Trigonometry

8

Chapter 8 Overview

This chapter introduces students to trigonometric ratios using right triangles. Lessons provide opportunities for students to discover and analyze these ratios and solve application problems using them. Students also explore the reciprocals of the basic trigonometric ratios sine, cosine, and tangent, along with their inverses to determine unknown angle measures. Deriving the Law of Sines and the Law of Cosines extends students' understanding of trigonometry to apply to all triangles.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
8.1	Introduction to Trigonometry	G.SRT.3 G.SRT.5 G.SRT.6	1	This lesson provides opportunities for students to explore the three basic ratios of trigonometry using triangle similarity. Questions ask students to measure the corresponding sides of similar right triangles with specific reference angles (45° and 30°) in order to draw the conclusion that the trigonometric ratios are the same given a specific reference angle.	X		X	X	
8.2	Tangent Ratio, Cotangent Ratio, and Inverse Tangent	G.SRT.3 G.SRT.5 G.SRT.6 G.SRT.8 G.MG.1	1	In this lesson, students explore the tangent ratio and connect this with the concept of slope. They also explore cotangent and inverse tangent. Questions ask students to use the ratios to solve for unknown side lengths and angle measures and to solve application problems.	X	X		X	
8.3	Sine Ratio, Cosecant Ratio, and Inverse Sine	G.SRT.8 G.MG.1	1	In this lesson, students explore the sine ratio and the reciprocal of the sine ratio, cosecant. Students also explore the inverse of sine. Questions ask students to use the ratios to solve for unknown side lengths and angle measures and to solve application problems.	X		X	X	

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
8.4	Cosine Ratio, Secant Ratio, and Inverse Cosine	G.SRT.8 G.MG.1	1	In this lesson, students explore the cosine ratio and the reciprocal of the cosine ratio, secant. Students also explore the inverse of cosine. Questions ask students to use the ratios to solve for unknown side lengths and angle measures and to solve application problems.	X			X	
8.5	Complement Angle Relationships	G.SRT.7 G.SRT.8 G.MG.1	1	This lesson focuses on the relationships among the trigonometric ratios studied with an emphasis on complementary reference angles. Questions ask students to analyze complement angle relationships and solve problems using these relationships.	X			X	
8.6	Deriving the Triangle Area Formula, the Law of Sines, and the Law of Cosines	G.SRT.9 G.SRT.10 G.SRT.11 G.MG.1	1	Students derive the Law of Sines and Law of Cosines in this lesson in order to apply trigonometric ratios to any triangle. Students solve problems using the laws and discuss when it is appropriate to use each.	X			X	

Skills Practice Correlation for Chapter 8

Lesson		Problem Set	Objectives
8.1	Introduction to Trigonometry		Vocabulary
		1 – 6	Determine the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ in triangles given a reference angle
		7 – 12	Determine the ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$ in triangles given a reference angle
		13 – 18	Determine the ratios $\frac{\text{opposite}}{\text{hypotenuse}}$, $\frac{\text{adjacent}}{\text{hypotenuse}}$, and $\frac{\text{opposite}}{\text{adjacent}}$ in triangles given a reference angle
		19 – 24	Calculate trigonometric ratios in similar triangles
8.2	Tangent Ratio, Cotangent Ratio, and Inverse Tangent		Vocabulary
		1 – 6	Calculate the tangent of indicated angles in triangles
		7 – 12	Calculate the cotangent of indicated angles in triangles
		13 – 18	Use a calculator to approximate tangent ratios
		19 – 24	Use a calculator to approximate cotangent ratios
		25 – 30	Calculate missing lengths in triangles using tangent and cotangent
		31 – 36	Calculate angle measures using inverse tangent
		37 – 44	Solve problems using tangent, cotangent, and inverse tangent
8.3	Sine Ratio, Cosecant Ratio, and Inverse Sine		Vocabulary
		1 – 6	Calculate the sine of indicated angles in triangles
		7 – 12	Calculate the cosecant of indicated angles in triangles
		13 – 18	Use a calculator to approximate sine ratios
		19 – 24	Use a calculator to approximate cosecant ratios
		25 – 30	Calculate missing lengths in triangles using sine and cosecant
		31 – 36	Calculate angle measures using inverse sine
		37 – 44	Solve problems using sine, cosecant, and inverse sine

Lesson		Problem Set	Objectives
8.4	Cosine Ratio, Secant Ratio, and Inverse Cosine		Vocabulary
		1 – 6	Calculate the cosine of indicated angles in triangles
		7 – 12	Calculate the secant of indicated angles in triangles
		13 – 18	Use a calculator to approximate cosine ratios
		19 – 24	Use a calculator to approximate secant ratios
		25 – 30	Calculate missing lengths in triangles using cosine and secant
		31 – 36	Calculate angle measures using inverse cosine
		37 – 44	Solve problems using cosine, secant, and inverse cosine
8.5	Complement Angle Relationships	1 – 6	Write given trigonometric ratios in two ways
		7 – 12	Determine the trigonometric ratio that can be used to solve for an unknown measure in triangles
		13 – 20	Use trigonometric ratios to solve angle of elevation problems
		21 – 28	Use trigonometric ratios to solve angle of depression problems
8.6	Deriving the Triangle Area Formula, the Law of Sines, and the Law of Cosines		Vocabulary
		1 – 6	Use trigonometry to determine the area of triangles
		7 – 12	Determine unknown side lengths of triangles using the Law of Sines
		13 – 18	Determine given angle measures using the Law of Sines
		19 – 24	Determine unknown side lengths of triangles using the Law of Cosines

8

8.1

Three Angle Measure

Introduction to Trigonometry

LEARNING GOALS

In this lesson, you will:

- Explore trigonometric ratios as measurement conversions.
- Analyze the properties of similar right triangles.

KEY TERMS

- reference angle
- opposite side
- adjacent side

ESSENTIAL IDEAS

- Similar right triangles are formed by dropping vertical line segments from the hypotenuse to the base of right triangles.
- Given the same reference angle, the ratios $\frac{\text{opposite}}{\text{hypotenuse}}$, $\frac{\text{adjacent}}{\text{hypotenuse}}$, and $\frac{\text{opposite}}{\text{adjacent}}$ are constant.
- The ratios $\frac{\text{opposite}}{\text{hypotenuse}}$, $\frac{\text{adjacent}}{\text{hypotenuse}}$, and $\frac{\text{opposite}}{\text{adjacent}}$ are the same for all $45°–45°–90°$ triangles and the same for all $30°–60°–90°$ triangles given the same reference angle.
- The slope of the hypotenuse of a $45°–45°–90°$ triangle and a $30°–60°–90°$ triangle is equal to the opposite-to-adjacent ratio.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-SRT Similarity, Right Triangles, and Trigonometry

Understand similarity in terms of similarity transformations

3. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

Prove theorems involving similarity

5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Define trigonometric ratios and solve problems involving right triangles

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

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Overview

Students drop vertical lines from different points on the hypotenuses of $45°-45°-90°$ and $30°-60°-90°$ triangles to form similar right triangles and determine the lengths of the sides. Then they convert the lengths into ratios and compare them. Students calculate the slope of the hypotenuse and realize it is the same as the opposite-to-adjacent ratio for both the $45°-45°-90°$ triangle and the $30°-60°-90°$ triangle. Students conclude that the ratios they studied are constant in similar right triangles, given the same reference angle. Students also discuss how these ratios change in general as the measure of the reference angle changes.

Warm Up

1. How many centimeters are in 1 meter?

There are 100 centimeters in 1 meter.

2. What ratio equal to 1 represents meters to centimeters?

$$\frac{1 \text{ meter}}{100 \text{ centimeters}}$$

3. Use the ratio written in Question 2 to convert 520 centimeters to meters.

$$\frac{520 \cancel{\text{ centimeters}}}{1} \cdot \frac{1 \text{ meter}}{100 \cancel{\text{ centimeters}}} = \frac{520 \text{ meters}}{100} = 5.2 \text{ meters}$$

8

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8.1

Three Angle Measure

Introduction to Trigonometry

LEARNING GOALS

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- Explore trigonometric ratios as measurement conversions.
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KEY TERMS

- reference angle
- opposite side
- adjacent side

"I've been workin' on the railroad, all the live long day." Can you hear that tune in your head? Can you sing the first two notes? Those two notes are separated by an interval called a perfect fourth. The two notes of a perfect fourth vibrate at different frequencies, and these frequencies are always in the ratio 4 : 3. That is, the higher note of a perfect fourth vibrates about 1.33 times faster than the lower note.

What about the first two notes of "Frosty the Snowman"? Can you sing those? The interval between these notes is called a minor third. The ratio of the higher note frequency to lower note frequency in a minor third is 6 : 5.

All of the intervals in a musical scale are constructed according to specific frequency ratios.

Problem 1

A 45°—45°—90° triangle is presented. Students draw 3 vertical lines from the hypotenuse to the base. Each vertical line forms a new triangle and students determine the lengths of the sides of each right triangle. Students use these measurements to determine ratios among the legs and hypotenuse. Using the 45°—45°—90° triangle, students conclude the ratios are approximately the same. Students discover that the ratios are constant given the same reference angle.

Grouping

- Discuss the information above Question 1 as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What is the length of the base of each right triangle?
- What is the height of each right triangle?
- Are the triangles formed by the vertical lines similar? How do you know?

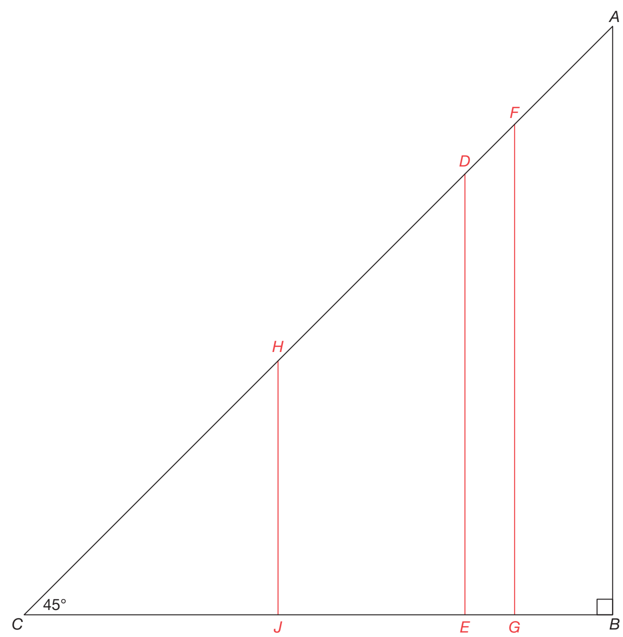
PROBLEM 1 Convert to Trigonometry!

You know that to convert between measurements you can multiply by a conversion ratio. For example, to determine the number of centimeters that is equivalent to 30 millimeters, you can multiply by $\frac{1 \text{ cm}}{10 \text{ mm}}$ because there are 10 millimeters in each centimeter:

$$30 \cancel{\text{mm}} \times \frac{1 \text{ cm}}{10 \cancel{\text{mm}}} = \frac{30 \text{ cm}}{10}$$
$$= 3 \text{ cm}$$

In trigonometry, you use conversion ratios too. These ratios apply to right triangles.

Triangle *ABC* shown is a 45°-45°-90° triangle.



1. Draw a vertical line segment, $\overline{DE}$, connecting the hypotenuse of triangle *ABC* with side $\overline{BC}$. Label the endpoint of the vertical line segment along the hypotenuse as point *D*. Label the other endpoint as point *E*.

   See diagram.

2. Explain how you know that triangle *ABC* is similar to triangle *DEC*.

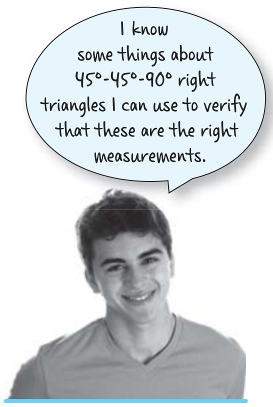
Triangle *ABC* ~ *DEC* by the AA Similarity Theorem.

Angle *C*, which measures 45°, is the same in both triangles, and one pair of corresponding angles each measure 90°.

3. Measure each of the sides of triangles *ABC* and *DEC* in millimeters. Record the approximate measurements.

Measures for triangle *DEC* will vary. Sample measures are shown.

Triangle *ABC*	Triangle *DEC*
Side $\overline{AB}$: 160 mm	Side $\overline{DE}$: 120 mm
Side $\overline{BC}$: 160 mm	Side $\overline{EC}$: 120 mm
Side $\overline{CA}$: 226 mm	Side $\overline{CD}$: 170 mm



You know that the hypotenuse of a right triangle is the side that is opposite the right angle. In trigonometry, the legs of a right triangle are often referred to as the *opposite side* and the *adjacent side*. These references are based on the angle of the triangle that you are looking at, which is called the **reference angle**. The **opposite side** is the side opposite the reference angle. The **adjacent side** is the side adjacent to the reference angle that is *not* the hypotenuse.

