that a circle is the set of all points in a plane that are equidistant from a given point, which is called the **center of the circle**. The distance from a point on the circle to the center is the **radius** of the circle. A circle is named by its center. For example, the circle seen in the Ferris wheel is circle *P*.

1. Use the circle to answer each question.

a. Name the circle.



The name of the circle is circle *O*.

b. Use a straightedge to draw $\overline{OB}$, a radius of circle *O*. Where are the endpoints located with respect to the circle?

Point *O* is at the center, and point *B* is on the circle.

- Are all chords passing through the center point of the circle considered diameters?
- If the length of the radius of a circle is 5 cm, what is the length of a diameter of the circle? How do you know?
- If the length of the diameter of a circle is 5 cm, what is the length of a radius of the circle? How do you know?

c. How many radii does a circle have? Explain your reasoning.

A circle has an infinite number of radii because there are an infinite number of points that make up the circle, each connecting to the center of the circle.

d. Use a straightedge to draw $\overline{AC}$. Then, use a straightedge to draw $\overline{BD}$. How are the line segments different? How are they the same?

Line segment *AC* passes through the center of the circle, but $\overline{BD}$ does not. Both endpoints of both segments are on the circle.

Both line segments *AC* and *BD* are *chords* of the circle. A **chord** is a line segment with each endpoint on the circle. Line segment *AC* is called a *diameter* of the circle. A **diameter** is a chord that passes through the center of the circle.

e. Why is $\overline{BD}$ not considered a diameter?

Line segment *BD* doesn't pass through the center of the circle.

f. How does the length of the diameter of a circle relate to the length of the radius?

The length of the diameter of a circle is two times the length of the radius.

g. Are all radii of the same circle, or of congruent circles, always, sometimes, or never congruent? Explain your reasoning.

All radii of the same circle, or of congruent circles, are always congruent. The radius is the distance from the center to the edge of the circle. A circle is determined by all of the points on a plane equidistant from a given point (center point). The equidistance establishes that all radii must be congruent if they are in the same circle or congruent circles.

Grouping

Have students complete Questions 2 through 8 with a partner. Then have students share their responses as a class.

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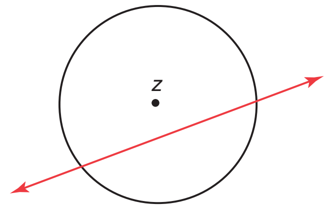
Guiding Questions for Share Phase, Questions 2 through 8

- What is the difference between a chord and a secant?
- Can a diameter also be a secant? Why or why not?
- Is a secant a line or a line segment?
- Is a chord a line or a line segment?
- Is a diameter a line or a line segment?
- Is a tangent a line or a line segment?
- How many tangents can be drawn to a circle?
- What do a tangent and a secant have in common?
- What do a diameter and a radius have in common?

A **secant of a circle** is a line that intersects a circle at exactly two points.

2. Draw a secant using the circle shown.



3. Maribel says that a chord is part of a secant. David says that a chord is different from a secant. Explain why Maribel and David are both correct.

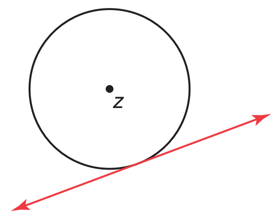
A secant is a line that intersects a circle at exactly two points; a chord is a line segment that intersects the circle at exactly two points.

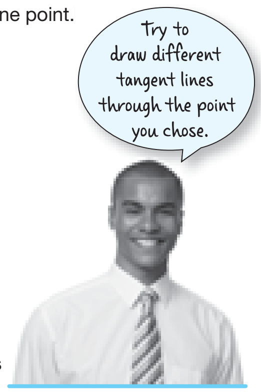
4. What is the longest chord in a circle?

The diameter is considered the longest chord in a circle because it passes through the center. The circle is widest between the edge of the circle and the center. Chords drawn either above or below the center point are connecting two points on the circle positioned closer to each other when compared to the endpoints of the diameter.

A **tangent of a circle** is a line that intersects a circle at exactly one point. The point of intersection is called the **point of tangency**.

5. Draw a tangent using circle *Z* shown.





6. Choose another point on the circle. How many tangent lines can you draw through this point?

Through any given point on a circle, there is only one line tangent to the circle.

7. Explain the difference between a secant and a tangent.

A secant is a line that intersects the circle at exactly two points, and a tangent is a line that intersects the circle at exactly one point.

8. Check the appropriate term(s) associated with each characteristic in the table shown.

Characteristic	Chord	Secant	Diameter	Radius	Tangent
A line		X			X
A line segment	X		X	X	
A line segment having both endpoints on the circle	X		X		
A line segment having one endpoint on the circle				X	
A line segment passing through the center of the circle			X		
A line intersecting a circle at exactly two points		X			
A line intersecting a circle at exactly one point					X

Problem 2

The terms central angle, inscribed angle, arc, major arc, minor arc, and semicircle are defined.

9

Grouping

- Ask a student to read aloud the definitions. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

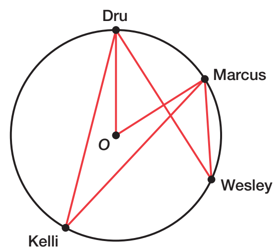
- What is the difference between a central angle and an inscribed angle?
- What do a central angle and an inscribed angle have in common?
- How would you describe the location of the arc cut by the central angle?
- How would you describe the location of the arc cut by the inscribed angles?
- What is the difference between a major arc and a minor arc?

PROBLEM 2 Sitting on the Wheel

A **central angle** is an angle whose vertex is the center of the circle.

An **inscribed angle** is an angle whose vertex is on the circle.

1. Four friends are riding a Ferris wheel in the positions shown.



a. Draw a central angle where Dru and Marcus are located on the sides of the angle.

b. Draw an inscribed angle where Kelli is the vertex and Dru and Marcus are located on the sides of the angle.

c. Draw an inscribed angle where Wesley is the vertex and Dru and Marcus are located on the sides of the angle.

d. Compare and contrast these three angles.

The endpoints located on the sides of the angles are the same points. Each angle has a different vertex. Two angles have vertices on the circle, and one angle has a vertex at the center point.

An **arc** of a circle is any unbroken part of the circumference of a circle. An arc is named using its two endpoints. The symbol used to describe arc AB is $\overset{\frown}{AB}$.

A **major arc** of a circle is the largest arc formed by a secant and a circle. It goes more than halfway around a circle.

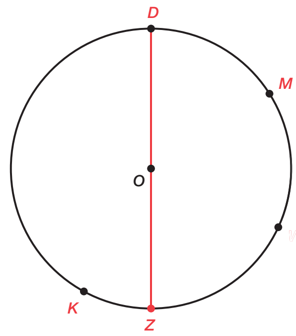
A **minor arc** of a circle is the smallest arc formed by a secant and a circle. It goes less than halfway around a circle.

A **semicircle** is exactly half of a circle.

To avoid confusion, three points are used to name semicircles and major arcs. The first point is an endpoint of the arc, the second point is any point at which the arc passes through and the third point is the other endpoint of the arc.

- How would you describe Dru's location on the Ferris wheel?
- How would you describe Marcus's location on the Ferris wheel?
- How would you describe Wesley's location on the Ferris wheel?
- How would you describe Kelli's location on the Ferris wheel?
- Which major arc is determined by minor arc *DM*?
- Which minor arc is determined by major arc *WDM*?

2. Use the same Ferris wheel from Question 1 to answer each question.



a. Label the location of each person with the first letter of his or her name.

b. Identify two different arcs and name them.

Answers will vary.

$\overset{\frown}{DM}$ and $\overset{\frown}{DW}$ are arcs.

c. Draw a diameter on the circle shown so that point *D* is an endpoint. Label the second endpoint as point *Z*. The diameter divided the circle into two semicircles.

d. Name each semicircle.

The first semicircle is $\overset{\frown}{DKZ}$, and the second semicircle is $\overset{\frown}{DMZ}$ or $\overset{\frown}{DWZ}$.

e. Name all minor arcs.

Minor arcs: $\overset{\frown}{DM}$, $\overset{\frown}{MW}$, $\overset{\frown}{WZ}$, $\overset{\frown}{ZK}$, $\overset{\frown}{KD}$, $\overset{\frown}{DW}$, $\overset{\frown}{MZ}$, $\overset{\frown}{WK}$, $\overset{\frown}{MK}$

f. Name all major arcs.

Major arcs: $\overset{\frown}{DMK}$, $\overset{\frown}{MKD}$, $\overset{\frown}{WKD}$, $\overset{\frown}{KDM}$, $\overset{\frown}{KDW}$, $\overset{\frown}{ZDM}$, $\overset{\frown}{ZDW}$, $\overset{\frown}{WDM}$, $\overset{\frown}{KWZ}$

9

Problem 3

Rigid motion and dilation are used to show all circles are similar. Students perform a translation and dilation to show circle A is similar to circle A'. The ratio of the radii of circles A and A' are equal to the absolute value of the scale factor.

Grouping

Have students follow the steps and complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- Do transformations preserve shape?
- Do dilations preserve shape?
- Would we get the same result using a smaller or larger scale factor?
- How would a different scale factor affect the result?
- If the circumference of a circle is divided by the length of the diameter, what is the result?
- If the circumference of a circle is divided by the length of the diameter, is this ratio the same for circles of all sizes?

PROBLEM 3 Are All Circles Similar?

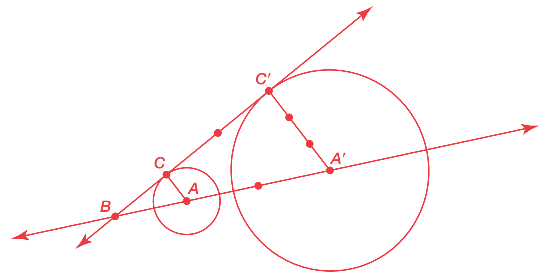
Recall that two figures are similar if there is a set of transformations that will move one figure exactly covering the other. To prove any two circles are similar, only a translation (slide) and a dilation (enlargement or reduction) are necessary. In this problem, you will use a point that is not on a circle as the center of dilation and a given scale factor to show any two circles are similar.

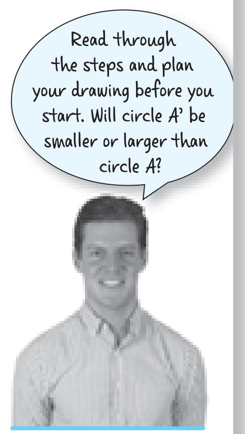
Step 1: Draw circle A.

Step 2: Locate point B not on circle A as the center of dilation.

Step 3: Dilate circle A by a scale factor of 3, locating points A' and C' such that $BA' = 3 \cdot BA$ and $BC' = 3 \cdot BC$

Step 4: Using radius $A'C'$, draw circle A'.





The ratio of the radii of circles A and A' are equal to the absolute value of the scale factor.

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1. Given any two circles, do you think you can always identify a dilation that maps one circle onto another?

   Yes. The center of dilation can be found by drawing rays connecting the corresponding points of any two circles as shown.

2. The scale factor is the ratio of $C'B$ to CB. Will this be true for any two circles?

   Yes. It will be true for any two circles because the second circle was drawn using the given scale factor by connecting the corresponding points of both circles to the center of dilation.

3. Can you conclude any two circles are always similar? Explain your reasoning.

   Yes. Any circle in a plane can be dilated onto any other circle in the plane, as shown. Therefore, the circles are always similar.

Talk the Talk

Students identify a diameter, radius, central angle, inscribed angle, minor arc, major arc, and semicircle in the diagram provided.

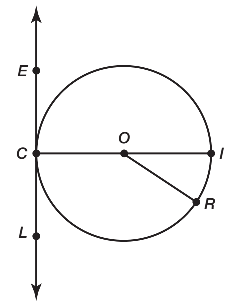
9

Grouping

Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

Talk the Talk

Use the diagram shown to answer Questions 1 through 7.



1. Name a diameter.
   The diameter is $\overline{CI}$.

2. Name a radius.
   A radius is $\overline{OI}$, $\overline{OC}$, or $\overline{OR}$.

3. Name a central angle.
   A central angle is $\angle ROI$ or $\angle COR$.

4. Name an inscribed angle.
   An inscribed angle is $\angle OCE$ or $\angle OCL$.

5. Name a minor arc.
   A minor arc is $\overset{\frown}{IR}$ or $\overset{\frown}{CR}$.

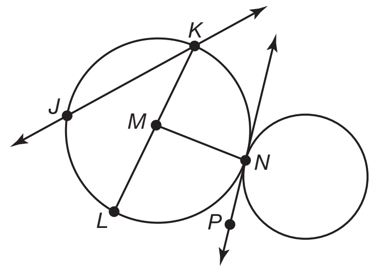
6. Name a major arc.
   The major arcs are $\overset{\frown}{CIR}$ or $\overset{\frown}{RCI}$.

7. Name a semicircle.
   A semicircle is $\overset{\frown}{CRI}$ or $\overset{\frown}{CI}$.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Use the diagram to match each notation with the term that provides the best description.



	Notation	Term
(D) 1.	$\overline{MN}$	A. point of tangency
(F) 2.	$\overleftrightarrow{JK}$	B. diameter
(E) 3.	Point M	C. tangent
(A) 4.	Point N	D. radius
(B) 5.	$\overline{LK}$	E. center of circle
(C) 6.	$\overleftrightarrow{PN}$	F. secant

9

9

Take the Wheel

9.2

Central Angles, Inscribed Angles, and Intercepted Arcs

LEARNING GOALS

In this lesson, you will:

- Determine the measures of arcs.
- Use the Arc Addition Postulate.
- Determine the measures of central angles and inscribed angles.
- Prove the Inscribed Angle Theorem.
- Prove the Parallel Lines–Congruent Arcs Theorem.

KEY TERMS

- degree measure of an arc
- adjacent arcs
- Arc Addition Postulate
- intercepted arc
- Inscribed Angle Theorem
- Parallel Lines–Congruent Arcs Theorem

ESSENTIAL IDEAS

- An intercepted arc is an arc of the circle formed by the intersection of the sides of an angle with the circle.
- The measure of a central angle is equal to the measure of its intercepted arc.
- The Arc Addition Postulate states: "The measure of an arc formed by two adjacent arcs is the sum of the measures of the two arcs."
- The Inscribed Angle Theorem states: "The measure of an inscribed angle is equal to half the measure of its intercepted arc."
- The Parallel Lines-Congruent Arcs Theorem states that parallel lines intercept congruent arcs on a circle.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Experiment with transformations in the plane

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

G-C Circles

Understand and apply theorems about circles

2. Identify and describe relationships among inscribed angles, radii, and chords.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

The terms degree measure of an arc and intercepted arc are defined. The Arc Addition Postulate is stated and used to determine the measure of arcs and angles of a circle associated with the arcs. Students use a two-column proof to prove the Inscribed Angle Theorem and the Parallel-Lines-Congruent Arcs Theorem. These theorems are used to determine the measures of arcs and angles of a circle in different situations.

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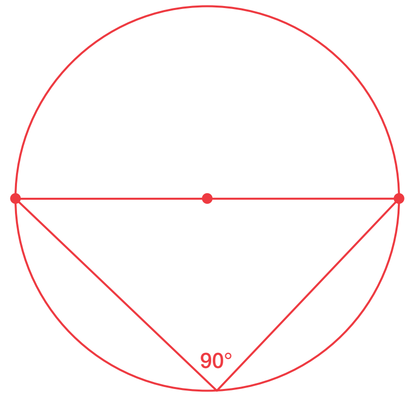
Warm Up

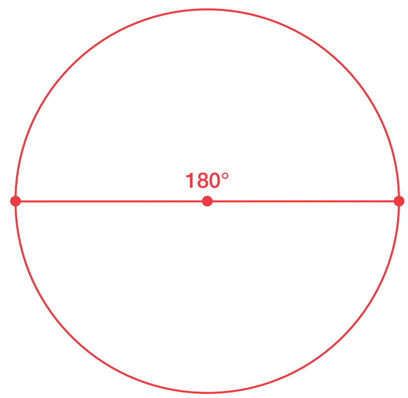
1. Draw a circle with an inscribed angle whose measure is equal to 90°.



2. Describe the arc determined by a 90° inscribed angle.

The arc determined by a 90° inscribed angle is half the circle, or a semicircle.

3. Draw a circle with a central angle whose measure is equal to 180°.



4. Describe the arc determined by a 180° central angle.

The arc determined by a 180° central angle is half the circle, or a semicircle.

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9.2

Take the Wheel

Central Angles, Inscribed Angles, and Intercepted Arcs

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LEARNING GOALS

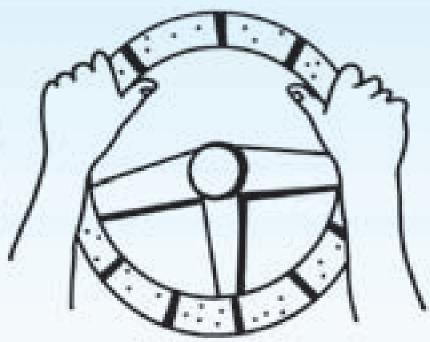
In this lesson, you will:

- Determine the measures of arcs.
- Use the Arc Addition Postulate.
- Determine the measures of central angles and inscribed angles.
- Prove the Inscribed Angle Theorem.
- Prove the Parallel Lines–Congruent Arcs Theorem.

KEY TERMS

- degree measure of an arc
- adjacent arcs
- Arc Addition Postulate
- intercepted arc
- Inscribed Angle Theorem
- Parallel Lines–Congruent Arcs Theorem

Before airbags were installed in car steering wheels, the recommended position for holding the steering wheel was the 10–2 position. Now, one of the recommended positions is the 9–3 position to account for the airbags. The numbers 10, 2, 9, and 3 refer to the numbers on a clock. So, the 10–2 position means that one hand is at 10 o'clock and the other hand is at 2 o'clock.



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Problem 1

Students calculate the measures of central angles using the positions of numbers on a clock and the placement of the clock hands as mapped to a steering wheel. The measure of a central angle is equal to the degree measure of its intercepted arc. Students conclude that if the measures of two central angles of the same circle or congruent circles are equal, then their corresponding minor arcs are congruent. And if the measures of two minor arcs of the same circle or congruent circles are equal, then their corresponding central angles are congruent.

Adjacent arcs are two arcs of the same circle sharing a common endpoint. Adjacent arcs are used to perform arc addition. The Arc Addition Postulate states that the measure of an arc formed by two adjacent arcs is the sum of the measures of the two arcs.

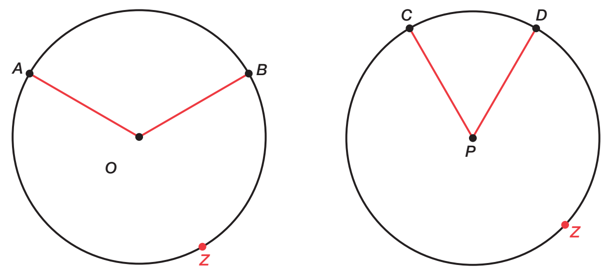
Students' familiarity with angle addition and segment addition will directly transfer to understanding the concept of arc addition. Students create and prove a conjecture about the measure of an inscribed angle related to the measure of its intercepted arc. The resulting theorem is called the Inscribed Angle Theorem and states that the measure of an inscribed angle is one half the measure of its intercepted arc. To prove this conjecture, it is necessary to prove three cases, locating the center of the circle inside, outside and on the inscribed angle. The steps are somewhat lengthy and it is suggested that you prove the conjecture as a class.

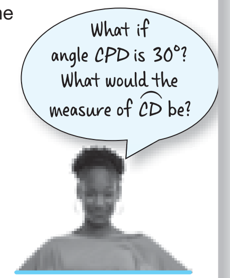
PROBLEM 1 Keep Both Hands on the Wheel

Recall that the degree measure of a circle is 360°.

Each minor arc of a circle is associated with and determined by a specific central angle. The **degree measure** of a minor arc is the same as the degree measure of its central angle. For example, if the measure of central angle *PRQ* is 30°, then the degree measure of its minor arc *PQ* is equal to 30°. Using symbols, this can be expressed as follows: If $\angle PRQ$ is a central angle and $m\angle PRQ = 30°$, then $m\overset{\frown}{PQ} = 30°$.

1. The circles shown represent steering wheels, and the points on the circles represent the positions of a person's hands.





For each circle, use the given points to draw a central angle. The hand position on the left is 10–2 and the hand position on the right is 11–1.

a. What are the names of the central angles?

The names of the central angles are $\angle AOB$ and $\angle CPD$.

b. Without using a protractor, determine the central angle measures. Explain your reasoning.

On a clock face, the measure of the angle between each consecutive pair of clock numbers, such as 12 and 1, is 30°. So, the measure of $\angle AOB$ is $30 \cdot 4 = 120°$, and the measure of $\angle CPD$ is $30° \cdot 2 = 60°$.

c. How do the measures of these angles compare?

The measure of $\angle AOB$ is twice the measure of $\angle CPD$.

Grouping

- Discuss the information and definition as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- Can the central angle be named in more than one way?
- How is thinking about a clock face helpful in determining the measure of the central angle?
- How many degrees are between each number on a clock face? How do you know?
- Why is it safer to have your hands farther apart on the steering wheel?
- How far apart is too far apart to place your hands on the wheel and therefore considered unsafe?
- What degree measure is associated with the entire circle?
- How did you determine the measure of the major arc?
- What is the relationship between the degree measure of the minor arc and the measure of the central angle associated with the arc?

d. Why do you think the hand position represented by the circle on the left is recommended and the hand position represented on the right is not recommended?

The hands should be spaced somewhat farther apart so that a driver can react to a situation quickly and steer out of harm's way. The hands should not be close together because it will be harder to quickly make a turn left or right.

e. Describe the measures of the minor arcs.

The measures of minor arcs are $m\overset{\frown}{AB} = 120°$ and $m\overset{\frown}{CD} = 60°$.

f. Plot and label point Z on each circle so that it does not lie between the endpoints of the minor arcs. Determine the measures of the major arcs that have the same endpoints as the minor arcs.

$m\overset{\frown}{AZB} = 240°$

$m\overset{\frown}{CZD} = 300°$

The angle around the entire circle is 360°. To determine the measure of the major arc, subtract the measure of the minor arc from 360°.

2. If the measures of two central angles of the same circle (or congruent circles) are equal, are their corresponding minor arcs congruent? Explain your reasoning.

Yes. Because the degree measure of a minor arc is defined as being equal to the measure of its central angle, then two congruent central angles of the same circle or congruent circles would determine arcs of equal measure. For example, if two angles of the same circle were each 46°, then the measure of each arc determined by the angles would also be equal to 46°.

3. If the measures of two minor arcs of the same circle (or congruent circles) are equal, are their corresponding central angles congruent? Explain your reasoning.

Yes. Because the degree measure of a minor arc is defined as equal to the measure of its central angle, then two congruent minor arcs of the same circle would determine central angles of equal measure. If two arcs of the same circle were each 46°, then the measure of each central angle determined by the minor arcs would also be equal to 46°.

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Grouping

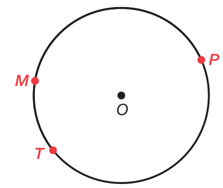
Have students complete Questions 4 through 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 9

- What do adjacent arcs have in common?
- How is the Arc Addition Postulate similar to the Angle Addition Postulate?
- How is the Arc Addition Postulate similar to the Segment Addition Postulate?
- What do an inscribed angle and its intercepted arc have in common?
- Do the inscribed angle and the central angle share the same intercepted arc?
- If the measure of a central angle is given, do you know the measure of its intercepted arc?
- If the measure of a central angle is given, do you know the measure of an inscribed angle that shares the same intercepted arc?
- Can two or more central angles share the same intercepted arc?
- Can two or more inscribed angles share the same intercepted arc?

Adjacent arcs are two arcs of the same circle sharing a common endpoint.

4. Draw and label two adjacent arcs on circle *O* shown.



Arc *MT* is adjacent to $\overset{\frown}{MP}$ because they share a common endpoint, point *M*.

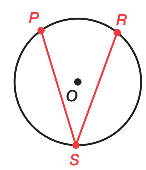
The **Arc Addition Postulate** states: "The measure of an arc formed by two adjacent arcs is the sum of the measures of the two arcs."

5. Apply the Arc Addition Postulate to the adjacent arcs you created.

$m\overset{\frown}{PMT} = m\overset{\frown}{PM} + m\overset{\frown}{MT}$

An *intercepted arc* is an arc associated with and determined by angles of the circle. An **intercepted arc** is a portion of the circumference of the circle located on the interior of the angle whose endpoints lie on the sides of an angle.

6. Consider circle *O*.

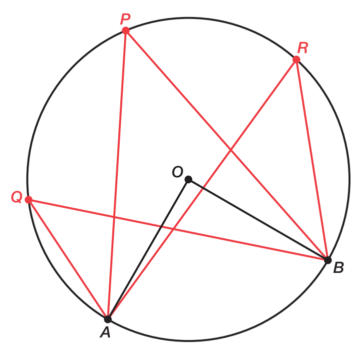


a. Draw inscribed $\angle PSR$ on circle *O*.

b. Name the intercepted arc associated with $\angle PSR$.

Arc *PR* is associated with $\angle PSR$.

7. Consider the central angle shown.



a. Use a straightedge to draw an inscribed angle that contains points *A* and *B* on its sides. Name the vertex of your angle point *P*. What do the angles have in common?

The angles intercept the same arc.

b. Use your protractor to measure the central angle and the inscribed angle. How is the measure of the inscribed angle related to the measure of the central angle and the measure of $\overset{\frown}{AB}$?

$m\angle APB = 45°$

$m\angle AOB = 90°$

The measure of the inscribed angle is half the measure of the central angle and half the measure of $\overset{\frown}{AB}$.

c. Use a straightedge to draw a different inscribed angle that contains points *A* and *B* on its sides. Name its vertex point *Q*. Measure the inscribed angle. How is the measure of the inscribed angle related to the measure of the central angle and the measure of $\overset{\frown}{AB}$?

$m\angle AQB = 45°$

The measure of the inscribed angle is half the measure of the central angle and half the measure of $\overset{\frown}{AB}$.

d. Use a straightedge to draw one more inscribed angle that contains points *A* and *B* on its sides. Name its vertex point *R*. Measure the inscribed angle. How is the measure of the inscribed angle related to the measure of the central angle and the measure of $\overset{\frown}{AB}$?

$m\angle ARB = 45°$

The measure of the inscribed angle is half the measure of the central angle and half the measure of $\overset{\frown}{AB}$.

8. What can you conclude about inscribed angles that have the same intercepted arc?
The measures of inscribed angles that have the same intercepted arc are congruent.

9. Dalia says that the measure of an inscribed angle is half the measure of the central angle that intercepts the same arc. Nate says that it is twice the measure. Sandy says that the inscribed angle is the same measure. Who is correct? Explain your reasoning.
Dalia is correct. The inscribed angles in circle *O* are all half the measure of the central angle that intercepts the same arc.

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Grouping

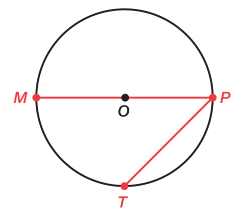
Complete Questions 10 and 11 to prove the Inscribed Angle Theorem as a class.

Guiding Questions for Share Phase, Questions 10 and 11

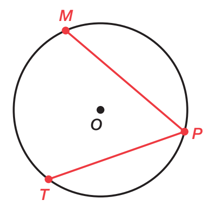
- Why do you think the proof of this theorem requires the proof of three different cases?
- Which theorems or postulates were helpful when proving case 1?
- Why are segments *OP* and *OT* congruent?
- What two angles in triangle *OPT* are congruent?
- If the $m\angle OTP = x$, what is $m\angle MOT$?
- If the $m\angle MOT = 2x$, what is the measure of its intercepted arc?
- Which theorems or postulates were helpful when proving case 2?
- How is the Arc Addition Postulate and the Angle Addition Postulate helpful in this proof?
- Which theorems or postulates were helpful when proving case 3?
- What radii are needed to prove this case?
- Why is diameter *RP* needed in this proof?
- How is the Isosceles Triangle Base Angle Theorem used in this situation?

10. Inscribed angles formed by two chords can be drawn three different ways with respect to the center of the circle.

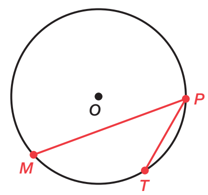
Case 1: Use circle *O* shown to draw and label inscribed $\angle MPT$ such that the center point lies on one side of the inscribed angle.



Case 2: Use circle *O* shown to draw and label inscribed $\angle MPT$ such that the center point lies on the interior of the inscribed angle.



Case 3: Use circle *O* shown to draw and label inscribed $\angle MPT$ such that the center point lies on the exterior of the inscribed angle.



- How is the Exterior Angle Theorem used in this situation?
- How is substitution used in this situation?
- How is the Arc Addition Postulate and the Angle Addition Postulate helpful in this proof?

11. To prove your Inscribed Angle Conjecture, you must prove each case in Question 10.

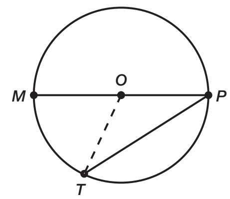
Case 1:

Given: $\angle MPT$ is inscribed in circle O.

$m\angle MPT = x$

Point O lies on diameter $\overline{PM}$.

Prove: $m\angle MPT = \frac{1}{2}m\overset{\frown}{MT}$



Statements	Reasons
1. $\angle MPT$ is inscribed in circle O. $m\angle MPT = x$ Point O lies on diameter $\overline{PM}$.	**1.** Given
2. Connect points O and T to form radius $\overline{OT}$.	**2.** Construction
3. $\overline{OT} \cong \overline{OP}$	3. All radii of the same circle are congruent.
4. $\angle MPT \cong \angle OTP$	4. Isosceles Triangle Base Angle Theorem
5. $m\angle OTP = x$	5. Substitution Property steps 1 and 4
6. $m\angle MOT = 2x$	6. Exterior Angle Theorem
7. $m\overset{\frown}{MT} = 2x$	7. A central angle is equal to the measure of its intercepted arc.
8. $\frac{1}{2}m\overset{\frown}{MT} = x$	8. Division Property of Equality
9. $\frac{1}{2}m\overset{\frown}{MT} = m\angle MPT$	9. Substitution Property steps 1 and 8

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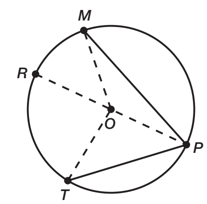
Case 2:



Given: $\angle MPT$ is inscribed in circle O.

Point O is in the interior of $\angle MPT$.

$m\angle MPO = x$

$m\angle TPO = y$

Prove: $m\angle MPT = \frac{1}{2}m\overset{\frown}{MT}$

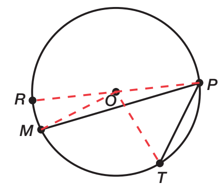
Statements	Reasons
1. $\angle MPT$ is inscribed in circle O. Point O is in the interior of $\angle MPT$. $m\angle MPO = x$ $m\angle TPO = y$	1. Given
2. Construct diameter $\overline{PR}$. Connect points O and T to form radius $\overline{OT}$. Connect points O and M to form radius $\overline{OM}$.	2. Construction
3. $\overline{OT} \cong \overline{OM} \cong \overline{OP}$	3. All radii of the same circle are congruent.
4. $\angle OMP \cong \angle MPO$, $\angle OTP \cong \angle OPT$	4. Isosceles Triangle Base Angle Theorem
5. $m\angle OMP = x$, $m\angle OTP = y$	5. Substitution Property steps 1 and 4
6. $m\angle MOR = 2x$ $m\angle TOR = 2y$	6. Exterior Angle Theorem
7. $m\overset{\frown}{MR} = 2x$ $m\overset{\frown}{TR} = 2y$	7. A central angle is equal to the measure of its arc.
8. $m\overset{\frown}{MT} = m\overset{\frown}{MR} + m\overset{\frown}{RT}$	8. Arc Addition Postulate
9. $m\overset{\frown}{MT} = 2x + 2y$	9. Substitution Property steps 7 and 8
10. $m\angle MPT = m\angle MPO + m\angle TPO$	10. Angle Addition Postulate
11. $m\angle MPT = x + y$	11. Substitution Property steps 1 and 10
12. $\frac{1}{2}m\overset{\frown}{MT} = x + y$	12. Division Property of Equality
13. $\frac{1}{2}m\overset{\frown}{MT} = m\angle MPT$	13. Substitution Property steps 11 and 12

Case 3:

Given: $\angle MPT$ is inscribed in circle O.

Point O is in the exterior of $\angle MPT$.

Prove: $m\angle MPT = \frac{1}{2}m\overset{\frown}{MT}$



Statements	Reasons
1. $\angle MPT$ is inscribed in circle O. Point O is in the exterior of $\angle MPT$.	1. Given
2. Construct diameter $\overline{PR}$. Connect points O and T to form radius $\overline{OT}$. Connect points O and M to form radius $\overline{OM}$.	2. Construction
3. $\overline{OT} \cong \overline{OM} \cong \overline{OP}$	3. All radii of the same circle are congruent.
4. $\angle OMP \cong \angle MPO$, $\angle OTP \cong \angle OPT$	4. Isosceles Triangle Base Angle Theorem
5. $m\angle OMP = m\angle MPO$, $m\angle OTP = m\angle OPT$	5. Definition of congruent angles
6. $m\angle ROT = m\angle OPT + m\angle OTP$	6. Exterior Angle Theorem
7. $m\angle ROT = 2m\angle OPT$	7. Substitution Property steps 5 and 6
8. $m\overset{\frown}{RT} = m\angle ROT$	8. A central angle is equal to the measure of its intercepted arc.
9. $m\overset{\frown}{RT} = 2m\angle OPT$	9. Substitution Property steps 7 and 8
10. $m\angle ROM = m\angle OMP + m\angle MPO$	10. Exterior Angle Theorem
11. $m\angle ROM = 2m\angle MPO$	11. Substitution Property steps 5 and 10
12. $m\overset{\frown}{RM} = m\angle ROM$	12. A central angle is equal to the measure of its intercepted arc.
13. $m\overset{\frown}{RM} = 2m\angle MPO$	13. Substitution Property steps 11 and 12
14. $m\overset{\frown}{MT} = m\overset{\frown}{RT} - m\overset{\frown}{RM}$	14. Arc Addition Postulate
15. $m\overset{\frown}{MT} = 2m\angle OPT - 2m\angle MPO$	15. Substitution Property steps 11, 13, 14
16. $\frac{1}{2}m\overset{\frown}{MT} = m\angle OPT - m\angle MPO$	16. Division Property of Equality
17. $m\angle MPT = \angle OPT - \angle MPO$	17. Angle Addition Postulate
18. $m\angle MPT = \frac{1}{2}m\overset{\frown}{MT}$	18. Substitution Property steps 1 and 5

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The **Inscribed Angle Theorem** states: "The measure of an inscribed angle is half the measure of its intercepted arc."