4. For triangles *ABC* and *DEC*, identify the opposite side, adjacent side, and hypotenuse, using angle *C* as the reference angle.

Triangle *ABC*

Side $\overline{AB}$ is the opposite side to angle *C*.

Side $\overline{BC}$ is the adjacent side to angle *C*.

The hypotenuse of triangle *ABC* is side $\overline{CA}$.

Triangle *DEC*

Side $\overline{DE}$ is the opposite side to angle *C*.

Side $\overline{EC}$ is the adjacent side to angle *C*.

The hypotenuse of triangle *DEC* is side $\overline{CD}$.

Grouping

- Discuss the information above Question 4 as a class.
- Have students complete Questions 4 through 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 9

- What numbers are you dividing to determine the ratio?
- Are these lengths exact or approximate?
- Are the ratios exact or approximate?
- Are all of the triangles formed by the vertical lines similar to the original triangle? How do you know?
- What ratios are approximately the same? What ratios are different?
- Does the placement of the vertical lines affect the ratios? Why or why not?
- What ratio corresponds to the slope ratio?

5. Determine each side length ratio for triangles *ABC* and *DEC*, using angle *C* as the reference angle. Write your answers as decimals rounded to the nearest thousandth.

a. $\dfrac{\text{side opposite } \angle C}{\text{hypotenuse}}$

Using angle *C* as the reference angle, the ratio of opposite side to hypotenuse for triangle *ABC* is $\frac{160}{226} \approx 0.708$.

Using angle *C* as the reference angle, the ratio of opposite side to hypotenuse for triangle *DEC* is $\frac{120}{170} \approx 0.706$.

b. $\dfrac{\text{side adjacent to } \angle C}{\text{hypotenuse}}$

Using angle *C* as the reference angle, the ratio of adjacent side to hypotenuse for triangle *ABC* is $\frac{160}{226} \approx 0.708$.

Using angle *C* as the reference angle, the ratio of adjacent side to hypotenuse for triangle *DEC*, is $\frac{120}{170} \approx 0.706$.

c. $\dfrac{\text{side opposite } \angle C}{\text{side adjacent to } \angle C}$

Using angle *C* as the reference angle, the ratio of opposite side to adjacent side for triangle *ABC* is $\frac{160}{160} = 1$.

Using angle *C* as the reference angle, the ratio of opposite side to adjacent side for triangle *DEC* is $\frac{120}{120} = 1$.

6. Draw two more vertical line segments, $\overline{FG}$ and $\overline{HJ}$, connecting the hypotenuse of triangle *ABC* with side $\overline{BC}$. Label the endpoints of the vertical line segments along the hypotenuse as points *F* and *H*. Label the other endpoints as points *G* and *J*.

a. Explain how you know that triangles *ABC*, *DEC*, *FGC*, and *HJC* are all similar.

Triangles *ABC*, *DEC*, *FGC*, and *HJC* are all similar by the AA Similarity Theorem. Angle *C*, which measures 45°, is the same in each triangle, and one angle in each triangle measures 90°.

b. Measure each of the sides of the two new triangles you created. Record the side length measurements for all four triangles in the table.

Triangle Name	Length of Side Opposite Angle *C*	Length of Side Adjacent to Angle *C*	Length of Hypotenuse
Triangle *ABC*	160 mm	160 mm	226 mm
Triangle *DEC*	120 mm	120 mm	170 mm
Triangle *FGC*	133 mm	133 mm	188 mm
Triangle *HJC*	69 mm	69 mm	97.5 mm

Measures for triangles *DEC*, *FGC*, and *HJC* will vary.

Sample measures are shown.

c. Determine each side length ratio for all four triangles using angle *C* as the reference angle.

Triangle Name	$\frac{\text{side opposite } \angle C}{\text{hypotenuse}}$	$\frac{\text{side adjacent to } \angle C}{\text{hypotenuse}}$	$\frac{\text{side opposite } \angle C}{\text{side adjacent to } \angle C}$
Triangle *ABC*	$\frac{160}{226} \approx 0.708$	$\frac{160}{226} \approx 0.708$	$\frac{160}{160} = 1$
Triangle *DEC*	$\frac{120}{170} \approx 0.706$	$\frac{120}{170} \approx 0.706$	$\frac{120}{120} = 1$
Triangle *FGC*	$\frac{133}{188} \approx 0.707$	$\frac{133}{188} \approx 0.707$	$\frac{133}{133} = 1$
Triangle *HJC*	$\frac{69}{97.5} \approx 0.708$	$\frac{69}{97.5} \approx 0.708$	$\frac{69}{69} = 1$

7. Compare the side length ratios of all four triangles in the table. What do you notice?

The ratio of the length of the side opposite angle *C* to the length of the hypotenuse is approximately 0.707 for all four triangles.

The ratio of the length of the side adjacent to angle *C* to the length of the hypotenuse is approximately 0.707 for all four triangles.

The ratio of the length of the side opposite angle *C* to the length of the side adjacent to angle *C* is 1 for all four triangles.

8. Compare your measurements and ratios with those of your classmates. What do you notice?

Even though we drew our line segments in different places in the triangle and came up with different measurements for the triangles' sides, the side length ratios in our tables were equal or very close to equal.

9. Calculate the slope of the hypotenuse in each of the four triangles. Explain how you determined your answers.

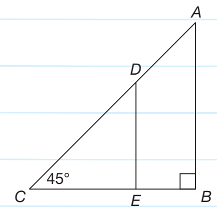
The slope of the hypotenuse in each triangle is the ratio $\frac{\text{rise}}{\text{run}}$. This ratio is the exact same as the ratio $\frac{\text{side opposite } \angle C}{\text{side adjacent to } \angle C}$ for each triangle, given angle *C* as the reference angle. Therefore, the hypotenuse in each of the four triangles has a slope of 1.

Grouping

Discuss the worked example and complete Question 10 as a class.

Given the same reference angle measure, are each of the ratios you studied constant in similar right triangles? You can investigate this question by analyzing similar right triangles without side measurements.

Consider triangles *ABC* and *DEC* shown. They are both 45°-45°-90° triangles.



Triangle *ABC* is similar to triangle *DEC* by the AA Similarity Theorem. This means that the ratios of the corresponding sides of the two triangles are equal.

$$\frac{CE}{CB} = \frac{CD}{CA}$$

Rewrite the proportion.

side adjacent to $\angle C$ → CE ; CB ← side adjacent to $\angle C$

$$\frac{CE}{CD} = \frac{CB}{CA}$$

hypotenuse → CD ; CA ← hypotenuse

So, the ratio $\frac{\text{side adjacent to reference angle}}{\text{hypotenuse}}$ is constant in similar right triangles given the same reference angle measure.

Problem 2

A 30°—60°—90° triangle is presented. As in Problem 1, students draw 3 vertical lines from the hypotenuse to the base. Each vertical line forms a new triangle and students determine the lengths of the sides of each right triangle. Students use these measurements to determine ratios among the legs and hypotenuse. Using the 30°—60°—90° triangle, students conclude the ratios are constant, as in the previous problem, given the same reference angle.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

10. Use triangle *ABC* in the worked example with reference angle *C* to verify that the ratios $\frac{\text{side opposite reference angle}}{\text{hypotenuse}}$ and $\frac{\text{side opposite reference angle}}{\text{side adjacent to reference angle}}$ are constant in similar right triangles. Show your work.

$\frac{DE}{AB} = \frac{CD}{CA}$, so $\frac{DE}{CD} = \frac{AB}{CA}$

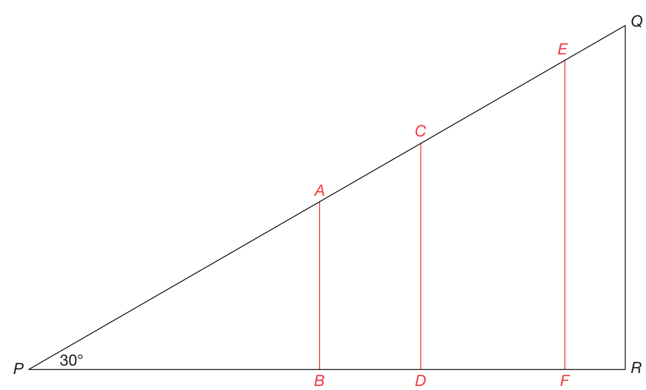
This shows that the ratio $\frac{\text{side opposite reference angle}}{\text{hypotenuse}}$ is always the same in similar right triangles.

$\frac{DE}{AB} = \frac{CE}{CB}$, so $\frac{DE}{CE} = \frac{AB}{CB}$

This shows that the ratio $\frac{\text{side opposite reference angle}}{\text{side adjacent to reference angle}}$ is always the same in similar right triangles.

PROBLEM 2 30°-60°-90°

Triangle *PQR* shown is a 30°-60°-90° triangle.



1. Draw three vertical line segments, $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$, connecting the hypotenuse of triangle *PQR* with side $\overline{PR}$. Label the endpoints of the vertical line segments along the hypotenuse as points *A*, *C*, and *E*. Label the other endpoints as points *B*, *D*, and *F*.

See diagram.

Guiding Questions for Share Phase, Questions 1 through 6

- What is the length of the base of each right triangle?
- What is the height of each right triangle?
- Are all of the triangles similar? Explain.
- What three ratios are the same for each triangle?

2. Measure each of the sides of the four similar right triangles in millimeters. Record the side length measurements in the table.

Triangle Name	Length of Side Opposite Angle *P*	Length of Side Adjacent to Angle *P*	Length of Hypotenuse
Triangle *PQR*	93 mm	162 mm	187 mm
Triangle *PEF*	83 mm	146 mm	168 mm
Triangle *PCD*	62 mm	107 mm	124 mm
Triangle *PAB*	46 mm	79 mm	91.5 mm

Measures for triangles *PEF*, *PCD*, and *PAB* will vary.

Sample measures are shown.

3. Determine each side length ratio for all four triangles using angle *P* as the reference angle.

Triangle Name	$\frac{\text{side opposite } \angle P}{\text{hypotenuse}}$	$\frac{\text{side adjacent to } \angle P}{\text{hypotenuse}}$	$\frac{\text{side opposite } \angle P}{\text{side adjacent to } \angle P}$
Triangle *PQR*	$\frac{93}{187} \approx 0.5$	$\frac{162}{187} \approx 0.866$	$\frac{93}{162} \approx 0.574$
Triangle *PEF*	$\frac{83}{168} \approx 0.49$	$\frac{146}{168} \approx 0.869$	$\frac{83}{146} \approx 0.568$
Triangle *PCD*	$\frac{62}{124} = 0.5$	$\frac{107}{124} \approx 0.863$	$\frac{62}{107} \approx 0.579$
Triangle *PAB*	$\frac{46}{91.5} \approx 0.503$	$\frac{79}{91.5} \approx 0.863$	$\frac{46}{79} \approx 0.582$

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4. Compare the side length ratios of all four triangles in the table. What do you notice?

The ratio of the length of the side opposite angle *P* to the length of the hypotenuse is approximately 0.5 for all four triangles.

The ratio of the length of the side adjacent to angle *P* to the length of the hypotenuse is approximately 0.86 for all four triangles.

The ratio of the length of the side opposite angle *P* to the length of the side adjacent to angle *P* is approximately 0.57 for all four triangles.

5. What conclusions can you draw from Problem 1 and Problem 2 about the three ratios you studied in 45°-45°-90° triangles and 30°-60°-90° triangles?

Answers will vary.

For each type of triangle, each of the ratios $\frac{\text{side opposite reference angle}}{\text{hypotenuse}}$, $\frac{\text{side adjacent to reference angle}}{\text{hypotenuse}}$, and $\frac{\text{side opposite reference angle}}{\text{side adjacent to reference angle}}$ is the same given a specific reference angle, no matter what the side measures of the triangle are.

6. Is each of the three ratios you studied in this lesson the same for any right triangles with congruent reference angles? Explain your reasoning.

Yes. If two or more right triangles have congruent reference angles, the triangles must be similar by the AA Similarity Theorem. This means that the ratios of the corresponding sides of the two triangles are equal. I have also proven that the ratios $\frac{\text{side opposite reference angle}}{\text{hypotenuse}}$, $\frac{\text{side adjacent to reference angle}}{\text{hypotenuse}}$, and $\frac{\text{side opposite reference angle}}{\text{side adjacent to reference angle}}$ must be equal.

Grouping

Have students complete Questions 7 and 8 with a partner. Then have students share their responses as a class.

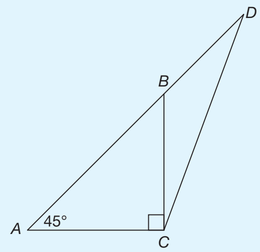
Guiding Questions for Share Phase, Questions 7 and 8

- Are the ratios you studied the same for non-similar triangles?
- Are the ratios you studied the same for non-right triangles?
- Does the reference angle measure matter when determining the ratios?

7. Explain why Alicia is incorrect.

Alicia

The ratio $\frac{BC}{AB}$ is equal to the ratio $\frac{DC}{AD}$, because the ratio $\frac{\text{side opposite } \angle A}{\text{hypotenuse}}$ is the same for both triangles ABC and ADC, given the reference angle A, which is 45°.



Alicia is incorrect because triangles *ABC* and *ADC* are not similar triangles and they are not both right triangles. Each of the three ratios $\frac{\text{side opposite } \angle A}{\text{hypotenuse}}$, $\frac{\text{side adjacent to } \angle A}{\text{hypotenuse}}$, and $\frac{\text{side opposite } \angle A}{\text{side adjacent to } \angle A}$ are only equal in similar right triangles with a given reference angle.

8. Is each of the three ratios you studied in this lesson the same for any triangles with congruent reference angles? Explain your reasoning.

No. The triangles must be right triangles, and the triangles must be similar. If the triangles are not similar, then the ratios I studied in this lesson, $\frac{\text{side opposite reference angle}}{\text{hypotenuse}}$, $\frac{\text{side adjacent to reference angle}}{\text{hypotenuse}}$, and $\frac{\text{side opposite reference angle}}{\text{side adjacent to reference angle}}$, would not be equal.

The three ratios you worked with in this lesson are very important to trigonometry and have special names and properties. You will learn more about these ratios in the next several lessons.

Talk the Talk

Students describe how the ratio changes as the reference angle measure increases for each ratio.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Talk the Talk

1. As the reference angle measure increases, what happens to each ratio? Explain your reasoning.

   a. $\frac{\text{opposite}}{\text{hypotenuse}}$

   The ratio increases because the opposite side increases in length as the measure of the labeled angle increases.

   b. $\frac{\text{adjacent}}{\text{hypotenuse}}$

   The ratio decreases because the adjacent side decreases in length as the measure of the labeled angle increases.

   c. $\frac{\text{opposite}}{\text{adjacent}}$

   The ratio increases because the opposite side increases as the adjacent side decreases.

Be prepared to share your solutions and methods.

Check for Students' Understanding

1. In this lesson, what was one purpose for dropping vertical lines in the special right triangles?

   Dropping the vertical lines in the special right triangles enabled us to form triangles similar to the original triangle.

2. In this lesson, what was one purpose for calculating the ratios $\frac{\text{opposite}}{\text{hypotenuse}}$ and $\frac{\text{adjacent}}{\text{hypotenuse}}$?

   Calculating the ratios helped us to realize that all triangles were similar because if the corresponding sides have equal ratios, the triangles must be similar.

3. In this lesson, what was one purpose for calculating the slope of the hypotenuse?

   After calculating the slope of the hypotenuse, we were able to conclude the ratio $\frac{\text{opposite}}{\text{adjacent}}$ was the same as the slope in each special right triangle.

8.2

The Tangent Ratio

Tangent Ratio, Cotangent Ratio, and Inverse Tangent

LEARNING GOALS

In this lesson, you will:

- Use the tangent ratio in a right triangle to solve for unknown side lengths.
- Use the cotangent ratio in a right triangle to solve for unknown side lengths.
- Relate the tangent ratio to the cotangent ratio.
- Use the inverse tangent in a right triangle to solve for unknown angle measures.

KEY TERMS

- rationalizing the denominator
- tangent (tan)
- cotangent (cot)
- inverse tangent

ESSENTIAL IDEAS

- The tangent (tan) of an acute angle in a right triangle is the ratio of the length of the side that is opposite the angle to the length of the side that is adjacent to the angle.
- The cotangent (cot) of an acute angle in a right triangle is the ratio of the length of the side that is adjacent to the angle to the length of the side that is opposite the angle.
- The inverse tangent (or arc tangent) of *x* is the measure of an acute angle whose tangent is *x*.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-SRT Similarity, Right Triangles, and Trigonometry

Understand similarity in terms of similarity transformations

3. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

Prove theorems involving similarity

5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Define trigonometric ratios and solve problems involving right triangles

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

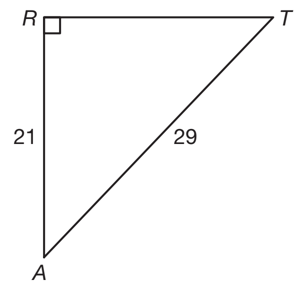
1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

The terms tangent, cotangent, and inverse tangent are introduced. Wheelchair ramps are the context for determining surface length and slope in right triangles. Applying the tangent ratio to similar triangles, students conclude that the value of the tangent of congruent angles of similar triangles is always congruent and the measure of an acute angle increases as the value of the tangent increases. Students write expressions based on the complementary relationship between the two acute angles in right triangles. Students prove algebraically $\cot A = \frac{1}{\tan A}$. When the arc tangent is introduced, students are able to use their calculators to solve for the measure of an acute angle in a right triangle. Calculators are used throughout this lesson to compute the value of the trigonometric ratios.

Warm Up

1. Solve for the length of $\overline{RT}$.



$$21^2 + RT^2 = 29^2$$
$$RT^2 = 29^2 - 21^2$$
$$= 841 - 441$$
$$= 400$$
$$= \sqrt{400} = 20$$

2. Write a ratio to compare the length of the side opposite $\angle A$ to the length of the side adjacent to $\angle A$. (Do not use the hypotenuse.)

$\frac{20}{21}$

3. Write a ratio to compare the length of the side adjacent to $\angle A$ to the length of the side opposite $\angle A$. (Do not use the hypotenuse.)

$\frac{21}{20}$

4. What is the difference between the ratios you wrote in Question 2 and Question 3?

Question 2 is asking for a ratio that is the reciprocal of the ratio asked for in Question 3. The numerator of the ratio in Question 1 is the denominator of the ratio in Question 2. And the denominator in Question 1 is the numerator in Question 2.

8

The Tangent Ratio

Tangent Ratio, Cotangent Ratio, and Inverse Tangent

8.2

LEARNING GOALS

In this lesson, you will:

- Use the tangent ratio in a right triangle to solve for unknown side lengths.
- Use the cotangent ratio in a right triangle to solve for unknown side lengths.
- Relate the tangent ratio to the cotangent ratio.
- Use the inverse tangent in a right triangle to solve for unknown angle measures.

KEY TERMS

- rationalizing the denominator
- tangent (tan)
- cotangent (cot)
- inverse tangent

When we talk about "going off on a tangent" in everyday life, we are talking about touching on a topic and then veering off to talk about something completely unrelated.

"Tangent" in mathematics has a similar meaning—a tangent line is a straight line that touches a curve at just one point. In this lesson, though, you will see that tangent is a special kind of ratio in trigonometry.

It's one you've already learned about!

Problem 1

Wheelchair ramps are represented using right triangles. Students sketch two different wheelchair ramps given the necessary measurements. Then they compute the surface length of each ramp and compare the ramps, concluding the right triangles are similar because the corresponding sides are proportional.

Grouping

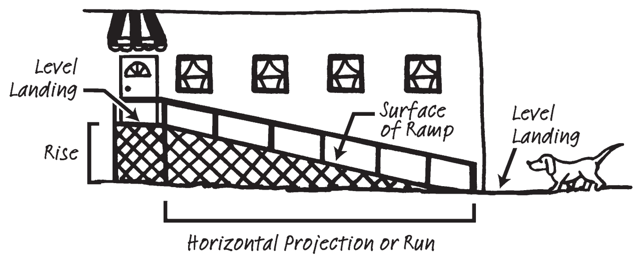
- Discuss the information above Question 1 as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- How would you determine the vertical rise of a wheelchair ramp?
- How is the vertical rise related to the horizontal run?
- Why do you suppose the maximum rise for any run is 30 inches?
- If the vertical rise if 2.5 feet, how did you determine the length of the horizontal run?
- Did you use the 1:12 ratio to determine the length of the horizontal run?
- How was the 1:12 ratio used to determine the length of the horizontal run?

PROBLEM 1 Wheelchair Ramps

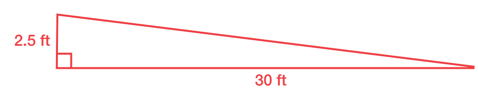
The maximum incline for a safe wheelchair ramp should not exceed a ratio of 1 : 12. This means that every 1 unit of vertical rise requires 12 units of horizontal run. The maximum rise for any run is 30 inches. The ability to manage the incline of the ramp is related to both its steepness and its length.



Troy decides to build 2 ramps, each with the ratio 1 : 12.

1. The first ramp extends from the front yard to the front porch. The vertical rise from the yard to the porch is 2.5 feet.

   a. Draw a diagram of the ramp. Include the measurements for the vertical rise and horizontal run of the ramp.



   The vertical rise is 2.5 feet. The horizontal run is 2.5(12), or 30 feet.

   b. Calculate the length of the surface of the ramp.

$$2.5^2 + 30^2 = c^2$$
$$6.25 + 900 = c^2$$
$$906.25 = c^2$$
$$\sqrt{906.25} = c$$
$$30.1 \approx c$$

   The length of the surface of the ramp is approximately 30.1 feet.

- Are the length of the hypotenuse and the surface length the same thing?
- How did you calculate the length of the hypotenuse or the surface length?
- What is the ratio of the vertical rises of the two triangles?
- What is the ratio of the horizontal runs of the two triangles?
- What is the ratio of the surface lengths of the two triangles?
- Are the corresponding sides of the two triangles proportional? How do you know?

2. The second ramp extends from the deck on the back of the house to the backyard. The vertical rise from the yard to the deck is 18 inches.

   a. Draw a diagram of the ramp. Include the measurements for the vertical rise and horizontal run of the ramp.

   The vertical rise is 18 inches. The horizontal run is 18(12), or 216 inches.

   b. Calculate the length of the surface of the ramp.

   $18^2 + 216^2 = c^2$

   $324 + 46{,}656 = c^2$

   $46{,}980 = c^2$

   $216.75 \approx c$

   The length of the surface of the ramp is approximately 216.75 inches or 18.1 feet.

3. Compare the two ramps. Are the triangles similar? Explain your reasoning.

   The ratio of the vertical rises of the two triangles is $\frac{2.5}{1.5} \approx 1.67$.

   The ratio of the horizontal runs of the two triangles is $\frac{360}{216} \approx 1.67$.

   The ratio of the surface lengths of the two triangles is $\frac{361.25}{216.75} \approx 1.67$.

   Yes. The triangles are similar because the corresponding sides are proportional.

4. Compare and describe the angles of inclination of the two ramps.

   The angles of inclination of the two ramps must be of equal measure because corresponding angles of similar triangles are congruent.

Problem 2

The tangent ratio is introduced and defined within the context of wheelchair ramps. Rise over run is equated with the slope or steepness of the ramp. Students deduce the ratios of proportional sides of similar triangles are equal to 1. Applying the tangent ratio to similar triangles, students conclude that the value of the tangent of congruent angles of similar triangles is constant and the measure of an acute angle increases as the value of the tangent increases. Calculators are used to compute the value of the tangent.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

- Are all wheelchair ramps right triangles? Why?
- What is the definition of slope?
- How is the concept of slope related to a wheelchair ramp?
- If two pair of corresponding angles are congruent, is that enough information to conclude the triangles are similar?

PROBLEM 2 Slope and Right Triangles

In the wheelchair ramp problem, Troy used 1 : 12 as the ratio of the rise of each ramp to the run of each ramp.

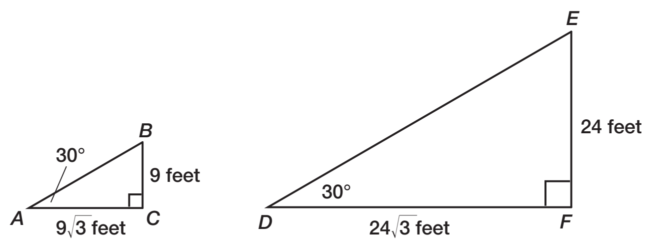
1. Describe the shape of each wheelchair ramp.

   Each ramp is in the shape of a right triangle.

2. What does the ratio of the rise of the ramp to the run of the ramp represent?

   The ratio represents the slope of the ramp.