Grouping

Have students complete Question 12 with a partner. Then have students share their responses as a class.

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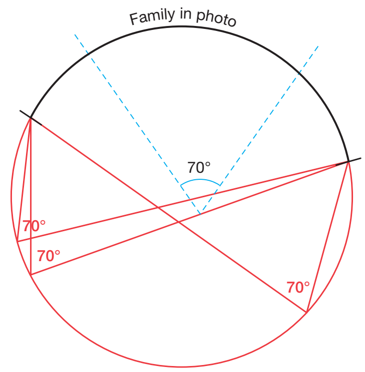
12. Aubrey wants to take a family picture. Her camera has a 70° field of view, but to include the entire family in the picture, she needs to cover a 140° arc. Explain what Aubrey needs to do to fit the entire family in the picture. Use the diagram to draw the solution.



Aubrey can use her location as the center of a circle. The Inscribed Angle Theorem says that the measure of an inscribed angle is half the measure of its intercepted arc. So, if Aubrey moves to a point on the circle, her camera's 70° field of view will be able to capture a 140° arc.

Problem 2

Students prove the Parallel Lines–Congruent Arc Conjecture stating that parallel lines intercept congruent arcs on a circle.

Grouping

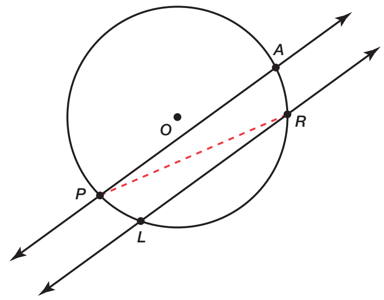
Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Which arcs are cut between chords *PA* and *LR* in circle *O*?
- Are lines *PA* and *LR* parallel to each other?
- Did you connect points *P* and *R* or points *L* and *A*?
- Which pair of interior angles was used in this proof?
- What can you conclude using the Inscribed Angle Theorem?
- How is substitution used to complete this proof?

PROBLEM 2 Parallel Lines Intersecting a Circle

Do parallel lines intersecting a circle intercept congruent arcs on the circle?



1. Create a proof for this conjecture.

   Given: $\overleftrightarrow{PA} \parallel \overleftrightarrow{LR}$

   Prove: $\overset{\frown}{PL} \cong \overset{\frown}{AR}$

Statements	Reasons
1. $\overleftrightarrow{PA} \parallel \overleftrightarrow{LR}$	1. Given
2. Connect points *P* and *R* to form inscribed $\angle APR$ and $\angle LRP$.	2. Construction
3. $\angle APR \cong \angle LRP$	3. Alternate Interior Angle Theorem
4. $m\angle APR = m\angle LRP$	4. Definition of congruent angles
5. $m\angle APR = \frac{1}{2}m\overset{\frown}{AR}$	5. Inscribed Angle Theorem
6. $m\angle LRP = \frac{1}{2}m\overset{\frown}{PL}$	6. Inscribed Angle Theorem
7. $\frac{1}{2}m\overset{\frown}{PL} = \frac{1}{2}m\overset{\frown}{AR}$	7. Substitution Property steps 4, 5, and 6
8. $m\overset{\frown}{PL} = m\overset{\frown}{AR}$	8. Multiplication
9. $\overset{\frown}{PL} \cong \overset{\frown}{AR}$	9. Definition of congruent arcs

You have just proven the Parallel Lines–Congruent Arcs Conjecture. It is now known as the Parallel Lines–Congruent Arcs Theorem which states that parallel lines intercept congruent arcs on a circle.

Talk the Talk

Students describe relationships among arcs, radii, and angles in different situations. They also determine the measure of angles or the measure of arcs associated with the angles of a circle.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

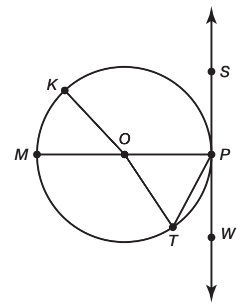
Talk the Talk

1. $\overline{MP}$ is a diameter of circle O.

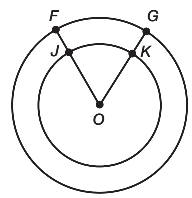
   If $m\overset{\frown}{MT} = 124°$, determine $m\angle TPW$.

   Explain your reasoning.

   In the diagram, $\overset{\frown}{MTP}$ is a semicircle, so $m\overset{\frown}{MTP} = 180°$. If $m\overset{\frown}{MTP} - m\overset{\frown}{MT} = m\overset{\frown}{TP}$, then $180° - 124° = 56°$, so $m\overset{\frown}{TP} = 56°$. So, $m\angle TPW = \frac{1}{2}m\overset{\frown}{TP}$, because the measure of an inscribed angle is equal to half the measure of its arc, so $m\angle TPW = 28°$.



2. Use the diagram shown to answer each question.



   a. Are radii $\overline{OJ}$, $\overline{OK}$, $\overline{OF}$, and $\overline{OG}$ all congruent? Explain your reasoning.

   No. The radii $\overline{OJ}$, $\overline{OK}$, $\overline{OF}$, and $\overline{OG}$ are not congruent because they are not all radii of the same or congruent circles. $\overline{OJ} \cong \overline{OK}$ and $\overline{OF} \cong \overline{OG}$ because radii of the same circles are congruent.

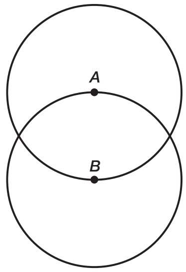
   b. Is $m\overset{\frown}{FG}$ greater than $m\overset{\frown}{JK}$? Explain your reasoning.

   No. The measure of $\overset{\frown}{FG}$ is not greater than $m\overset{\frown}{JK}$. The arcs are the same measure because they are determined by the same central angle.

   c. If $m\angle FOG = 57°$, determine $m\overset{\frown}{JK}$ and $m\overset{\frown}{FG}$. Explain your reasoning.

   No. The measure of $\overset{\frown}{JK}$ and $m\overset{\frown}{FG}$ are both equal to 57° because a central angle is equal to the measure of its arc. Both arcs share the same central angle.

3. DeJaun told Thomas there was not enough information to determine whether circle A was congruent to circle B. He said they would have to know the length of a radius in each circle to determine whether the circles were congruent. Thomas explained to DeJaun why he was incorrect. What did Thomas say to DeJaun?

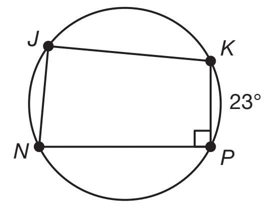


Thomas connected points A and B to show DeJaun how $\overline{AB}$ was a radius that is shared by both circles. Using the Reflexive Property, $\overline{AB} \cong \overline{AB}$, Thomas concluded circle A and circle B must have congruent radii.

Be prepared to share your solutions and methods.

Check for Students' Understanding

What can you conclude about the diagram? Explain your reasoning.



The measure of $\overset{\frown}{NJK}$ is 180°, because the inscribed angle determining this angle is a right angle and an inscribed angle is half the measure of its intercepted arc.

The measure of $\overset{\frown}{NPK}$ is 180°, because a circle is 360° and this arc and the measure of $\overset{\frown}{NJK}$ determine the entire circle.

The measure of $\overset{\frown}{NP}$ is 157° because the measure of $\overset{\frown}{NP}$ plus the measure of $\overset{\frown}{PK}$ is equal to the measure of $\overset{\frown}{NPK}$ using the Arc Addition Postulate.

The measure of inscribed angle J is equal to 90° because an inscribed angle is half the measure of its intercepted arc.

9.3

Manhole Covers

Measuring Angles Inside and Outside of Circles

LEARNING GOALS

In this lesson, you will:

- Determine measures of angles formed by two chords.
- Determine measures of angles formed by two secants.
- Determine measures of angles formed by a tangent and a secant.
- Determine measures of angles formed by two tangents.
- Prove the Interior Angles of a Circle Theorem.
- Prove the Exterior Angles of a Circle Theorem.
- Prove the Tangent to a Circle Theorem

KEY TERMS

- Interior Angles of a Circle Theorem
- Exterior Angles of a Circle Theorem
- Tangent to a Circle Theorem

ESSENTIAL IDEAS

- The Interior Angles of a Circle Theorem states: "If an angle is formed by two intersecting chords or secants of a circle such that the vertex of the angle is in the interior of the circle, then the measure of the angle is half of the sum of the measures of the arcs intercepted by the angle and its vertical angle."
- The Exterior Angles of a Circle Theorem states: "If an angle is formed by two intersecting chords or secants of a circle such that the vertex of the angle is in the exterior of the circle, then the measure of the angle is half of the difference of the measures of the arcs intercepted by the angle."
- The Tangent to a Circle Theorem states: "A line drawn tangent to a circle is perpendicular to a radius of the circle drawn to the point of tangency."

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-C Circles

Understand and apply theorems about circles

2. Identify and describe relationships among inscribed angles, radii, and chords.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

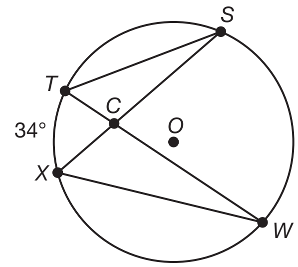
Overview

Students explore and prove theorems for determining the measures of angles located on the inside and outside of a circle. A proof by contradiction is provided to show a perpendicular relationship exists when a radius of a circle is drawn to a point of tangency. Students use these theorems to solve problem situations.

Warm Up

1. What can you conclude about $m\angle TSX$?

   $m\angle TSX = 17°$



2. What can you conclude about $m\angle TWX$?

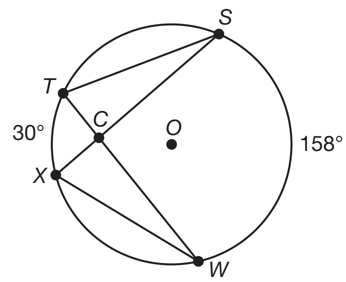
   $m\angle TWX = 17°$

3. What can you conclude about $m\angle SXW$ and $m\angle STW$?

   The $m\angle SXW$ and the $m\angle STW$ are congruent because they are both inscribed angles and share the same intercepted arc and they are half the measure of arc *SW*.

4. In circle *O*, determine $m\angle XCW$ and explain how it was calculated.

   The $m\angle WXS = 79°$, because it is an inscribed angle which is half the measure of its intercepted arc (158°). The $m\angle TWX = 15°$, because it is an inscribed angle which is half the measure of its intercepted arc (30°). The sum of the three interior angles of a triangle is equal to 180°, so the third angle, $\angle XCW$ in triangle *XCW* has to have a measure of 86°.



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9.3

Manhole Covers

Measuring Angles Inside and Outside of Circles

LEARNING GOALS

In this lesson, you will:

- Determine measures of angles formed by two chords.
- Determine measures of angles formed by two secants.
- Determine measures of angles formed by a tangent and a secant.
- Determine measures of angles formed by two tangents.
- Prove the Interior Angles of a Circle Theorem.
- Prove the Exterior Angles of a Circle Theorem.
- Prove the Tangent to a Circle Theorem

KEY TERMS

- Interior Angles of a Circle Theorem
- Exterior Angles of a Circle Theorem
- Tangent to a Circle Theorem

Manhole covers are heavy removable plates that are used to cover maintenance holes in the ground. Most manhole covers are circular and can be found all over the world. The tops of these covers can be plain or have beautiful designs cast into their tops.

Problem 1

Manhole covers are circular and the points located around the rim are connected to form chords and angles. Students calculate the measures of inscribed angles using the measures of their intercepted arcs.

Students write an expression to calculate the measure of an angle drawn in the interior of a circle using the angle's intercepted arc and its vertical angle's intercepted arc (half the sum).

Students prove that the measure of an angle formed by two intersecting chords or secants such that the vertex of the angle is in the interior of the circle is equal to half the sum of the measures of the arcs intercepted by the angle and its vertical angle.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

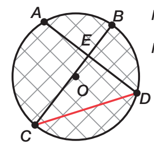
Guiding Questions for Share Phase, Questions 1 and 2

- Is $\angle BED$ a central angle? Why not?
- Is $\angle BED$ an inscribed angle? Why not?
- With respect to the circle, how would you describe the location of the vertex of $\angle BED$?
- Arc BD is associated with which inscribed angle?
- Arc AC is associated with which inscribed angle?
- Exterior $\angle BED$ is associated with which two remote interior angles of triangle CED?
- Exterior $\angle KFH$ is associated with which two remote interior angles of triangle FEH?
- Which arc is associated with $\angle E$?

PROBLEM 1 Inside the Circle

The vertex of an angle can be located inside of a circle, outside of a circle, or on a circle. In this lesson, you will explore these locations and prove theorems related to each situation.

1. Circle O shows a simple manhole cover design.



$m\overset{\frown}{BD} = 70°$

$m\overset{\frown}{AC} = 110°$

a. Consider $\angle BED$. How is this angle different from the angles that you have seen so far in this chapter? How is this angle the same?

The angle is different because its vertex is neither at the center of the circle nor on the circle. The angle is the same because its sides intersect the circle.

b. Can you determine the measure of $\angle BED$ with the information you have so far? If so, how? Explain your reasoning.

Answers will vary.

c. Draw chord CD. Use the information given in the figure to name the measures of any angles that you do know. Explain your reasoning.

Because the intercepted arc of $\angle BCD$ is $\overset{\frown}{BD}$ and $m\overset{\frown}{BD} = 70°$, $m\angle BCD = 35°$.
Because the intercepted arc of $\angle ADC$ is $\overset{\frown}{AC}$ and $m\overset{\frown}{AC} = 110°$, $m\angle ADC = 55°$.

d. How does $\angle BED$ relate to $\triangle CED$?

Angle BED is an exterior angle of the triangle.

e. Write a statement showing the relationship between $m\angle BED$, $m\angle EDC$, and $m\angle ECD$.

$m\angle EDC + m\angle ECD = m\angle BED$

f. What is the measure of $\angle BED$?

The measure of $\angle BED$ is $55° + 35° = 90°$.

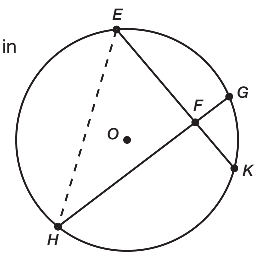
- Which arc is associated with $\angle H$?
- What is the Inscribed Angle Theorem?
- How is the Inscribed Angle Theorem helpful in proving this theorem?

It appears that the measure of an interior angle in a circle is equal to half of the sum of the measures of the arcs intercepted by the angle and its vertical angle. This observation can be stated as a theorem and proven.

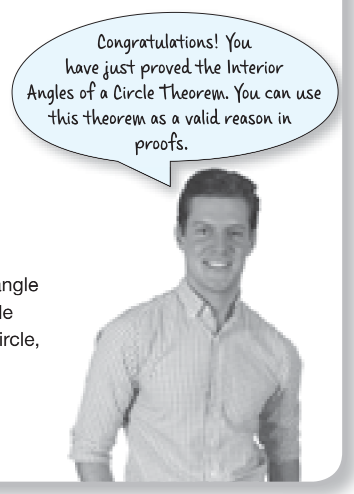
2. Prove the Interior Angles of a Circle Theorem.

Given: Chords EK and GH intersect at point F in circle O.

Prove: $m\angle KFH = \frac{1}{2}(m\overset{\frown}{HK} + m\overset{\frown}{EG})$



Statements	Reasons
1. Chords $\overline{EK}$ and $\overline{GH}$ intersect at point F in circle O.	1. Given
2. Connect points E and H to form chord $\overline{EH}$.	2. Construction
3. $m\angle KFH = m\angle E + m\angle H$	3. Exterior Angle Theorem
4. $m\angle E = \frac{1}{2}m\overset{\frown}{HK}$	4. Inscribed Angle Theorem
5. $m\angle H = \frac{1}{2}m\overset{\frown}{EG}$	5. Inscribed Angle Theorem
6. $m\angle KFH = \frac{1}{2}m\overset{\frown}{HK} + \frac{1}{2}m\overset{\frown}{EG}$	6. Substitution Property steps 3, 4, and 5
7. $m\angle KFH = \frac{1}{2}(m\overset{\frown}{HK} + m\overset{\frown}{EG})$	7. Distributive Property



The **Interior Angles of a Circle Theorem** states: "If an angle is formed by two intersecting chords or secants of a circle such that the vertex of the angle is in the interior of the circle, then the measure of the angle is half of the sum of the measures of the arcs intercepted by the angle and its vertical angle."

Problem 2

Students draw an angle in the exterior of a circle formed by a secant and a tangent, two secants, and two tangents. They use these diagrams to prove the Exterior Angles of the Circle Theorem. This theorem states that the measure of an angle drawn in the exterior of a circle is half the difference of the measures of the arcs intercepted by the angle. To prove this, it is necessary to prove three cases. The steps are somewhat lengthy and it is suggested that you prove the conjecture as a class.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Do the secants intersect on the circle, inside the circle, or outside the circle?
- Arc *KM* is associated with which inscribed angle?
- Arc *LN* is associated with which inscribed angle?
- Exterior $\angle KNM$ is associated with which two remote interior angles of triangle *CED*?
- Which arc is associated with $\angle LKN$?
- Which arc is associated with $\angle KNM$?

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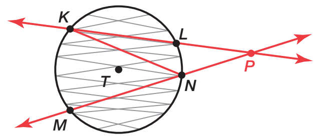
PROBLEM 2 Outside the Circle

1. Circle *T* shows another simple manhole cover design.



$m\overset{\frown}{KM} = 80°$
$m\overset{\frown}{LN} = 30°$

a. Consider $\overline{KL}$ and $\overline{MN}$. Use a straightedge to draw secants that coincide with each line segment. Where do the secants intersect? Label this point as point *P* on the figure.

The secant lines intersect outside the circle.

b. Draw chord $\overline{KN}$. Can you determine the measure of $\angle KPM$ with the information you have so far? If so, how? Explain your reasoning.

Answers will vary.

c. Use the information given in the figure to name the measures of any angles that you do know. Explain how you determined your answers.

Because the intercepted arc of $\angle LKN$ is $\overset{\frown}{LN}$ and $m\overset{\frown}{LN} = 30°$, $m\angle LKN = 15°$.
Because the intercepted arc of $\angle KNM$ is $\overset{\frown}{KM}$ and $m\overset{\frown}{KM} = 80°$, $m\angle KNM = 40°$.

d. How does $\angle KPN$ relate to $\triangle KPN$?

Angle *KPN* is an interior angle of $\triangle KPN$.

e. Write a statement showing the relationship between $m\angle KPN$, $m\angle NKP$, and $m\angle KNM$.

$m\angle KPN + m\angle NKP = m\angle KNM$

f. What is the measure of $\angle KPN$?

The measure of $\angle KPN$ is $40° - 15° = 25°$.

g. Describe the measure of $\angle KPM$ in terms of the measures of both arcs intercepted by $\angle KPM$.

The measure of $\angle KPM$ is half of the difference of the measures of $\overset{\frown}{KM}$ and $\overset{\frown}{LN}$.

Grouping

Have students complete Question 2 with a partner. Then have students share their responses as a class.

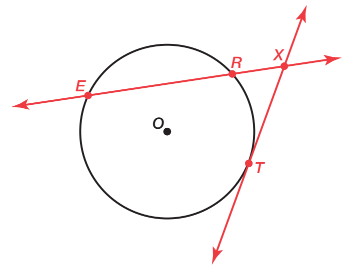
Guiding Questions for Share Phase, Question 2

- Is the angle associated with the intersection of a secant and tangent always an exterior angle?
- Is the angle associated with the intersection of two secants always an exterior angle?
- Is the angle associated with the intersection of two tangents always an exterior angle?

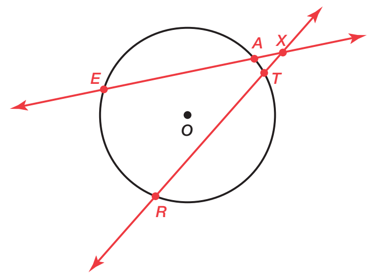
It appears that the measure of an exterior angle of a circle is equal to half of the difference of the arc measures that are intercepted by the angle. This observation can be stated as a theorem and proved.

2. An angle with a vertex located in the exterior of a circle can be formed by a secant and a tangent, two secants, or two tangents.

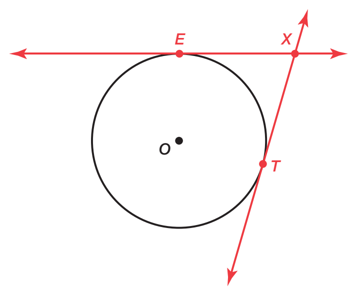
a. Case 1: Use circle *O* shown to draw and label an exterior angle formed by a secant and a tangent.



b. Case 2: Use circle *O* shown to draw and label an exterior angle formed by two secants.



c. Case 3: Use circle *O* shown to draw and label an exterior angle formed by two tangents.



Grouping

Complete Question 3 to prove the three cases associated with the Exterior Angles of a Circle Conjecture as a class.

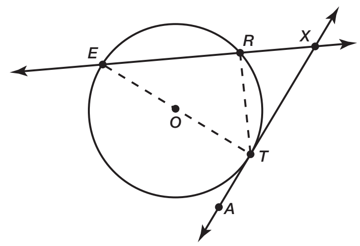
Guiding Questions for Share Phase, Question 3

- Which two remote interior angles are associated with exterior angle *ETA*?
- What can you conclude using the Inscribed Angle Theorem?
- How is proving case 2 similar to proving case 1?
- In case 3, which two points did you connect to form a triangle?
- What is the name of the exterior angle and the two remote interior angles in this situation?
- How is case 3 similar to cases 1 and 2?

To prove the Exterior Angles of a Circle Conjecture previously stated, you must prove each of the three cases.

3. Prove each case of the Exterior Angles of a Circle Conjecture.

a. Case 1

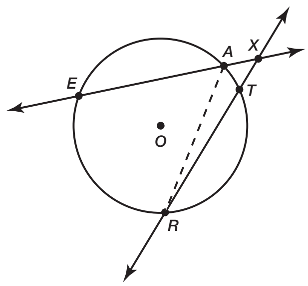


Given: Secant *EX* and tangent *TX* intersect at point *X*.

Prove: $m\angle EXT = \frac{1}{2}(m\overset{\frown}{ET} - m\overset{\frown}{RT})$

Statements	Reasons
1. Secant *EX* and tangent *TX* intersect at point *X*.	**1.** Given
2. Connect points *E* and *T* to form chord *ET*. Connect points *R* and *T* to form chord *RT*.	**2.** Construction
3. $m\angle ETA = m\angle TEX + m\angle EXT$	**3.** Exterior Angle Theorem
4. $m\angle ERT = \frac{1}{2}m\overset{\frown}{ET}$	**4.** Inscribed Angle Theorem
5. $m\angle ETA = \frac{1}{2}m\overset{\frown}{ET}$	**5.** Two inscribed angles have the same intercepted arc.
6. $m\angle TEX = \frac{1}{2}m\overset{\frown}{RT}$	**6.** Inscribed Angle Theorem
7. $\frac{1}{2}m\overset{\frown}{ET} = \frac{1}{2}m\overset{\frown}{RT} + m\angle EXT$	**7.** Substitution Property steps 3, 4, and 5
8. $\frac{1}{2}m\overset{\frown}{ET} - \frac{1}{2}m\overset{\frown}{RT} = m\angle EXT$	**8.** Subtraction
9. $m\angle EXT = \frac{1}{2}(m\overset{\frown}{ET} - m\overset{\frown}{RT})$	**9.** Distributive Property

b. Case 2



Given: Secants EX and RX intersect at point X.

Prove: $m\angle EXR = \frac{1}{2}(m\overset{\frown}{ER} - m\overset{\frown}{AT})$

Statements	Reasons
1. Secants EX and RX intersect at point X.	**1.** Given
2. Connect points A and R to form chord AR.	**2.** Construction
3. $m\angle EAR = m\angle ARX + m\angle EXR$	**3.** Exterior Angle Theorem
4. $m\angle EAR = \frac{1}{2}m\overset{\frown}{ER}$	**4.** Inscribed Angle Theorem
5. $m\angle ARX = \frac{1}{2}m\overset{\frown}{AT}$	**5.** Inscribed Angle Theorem
6. $\frac{1}{2}m\overset{\frown}{ER} = \frac{1}{2}m\overset{\frown}{AT} + m\angle EXR$	**6.** Substitution Property steps 3, 4, and 5
7. $\frac{1}{2}m\overset{\frown}{ER} - \frac{1}{2}m\overset{\frown}{AT} = m\angle EXR$	**7.** Subtraction
8. $m\angle EXR = \frac{1}{2}(m\overset{\frown}{ER} - m\overset{\frown}{AT})$	**8.** Distributive Property

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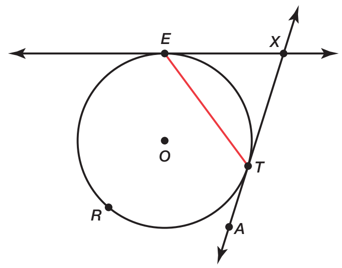
c. **Case 3**



Given: Tangents *EX* and *AX* intersect at point *X*.

Prove: $m\angle EXT = \frac{1}{2}(m\widehat{ERT} - m\widehat{ET})$

Statements	Reasons
1. Tangents *EX* and *AX* intersect at point *X*.	**1.** Given
2. Connect points *E* and *T* to form chord $\overline{ET}$.	**2.** Construction
3. $m\angle ETA = m\angle TEX + m\angle EXT$	**3.** Exterior Angle Theorem
4. $m\angle TEX = \frac{1}{2}m\widehat{ET}$	**4.** Inscribed Angle Theorem
5. $m\angle ETA = \frac{1}{2}m\widehat{ERT}$	**5.** Two inscribed angles have the same intercepted arc.
6. $\frac{1}{2}m\widehat{ERT} = \frac{1}{2}m\widehat{ET} + m\angle EXT$	**6.** Substitution Property steps 3, 4, and 5
7. $\frac{1}{2}m\widehat{ERT} - \frac{1}{2}m\widehat{ET} = m\angle EXT$	**7.** Subtraction
8. $m\angle EXT = \frac{1}{2}(m\widehat{ERT} - m\widehat{ET})$	**8.** Distributive Property

The **Exterior Angles of a Circle Theorem** states: "If an angle is formed by two intersecting chords or secants of a circle such that the vertex of the angle is in the exterior of the circle, then the measure of the angle is half of the difference of the measures of the arcs intercepted by the angle."

Problem 3

An angle formed by a diameter and a tangent has a vertex on the circle. This angle is considered an inscribed angle, so the measure of the angle is determined by half the measure of its intercepted arc.

The intercepted arc in this case is a semicircle, so students conclude the angle formed at the point of tangency is a right angle, and the lines forming this angle are perpendicular to each other. The Tangent to a Circle Theorem is proven by contradiction and the steps of the proof are provided for a classroom discussion. Students use the theorem to solve problem situations.

Grouping

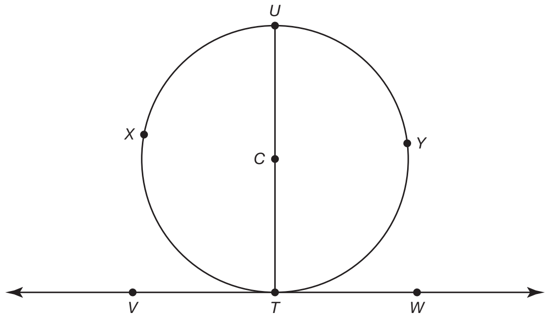
- Have students complete Question 1 with a partner. Then have students share their responses as a class.
- Discuss the indirect proof of the Tangent to a Circle Theorem in the worked example as a class.

Guiding Questions for Share Phase, Question 1

- Is chord UT a diameter of circle C?
- Does a diameter always divide a circle into two semicircles?
- If an intercepted arc is a semicircle, what can you conclude about the inscribed angle associated with this arc?

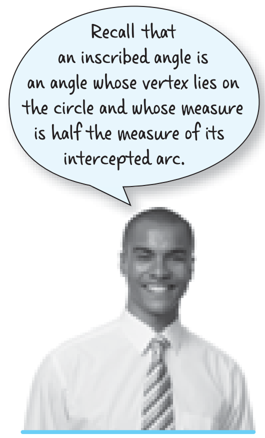
PROBLEM 3 Vertex On the Circle

1. Consider $\angle UTV$ with vertex located on circle C. Line VW is drawn tangent to circle C at point T.



a. Determine $m\overset{\frown}{UXT}$ and $m\overset{\frown}{UYT}$. Explain your reasoning.

$m\overset{\frown}{UXT} = 180°$ and $m\overset{\frown}{UYT} = 180°$ because segment UT is a diameter of circle C and arcs $\overset{\frown}{UXT}$ and $\overset{\frown}{UYT}$ are semicircles.



b. Determine $m\angle UTV$ and $m\angle UTW$. Explain your reasoning.

$m\angle UTV = 90°$ and $m\angle UTW = 90°$ because the measure of an inscribed angle is equal to half the measure of its intercepted arc.

It appears that when a line is drawn tangent to a circle, the angles formed at the point of tangency are right angles and therefore the radius drawn to the point of tangency is perpendicular to the tangent line.

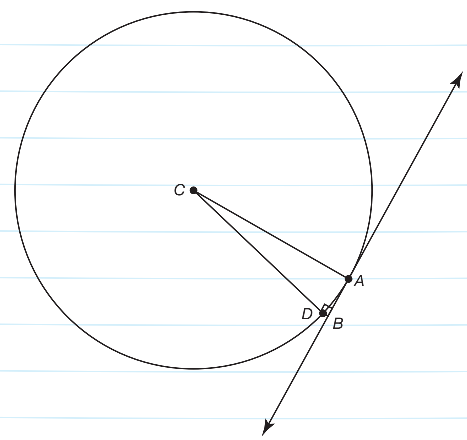
This observation can be proved and stated as a theorem.

Guiding Questions for Discuss Phase

- Does the location of the right angle in a right triangle determine the location of the hypotenuse?
- Is the hypotenuse always the longest side of a right triangle?
- How can *CA* (radius) be longer than *CB*, if $CB = CD$ (radius) $+ DB$?

The proof of this theorem is done by contradiction. Recall that a proof by contradiction begins with an assumption. Using the assumption and its implications, we arrive at a contradiction. When this happens, the proof is complete.

Line segment *CA* is a radius of circle *C*. Point *A* is the point at which the radius intersects the tangent line.



Step 1: Assumption: The tangent line is not perpendicular to the radius ($\overline{CA}$) of the circle.

Step 2: Point *B*, another point on the tangent line, is the point at which *CB* (line segment over this) is perpendicular to the tangent line.

Step 3: Consider right triangle *CBA* with hypotenuse *CA* and leg *CB*, so $CA > CB$.

Step 4: Impossible!! $CB > CA$ because $CB =$ length of radius (CD) $+ DB$.

The assumption is incorrect; therefore, the tangent line is perpendicular to the radius ($\overline{CA}$) of the circle.

This completes the proof of the Tangent to a Circle Theorem.

The **Tangent to a Circle Theorem** states: "A line drawn tangent to a circle is perpendicular to a radius of the circle drawn to the point of tangency."

Grouping

Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 2 and 3

- Which segment represents the distance Molly can see on the horizon?
- How many miles are in 29,034 feet?
- How can the Pythagorean Theorem be used to solve this problem?
- What algebraic expression represents the hypotenuse in this situation?
- Which variable are you solving for in this situation?

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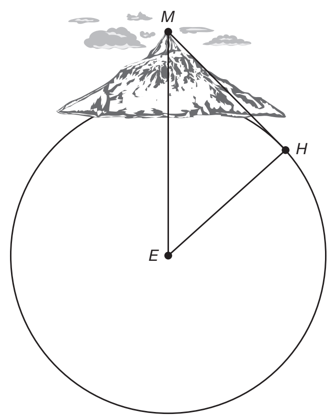
2. Molly is standing at the top of Mount Everest, which has an elevation of 29,029 feet. Her eyes are 5 feet above ground level. The radius of Earth is approximately 3960 miles. How far can Molly see on the horizon?



$29{,}034 \text{ feet} \cdot \frac{1 \text{ mile}}{5280 \text{ feet}} \approx 5.5 \text{ miles}$

$HM^2 + HE^2 = ME^2$

$HM^2 + (3960)^2 = (3960 + 5.5)^2$

$HM^2 + 15{,}681{,}600 = 15{,}725{,}190.25$

$HM^2 = 43{,}590.25$

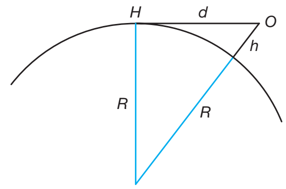
$HM = \sqrt{43{,}590.25} \approx 208.78$

Molly can see approximately 208.78 miles on the horizon.

9

3. When you are able to see past buildings and hills or mountains—when you can look all the way to the horizon, how far is that? You can use the Pythagorean Theorem to help you tell.

Imagine you are standing on the surface of the Earth and you have a height of h. The distance to the horizon is given by d in the diagram shown, and R is the radius of Earth.



Using your height, create a formula you can use to determine how far away the horizon is.

$d^2 + R^2 = (R + h)^2$

$d^2 + R^2 = R^2 + 2Rh + h^2$

$d^2 = 2Rh + h^2$

$d^2 = h(2R + h)$

$d = \sqrt{h(2R + h)}$

Problem 4

Students use the new theorems to solve for unknown measurements.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Arc RT is associated with which inscribed angle?
- Which theorem is helpful when solving for the measure of arc RT?
- Is there enough information to determine the measure of the four interior angles in this situation?
- Is there enough information to determine the measure of the arcs associated with angles 2 and 4?
- Is there enough information to determine the sum of the measures of the arcs associated with angles 2 and 4?
- If $m\angle AED = 80°$, can you determine the measures of the other three interior angles?
- Which theorem is helpful when solving for the measure of arc CD?
- Is there more than one way to solve this problem? Can a different theorem be used?

9

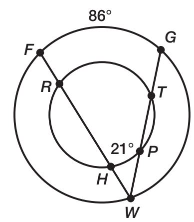
PROBLEM 4 Determine the Measures

1. Use the diagrams shown to determine the measures of each.

   a. Determine $m\overset{\frown}{RT}$.

   $m\overset{\frown}{FG} = 86°$

   $m\overset{\frown}{HP} = 21°$



$m\angle W = \frac{1}{2}m\overset{\frown}{FG}$

$m\angle W = \frac{1}{2}(86°) = 43°$

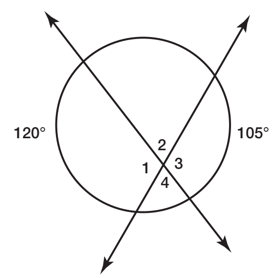
$43° = \frac{1}{2}(m\overset{\frown}{RT} - m\overset{\frown}{HP})$

$43° = \frac{1}{2}(m\overset{\frown}{RT} - 21°)$

$86° = m\overset{\frown}{RT} - 21°$

$m\overset{\frown}{RT} = 107°$

   b. Using the given information, what additional information can you determine about the diagram?