3. Analyze the triangles shown.



   a. Verify the triangles are similar. Explain your reasoning.

   $\angle A \cong \angle D$ and $\angle C \cong \angle F$.

   By the AA Similarity Postulate, the triangles are similar.

   b. Calculate the ratio of the rise to the run for each triangle. How do the ratios compare?

   Triangle *ABC*: $\frac{9}{9\sqrt{3}} = \frac{1}{\sqrt{3}}$

   Triangle *DEF*: $\frac{24}{24\sqrt{3}} = \frac{1}{\sqrt{3}}$

   The ratios are equal.

- Do you need to remove the radical from the denominator of a fraction to determine if the ratios are the same? Why or why not?
- How do you remove the radical from the denominator of a fraction?

A standard mathematical convention is to write fractions so that there are no irrational numbers in the denominator. **Rationalizing the denominator** is the process of rewriting a fraction so that no irrational numbers are in the denominator.

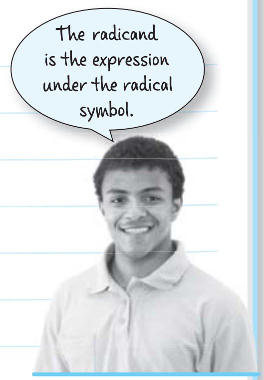
To rationalize the denominator of a fraction involving radicals, multiply the fraction by a form of 1 so that the product in the denominator includes a perfect square radicand. Then simplify, if possible.

Example 1:

$$\frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{\sqrt{4}}$$
$$= \frac{10\sqrt{2}}{2}$$
$$= 5\sqrt{2}$$

Example 2:

$$\frac{3}{5\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{5\sqrt{9}}$$
$$= \frac{3\sqrt{3}}{5 \cdot 3}$$
$$= \frac{3\sqrt{3}}{15}$$
$$= \frac{\sqrt{3}}{5}$$

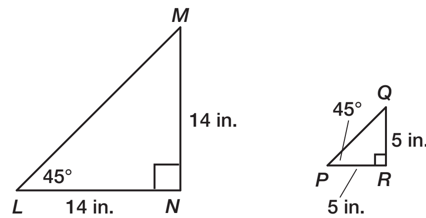


4. Rewrite your answers in Question 3, part (b), by rationalizing the denominators. Show your work.

The ratio of the rise to run for each triangle is $\frac{\sqrt{3}}{3}$.

$$\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{9}}$$
$$= \frac{\sqrt{3}}{3}$$

5. Analyze the triangles shown.



a. Verify the triangles are similar. Explain your reasoning.

Angle $L \cong \angle P$ and $\angle N \cong \angle R$. By the AA Similarity Postulate, the triangles are similar.

b. Calculate the ratio of the rise to the run for each triangle. How do the ratios compare?

Triangle *LMN*: $\frac{14}{14} = \frac{1}{1}$

Triangle *PQR*: $\frac{5}{5} = \frac{1}{1}$

The ratios are equal.

6. What can you conclude about the ratios of the rise to the run in similar right triangles?

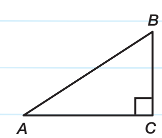
The ratios of the rise to the run in similar right triangles are equal.

Grouping

Discuss the worked example and complete Question 7 as a class.

The **tangent (tan)** of an acute angle in a right triangle is the ratio of the length of the side that is opposite the reference angle to the length of the side that is adjacent to the reference angle. The expression "tan *A*" means "the tangent of $\angle A$."

Consider $\angle A$ in the right triangle shown.



The tangent ratio describes the relationship between $\angle A$, the side opposite $\angle A$, and the side adjacent to $\angle A$.

$$\tan A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A} = \frac{BC}{AC}$$

Grouping

Have students complete Questions 8 through 14 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 8 through 14

- Can the tangent ratio ever have a value that is greater than 1? How?
- Can the tangent ratio ever have a value that is equal to 1? How?
- Can the tangent ratio ever have a value that is less than 1? How?
- Are the tangent values of the angles always the same if the triangles are similar? Why?
- Are the tangent values of complementary angles always reciprocals of each other? Why?
- Is the value of tan 4° the slope of the ramp?
- What is the value of tan 4°?
- Is your calculator in the degree mode?
- What is the decimal equivalent of the fraction $\frac{1}{12}$?
- How does the decimal 0.083 compare to the decimal 0.07?

7. Complete the ratio that represents the tangent of $\angle B$.

$$\tan B = \frac{\text{length of side opposite } \angle B}{\text{length of side adjacent to } \angle B} = \frac{\boxed{AC}}{\boxed{BC}}$$

8. Determine the tangent values of all the acute angles in the right triangles from Questions 3 and 5.

Triangle *ABC*: $\tan 30° = \frac{9}{9\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\tan 60° = \frac{9\sqrt{3}}{9} = \frac{\sqrt{3}}{1}$

Triangle *DEF*: $\tan 30° = \frac{24}{24\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

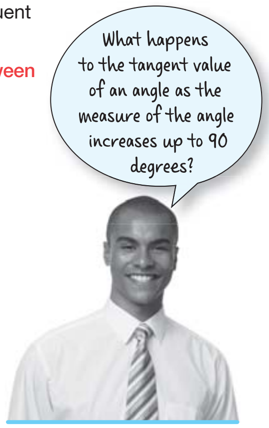
$\tan 60° = \frac{24\sqrt{3}}{24} = \frac{\sqrt{3}}{1}$

Triangle *LMN*: $\tan 45° = \frac{14}{14} = \frac{1}{1}$

Triangle *PQR*: $\tan 45° = \frac{5}{5} = \frac{1}{1}$

9. What can you conclude about the tangent values of congruent angles in similar triangles?

The tangent values of the angles are always the same between two similar triangles.



10. Consider the tangent values in Question 8. In each triangle, compare tan 30° to tan 60°. What do you notice? Why do you think this happens?

The tangent values are reciprocals of each other.

The side adjacent to one angle in a triangle is the side that is opposite the other angle and vice versa.

11. A proposed wheelchair ramp is shown.

a. What information about the ramp is required to show that the ramp meets the safety rules?

The slope of the ramp needs to be known. The ratio of the vertical rise to the horizontal rise must not exceed 1 : 12.

b. Write a decimal that represents the greatest value of the slope of a safe ramp.

The decimal that represents the greatest value of the slope is $0.08\overline{3}$.

$1 : 12 = \frac{1}{12} = 0.08\overline{3}$

c. If you calculate the value of tan 4°, how can you use this value to determine whether the ramp meets the safety rules?

The value of tan 4° is the slope of the ramp.

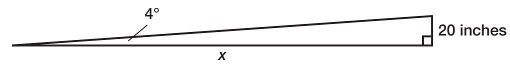
d. Use a calculator to determine the value of tan 4°. Round your answer to the nearest hundredth.

$\tan 4° \approx 0.07$

e. What is the ratio of the rise of the proposed ramp to its run? Is the ramp safe?

The ratio of the rise to the run is approximately 0.07. Because the slope can be no more than 1 : 12, or $0.08\overline{3}$, the ramp is safe.

12. Another proposed wheelchair ramp is shown. What is the run of the ramp? If necessary, round your answer to the nearest inch.



$\tan 4° = \dfrac{20}{x}$

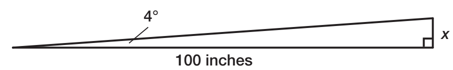
$x(\tan 4°) = 20$

$x = \dfrac{20}{\tan 4°}$

$x \approx 286$

The ramp has a run of about 286 inches.

13. Another proposed wheelchair ramp is shown. What is the rise of this ramp? If necessary, round your answer to the nearest inch.



$\tan 4° = \dfrac{x}{100}$

$100(\tan 4°) = x$

$x \approx 7$

The ramp has a rise of about 7 inches.

14. If other ramps that have a 4° angle with different side measurements are drawn by extending or shortening the rise and run, will the tangent ratios always be equivalent? Explain your reasoning.

Yes. The other right triangles, or ramps, would be similar because of the AA Similarity Theorem. That means that the ratios of the sides of any of those right triangles would be the same. Therefore, the tangent ratios will also be equivalent.

Problem 3

Students write algebraic expressions representing tangent ratios using two similar triangles. This problem is different from previous problems because the measures of the angles and the measures of the sides of the two similar triangles are all unknown. The expressions are based on a complementary relationship between the two acute angles in each right triangle.

Grouping

Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

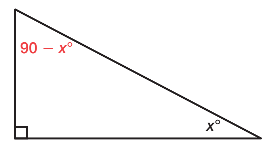
- What is the sum of the measures of the three interior angles in any triangle?
- What is the sum of the measures of the two acute angles in any right triangle?
- Is the complement of an angle measuring $x°$ written $(x° - 90)$ or $(90 - x°)$? Why?
- What is the difference between $(x° - 90)$ and $(90 - x°)$?

PROBLEM 3 Generally Speaking . . .

In the previous problems, you used the measure of an acute angle and the length of a side in a right triangle to determine the unknown length of another side.

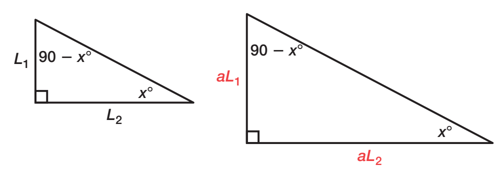
Consider a right triangle with acute angles of unknown measures and sides of unknown lengths. Do you think the same relationships will be valid?

1. If one acute angle of a right triangle has a measure of x degrees, what algebraic expression represents the measure of the second acute angle? Label this angle and explain your reasoning.



The second acute angle is represented by the algebraic expression $90 - x$ because the sum of the measures of the three interior angles of a triangle is 180 degrees.

2. Suppose two right triangles are similar and each triangle contains an acute angle that measures $x°$, as shown.



a. If the side opposite the acute angle measuring $x°$ in the first triangle is of length L_1, what algebraic expression represents the length of the side opposite the acute angle measuring $x°$ in the second triangle?

The length of the side opposite the acute angle measuring $x°$ in the second right triangle is represented by the algebraic expression aL_1, where a represents the ratio of the corresponding sides of the similar triangles.

b. If the side opposite the acute angle measuring $90 - x°$ in the first triangle is of length L_2, what algebraic expression represents the length of the side opposite the acute angle measuring $90 - x°$ in the second triangle?

The length of the side opposite the acute angle measuring $90 - x°$ in the second right triangle is represented by the algebraic expression aL_2, where a represents the ratio of the corresponding sides of the similar triangles.

- Does $\tan x°$ equal $\frac{L_1}{L_2}$ or $\frac{L_2}{L_1}$? How do you know?
- How is the expression representing $\tan x°$ in the first triangle different than the expression representing $\tan x°$ in the second triangle?

Problem 4

The cotangent ratio is introduced as a reciprocal relationship of the tangent ratio. Students prove algebraically $\cot A = \frac{1}{\tan A}$. To compute the cotangent on a calculator, $\frac{1}{\tan A}$ can be used. Students use the tangent and cotangent ratios to solve for unknown measurements.

Grouping

- Discuss the information above Question 1 as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- How can the tan A in the denominator be rewritten as a ratio?
- How do you remove a fraction from the denominator?

3. Write an expression to represent the tangent of the angle measuring $x°$ in the first triangle.

$\tan x° = \frac{L_1}{L_2}$

4. Write an expression to represent the tangent of the angle measuring $x°$ in the second triangle.

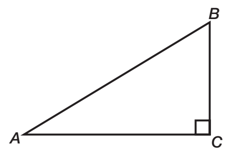
$\tan x° = \frac{aL_1}{aL_2}$

5. Write a proportion to represent the relationship between the two triangles in terms of the tangents of the angle measuring $x°$.

$\frac{L_1}{L_2} = \frac{aL_1}{aL_2}$

PROBLEM 4 Cotangent Ratio

The **cotangent (cot)** of an acute angle in a right triangle is the ratio of the length of the side that is adjacent to the angle to the length of the side that is opposite the angle. The expression "cot A" means "the cotangent of $\angle A$."



1. Complete the ratio that represents the cotangent of $\angle A$.

$$\cot A = \frac{\text{length of side adjacent to } \angle A}{\text{length of side opposite } \angle A} = \frac{\boxed{AC}}{\boxed{BC}}$$

2. Prove algebraically that the cotangent of $A = \frac{1}{\tan A}$.

$\cot A = \frac{1}{\tan A}$

$$\frac{1}{\tan A} = \frac{1}{\frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A}}$$

$$= \frac{\text{length of side adjacent to } \angle A}{\text{length of side opposite } \angle A}$$

$= \cot A$

- How can the ratio $\frac{\text{length of side adjacent to } \angle A}{\text{length of side opposite } \angle A}$ be rewritten?
- As an acute angle increases in measure, what happens to the denominator of the ratio?
- As the denominator of the ratio increases, what happens to the fractional value?