$m\angle 1 = \frac{1}{2}(120° + 105°) = 112.5°$

$m\angle 2 = 180° - 112.5° = 67.5°$

$m\angle 3 = m\angle 1 = 112.5°$

$m\angle 4 = m\angle 2 = 67.5°$

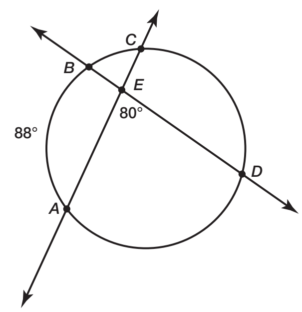
The sum of the two unknown arcs is $360° - 120° - 105° = 135°$.

- Angle EXT is what kind of angle, with respect to the circle?
- To determine the measure of angle EXT, which arc measures are needed?
- Knowing the measure of $\angle ERT$ helps to determine which minor arc?
- Knowing the measure of the minor arc helps to determine the measure of which other arc?

c. Determine $m\overset{\frown}{CD}$.

$m\overset{\frown}{AB} = 88°$

$m\angle AED = 80°$



$m\angle BEA = 180° - 80° = 100°$

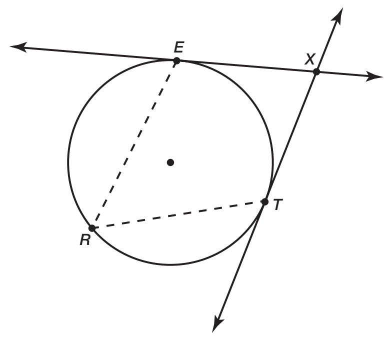
$m\angle BEA = \frac{1}{2}(88° + m\overset{\frown}{CD})$

$100° = \frac{1}{2}(88° + m\overset{\frown}{CD})$

$200° = 88° + m\overset{\frown}{CD}$

$m\overset{\frown}{CD} = 112°$

d. Explain how knowing $m\angle ERT$ can help you determine $m\angle EXT$.

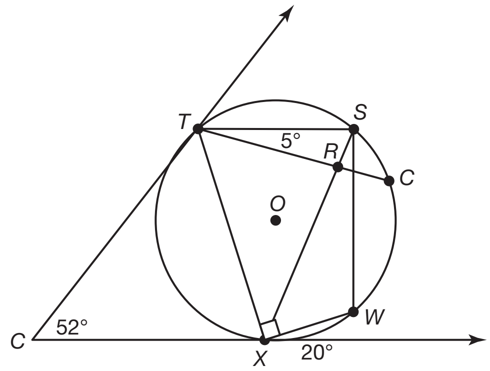


If I know the measure of inscribed $\angle ERT$, that will give me the measure of arc ET, because the measure of an inscribed angle is one-half the measure of its intercepted arc. Then, knowing the measure of arc ET, I can subtract that measure from 180° to get the measure of major arc ERT. Finally, I can use the measures of arcs ERT and ET to calculate the measure of $\angle EXT$.

Be prepared to share your solutions and methods.

Check for Students' Understanding

What additional information can you conclude about the measures of the arcs and angles in the diagram? Explain.



- The measure of $\overparen{TSW}$ is 180°, because the measure of the inscribed angle with this intercepted arc is a right angle and the measure of an inscribed angle is equal to half the measure of its intercepted arc.
- The measure of $\overparen{TWX}$ is 200°, using the Arc Addition Postulate.
- The measure of $\overparen{TX}$ is 96°, because the measure of an exterior angle is half the difference of the measure of its intercepted arcs.
- The measure of $\overparen{SC}$ is 10°, because the measure of the inscribed angle with this intercepted arc is 5° and the measure of an inscribed angle is equal to half the measure of its intercepted arc.
- The $m\angle TRX = 53°$ because the measure of an interior angle is half the sum of the measure of its intercepted arc and its vertical angles intercepted arc.
- The $m\angle SRC = 53°$ because vertical angles are congruent.
- The $m\angle TRS = 127°$ because it forms a linear pair with $\angle SRC$.
- The $m\angle XRC = 127°$ because it forms a linear pair with $\angle SRC$.
- The $m\angle SXW = 10°$ because the measure of the inscribed angle intercepted arc is 20° and the measure of an inscribed angle is equal to half the measure of its intercepted arc.

9

Color Theory

Chords

9.4

LEARNING GOALS

In this lesson, you will:

- Determine the relationships between a chord and a diameter of a circle.
- Determine the relationships between congruent chords and their minor arcs.
- Prove the Diameter–Chord Theorem.
- Prove the Equidistant Chord Theorem.
- Prove the Equidistant Chord Converse Theorem
- Prove the Congruent Chord–Congruent Arc Theorem.
- Prove the Congruent Chord–Congruent Arc Converse Theorem.
- Prove the Segment–Chord Theorem.

KEY TERMS

- Diameter–Chord Theorem
- Equidistant Chord Theorem
- Equidistant Chord Converse Theorem
- Congruent Chord–Congruent Arc Theorem
- Congruent Chord–Congruent Arc Converse Theorem
- segments of a chord
- Segment–Chord Theorem

ESSENTIAL IDEAS

- The Diameter-Chord Theorem states: "When the diameter of a circle is perpendicular to a chord, the diameter bisects the chord and bisects the arc determined by the chord."
- The Equidistant Chord Theorem states: "When two chords of the same circle or congruent circles are congruent, they are equidistant from the center of the circle."
- The Equidistant Chord Converse Theorem states: "When two chords of the same circle or congruent circles are equidistant from the center of the circle, the chords are congruent."
- The Congruent Chord-Congruent Arc Theorem states: "When two chords of the same circle or congruent circles are congruent, their corresponding arcs are congruent."
- The Congruent Chord-Congruent Arc Converse Theorem states: "When two arcs of the same circle or congruent circles are congruent, their corresponding chords are congruent.
- The Segment-Chord Theorem states: "When two chords in a circle intersect, the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the second chord."

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-C Circles

Understand and apply theorems about circles

2. Identify and describe relationships among inscribed angles, radii, and chords.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

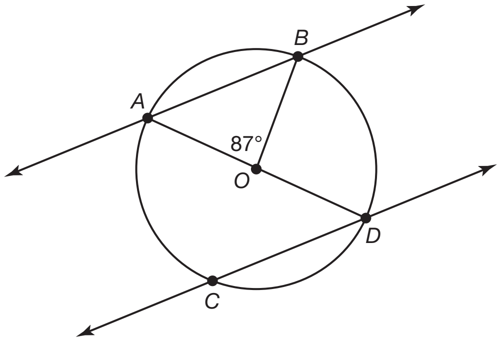
9

Overview

Students explore and prove theorems related to chords of a circle and segments of chords. The converses of two theorems are also proven which enables students to express the theorems as biconditional statements.

Warm Up

1. Determine $m\overset{\frown}{CD}$ and explain how it was calculated.



$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

The $m\angle BOD = 93°$, because $\angle AOB$ and $\angle BOD$ are a linear pair. The $m\overset{\frown}{AB} = 87°$ and $m\overset{\frown}{BD} = 93°$ because a central angle is equal to the measure of its intercepted arc. Parallel lines determine congruent intercepted arcs so $m\overset{\frown}{AC} = m\overset{\frown}{BD}$. The entire circle is 360° so $m\overset{\frown}{CD} = 360° - 93° - 93° - 87° = 87°$.

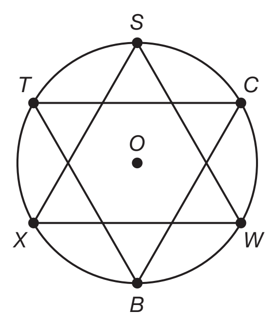
2. Is $\overset{\frown}{AB} \cong \overset{\frown}{CD}$?

Yes, the measure of both arcs is equal to 87°, so the arcs are congruent.

3. If $m\angle AOB$ was changed to 85°, would $\overset{\frown}{AB} \cong \overset{\frown}{CD}$?

Yes, the measure of both arcs is equal to 85°, so the arcs are congruent.

4. Determine $m\angle S + m\angle W + m\angle X$ using what you know about inscribed angles and intercepted arcs.



The $m\angle S + m\angle W + m\angle X = 180°$ because the sum of the intercepted arcs of the three inscribed angles is the entire circle which is equal to 360° and the measure of an inscribed angle is equal to half the measure of its intercepted arc.

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9.4

Color Theory

Chords

9

LEARNING GOALS

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- Prove the Equidistant Chord Theorem.
- Prove the Equidistant Chord Converse Theorem.
- Prove the Congruent Chord–Congruent Arc Theorem.
- Prove the Congruent Chord–Congruent Arc Converse Theorem.
- Prove the Segment–Chord Theorem.

KEY TERMS

- Diameter–Chord Theorem
- Equidistant Chord Theorem
- Equidistant Chord Converse Theorem
- Congruent Chord–Congruent Arc Theorem
- Congruent Chord–Congruent Arc Converse Theorem
- segments of a chord
- Segment–Chord Theorem

Color theory is a set of rules that is used to create color combinations. A color wheel is a visual representation of color theory.

The color wheel is made of three different kinds of colors: primary, secondary, and tertiary. Primary colors (red, blue, and yellow) are the colors you start with. Secondary colors (orange, green, and purple) are created by mixing two primary colors. Tertiary colors (red-orange, yellow-orange, yellow-green, blue-green, blue-purple, red-purple) are created by mixing a primary color with a secondary color.

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Problem 1

Through a series of questions, students conclude that the perpendicular bisector of a chord passes through the center of the circle. Students explore how the diameter of a circle bisects a chord and bisects the arc determined by the chord. They write a conjecture and prove that if the diameter of a circle is perpendicular to a chord, then the diameter bisects the chord and bisects the arc determined by the chord.

Next, students explore the relationship between two congruent chords of the same circle and their equidistance to the center of the circle. They write a conjecture and prove it. Then they prove the converse of the theorem and write the theorems as a biconditional statement.

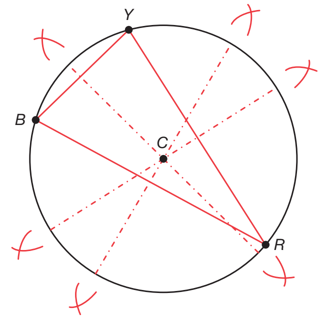
Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

PROBLEM 1 Chords and Diameters

Chords and their perpendicular bisectors lead to several interesting conclusions. In this lesson, we will prove theorems to identify these special relationships.

1. Consider circle C with points B, Y, and R.



   a. Draw chord $\overline{YR}$.

   b. Construct the perpendicular bisector of chord YR.

   c. Draw chord $\overline{BR}$.

   d. Construct the perpendicular bisector of chord BR.

   e. Draw chord $\overline{BY}$.

   f. Construct the perpendicular bisector of chord BY.

   g. What do you notice about the relationship between the perpendicular bisectors of a chord and the center point of the circle?

   The perpendicular bisector of each chord passes through the center point of the circle.

The perpendicular bisector of a chord appears to also bisect the chord's intercepted arc. This observation can be proved and stated as a theorem.

Guiding Questions for Share Phase, Question 1

- Which arc is associated with inscribed $\angle YBR$?
- What appears to be true about the perpendicular bisector of chord YR and arc YR?
- Which arc is associated with inscribed $\angle BRY$?
- What appears to be true about the perpendicular bisector of chord BY and arc BY?
- Which arc is associated with inscribed $\angle BYR$?
- What appears to be true about the perpendicular bisector of chord BR and arc BR?

Guiding Questions for Share Phase, Question 2

- What is the definition of perpendicular lines?
- What is true about the radii of the same circle?
- Do the right triangles share a common side?
- What triangle congruence theorem is helpful in this situation?
- Which corresponding parts of congruent triangles will help show segment *MI* bisected segment *DA*?
- Which corresponding parts of congruent triangles will help show segment *MI* bisected arc *DA*?

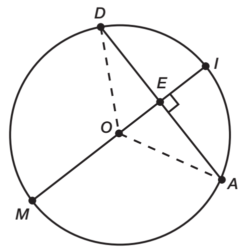
2. Prove the Diameter–Chord Conjecture.

Given: $\overline{MI}$ is a diameter of circle *O*.

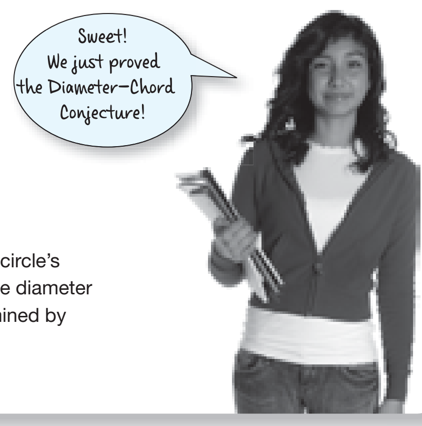
$\overline{MI} \perp \overline{DA}$

Prove: $\overline{MI}$ bisects $\overline{DA}$.

$\overline{MI}$ bisects $\overset{\frown}{DA}$.



Statements	Reasons
1. $\overline{MI}$ is a diameter of circle *O*. $\overline{MI} \perp \overline{DA}$	1. Given
2. Connect points *O* and *D* to form chord $\overline{OD}$. Connect points *O* and *A* to form chord $\overline{OA}$.	2. Construction
3. $\angle OED$ and $\angle OEA$ are right angles.	3. Definition of perpendicular lines
4. $\triangle OED$ and $\triangle OEA$ are right triangles.	4. Definition of right triangles
5. $\overline{OD} \cong \overline{OA}$	5. All radii of the same circle or congruent circles are congruent.
6. $\overline{OE} \cong \overline{OE}$	6. Reflexive Property
7. $\triangle OED \cong \triangle OEA$	7. HL Congruence Theorem
8. $\overline{ED} \cong \overline{EA}$	8. CPCTC
9. $\overline{MI}$ bisects $\overline{DA}$	9. Definition of bisection
10. $\angle DOE \cong \angle AOE$	10. CPCTC
11. $\overset{\frown}{ID} \cong \overset{\frown}{IA}$	11. Congruent central angles of the same circle or congruent circles determine congruent corresponding arcs.
12. $\overline{MI}$ bisects $\overset{\frown}{DA}$	12. Definition of bisection



The **Diameter–Chord Theorem** states: "If a circle's diameter is perpendicular to a chord, then the diameter bisects the chord and bisects the arc determined by the chord."

Grouping

Have students complete Question 3 with a partner. Then have students share their responses as a class.

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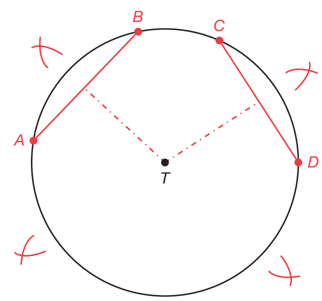
Guiding Questions for Share Phase, Question 3

- How do you know the chords you have drawn are congruent?
- Should the perpendicular bisector of each chord pass through the center point?
- Does the perpendicular bisector of each chord pass through the center point?
- Do the perpendicular bisectors of the two chords intersect at the center point?
- How can a compass be used to show the two chords are equidistant from the center point?
- Do the chords appear to be equidistant from the center point?

3. Use circle T to draw two congruent chords that are not parallel to each other and do not pass through the center point of the circle.



a. Construct the perpendicular bisector of each chord.

b. Use your compass to compare the distance each chord is from the center point of the circle.

Congruent chords appear to be equidistant from the center point of the circle. This observation can be proved and stated as a theorem.

Grouping

Have students complete Questions 4 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 6

- Which set of triangles can be used to show $OE = OI$?
- Is there more than one set of triangles that can be used to show $OE = OI$?
- Triangles COH and DOR can be proven congruent using which triangle congruency theorem?
- Triangles OEH and OID can be proven congruent using which triangle congruency theorem?
- Which set of triangles can be used to show $CH = DR$?
- Is there more than one set of triangles that can be used to show $CH = DR$?
- Triangles OHE, OCE, ODI, and ORI can be proven congruent using which triangle congruency theorem?
- How is the Segment Addition Postulate helpful when proving this theorem?
- What phrase changes a conditional statement into a biconditional statement?

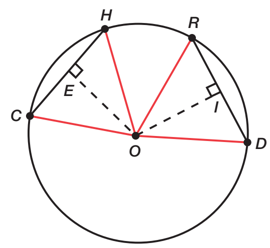
4. Prove the Equidistant Chord Conjecture.

Given: $\overline{CH} \cong \overline{DR}$

$\overline{OE} \perp \overline{CH}$

$\overline{OI} \perp \overline{DR}$

Prove: $\overline{CH}$ and $\overline{DR}$ are equidistant from the center point.



Statements	Reasons
1. $\overline{CH} \cong \overline{DR}$ $\overline{OE} \perp \overline{CH}$ $\overline{OI} \perp \overline{DR}$	1. Given
2. Connect points O and H, O and C, O and D, O and R to form radii OH, OC, OD, and OR, respectively.	2. Construction
3. $\overline{OH} \cong \overline{OC} \cong \overline{OD} \cong \overline{OR}$	3. All radii of the same circle are congruent.
4. $\triangle COH \cong \triangle DOR$	4. SSS Congruence Theorem
5. $\angle D \cong \angle H$	5. CPCTC
6. $\angle OEH \cong \angle OID$ are right angles.	6. Definition of perpendicular lines
7. $\angle OEH \cong \angle OID$	7. All right angles are congruent.
8. $\triangle OEH \cong \triangle OID$	8. AAS Congruence Theorem
9. $\overline{OE} \cong \overline{OI}$	9. CPCTC
10. $OE = OI$	10. Definition of congruent segments
11. $\overline{CH}$ and $\overline{DR}$ are equidistant from the center point.	11. Definition of equidistance

The **Equidistant Chord Theorem** states: "If two chords of the same circle or congruent circles are congruent, then they are equidistant from the center of the circle."