Grouping

Have students complete Questions 4 through 6 with a partner. Then have students share their responses as a class.

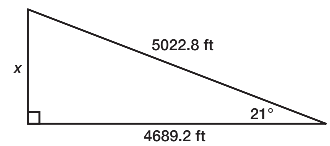
Guiding Questions for Share Phase, Questions 4 through 6

- How do you input cot 21° into a calculator?
- If several right triangles contain a 21° angle, what else do they have in common?

3. As the measure of an acute angle increases, the tangent value of the acute angle increases. Explain the behavior of the cotangent value of an acute angle as the measure of the acute angle increases.

As the measure of an acute angle increases, its cotangent value decreases because the denominator of the ratio increases, causing the fractional value to decrease.

4. A ski slope at Snowy Valley has an average angle of elevation of 21°.



a. Calculate the vertical height of the ski slope x using the cotangent ratio.

$$\cot 21° = \frac{4689.2}{x}$$
$$x(\cot 21°) = 4689.2$$
$$x\left(\frac{1}{\tan 21°}\right) = 4689.2$$
$$x = 4689.2(\tan 21°)$$
$$x = 4689.2(0.3839)$$
$$x \approx 1800.2$$

The unknown side length is ≈ 1800.2 feet.

b. Calculate the vertical height of the ski slope x using the tangent ratio.

$$\tan 21° = \frac{x}{4689.2}$$
$$4689.2(\tan 21°) = x$$
$$x = 4689.2(\tan 21°)$$
$$x = 4689.2(0.3839)$$
$$x \approx 1800.2$$

The unknown side length is ≈ 1800.2 feet.

c. Which ratio did you prefer to use when calculating the value of x? Explain your reasoning.

Answers will vary.

I preferred using the tangent ratio because I did not have to use fractional notation to solve for x, and it required fewer steps to calculate.

5. Are all right triangles that contain a 21° angle similar? Why or why not?

Yes. The right triangles will have to be similar because of the AA Similar Triangle Theorem.

6. If other right triangles containing a 21° angle with different side measurements are drawn by extending or shortening the rise and run, will the cotangent ratios always be equivalent?

Yes. The other right triangles will have to be similar because of the AA Similar Triangle Theorem, therefore the cotangent ratios will also be equivalent.

Problem 5

The inverse tangent, or the arc tangent, is introduced and students are now able use their calculators to solve for the measure of an acute angle in a right triangle when the length of the adjacent side and the length of the opposite side are known.

Grouping

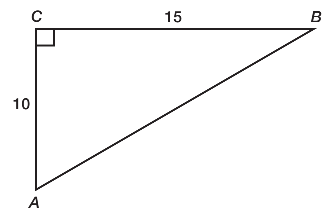
- Discuss the information above Question 1 as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

PROBLEM 5 Inverse Tangent

The **inverse tangent** (or arc tangent) of x is defined as the measure of an acute angle whose tangent is x. If you know the length of any two sides of a right triangle, it is possible to compute the measure of either acute angle by using the inverse tangent, or the $\tan^{-1}$ button on a graphing calculator.

In right triangle ABC, if $\tan A = x$, then $\tan^{-1} x = m\angle A$.

1. Consider triangle ABC shown.



a. If $\tan A = \frac{15}{10}$, then calculate $\tan^{-1}\left(\frac{15}{10}\right)$ to determine $m\angle A$.

$m\angle A = \tan^{-1}\left(\frac{15}{10}\right) \approx 56.31°$

b. Determine the ratio for $\tan B$, and then use $\tan^{-1}(\tan B)$ to calculate $m\angle B$.

$\tan B = \frac{10}{15} = \frac{2}{3}$

$m\angle B = \tan^{-1}\left(\frac{2}{3}\right) \approx 33.69°$

Guiding Questions for Share Phase, Questions 1 through 3

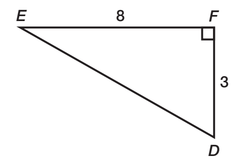
- Where is arc tan or $\tan^{-1}$ found on your calculator?
- Is the ratio for $\tan B$ $\frac{10}{15}$ or $\frac{15}{10}$?
- How is the ratio for $\tan B$ used to determine $m\angle B$?
- Is the ratio for $\tan E$ $\frac{3}{8}$ or $\frac{8}{3}$?
- How is the ratio for $\tan E$ used to determine $m\angle E$?
- Is the ratio for $\tan x$ $\frac{37}{85}$ or $\frac{85}{37}$?

c. Add $m\angle A$ and $m\angle B$. Does your sum make sense in terms of the angle measures of a triangle?

$56.31° + 33.69° = 90°$

Yes. The sum of the two acute angles in a right triangle will always be 90°.

2. Calculate $m\angle E$.



$\tan E = \frac{3}{8}$

$m\angle E = \tan^{-1}\left(\frac{3}{8}\right) \approx 20.56°$

3. Movable bridges are designed to open waterways for large boats and barges. When the bridge moves, all vehicle traffic stops.

The maximum height of the open bridge deck of the movable bridge shown is 37 feet above the water surface. The waterway width is 85 feet. Calculate the angle measure formed by the movement of the bridge.

$\tan x = \frac{37}{85}$

$m\angle x = \tan^{-1}\left(\frac{37}{85}\right) \approx 23.52°$

Problem 6

Students use the tangent ratio to determine the percentage grade on a road sign given the angle of elevation, and the inverse tangent function to determine the angle of elevation on a road with the percentage grade given.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

PROBLEM 6 **Road Grades**

Many mountainous areas have road signs like this sign that refer to the percentage grade for the road. An 8% grade, for example, means that the altitude changes by 8 feet for each 100 feet of horizontal distance.

1. To determine the percentage grade that should be put on a road sign where the angle of elevation of the road is 9°, what function would be most helpful?

   The tangent function would be helpful.

2. To determine the angle of elevation of a road with a percentage grade of 7%, what function would be most helpful?

   The inverse tangent function would be helpful.

3. Determine the angle of elevation of a road with a percentage grade of 6%.

   $\tan^{-1}(0.06) \approx 3.43°$

   The angle of elevation for a 6% road grade is approximately 3°.

4. Determine the percentage road grade that should be put on a road sign where the angle of elevation is 10°.

   $\tan(10) \approx 0.1763 \approx 17.6\%$

   When the angle of elevation is 10°, the percentage on the road sign should read 18%.

5. Does the image in the sign accurately represent an 8% grade? Explain how you can determine the answer.

   No. The percentage grade represented in the sign is approximately 46.7%. The rise measures approximately 14 mm, and the run measures approximately 30 mm ($14 \div 30 \approx 0.4666\ldots$).

6. What is the approximate angle of elevation that is actually shown in the sign?

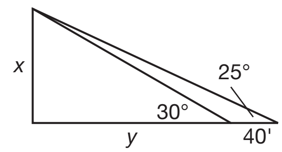
   $\tan^{-1}(0.467) \approx 25.03°$

Be prepared to share your solutions and methods.

8

Check for Students' Understanding

Bobby is standing near a lighthouse. He measured the angle formed from where he stood to the top of the lighthouse and it was 30°. Then he backed up 40 feet and measured the angle again and it was 25°. Solve for the height of the lighthouse.



$\tan 30° = \frac{x}{y}$ $\tan 25° = \frac{x}{y + 40}$

$x = y \tan 30°$

$\tan 25° = \frac{y \tan 30°}{y + 40}$

$y \tan 30° = y \tan 25° + 40 \tan 25°$

$y \tan 30° - y \tan 25° = 40 \tan 25°$

$y(\tan 30° - \tan 25°) = 40 \tan 25°$

$y = \frac{40 \tan 25°}{\tan 30° - \tan 25°}$

$y \approx 31.8'$

$\tan 30° = \frac{x}{31.8}$

$x = 31.8 \tan 30° \approx 18.4'$

The height of the lighthouse is approximately 18.4 feet.

8.3

The Sine Ratio

Sine Ratio, Cosecant Ratio, and Inverse Sine

LEARNING GOALS

In this lesson, you will:

- Use the sine ratio in a right triangle to solve for unknown side lengths.
- Use the cosecant ratio in a right triangle to solve for unknown side lengths.
- Relate the sine ratio to the cosecant ratio.
- Use the inverse sine in a right triangle to solve for unknown angle measures.

KEY TERMS

- sine (sin)
- cosecant (csc)
- inverse sine

ESSENTIAL IDEAS

- The sine (sin) of an acute angle in a right triangle is the ratio of the length of the side that is opposite the angle to the length of the hypotenuse.
- The cosecant (csc) of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is opposite the angle.
- The inverse sine (or arc sine) of x is the measure of an acute angle whose sine is x.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-SRT Similarity, Right Triangles, and Trigonometry

Define trigonometric ratios and solve problems involving right triangles

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

G-MG Modeling with Geometry

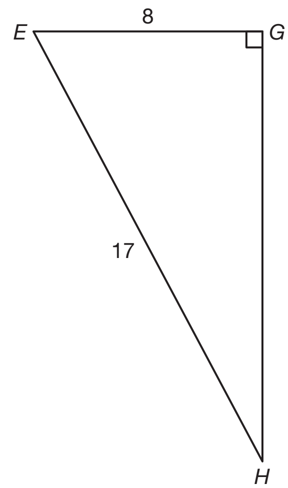
Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

The terms sine, cosecant, and inverse sine are introduced. Golf clubs are the context for determining the sine ratio in right triangles. Students conclude that as the acute angle increases in measure, the sine ratio increases in value, and the value of sine will always be less than 1 because the hypotenuse (the denominator in the sine ratio) is the longest side of the right triangle. Students prove algebraically $\csc A = \frac{1}{\sin A}$. When the arc sine is introduced, students use their calculators to solve for the measure of an acute angle in a right triangle. Calculators are used throughout this lesson to compute the value of the trigonometric ratios.

Warm Up



1. Calculate the value of tan H.

$8^2 + GH^2 = 17^2$

$GH^2 = 17^2 - 8^2$

$= 289 - 64$

$= 225$

$= \sqrt{225} = 15$

Tan $H = \frac{8}{15} \approx 0.5\overline{33}$

2. Calculate the value of cot H.

cot $H = \frac{15}{8} = 1.875$

3. Calculate $m\angle H$.

$\tan^{-1} H = \frac{8}{15}$

$\approx 28.07°$

4. Calculate $m\angle E$.

$m\angle E \approx 180° - 90° - 28.07° \approx 61.93°$

The Sine Ratio

Sine Ratio, Cosecant Ratio, and Inverse Sine

8.3

LEARNING GOALS

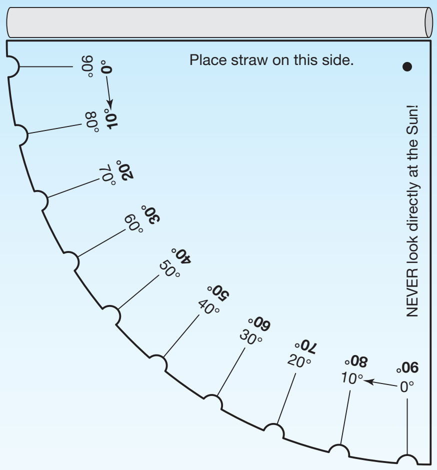
In this lesson, you will:

- Use the sine ratio in a right triangle to solve for unknown side lengths.
- Use the cosecant ratio in a right triangle to solve for unknown side lengths.
- Relate the sine ratio to the cosecant ratio.
- Use the inverse sine in a right triangle to solve for unknown angle measures.

KEY TERMS

- sine (sin)
- cosecant (csc)
- inverse sine

Measuring angles on paper is easy when you have a protractor. But what about measuring angles in the real world? You can build an astrolabe (pronounced uh-STRAW-luh-bee) to help you.



Copy and cut out the astrolabe shown (without the straw). You will probably want to glue the astrolabe to cardboard or heavy paper before cutting it out.

Cut a drinking straw to match the length of the one shown. Tape the straw to the edge labeled "Place straw on this side" so that it rests on the astrolabe as shown.

Poke a hole through the black dot shown and pass a string through this hole. Knot the string or tape it so that it stays in place.

Finally, tie a weight to the end of the string. You're ready to go!

Problem 1

The face of a golf club forms a right triangle and the sine ratio is introduced within this context. Students determine the sine ratio for different triangles and conclude the value of the sine ratio will always be less than 1 because the hypotenuse, the longest side in a right triangle, is the denominator of the ratio. Students also deduce that the sine of an acute angle increases as its measure increases. Calculators are used to compute the value of the sine ratio.

Grouping

- Discuss the information above Question 1 as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

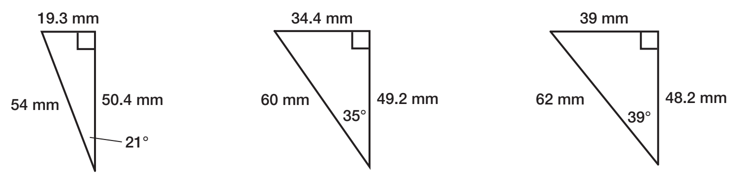
Guiding Questions for Share Phase, Questions 1 through 3

- What is the relationship between the measure of the angle and the distance the ball will travel?
- What is the relationship between the measure of the angle and the height of the ball?
- If the measure of the angle is very small, will the ball travel a longer distance?

PROBLEM 1 Fore!

Each golf club in a set of clubs is designed to cause the ball to travel different distances and different heights. One design element of a golf club is the angle of the club face.

You can draw a right triangle that is formed by the club face angle. The right triangles formed by different club face angles are shown.



1. How do you think the club face angle affects the path of the ball?

A greater angle measure makes the ball go higher and travel a shorter distance than a lesser angle measure.

2. For each club face angle, write the ratio of the side length opposite the given acute angle to the length of the hypotenuse. Write your answers as decimals rounded to the nearest hundredth.

21° club face angle: $\frac{19.3}{54} \approx 0.36$

35° club face angle: $\frac{34.4}{60} \approx 0.57$

39° club face angle: $\frac{39}{62} \approx 0.63$

- If the measure of the angle is very large, will the ball travel to a greater height?
- Which club face (angle measure) has a ratio that is the largest value?
- Which club face (angle measure) has a ratio that is the smallest value?

Grouping

- Have a student read the information above Question 4. Complete Question 4 as a class.
- Have students complete Questions 5 through 10 with a partner. Then have students share their responses as a class.

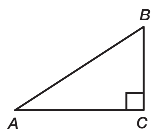
Guiding Questions for Share Phase, Questions 5 through 10

- Which angle measure has the greatest sine value? the least sine value?
- What would be true of the side lengths of a triangle if the sine of an acute angle was greater than 1?
- Which angles have irrational sine values?
- Which angles have rational sine values?

3. What happens to this ratio as the angle measure gets larger?

The ratio gets larger as the angle measure gets larger.

The **sine (sin)** of an acute angle in a right triangle is the ratio of the length of the side that is opposite the angle to the length of the hypotenuse. The expression "sin *A*" means "the sine of $\angle A$."



4. Complete the ratio that represents the sine of $\angle A$.

$$\sin A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}} = \frac{\boxed{BC}}{\boxed{AB}}$$

5. For each triangle in Problem 1, calculate the sine value of the club face angle. Then calculate the sine value of the other acute angle. Round your answers to the nearest hundredth.

$\sin 21° = \frac{19.3}{54} \approx 0.36$ $\sin 39° = \frac{39}{62} \approx 0.63$ $\sin 55° = \frac{49.2}{60} \approx 0.82$

$\sin 35° = \frac{34.4}{60} \approx 0.57$ $\sin 51° = \frac{48.2}{62} \approx 0.78$ $\sin 69° = \frac{50.4}{54} \approx 0.93$

6. What do the sine values of the angles in Question 5 all have in common?

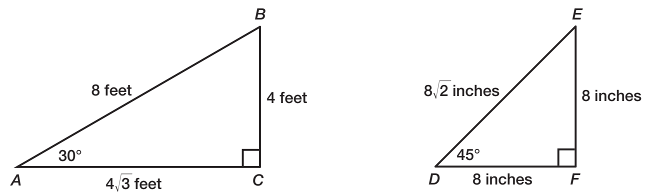
The values are all less than 1.

7. Jun says that the sine value of every acute angle is less than 1. Is Jun correct? Explain your reasoning.

Yes. The sine of an acute angle is the ratio of the length of a side of a right triangle to the length of the hypotenuse. Because the hypotenuse is the longest side in a right triangle, the ratio will always be less than 1.

8. What happens to the sine values of an angle as the measure of the angle increases?
The sine values of an angle increase as the measure of the angle increases.

9. Use the right triangles shown to calculate the values of sin 30°, sin 45°, and sin 60°.



$\sin 30° = \frac{4}{8} = \frac{1}{2}$ $\sin 45° = \frac{8}{8\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\sin 60° = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$

10. A golf club has a club face angle A for which $\sin A \approx 0.45$. Estimate the measure of $\angle A$. Use a calculator to verify your answer.
Because $\sin 21° \approx 0.36$ and $\sin 30° = 0.5$, $m\angle A$ should be between 21° and 30°. The actual sine value of angle A is about 27 degrees.

Problem 2

The cosecant ratio is introduced as a reciprocal relationship of the sine ratio. Students prove algebraically $\csc A = \frac{1}{\sin A}$. To compute the cosecant on a calculator, $\frac{1}{\sin A}$ can be used. Students use the sine and cosecant ratios to solve for unknown measurements.

Grouping

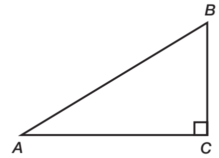
- Discuss the information above Question 1 as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

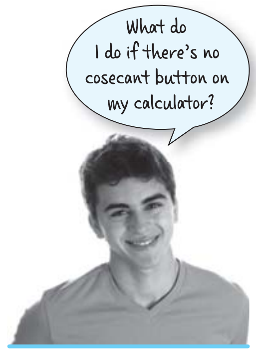
Guiding Questions for Share Phase, Questions 1 through 3

- How can the sin *A* in the denominator be rewritten as a ratio?
- How do you remove a fraction from the denominator?
- How can the ratio $\dfrac{\textit{length of hypotenuse}}{\textit{length of side opposite } \angle A}$ be rewritten?
- As an acute angle increases in measure, what happens to the denominator of the cosecant ratio?
- As the denominator of the cosecant ratio increases, what happens to the fractional value?

PROBLEM 2 Cosecant Ratio

The **cosecant (csc)** of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is opposite the angle. The expression "csc *A*" means "the cosecant of ∠*A*."





1. Complete the ratio that represents the cosecant of ∠*A*.

$$\csc A = \frac{\text{length of hypotenuse}}{\text{length of side opposite } \angle A} = \frac{\boxed{AB}}{\boxed{BC}}$$

2. Prove algebraically that the cosecant of $A = \frac{1}{\sin A}$.

$$\csc A = \frac{1}{\sin A}$$

$$\frac{1}{\sin A} = \frac{1}{\frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}}}$$

$$= \frac{\text{length of hypotenuse}}{\text{length of side opposite } \angle A}$$

$$= \csc A$$

3. As the measure of an acute angle increases, the sine value of the acute angle increases. Explain the behavior of the cosecant value of an acute angle as the measure of the acute angle increases.

As the measure of an acute angle increases, its cosecant value decreases because the denominator of the ratio increases, causing the fractional value to decrease.

Problem 3

The inverse sine, or the arc sine, is introduced and students are now able use their calculators to solve for the measure of an acute angle in a right triangle when the length of the opposite side and the length of the hypotenuse are known.

Grouping

- Discuss the information above Question 1 as a class.
- Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- Where is arc sine or $\sin^{-1}$ found on your calculator?
- Is the ratio for sin B $\frac{11}{25}$ or $\frac{25}{11}$?
- How is the ratio for sin B used to determine $m\angle B$?
- Is the ratio for sin B $\frac{5}{12}$ or $\frac{12}{5}$?
- How is the ratio for sin B used to determine $m\angle E$?
- Is the ratio for sin x $\frac{30}{42}$ or $\frac{42}{30}$?

PROBLEM 3 Inverse Sine

The **inverse sine** (or arc sine) of x is defined as the measure of an acute angle whose sine is x. If you know the length of any two sides of a right triangle, it is possible to calculate the measure of either acute angle by using the inverse sine, or $\sin^{-1}$ button on a graphing calculator.

In right triangle ABC, if $\sin A = x$, then $\sin^{-1} x = m\angle A$.

1. In right triangle ABC, if $\sin A = \frac{2}{5}$, calculate $\sin^{-1}\left(\frac{2}{5}\right)$ to determine $m\angle A$.

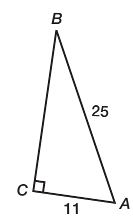
   $m\angle A = \sin^{-1}\left(\frac{2}{5}\right) \approx 23.58°$

2. Determine the ratio for sin B, and then use $\sin^{-1}(\sin B)$ to calculate $m\angle B$.

   $\sin B = \frac{11}{25}$

   $m\angle B = \sin^{-1}\left(\frac{11}{25}\right)$

   $\approx 26.10°$

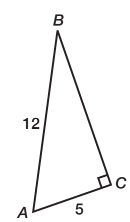


3. Calculate $m\angle B$.

   $\sin B = \frac{5}{12}$

   $m\angle B = \sin^{-1}\left(\frac{5}{12}\right)$

   $\approx 24.62°$



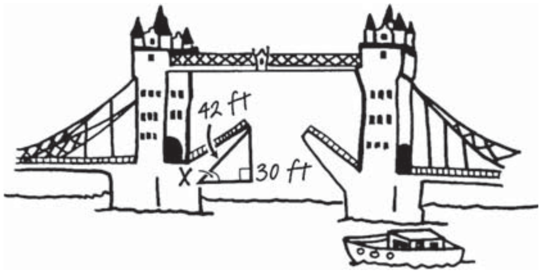
4. The movable bridge shown is called a double-leaf Bascule bridge. It has a counterweight that continuously balances the bridge deck, or "leaf," throughout the entire upward swing, providing an open waterway for boat traffic. The counterweights on double-leaf bridges are usually located below the bridge decks.

The length of one leaf, or deck, is 42 feet. The maximum height of an open leaf is 30 feet. Calculate the measure of the angle formed by the movement of the bridge.

$\sin x = \frac{30}{42}$

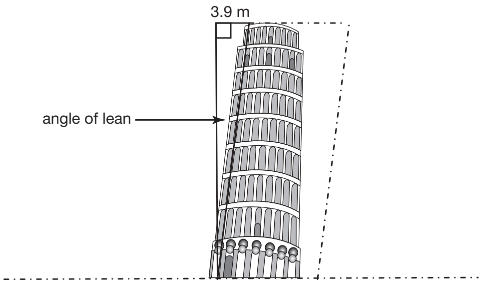
$m\angle x = \sin^{-1}\left(\frac{30}{42}\right)$

$\approx 45.58°$



5. The Leaning Tower of Pisa is a tourist attraction in Italy. It was built on unstable land and, as a result, it really does lean!

The height of the tower is approximately 55.86 meters from the ground on the low side and 56.7 meters from the ground on the high side. The top of the tower is displaced horizontally 3.9 meters as shown.



Determine the angle at which the tower leans.

I used an average height of 56 meters.

$\sin A = \frac{3.9}{56} \approx 0.0696$

$\sin^{-1}(0.0696) \approx 3.99°$

The tower of Pisa leans approximately 3.99°.

Talk the Talk

Students match and label each of five diagrams with the appropriate situation and identify the ratio that would be most helpful when solving for the unknown measurement.

Grouping

Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Talk the Talk

Match each diagram with the appropriate situation and identify the trigonometric function that would be most helpful in answering each question. Then calculate the unknown measurement.

Diagram A

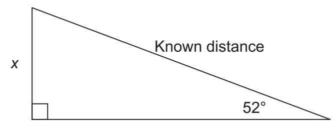


Diagram B

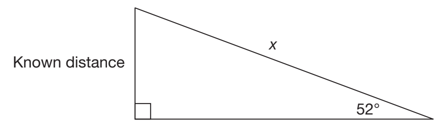


Diagram C

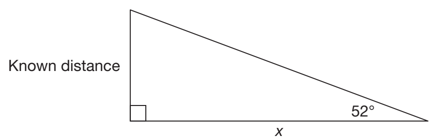


Diagram D

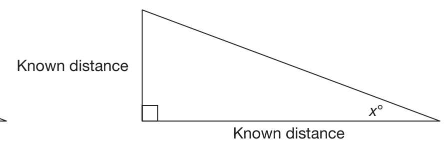
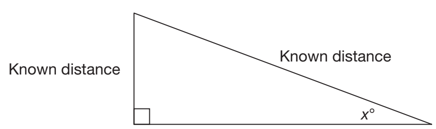
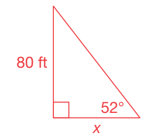


Diagram E



1. A building is 80 feet high. An observer stands an unknown distance away from the building and, looking up to the top of the building, notes that the angle of elevation is 52°. Determine the distance from the base of the building to the observer.