9

The **Equidistant Chord Converse Theorem** states: "If two chords of the same circle or congruent circles are equidistant from the center of the circle, then the chords are congruent."

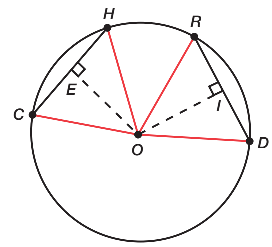
5. Prove the Equidistant Chord Converse Theorem.

Given: $OE = OI$ ($\overline{CH}$ and $\overline{DR}$ are equidistant from the center point.)

$\overline{OE} \perp \overline{CH}$

$\overline{OI} \perp \overline{DR}$

Prove: $\overline{CH} \cong \overline{DR}$



Statements	Reasons
1. $OE = OI$ $\overline{OE} \perp \overline{CH}$ $\overline{OI} \perp \overline{DR}$	**1.** Given
2. Connect points O and H, O and C, O and D, O and R to form radii $\overline{OH}$, $\overline{OC}$, $\overline{OD}$, and $\overline{OR}$, respectively.	**2.** Construction
3. $\overline{OE} \cong \overline{OI}$	**3.** Definition of congruent segments
4. $\overline{OH} \cong \overline{OC} \cong \overline{OD} \cong \overline{OR}$	**4.** All radii of the same circle are congruent.
5. $\angle OEH$, $\angle OEC$, $\angle OID$, and $\angle OIR$ are right angles.	**5.** Definition of perpendicular lines
6. $\triangle OHE$, $\triangle OCE$, $\triangle ODI$, and $\triangle ORI$ are right triangles.	**6.** Definition of right triangle
7. $\triangle OHE \cong \triangle OCE \cong \triangle ODI \cong \triangle ORI$	**7.** HL Congruence Theorem
8. $\overline{HE} \cong \overline{EC} \cong \overline{DI} \cong \overline{IR}$	**8.** CPCTC
9. $HE = EC = DI = IR$	**9.** Definition of congruent segments
10. $HE + EC = CH$ $DI + IR = DR$	**10.** Segment Addition Postulate
11. $CH = DR$	**11.** Substitution Property steps 10 and 11
12. $\overline{CH} \cong \overline{DR}$	**12.** Definition of congruent segments

6. Write the Equidistant Chord Theorem and the Equidistant Chord Converse Theorem as a biconditional statement.

Two chords of the same circle or congruent circles are equidistant from the center of the circle if and only if the chords are congruent.

Problem 2

A piece of broken plate gives students the opportunity to use their knowledge of chords to determine the diameter of the plate. Students first explain how to locate the center of a circle using two chords and their perpendicular bisectors, and then double the length of the radius to determine the length of the diameter.

Grouping

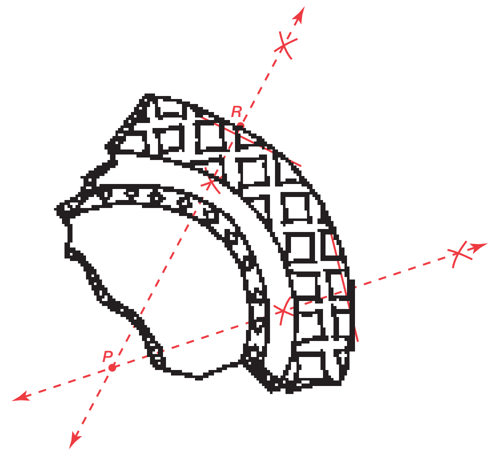
Have students complete the problem with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Problem 2

- Is the piece of plate large enough to contain two chords?
- What location describes the intersection of the perpendicular bisectors of two chords?
- If you can locate the center point of the plate, can you determine the length of the radius and the length of the diameter?

PROBLEM 2 That Darn Kitty!

A neighbor gave you a plate of cookies as a housewarming present. Before you could eat a single cookie, the cat jumped onto the kitchen counter and knocked the cookie plate onto the floor, shattering it into many pieces. The cookie plate will need to be replaced and returned to the neighbor. Unfortunately, cookie plates come in various sizes and you need to know the exact diameter of the broken plate. It would be impossible to reassemble all of the broken pieces, but one large chunk has remained intact as shown.



You think that there has to be an easy way to determine the diameter of the broken plate. As you sit staring at the large piece of the broken plate, your sister Sarah comes home from school. You update her on the latest crisis, and she begins to smile. Sarah tells you not to worry because she learned how to solve for the diameter of the plate in geometry class today. She gets a piece of paper, a compass, a straightedge, a ruler, and a marker out of her backpack and says, "Watch this!"

What does Sarah do? Describe how she can determine the diameter of the plate with the broken piece. Then, show your work on the broken plate shown.

First, Sarah places the broken plate on a piece of paper. Using the marker and a straightedge, she draws two chords on the broken chunk of plate. Then, she constructs the perpendicular bisector of each chord. The point at which the two perpendicular bisectors intersect is the center of the plate. Now that Sarah knows where the center is, she uses the ruler to measure the distance from the point of intersection (*P*) to any point on the edge of the plate (*R*) to determine the radius of the plate. Finally, she doubles the radius to determine the diameter of the plate.

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Problem 3

Students write a conjecture about congruent chords in the same circle or congruent circles determining congruent arcs and prove it. Then they prove the converse of the theorem and write the theorems as a biconditional statement.

Grouping

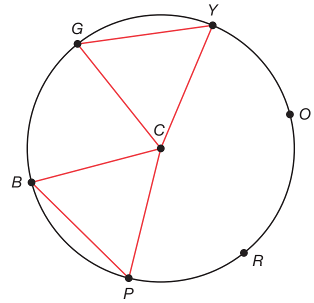
Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- What are the primary colors?
- What are the secondary colors?
- What are the names of the central angles associated with the two chords?
- What are the names of the arcs associated with the two chords?
- If the central angles are congruent, is that enough information to determine the arcs congruent?
- If the central angles are congruent, is that enough information to determine the chords congruent?

PROBLEM 3 Chords and Arcs

1. Consider circle C shown.



a. Draw two congruent chords.

Answers will vary.

I drew congruent chords $\overline{GY}$ and $\overline{BP}$.

b. Draw four radii by connecting the endpoints of each chord with the center point of the circle.

The two central angles formed by each pair of radii appear to be congruent; therefore, the minor arcs associated with each central angle are also congruent.

This observation can be proved and stated as a theorem.

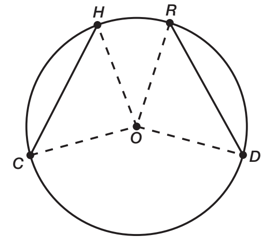
Grouping

Have students complete Question 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- What are the names of the triangles formed by the radii?
- Which triangle congruence theorem is helpful in this situation?
- Which corresponding parts of the congruent triangles would be helpful in proving this theorem?

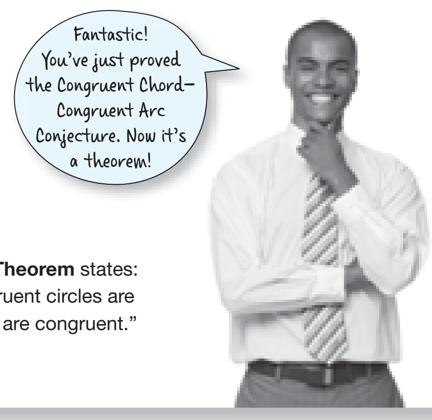
2. Prove the Congruent Chord–Congruent Arc Theorem.



Given: $\overline{CH} \cong \overline{DR}$

Prove: $\overset{\frown}{CH} \cong \overset{\frown}{DR}$

Statements	Reasons
1. $\overline{CH} = \overline{DR}$	1. Given
2. Connect points *O* and *H*, *O* and *C*, *O* and *D*, *O* and *R* to form radii *OH*, *OC*, *OD*, and *OR*, respectively.	2. Construction
3. $\overline{OH} \cong \overline{OC} \cong \overline{OD} \cong \overline{OR}$	3. All radii of the same circle are congruent.
4. $\triangle COH \cong \triangle DOR$	4. SSS Congruence Theorem
5. $\angle COH \cong \angle DOR$	5. CPCTC
6. $\overset{\frown}{CH} \cong \overset{\frown}{DR}$	6. Congruent central angles of the same circle or congruent circles determine congruent arcs.



The **Congruent Chord–Congruent Arc Theorem** states: "If two chords of the same circle or congruent circles are congruent, then their corresponding arcs are congruent."

Grouping

Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

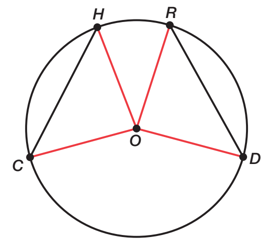
9

Guiding Questions for Share Phase, Questions 3 and 4

- What are the names of the triangles formed by the radii?
- Which triangle congruence theorem is helpful in this situation?
- Which corresponding parts of the congruent triangles would be helpful in proving this theorem?

The **Congruent Chord–Congruent Arc Converse Theorem** states: "If two arcs of the same circle or congruent circles are congruent, then their corresponding chords are congruent."

3. Prove the Congruent Chord–Congruent Arc Converse Theorem.



Given: $\overset{\frown}{CH} \cong \overset{\frown}{DR}$

Prove: $\overline{CH} \cong \overline{DR}$

Statements	Reasons
1. $\overset{\frown}{CH} \cong \overset{\frown}{DR}$	1. Given
2. Connect points O and H, O and C, O and D, and O and R to form radii OH, OC, OD, and OR, respectively.	2. Construction
3. $\overline{OH} \cong \overline{OC} \cong \overline{OD} \cong \overline{OR}$	3. All radii of the same circle are congruent.
4. $\angle COH \cong \angle DOR$	4. Congruent minor arcs of the same circle or congruent circles determine congruent central angles.
5. $\triangle COH \cong \triangle DOR$	5. SAS Congruence Theorem
6. $\overline{CH} \cong \overline{DR}$	6. CPCTC

4. Write the Congruent Chord–Congruent Arc Theorem and the Congruent Chord-Congruent Arc Converse Theorem as a biconditional statement.

Two chords of the same circle or congruent circles are congruent if and only if their corresponding arcs are congruent.

Problem 4

Students write and prove a conjecture stating that if two chords in a circle intersect, then the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the second chord. As a class, write and prove this conjecture.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

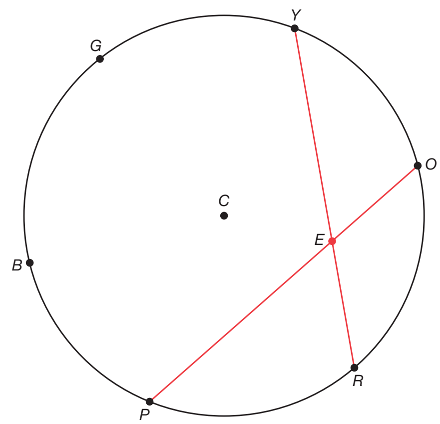
Guiding Questions for Share Phase, Question 1

- What are the names of the two chords?
- What are the lengths of the two segments on the first chord?
- What are the lengths of the two segments on the second chord?
- What is the product of the lengths of the two segments on the first chord?
- What is the product of the lengths of the two segments on the second chord?
- What is the relationship between the two products?

PROBLEM 4 Segments on Chords

Segments of a chord are the segments formed on a chord when two chords of a circle intersect.

1. Consider circle *C*.



a. Draw two intersecting chords such that one chord connects two primary colors and the second chord connects to secondary colors.

b. Label the point at which the two chords intersect point *E*.

c. Use a ruler to measure the length of each segment on the two chords.

The product of the lengths of the segments on the first chord appears to be equal to the product of the lengths of the segments on the second chord.

This observation can be proved and stated as a theorem.

Guiding Questions for Share Phase, Question 2

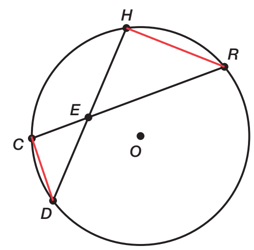
- What are the names of the two triangles formed by drawing the two chords?
- What arc is associated with inscribed $\angle C$?
- What arc is associated with inscribed $\angle H$?
- What is the relationship between $\angle C$ and $\angle H$?
- What is the relationship between $\angle CED$ and $\angle HER$?
- What triangle similarity theorem is helpful when determining the two triangles similar?
- Which proportion is helpful when proving this theorem?

9

2. Prove the Segment-Chord Conjecture.

Given: Chords HD and RC intersect at point E in circle O.

Prove: $EH \cdot ED = ER \cdot EC$

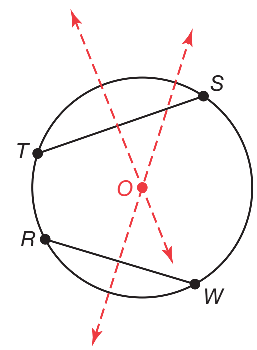


Statements	Reasons
1. Chords HD and RC intersect at point E in circle O.	1. Given
2. Connect points C and D to form $\triangle CED$. Connect points H and R to form $\triangle HER$.	2. Construction
3. $m\angle C = \frac{1}{2}m\overset{\frown}{DR}$ $m\angle H = \frac{1}{2}m\overset{\frown}{DR}$	3. Inscribed Angle Theorem
4. $m\angle C \cong m\angle H$	4. Substitution Property step 3
5. $\angle C \cong \angle H$	5. Definition of congruent angles
6. $\angle CED \cong \angle HER$	6. Vertical Angle Theorem
7. $\triangle CED \sim \triangle HER$	7. AA Similarity Postulate
8. $\frac{EC}{EH} = \frac{ED}{ER}$	8. Corresponding sides of similar triangles are proportional.
9. $EH \cdot ED = ER \cdot EC$	9. Multiplication

The **Segment–Chord Theorem** states that "if two chords in a circle intersect, then the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the second chord."

Be prepared to share your solutions and methods.

Check for Students' Understanding



1. Use a compass, a straightedge and chords *TS* and *RW* to locate the center of the circle. Which theorem did you use?

   Construct the perpendicular bisector of each chord. The intersection of the two perpendicular bisectors is the center of the circle.

2. If chord *TS* and chord *RW* are congruent, what additional information does this tell you?

   If the chords are congruent, then arc *TS* and arc *RW* are congruent arcs and chord *TS* and chord *RW* are equidistant from the center of the circle.

3. If chord *TS* and chord *RW* are not congruent, what additional information does this tell you?

   If the chords are not congruent, then arc *TS* and arc *RW* are not congruent arcs and chord *TS* and chord *RW* are not equidistant from the center of the circle.

4. Connect points *T* and *W* to form chord *TW*. Connect points *R* and *S* to form chord *RS*. Label the point at which these new chords intersect point *X*. What do you know about the length of the segments on chords *RS* and *TW*?

   The length of segment *TX* times the length of segment *WX* equals the length of segment *RX* times the length of segment *SX*.

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9.5

Solar Eclipses

Tangents and Secants

LEARNING GOALS

In this lesson, you will:

- Determine the relationship between a tangent line and a radius.
- Determine the relationship between congruent tangent segments.
- Prove the Tangent Segment Theorem.
- Prove the Secant Segment Theorem.
- Prove the Secant Tangent Theorem.

KEY TERMS

- tangent segment
- Tangent Segment Theorem
- secant segment
- external secant segment
- Secant Segment Theorem
- Secant Tangent Theorem

ESSENTIAL IDEAS

- A tangent segment is a line segment formed by connecting a point outside of the circle to a point of tangency.
- A secant segment is a line segment formed when two secants intersect outside a circle. It begins at the point at which the two secants intersect, continues into the circle, and ends at the point at which the secant exits the circle.
- An external secant segment is the portion of each secant segment that lies on the outside of the circle. It begins at the point at which the two secants intersect and ends at the point where the secant enters the circle.
- The Tangent Segment Theorem states: "Two tangent segments drawn from the same point on the exterior of a circle are congruent."
- The Secant Segment Theorem states: "If two secants intersect in the exterior of a circle, then the product of the lengths of the secant segment and its external secant segment is equal to the product of the lengths of the second secant segment and its external secant segment."
- The Secant Tangent Theorem states: "If a tangent and a secant intersect in the exterior of a circle, then the product of the lengths of the secant segment and its external secant segment is equal to the square of the length of the tangent segment."

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-C Circles

Understand and apply theorems about circles

4. Construct a tangent line from a point outside a given circle to the circle.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

The terms tangent segment, secant segment, and external secant segment are introduced. Students explore and prove theorems related to tangents and secants of a circle.

9

Warm Up

1. If the measures of two angles are equal, are the angles congruent?

Yes, that is the definition of congruent angles.

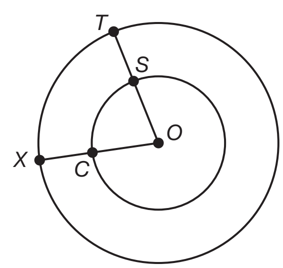
2. If the measures of line segments are equal, are the line segments congruent?

Yes, that is the definition of congruent line segments.

3. If the measures of two arcs are equal, are the arcs congruent?

No, two arcs that are congruent must have the same length. If the measures of the arcs are the same, it does not imply the length of the arcs are the same unless the arcs are of the same circle or congruent circles.