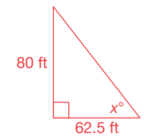
Diagram C best models this scenario.



$\tan 52° = \frac{80}{x}$ or $\cot 52° = \frac{x}{80}$

2. A building is 80 feet high. An observer stands 62.5 feet away from the building and looking up to the top of the building he ponders the measure of the angle of elevation. Determine the measure of the angle of elevation.

Diagram D best models this scenario.

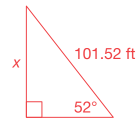


$\tan x° = \frac{80}{62.5}$

$m\angle x = \tan^{-1}\left(\frac{80}{62.5}\right)$

3. An observer stands an unknown distance away from a building and looking up to the top of the building notes that the angle of elevation is 52°. He also knows the distance from where he is standing to the top of the building is 101.52 feet. Determine the height of the building.

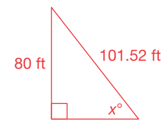
Diagram A best models this scenario.



$\sin 52° = \frac{x}{101.52}$ or $\csc 52° = \frac{101.52}{x}$

4. A building is 80 feet high. An observer stands an unknown distance away from the building and looking up to the top of the building he ponders the measure of the angle of elevation. He also knows the distance from where he is standing to the top of the building is 101.52 feet. Determine the measure of the angle of elevation.

Diagram E best models this scenario.

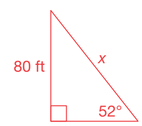


$\sin x° = \frac{80}{101.52}$

$m\angle x = \sin^{-1}\left(\frac{80}{101.52}\right)$

5. A building is 80 feet high. An observer stands an unknown distance away from the building and looking up to the top of the building notes that the angle of elevation is 52°. Determine the distance from the observer to the top of the building.

Diagram B best models this scenario.

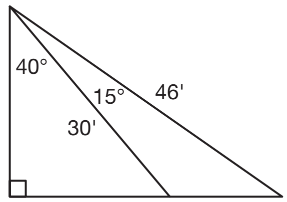


$\sin 52° = \frac{80}{x}$ or $\csc 52° = \frac{x}{80}$

Be prepared to share your solutions and methods.

Check for Students' Understanding

Two cables supporting the center pole of a circus tent are both connected at the top of the pole and are staked into the ground several feet apart. The length of the first cable is 30 feet and the length of the second cable is 46 feet. The angle formed by the pole and the first cable is 40°. The angle formed by the pole and the second cable is 55°. Calculate the height of center pole, and the distance between the two stakes.



$\sin 40° = \frac{x}{30}$

$30 \sin 40° = x$

$x \approx 19.28'$

The distance from the pole to the first cable is approximately 19.28′.

$\sin 55° = \frac{x}{46}$

$46 \sin 55° = x$

$x \approx 37.68'$

The distance from the pole to the second cable is approximately 37.68′.
The distance between the two stakes is approximately 18.4′. (37.68′ − 19.28′)

$\tan 40° = \frac{19.28}{x}$

$x \tan 40 = 19.28$

$x \approx \frac{19.28}{\tan 40°} \approx 22.98'$

8.4

The Cosine Ratio

Cosine Ratio, Secant Ratio, and Inverse Cosine

LEARNING GOALS

In this lesson, you will:

- Use the cosine ratio in a right triangle to solve for unknown side lengths.
- Use the secant ratio in a right triangle to solve for unknown side lengths.
- Relate the cosine ratio to the secant ratio.
- Use the inverse cosine in a right triangle to solve for unknown angle measures.

KEY TERMS

- cosine (cos)
- secant (sec)
- inverse cosine

ESSENTIAL IDEAS

- The cosine (cos) of an acute angle in a right triangle is the ratio of the length of the side that is adjacent to the angle to the length of the hypotenuse.
- The secant (sec) of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is adjacent to the angle.
- The inverse cosine (or arc cosine) of x is the measure of an acute angle whose cosine is x.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-SRT Similarity, Right Triangles, and Trigonometry

Define trigonometric ratios and solve problems involving right triangles

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

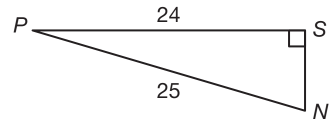
1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

The terms cosine, secant, and inverse cosine are introduced. Guy wires supporting a radio tower are the context for determining the cosine ratio in right triangles. Students conclude that as the acute angle increases in measure, the cosine ratio decreases in value, and the value of cosine will always be less than 1 because the hypotenuse (the denominator in the cosine ratio) is the longest side of the right triangle. Students prove algebraically $\sec A = \frac{1}{\cos A}$. When the arc cosine is introduced, students use their calculators to solve for the measure of an acute angle in a right triangle. Calculators are used throughout this lesson to compute the value of the trigonometric ratios.

Warm Up

1. Calculate the value of sin P.



$24^2 + SN^2 = 25^2$

$SN^2 = 25^2 - 24^2$

$= 625 - 576$

$= 49$

$= \sqrt{49} = 7$

$\sin P = \frac{7}{25} = 0.28$

2. Calculate the value of csc P.

$\csc P = \frac{25}{7}$

≈ 3.57

3. Calculate $m\angle P$.

$\sin^{-1} P = \frac{7}{25}$

$\approx 16.26°$

4. Calculate $m\angle N$.

$m\angle N \approx 180° - 90° - 16.26° \approx 73.74°$

8

8.4

The Cosine Ratio

Cosine Ratio, Secant Ratio, and Inverse Cosine

LEARNING GOALS

In this lesson, you will:

- Use the cosine ratio in a right triangle to solve for unknown side lengths.
- Use the secant ratio in a right triangle to solve for unknown side lengths.
- Relate the cosine ratio to the secant ratio.
- Use the inverse cosine in a right triangle to solve for unknown angle measures.

KEY TERMS

- cosine (cos)
- secant (sec)
- inverse cosine

The applications of trigonometry are tremendous. Engineering, acoustics, architecture, physics . . . you name it, they probably use it.

One important application of trigonometry can be found in finding things—specifically, where you are in the world using GPS, or the Global Positioning System. This system employs about two dozen satellites communicating with a receiver on Earth. The receiver talks to 4 satellites at the same time, uses trigonometry to calculate the information received, and then tells you where on Earth you are.

Problem 1

Guy wires attached to the top of a radio tower form right triangles and the cosine ratio is introduced within this context. Students deduce why the cosine ratio is always a value less than 1. Applying the cosine ratio to right triangles that are not similar, students notice that the measure of an acute angle increases as the value of the cosine decreases. A calculator is used to compute the value of the cosine ratio. A relationship between sine, cosine, and tangent is explored, $\frac{\sin A}{\cos A} = \tan A$.

Grouping

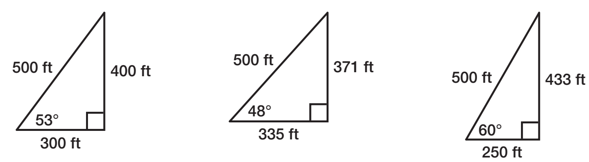
Discuss the information about guy wires and complete Questions 1 and 2 as a class.

PROBLEM 1 Making a Tower Stable

A "guy wire" is used to provide stability to tall structures like radio towers. Guy wires are attached near the top of a tower and are attached to the ground.

A guy wire and its tower form a right triangle. It is important that all guy wires form congruent triangles so that the tension on each wire is the same.

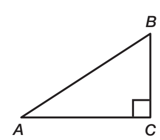
1. Each triangle shown represents the triangle formed by a tower and guy wire. The angle formed by the wire and the ground is given in each triangle.



For each acute angle formed by the wire and the ground, write the ratio of the length of the side adjacent to the angle to the length of the hypotenuse. Write your answers as decimals rounded to the nearest hundredth if necessary.

53° angle: $\frac{300}{500} = 0.6$ 48° angle: $\frac{335}{500} \approx 0.67$ 60° angle: $\frac{250}{500} = 0.5$

The **cosine (cos)** of an acute angle in a right triangle is the ratio of the length of the side that is adjacent to the angle to the length of the hypotenuse. The expression "cos *A*" means "the cosine of $\angle A$."



2. Complete the ratio to represent the cosine of $\angle A$.

$$\cos A = \frac{\text{length of side adjacent to } \angle A}{\text{length of hypotenuse}} = \frac{\boxed{AC}}{\boxed{AB}}$$

Grouping

Have students complete Questions 3 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 through 7

- What is the relationship between the measure of the angle and the height of the radio tower?
- Are the right triangles formed by the radio towers similar? How do you know?
- If the measure of the angle is very small, what information does this give you about the tower?
- How is the cosine ratio similar to the sine ratio?
- How is the cosine ratio different than the sine ratio?
- Why do you suppose the cosine value of the acute angle decreases as the measure of the angle increases?
- How do you remove the radical from the denominator of a fraction?
- How does the value of sin 30° compare the value of cos 60°?
- How does the value of sin 60° compare the value of cos 30°?
- How does the value of sin 45° compare the value of cos 45°?

3. For each triangle in Question 1, calculate the cosine value of the angle made by the guy wire and the ground. Then calculate the cosine value of the other acute angle. Round your answers to the nearest hundredth if necessary.

$\cos 53° = \frac{300}{500} = 0.6$ $\quad$ $\cos 48° = \frac{335}{500} = 0.67$ $\quad$ $\cos 60° = \frac{250}{500} = 0.5$

$\cos 37° = \frac{400}{500} = 0.8$ $\quad$ $\cos 42° = \frac{371}{500} = 0.74$ $\quad$ $\cos 30° = \frac{433}{500} = 0.87$

4. What do the cosine values of the angles in Question 3 all have in common?

The values are all less than 1.

5. Is the cosine value of every acute angle less than 1? Explain your reasoning.

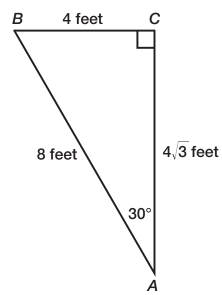
Yes.

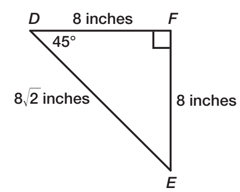
The cosine value of an acute angle is the ratio of the length of a side of a right triangle to the length of the hypotenuse. Because the hypotenuse is the longest side in a right triangle, the ratio will always be less than 1.

6. What happens to the cosine value of an angle as the measure of the angle increases?

The cosine value of an angle decreases as the measure of the angle increases.

7. Use the right triangles shown to calculate the values of cos 30°, cos 45°, and cos 60°. Show all your work.





$$\cos 30° = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$$

$$\cos 45° = \frac{8}{8\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 60° = \frac{4}{8} = \frac{1}{2}$$

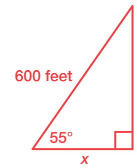
Grouping

Have students complete Questions 8 through 13 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 8 through 13

- When calculating the number of feet from the tower's base to where the wire is attached to the ground, where did you locate the variable on the diagram, in terms of the given acute angle?
- What equation did you use to solve for the unknown measurement?
- How was the Pythagorean Theorem helpful when solving for the unknown measurement in this situation?
- Why was the Pythagorean Theorem needed to solve for the value of cos A?
- Why do you suppose $\frac{\sin A}{\cos A} = \tan A$?
- What do you suppose $\frac{\cos A}{\sin A}$ is equal to?

8. A guy wire is 600 feet long and forms a 55° angle with the ground. First, draw a diagram of this situation. Then, calculate the number of feet from the tower's base to where the wire is attached to the ground.



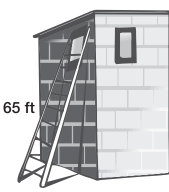
$\cos 55° = \frac{x}{600}$

$600 \cos 55° = x$

$x \approx 344$

The guy wire is attached to the ground about 344 feet from the tower's base.

9. Firemen are climbing a 65′ ladder to the top of a 56′ building. Calculate the distance from the bottom of the ladder to the base of the building, and use the cosine ratio to compute the measure of the angle formed where the ladder touches the top of the building.



$56^2 + b^2 = 65^2$

$b^2 = 65^2 - 56^2$

$b^2 = 4225 - 3136$

$b^2 = 1089$

$b = \sqrt{1089} = 33$

The distance from the bottom of the ladder to the base of the building is 33 feet.

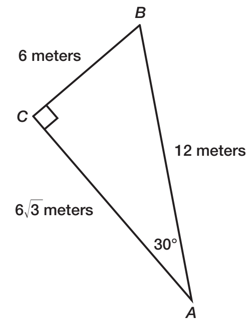
$\cos A = \frac{56}{65}$

$\cos A \approx 0.862$

$m\angle A \approx \cos^{-1} 0.862 \approx 30.46°$

The measure of the angle formed where the ladder touches the top of the building is approximately 30.46°.