4. Alicia explains to her geometry partner that $\overset{\frown}{SC}$ is congruent to $\overset{\frown}{TX}$. How did Alicia arrive at this conclusion? Is Alicia correct? Explain.



Alicia is not correct. She thinks that because both arcs have the same central angle, and the central angle determines the measure of the arc, the arcs must be congruent. But this is only true when the circles are the same circle or congruent circles. That is not the case in this diagram. The radii of each circle are different lengths. The arcs may be the same measure but that does not imply that the arcs are congruent.

9

9

9

9.5

Solar Eclipses

Tangents and Secants

LEARNING GOALS

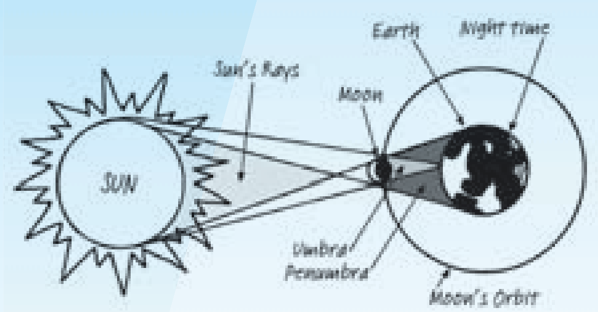
In this lesson, you will:

- Determine the relationship between a tangent line and a radius.
- Determine the relationship between congruent tangent segments.
- Prove the Tangent Segment Theorem.
- Prove the Secant Segment Theorem.
- Prove the Secant Tangent Theorem.

KEY TERMS

- tangent segment
- Tangent Segment Theorem
- secant segment
- external secant segment
- Secant Segment Theorem
- Secant Tangent Theorem

Total solar eclipses occur when the moon passes between Earth and the sun. The position of the moon creates a shadow on the surface of Earth.



A pair of tangent lines forms the boundaries of the umbra, the lighter part of the shadow. Another pair of tangent lines forms the boundaries of the penumbra, the darker part of the shadow.

Problem 1

Students follow the steps to construct a tangent line to a circle through a point outside of the circle. A compass and straightedge are needed.

9

Grouping

Have students complete Steps 1 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Problem 1

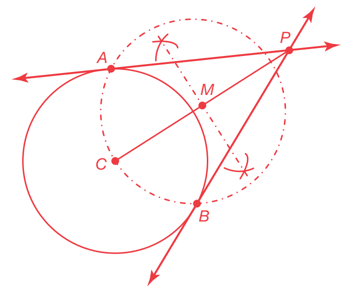
How many lines can be constructed through point *P* tangent to circle *C*?

PROBLEM 1 Constructing a Line Tangent to a Circle

Previously, you proved that when a tangent line is drawn to a circle, a radius of the circle drawn to the point of tangency is perpendicular to the tangent line. This lesson focuses on tangent lines drawn to a circle from a point outside the circle.

Follow these steps to construct a tangent line to a circle through a point outside of the circle.

Step 1: Draw a circle with center point *C* and locate point *P* outside of the circle.



Step 2: Draw line segment *PC*.

Step 3: Construct the perpendicular bisector of line segment *PC*.

Step 4: Label the midpoint of the perpendicular bisector of line segment *PC* point *M*.

Step 5: Adjust the radius of your compass to the distance from point *M* to point *C*.

Step 6: Place the compass point on point *M*, and cut two arcs that intersect circle *C*.

Step 7: Label the two points at which the arcs cut through circle *C* point *A* and point *B*.

Step 8: Connect point *P* and *A* to form tangent line *PA* and connect point *P* and *B* to form tangent line *PB*.

Line *PA* and line *PB* are tangent to circle *C*.

Problem 2

The scenario is about a total solar eclipse. The Sun and the Earth are connected by line segments tangent to both circular objects. The term tangent segment is defined and students use a compass to show that two tangent segments drawn to the same circle from the same point outside of the circle are congruent. Students prove and apply the Tangent Segment Theorem to determine unknown measures.

Grouping

- Discuss the information and diagram above Question 1 as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

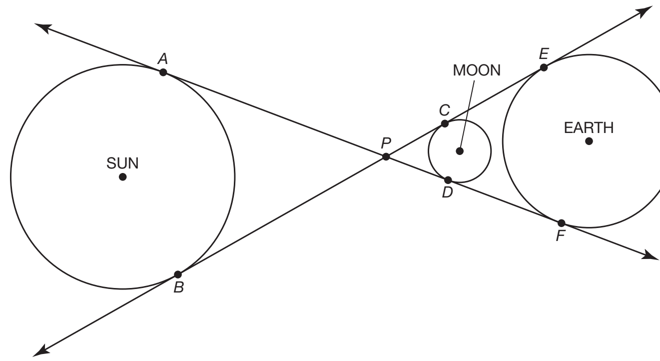
Guiding Questions for Share Phase, Questions 1 through 3

- Tangent segments *PA* and *PB* are associated with which heavenly body?
- Is the length of tangent segment *PA* equal to, less than, or greater than the length of tangent segment *PB*?
- Tangent segments *PC* and *PD* are associated with which heavenly body?

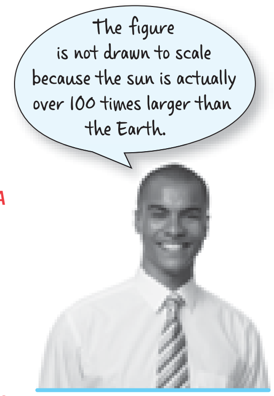
PROBLEM 2 Tangent Segments

For the purposes of the problem situation, the Moon, the Sun, and Earth are represented by circles of different sizes.

Consider point *P* located outside of the Moon, Earth, and the Sun. Lines *AF* and *BE* are drawn tangent to the Moon, Earth, and the Sun as shown.



A **tangent segment** is a line segment formed by connecting a point outside of the circle to a point of tangency.



1. Identify the two tangent segments drawn from point *P* associated with the Sun. Then, use a compass to compare the length of the two segments.

   The tangent segments associated with the Sun are line segments *PA* and *PB*. The length of line segment *PA* is equal to the length of line segment *PB*.

2. Identify the tangent segments drawn from point *P* associated with the Moon. Then, use a compass to compare the length of the two line segments.

   The tangent segments associated with the Moon are line segments *PC* and *PD*. The length of line segment *PC* is equal to the length of line segment *PD*.

3. Identify the tangent segments drawn from point *P* associated with the Earth. Then, use a compass to compare the length of the two line segments.

   The tangent segments associated with the Earth are line segments *PE* and *PF*. The length of line segment *PE* is equal to the length of line segment *PF*.

- Is the length of tangent segment *PC* equal to, less than, or greater than the length of tangent segment *PD*?
- Tangent segments *PE* and *PF* are associated with which heavenly body?
- Is the length of tangent segment *PE* equal to, less than, or greater than the length of tangent segment *PF*?

Grouping

Have students complete Question 4 with a partner. Then have students share their responses as a class.

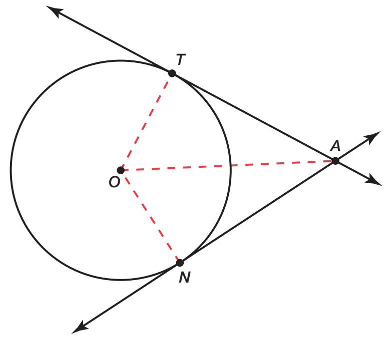
Guiding Questions for Share Phase, Question 4

- Connecting which segments in the diagram help to form two triangles?
- Do the triangles share a common side? Which side?
- Are the triangles right triangles? How do you know?
- Which triangle congruency theorem is helpful when proving the two right triangles congruent?

It appears that two tangent segments drawn to the same circle from the same point outside of the circle are congruent.

This observation can be proved and stated as a theorem.

4. Prove the Tangent Segment Conjecture.



Given: $\overleftrightarrow{AT}$ is tangent to circle O at point T.
$\overleftrightarrow{AN}$ is tangent to circle O at point N.

Prove: $\overline{AT} \cong \overline{AN}$

Statements	Reasons
1. $\overleftrightarrow{AT}$ is tangent to circle O at point T. $\overleftrightarrow{AN}$ is tangent to circle O at point N.	1. Given
2. Connect points O and T to form radius $\overline{OT}$. Connect points O and N to form radius $\overline{ON}$. Connect points O and A to form $\overline{OA}$.	2. Construction
3. $\overline{OT} \cong \overline{ON}$	3. Radii of the same circle are congruent.
4. $\overline{ON} \perp \overleftrightarrow{AN}$ $\overline{OT} \perp \overleftrightarrow{AT}$	4. A tangent is perpendicular to a radius drawn to the point of tangency.
5. $\angle ONA$ and $\angle OTA$ are right angles.	5. Definition of perpendicular lines
6. $\triangle ONA$ and $\triangle OTA$ are right triangles.	6. Definition of right triangles
7. $\overline{OA} \cong \overline{OA}$	7. Reflexive Property
8. $\triangle ONA \cong \triangle OTA$	8. HL Congruency Theorem
9. $\overline{AT} \cong \overline{AN}$	9. CPCTC

The **Tangent Segment Theorem** states: "If two tangent segments are drawn from the same point on the exterior of a circle, then the tangent segments are congruent.

Grouping

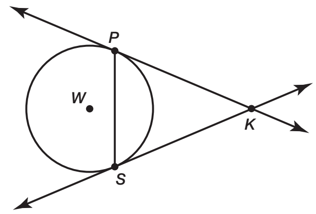
Have students complete Questions 5 and 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 and 6

- Is triangle KPS an isosceles triangle? How do you know?
- Is $\angle KPS$ congruent to $\angle KSP$? How do you know?
- Which theorems are helpful in determining $m\angle KPS$?

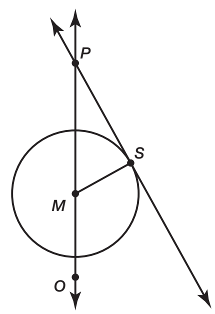
9

5. In the figure, $\overleftrightarrow{KP}$ and $\overleftrightarrow{KS}$ are tangent to circle W and $m\angle PKS = 46°$. Calculate $m\angle KPS$. Explain your reasoning.



Because the tangents $\overline{KP}$ and $\overline{KS}$ are both drawn from point K outside the circle, $\overline{KP} \cong \overline{KS}$. So, $\triangle KPS$ is isosceles. This means that $m\angle KPS = m\angle KSP$. Because $m\angle PKS = 46°$ and $m\angle PKS + m\angle KPS + m\angle KSP = 180°$, it follows that $2m\angle KPS = 180° - 46° = 134°$. So, $m\angle KPS = 134° \div 2 = 67°$.

6. In the figure, $\overleftrightarrow{PS}$ is tangent to circle M and $m\angle SMO = 119°$. Calculate $m\angle MPS$. Explain your reasoning.



Because $\overleftrightarrow{PS}$ is tangent to circle M and $\overline{SM}$ is a radius, $\overleftrightarrow{PS} \perp \overline{SM}$. So, $m\angle MSP = 90°$. In $\triangle MSP$, $m\angle MPS + m\angle MSP = m\angle SMO$ because $\angle SMO$ is an exterior angle of $\triangle MPS$ and $\angle MPS$ and $\angle MSP$ are nonadjacent interior angles. So $m\angle MPS + 90° = 119°$. Therefore, $m\angle MPS = 29°$.

Problem 3

A secant segment and an external secant segment are defined. Students prove the Secant Segment Theorem, which states that when two secants intersect in the exterior of a circle, the product of the lengths of the secant segment and its external secant segment is equal to the product of the lengths of the second secant segment and its external secant segment.

Students also prove the Secant Tangent Theorem, which states that when a tangent and a secant intersect in the exterior of a circle, the product of the lengths of the secant segment and its external secant segment is equal to the square of the length of the tangent segment.

9

Grouping

- Discuss the information and complete Question 1 as a class.
- Have students complete Question 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Discuss Phase, Question 1

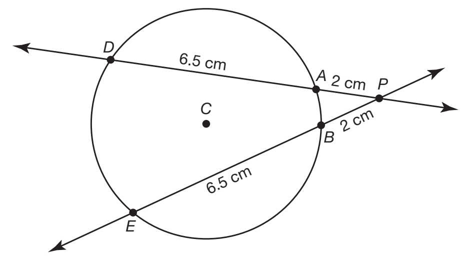
- If $m\angle DPE$ increased, would the external secant segment become longer or shorter? Why?
- If $m\angle DPE$ decreased, would the external secant segment become longer or shorter? Why?

PROBLEM 3 Secant Segments

A **secant segment** is the line segment formed when two secants intersect outside a circle. A secant segment begins at the point at which the two secants intersect, continues into the circle, and ends at the point at which the secant exits the circle.

An **external secant segment** is the portion of each secant segment that lies on the outside of the circle. It begins at the point at which the two secants intersect and ends at the point where the secant enters the circle.

1. Consider circle C with the measurements as shown.



The vertex of $\angle DPE$ is located outside of circle C. Because this angle is formed by the intersection of two secants, each secant line contains a secant segment and an external secant segment.

a. Identify the two secant segments.

The two secant segments are line segments PD and BE.

b. Identify the two external secant segments.

The two external secant segments are line segments PA and PB.

It appears that the product of the lengths of the segment and its external secant segment is equal to the product of the lengths of the second secant segment and its external secant segment.

This observation can be proved and stated as a theorem.

- As the portion of the secant inside the circle becomes longer, how does this affect the measure of the exterior angle P?
- As the portion of the secant inside the circle becomes shorter, how does this affect the measure of the exterior angle P?
- Do you think the product of the lengths of the secant and its external secant segment will change if the measure of the exterior angle increases?
- Do you think the product of the lengths of the secant and its external secant segment will change if the measure of the exterior angle decreases?

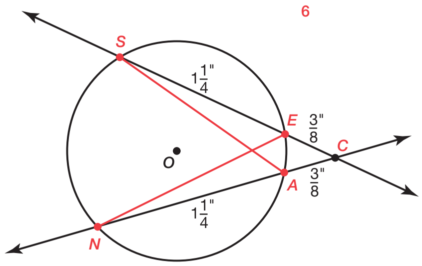
Guiding Questions for Share Phase, Question 2

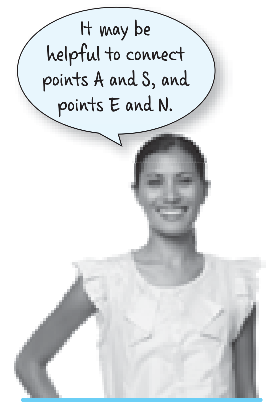
- Which arc is associated with inscribed $\angle S$?
- Which arc is associated with inscribed $\angle N$?
- Proving which set of triangles similar would help to prove this theorem?
- What proportional statement would support the prove statement?

2. Prove the Secant Segment Conjecture.

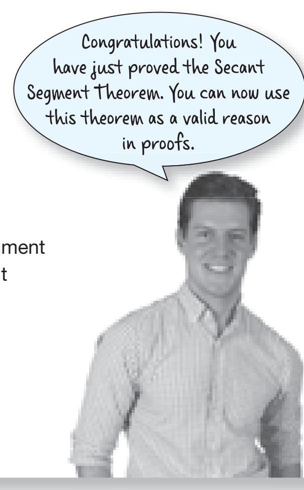
Given: Secants CS and CN intersect at point C in the exterior of circle O.

Prove: $CS \cdot CE = CN \cdot CA$





Statements	Reasons
1. Secants CS and CN intersect at point C in the exterior of circle O.	1. Given
2. Connect points A and S to form $\triangle CAS$. Connect points E and N to form $\triangle CEN$.	2. Construction
3. $m\angle S = \frac{1}{2}m\overparen{AE}$ $m\angle N = \frac{1}{2}m\overparen{AE}$	3. Inscribed Angle Theorem
4. $m\angle S = m\angle N$	4. Substitution Property step 3
5. $\angle S \cong \angle N$	5. Definition of congruent angles
6. $\angle C \cong \angle C$	6. Reflexive Property
7. $\triangle CAS \cong \triangle CEN$	7. AA Similarity Postulate
8. $\frac{CS}{CN} = \frac{CA}{CE}$	8. Corresponding sides of similar triangles are proportional.
9. $CS \cdot CE = CN \cdot CA$	9. Multiplication



The **Secant Segment Theorem** states: "If two secants intersect in the exterior of a circle, then the product of the lengths of the secant segment and its external secant segment is equal to the product of the lengths of the second secant segment and its external secant segment."

Grouping

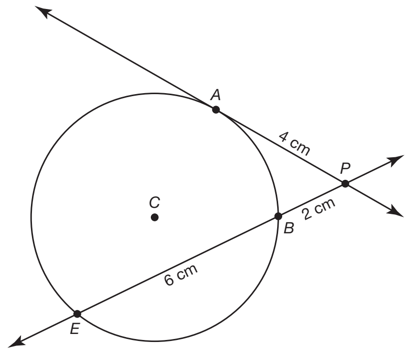
Discuss and complete Question 3 as a class.

Guiding Questions for Discuss Phase, Question 3

9

- What is the length of the secant segment in this diagram?
- What is the length of the external secant segment in this diagram?
- What is the length of the tangent segment in this diagram?
- Does the product of the lengths of the secant segment and its external secant segment equal the length of the tangent segment?
- Does the product of the lengths of the secant segment and its external secant segment equal the square of the length of the tangent segment?

3. Consider circle C with the measurements as shown.



The vertex of $\angle APE$ is located outside of circle C. Because this angle is formed by the intersection of a secant and a tangent, the secant line contains a secant segment and an external secant segment whereas the tangent line contains a tangent segment.

a. Identify the secant segment.

The secant segment is line segment PE.

b. Identify the external secant segment.

The external secant segment is line segment PB.

c. Identify the tangent segment.

The tangent segment is line segment PA.

It appears that the product of the lengths of the segment and its external secant segment is equal to the square of the length of the tangent segment.

This observation can be proved and stated as a theorem.

Grouping

Have students complete Question 4 with a partner. Then have students share their responses as a class.

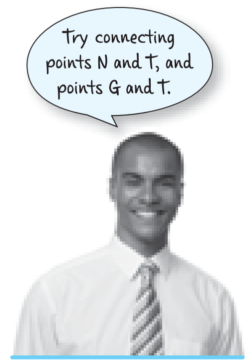
Guiding Questions for Share Phase, Question 4

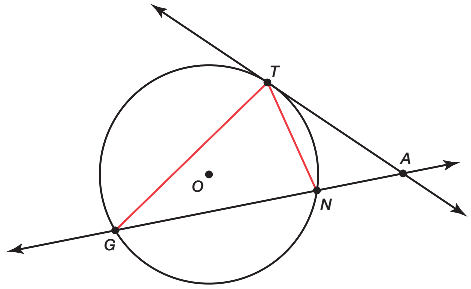
- What angle do triangles *TAN* and *GAT* have in common?
- Which arc is associated with inscribed $\angle G$?
- Which arc is associated with inscribed $\angle TNA$?
- What is the relationship between two inscribed angles sharing the same arc?
- Which triangle similarity theorem is helpful when proving triangle *TAN* and *GAT* similar?
- Which statement of proportionality is helpful in this situation?
- Will the proportionality statement support the prove statement?

4. Prove the Secant Tangent Conjecture.

Given: Tangent *AT* and secant *AG* intersect at point *A* in the exterior of circle *O*.

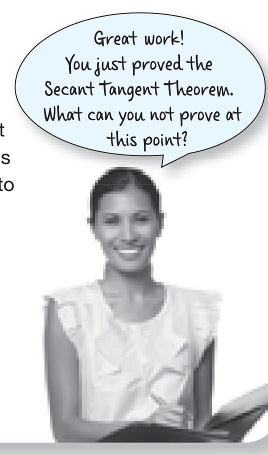
Prove: $(AT)^2 = AG \cdot AN$





Statements	Reasons
1. Tangent *AT* and secant *AG* intersect at point *A* in the exterior of circle *O*.	1. Given
2. Connect points *N* and *T* to form $\triangle TAN$. Connect points *G* and *T* to form $\triangle GAT$.	2. Construction
3. $m\angle G = \frac{1}{2} m\overset{\frown}{TN}$ $m\angle TNA = \frac{1}{2} m\overset{\frown}{TN}$	3. Inscribed Angle Theorem
4. $m\angle G = m\angle TNA$	4. Substitution Property step 3
5. $\angle G \cong \angle TNA$	5. Definition of congruent angles
6. $\angle A \cong \angle A$	6. Reflexive Property
7. $\triangle TAN \cong \triangle GAT$	7. AA Similarity Postulate
8. $\frac{AT}{AG} = \frac{AN}{AT}$	8. Corresponding sides of similar triangles are proportional.
9. $(AT)^2 = AG \cdot AN$	9. Multiplication

The **Secant Tangent Theorem** states: "If a tangent and a secant intersect in the exterior of a circle, then the product of the lengths of the secant segment and its external secant segment is equal to the square of the length of the tangent segment."



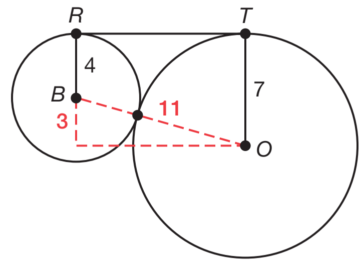
Be prepared to share your solutions and methods.

9

Check for Students' Understanding

9

- Circle B and circle O are tangent circles.
- The length of radius BR is 4.
- The length of radius OT is 7.
- Segment RT is a common tangent.



Calculate the length of segment RT.

Draw line segments to create a rectangle, use the Pythagorean Theorem to solve for the length of segment RT.

$$3^2 + RT^2 = 11^2$$

$$9 + RT^2 = 121$$

$$RT^2 = 112$$

$$RT = \sqrt{112} = 4\sqrt{7} \approx 10.58$$

Chapter 9 Summary

KEY TERMS

- center of a circle (9.1)
- radius (9.1)
- chord (9.1)
- diameter (9.1)
- secant of a circle (9.1)
- tangent of a circle (9.1)
- point of tangency (9.1)
- central angle (9.1)
- inscribed angle (9.1)
- arc (9.1)
- major arc (9.1)
- minor arc (9.1)
- semicircle (9.1)
- degree measure of an arc (9.2)
- adjacent arcs (9.2)
- intercepted arc (9.2)
- segments of a chord (9.4)
- tangent segment (9.5)
- secant segment (9.5)
- external secant segment (9.5)

POSTULATES AND THEOREMS

- Arc Addition Postulate (9.2)
- Inscribed Angle Theorem (9.2)
- Parallel Lines-Congruent Arcs Theorem (9.2)
- Interior Angles of a Circle Theorem (9.3)
- Exterior Angles of a Circle Theorem (9.3)
- Tangent to a Circle Theorem (9.3)
- Diameter-Chord Theorem (9.4)
- Equidistant Chord Theorem (9.4)
- Equidistant Chord Converse Theorem (9.4)
- Congruent Chord-Congruent Arc Theorem (9.4)
- Congruent Chord-Congruent Arc Converse Theorem (9.4)
- Segment-Chord Theorem (9.4)
- Tangent Segment Theorem (9.5)
- Secant Segment Theorem (9.5)
- Secant Tangent Theorem (9.5)

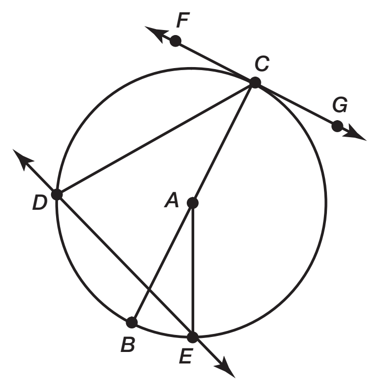
9

9.1 Identifying Parts of a Circle

A circle is the set of all points in a plane that are equidistant from a given point. The following are parts of a circle.

- The center of a circle is a point inside the circle that is equidistant from every point on the circle.
- A radius of a circle is a line segment that is the distance from a point on the circle to the center of the circle.
- A chord is a segment whose endpoints are on a circle.
- A diameter of a circle is a chord across a circle that passes through the center.
- A secant is a line that intersects a circle at exactly two points.
- A tangent is a line that intersects a circle at exactly one point, and this point is called the point of tangency.
- A central angle is an angle of a circle whose vertex is the center of the circle.
- An inscribed angle is an angle of a circle whose vertex is on the circle.
- A major arc of a circle is the largest arc formed by a secant and a circle.
- A minor arc of a circle is the smallest arc formed by a secant and a circle.
- A semicircle is exactly half a circle.

Examples



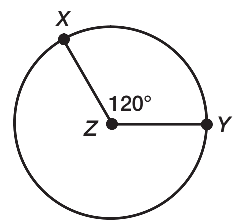
- Point *A* is the center of circle *A*.
- Segments *AB*, *AC*, and *AE* are radii of circle *A*.
- Segment *BC* is a diameter of circle *A*.
- Segments *BC*, *DC*, and *DE* are chords of circle *A*.
- Line *DE* is a secant of circle *A*.
- Line *FG* is a tangent of circle *A*, and point *C* is a point of tangency.
- Angle *BAE* and angle *CAE* are central angles.
- Angle *BCD* and angle *CDE* are inscribed angles.
- Arcs *BDE*, *CDE*, *CED*, *DCE*, and *DCB* are major arcs.
- Arcs *BD*, *BE*, *CD*, *CE*, and *DE* are minor arcs.
- Arc *BDC* and arc *BEC* are semicircles.

9.2 Determining Measures of Arcs

The degree measure of a minor arc is the same as the degree measure of its central angle.

Example

In circle Z, $\angle XZY$ is a central angle measuring $120°$. So, $m\overset{\frown}{XY} = 120°$.

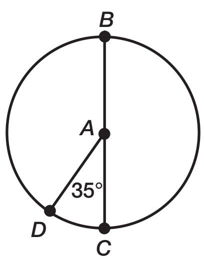


9.2 Using the Arc Addition Postulate

Adjacent arcs are two arcs of the same circle sharing a common endpoint. The Arc Addition Postulate states: "The measure of an arc formed by two adjacent arcs is equal to the sum of the measures of the two arcs."

Example

In circle A, arcs BC and CD are adjacent arcs. So, $m\overset{\frown}{BCD} = m\overset{\frown}{BC} + m\overset{\frown}{CD} = 180° + 35° = 215°$.



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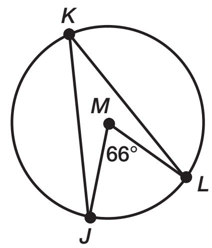
9.2 Using the Inscribed Angle Theorem

The Inscribed Angle Theorem states: “The measure of an inscribed angle is one half the measure of its intercepted arc.”

Example

In circle M, $\angle JKL$ is an inscribed angle whose intercepted arc JL measures 66°.

So, $m\angle JKL = \frac{1}{2}(m\overset{\frown}{JL}) = \frac{1}{2}(66°) = 33°$.

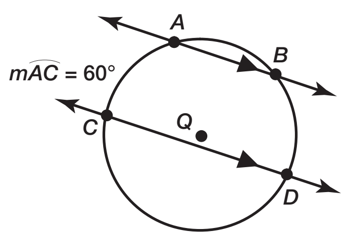


9.2 Using the Parallel Lines–Congruent Arcs Theorem

The Parallel Lines-Congruent Arcs Theorem states: “Parallel lines intercept congruent arcs on a circle.”

Example

Lines AB and CD are parallel lines on circle Q and $m\overset{\frown}{AC} = 60°$. So, $m\overset{\frown}{AC} = m\overset{\frown}{BD}$, and $m\overset{\frown}{BD} = 60°$.

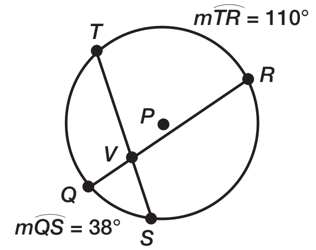


9.3 Using the Interior Angles of a Circle Theorem

The Interior Angles of a Circle Theorem states: "If an angle is formed by two intersecting chords or secants such that the vertex of the angle is in the interior of the circle, then the measure of the angle is half the sum of the measures of the arcs intercepted by the angle and its vertical angle."

Example

In circle P, chords QR and ST intersect to form vertex angle TVR and its vertical angle QVS. So, $m\angle TVR = \frac{1}{2}(m\widehat{TR} + m\widehat{QS}) = \frac{1}{2}(110° + 38°) = \frac{1}{2}(148°) = 74°$.



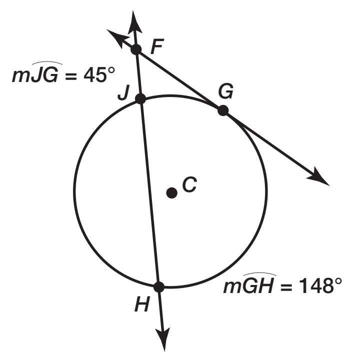
9.3 Using the Exterior Angles of a Circle Theorem

The Exterior Angles of a Circle Theorem states: "If an angle is formed by two intersecting secants, two intersecting tangents, or an intersecting tangent and secant such that the vertex of the angle is in the exterior of the circle, then the measure of the angle is half the difference of the measures of the arc(s) intercepted by the angle."

Example

In circle C, secant FH and tangent FG intersect to form vertex angle GFH.

So, $m\angle GFH = \frac{1}{2}(m\widehat{GH} - m\widehat{JG}) = \frac{1}{2}(148° - 45°) = \frac{1}{2}(103°) = 51.5°$.



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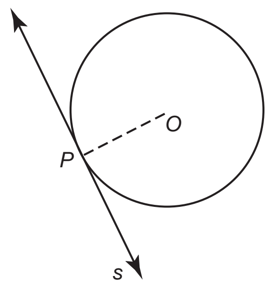
9.3 Using the Tangent to a Circle Theorem

The Tangent to a Circle Theorem states: "A line drawn tangent to a circle is perpendicular to a radius of the circle drawn to the point of tangency."

Example

Radius $\overline{OP}$ is perpendicular to the tangent line s.

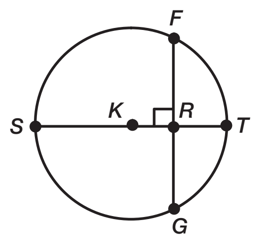


9.4 Using the Diameter-Chord Theorem

The Diameter-Chord Theorem states: "If a circle's diameter is perpendicular to a chord, then the diameter bisects the chord and bisects the arc determined by the chord."

Example

In circle K, diameter $\overline{ST}$ is perpendicular to chord $\overline{FG}$. So $FR = GR$ and $m\overset{\frown}{FT} = m\overset{\frown}{GT}$.



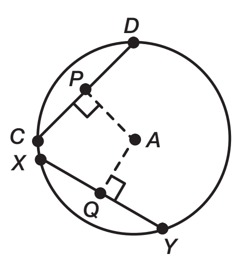
9.4 Using the Equidistant Chord Theorem and the Equidistant Chord Converse Theorem

The Equidistant Chord Theorem states: "If two chords of the same circle or congruent circles are congruent, then they are equidistant from the center of the circle."

The Equidistant Chord Converse Theorem states: "If two chords of the same circle or congruent circles are equidistant from the center of the circle, then the chords are congruent."

Example

In circle A, chord $\overline{CD}$ is congruent to chord $\overline{XY}$. So $PA = QA$.



9.4 Using the Congruent Chord–Congruent Arc Theorem and the Congruent Chord–Congruent Arc Converse Theorem

The Congruent Chord–Congruent Arc Theorem states: "If two chords of the same circle or congruent circles are congruent, then their corresponding arcs are congruent."

The Congruent Chord–Congruent Arc Converse Theorem states: "If two arcs of the same circle or congruent circles are congruent, then their corresponding chords are congruent."

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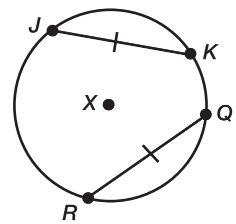
Example

In circle X, chord $\overline{JK}$ is congruent to chord $\overline{QR}$. So $m\overset{\frown}{JK} = m\overset{\frown}{QR}$.



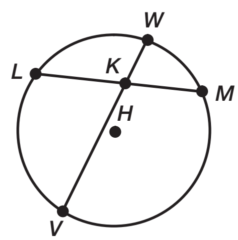
9.4 Using the Segment-Chord Theorem

Segments of a chord are the segments formed on a chord when two chords of a circle intersect.

The Segment-Chord Theorem states: "If two chords in a circle intersect, then the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the second chord."

Example

In circle H, chords $\overline{LM}$ and $\overline{VW}$ intersect to form $\overline{LK}$ and $\overline{MK}$ of chord $\overline{LM}$ and $\overline{WK}$ and $\overline{VK}$ of chord $\overline{VW}$. So $LK \cdot MK = WK \cdot VK$.



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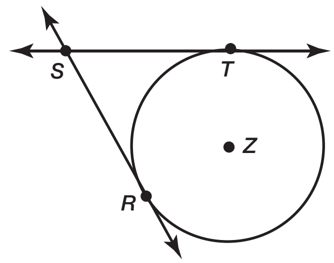
9.5 Using the Tangent Segment Theorem

A tangent segment is a segment formed from an exterior point of a circle to the point of tangency.

The Tangent Segment Theorem states: "If two tangent segments are drawn from the same point on the exterior of a circle, then the tangent segments are congruent."

Example

In circle Z, tangent segments $\overline{SR}$ and $\overline{ST}$ are both drawn from point S outside the circle. So, $SR = ST$.



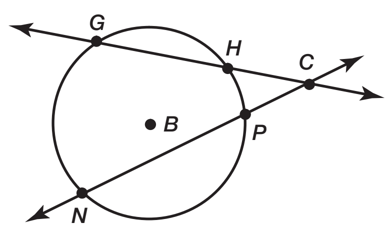
9.5 Using the Secant Segment Theorem

A secant segment is a segment formed when two secants intersect in the exterior of a circle. An external secant segment is the portion of a secant segment that lies on the outside of the circle.

The Secant Segment Theorem states: "If two secant segments intersect in the exterior of a circle, then the product of the lengths of one secant segment and its external secant segment is equal to the product of the lengths of the second secant segment and its external secant segment."

Example

In circle B, secant segments $\overline{GH}$ and $\overline{NP}$ intersect at point C outside the circle. So, $GC \cdot HC = NC \cdot PC$.

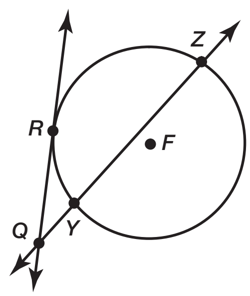


9.5 Using the Secant Tangent Theorem

The Secant Tangent Theorem states: "If a tangent and a secant intersect in the exterior of a circle, then the product of the lengths of the secant segment and its external secant segment is equal to the square of the length of the tangent segment."

Example

In circle F, tangent $\overline{QR}$ and secant $\overline{YZ}$ intersect at point Q outside the circle.
So, $QY \cdot QZ = QR^2$.



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