10. For the triangle shown, calculate the values of sin 30°, cos 30°, and tan 30°.



$\sin 30° = \frac{6}{12} = \frac{1}{2}$

$\cos 30° = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2}$

$\tan 30° = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

11. Calculate the value of $\frac{\sin 30°}{\cos 30°}$.

$$\frac{\sin 30°}{\cos 30°} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2} \div \frac{\sqrt{3}}{2}$$

$$= \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

12. What do you notice about the value of $\frac{\sin 30°}{\cos 30°}$?

The value of $\frac{\sin 30°}{\cos 30°}$ is equal to tan 30°.

13. Do you think that the relationship between the sine, cosine, and tangent values of an angle is true for any angle? Explain your reasoning.

Yes. The sine value of an angle is the ratio of the length of the opposite side to the length of the hypotenuse and the cosine value of an angle is the ratio of the length of the adjacent side to the length of the hypotenuse. So the quotient of these ratios is the ratio of the length of the opposite side to the length of the adjacent side, which is the tangent value of an angle.

Problem 2

The secant ratio is introduced as a reciprocal relationship of the cosine ratio. Students prove algebraically $\sec A = \frac{1}{\cos A}$. To compute the secant on a calculator, $\frac{1}{\cos A}$ can be used. Students use the cosine and secant ratios to solve for unknown measurements.

Grouping

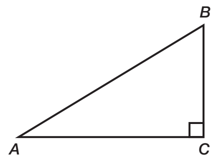
- Discuss the information above Question 1 as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- How can the cos *A* in the denominator be rewritten as a ratio?
- How do you remove a fraction from the denominator?
- How can the ratio $\frac{\textit{length of hypotenuse}}{\textit{length of the side adjacent to } \angle A}$ be rewritten?
- As an acute angle increases in measure, what happens to the denominator of the ratio?
- As the denominator of the ratio decreases, what happens to the fractional value?

PROBLEM 2 Secant Ratio

The **secant (sec)** of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is adjacent to the angle. The expression "sec *A*" means "the secant of $\angle A$."



1. Complete the ratio to represent the secant of $\angle A$.

$$\sec A = \frac{\text{length of hypotenuse}}{\text{length of side adjacent to } \angle A} = \frac{\boxed{AB}}{\boxed{AC}}$$

2. Prove algebraically that the secant of $A = \frac{1}{\cos A}$.

$$\sec A = \frac{1}{\cos A}$$

$$\frac{1}{\cos A} = \frac{1}{\frac{\text{length of the side adjacent to } \angle A}{\text{length of the hypotenuse}}}$$

$$= \frac{\text{length of hypotenuse}}{\text{length of the side adjacent to } \angle A}$$

$$= \sec A$$

3. As the measure of an acute angle increases, the cosine value of the acute angle decreases. Explain the behavior of the secant value of an acute angle as the acute angle increases.

   As an acute angle increases, its secant value increases because the denominator of the ratio decreases, causing the fractional value to increase.

4. If there is no "sec" button on your graphing calculator, how can you compute the secant?

   To compute the secant on a graphing calculator, express the secant as $\frac{1}{\cos A}$.

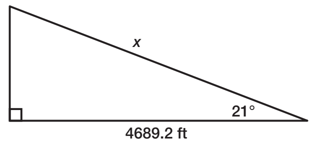
Grouping

Have students complete Question 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- Was it easier to use the cosine ratio or the secant ratio to solve for the length of the ski slope? Why?
- Why can both cosine and secant be used to determine the length of the slope?

5. The diagram shows the measurements for the ski slope in the first lesson.



a. Use the cosine function to determine the length of the slope.

$\cos 21° = \frac{4689.2}{x}$

$x(\cos 21°) = 4689.2$

$x = \frac{4689.2}{\cos 21°}$

$x \approx 5022.81$

The ski slope is approximately 5022.81 feet long.

b. Use the secant function to determine the length of the slope.

$\sec 21° = \frac{x}{4689.2}$

$4689.2(\sec 21°) = x$

$4689.2\left(\frac{1}{\cos 21°}\right) = x$

$x \approx 5022.81$

The ski slope is approximately 5022.81 feet long.

Problem 3

The inverse cosine is introduced and students are now able use their calculators to solve for the measure of an acute angle in a right triangle if the length of the adjacent side and the length of the hypotenuse is known.

Grouping

Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- Where is arc cosine or $\cos^{-1}$ found on your calculator?
- Is the ratio for cos B $\frac{16}{18}$ or $\frac{18}{16}$?
- How is the ratio for cos B used to determine $m\angle B$?
- Is the ratio for cos B $\frac{5}{8}$ or $\frac{8}{5}$?
- How is the ratio for cos B used to determine $m\angle B$?
- Is the ratio for cos x $\frac{95}{80}$ or $\frac{80}{95}$?

PROBLEM 3 Inverse Cosine

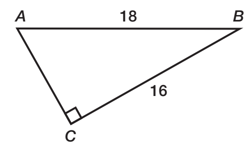
The **inverse cosine** (or arc cosine) of x is defined as the measure of an acute angle whose cosine is x. If you know the length of any two sides of a right triangle, it is possible to compute the measure of either acute angle by using the inverse cosine, or $\cos^{-1}$ button on a graphing calculator.

In right triangle ABC, if $\cos A = x$, then $\cos^{-1} x = m\angle A$.

1. In right triangle ABC, if $\cos A = \frac{2}{7}$, calculate $\cos^{-1}\left(\frac{2}{7}\right)$ to determine $m\angle A$.

   $m\angle A = \cos^{-1}\left(\frac{2}{7}\right) \approx 73.40°$

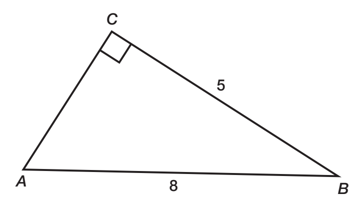
2. Determine the ratio for cos B, and then use $\cos^{-1}(\cos B)$ to calculate $m\angle B$.



   $\cos B = \frac{16}{18}$

   $m\angle B = \cos^{-1}\left(\frac{16}{18}\right) \approx 27.27°$

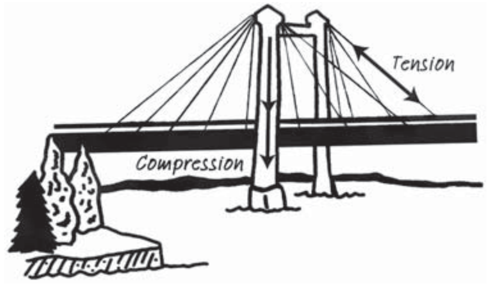
3. Calculate $m\angle B$.



   $\cos B = \frac{5}{8}$

   $m\angle B = \cos^{-1}\left(\frac{5}{8}\right) \approx 51.32°$

4. A typical cable-stayed bridge is a continuous girder with one or more towers erected above piers in the middle of the span. From these towers, cables stretch down diagonally (usually to both sides) and support the girder. Tension and compression are calculated into the design of this type of suspension bridge.



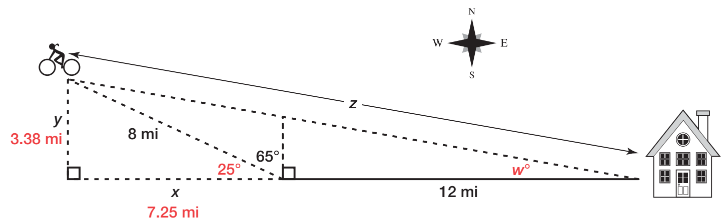
One cable is 95 feet. The span on the deck of the bridge from that cable to the girder is 80 feet. Calculate the angle formed by the deck and the cable.

$\cos x = \frac{80}{95}$

$m\angle x = \cos^{-1}\left(\frac{80}{95}\right) \approx 32.66°$

The angle formed by the deck and the cable is approximately 32.66°.

5. Diane is training for a charity bicycle marathon. She leaves her house at noon and heads due west, biking at an average rate of 4 miles per hour. At 3 PM she changes course to N 65°W as shown. Determine the bike's distance from Diane's home at 5 PM.



$$\cos 25° = \frac{x}{8}$$

$$8(\cos 25°) = x$$

$$x \approx 7.25$$

$$\sin 25° = \frac{y}{8}$$

$$8(\sin 25°) = y$$

$$y \approx 3.38$$

$$\tan w° = \frac{y}{x + 12}$$

$$= \frac{3.38}{7.25 + 12}$$

$$= \frac{3.38}{19.25} \approx 0.1755844156$$

$$w = \text{arc tan } 0.1755844156 \approx 9.96°$$

$$\sin 9.96° = \frac{3.38}{z}$$

$$z(\sin 9.96°) \approx 3.38$$

$$z = \frac{3.38}{\sin 9.96°} \approx 19.54$$

At 5 PM, Diane is 19.54 miles from home.

Talk the Talk

Students match each trigonometric ratio with the appropriate abbreviation and description. Last, they determine which ratio can be used to solve different situations that are described.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Talk the Talk

1. Match each trigonometric function with the appropriate abbreviation.

1. Sine	(G)	A. $\cos^{-1}$
2. Cosine	(F)	B. cot
3. Tangent	(H)	C. csc
4. Cosecant	(C)	D. $\tan^{-1}$
5. Secant	(I)	E. $\sin^{-1}$
6. Cotangent	(B)	F. cos
7. Arc tan	(D)	G. sin
8. Arc sin	(E)	H. tan
9. Arc cos	(A)	I. sec

2. Match each trigonometric function with the appropriate description.

1. Sin	(C)	A. $\frac{hypotenuse}{opposite}$
2. Cos	(F)	B. $\frac{hypotenuse}{adjacent}$
3. Tan	(D)	C. $\frac{opposite}{hypotenuse}$
4. Csc	(A)	D. $\frac{opposite}{adjacent}$
5. Sec	(B)	E. $\frac{adjacent}{opposite}$
6. Cot	(E)	F. $\frac{adjacent}{hypotenuse}$

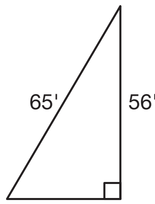
3. Given the known information and the solution requirement, determine which function can be used to solve each situation.

Known Information	Solution Requirement	Function Used
• Hypotenuse • Opposite	Measure of reference angle	$\sin^{-1}$
• Opposite • Acute angle measure	Hypotenuse	tan or cot
• Hypotenuse • Acute angle measure	Adjacent	cos or sec
• Opposite • Adjacent	Measure of reference angle	$\tan^{-1}$
• Hypotenuse • Acute angle measure	Opposite	sin or csc
• Hypotenuse • Adjacent	Measure of reference angle	$\cos^{-1}$

Be prepared to share your solutions and methods.

Check for Students' Understanding

Firemen are climbing a 65 foot ladder to the top of a building that is 56 feet tall. Calculate the distance from the bottom of the ladder to the base of the building and use the cosine ratio to compute the measure of the angle formed where the ladder touches the top of the building.



$56^2 + b^2 = 65^2$

$b^2 = 65^2 - 56^2$

$b^2 = 4225 - 3136$

$b^2 = 1089$

$b = \sqrt{1089} = 33'$

The distance from the bottom of the ladder to the base of the building is 33′.

$\cos A = \frac{56}{65}$

$\cos A \approx .862$

$m\angle A = \cos^{-1} .862 \approx 30.46°$

The measure of the angle formed where the ladder touches the top of the building is approximately 30.46°.

8

8.5

We Complement Each Other!

Complement Angle Relationships

LEARNING GOALS

In this lesson, you will:

- Explore complement angle relationships in a right triangle.
- Solve problems using complement angle relationships.

ESSENTIAL IDEAS

- When $\angle A$ and $\angle B$ are acute angles in a right triangle, $\sin \angle A = \cos \angle B$ and $\cos \angle A = \sin \angle B$.
- When $\angle A$ and $\angle B$ are acute angles in a right triangle, $\csc \angle A = \sec \angle B$ and $\sec \angle A = \csc \angle B$.
- When $\angle A$ and $\angle B$ are acute angles in a right triangle, $\tan \angle A = \cot \angle B$ and $\cot \angle A = \tan \angle B$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-SRT Similarity, Right Triangles, and Trigonometry

Define trigonometric ratios and solve problems involving right triangles

7. Explain and use the relationship between the sine and cosine of complementary angles.

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

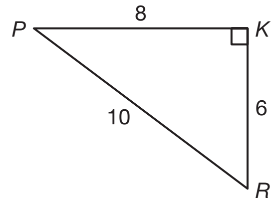
Overview

Students explore the complementary relationships involved with trigonometric ratios and use them to solve application problems. The Pythagorean Theorem in conjunction with complementary relationships is used to determine the value of the six trigonometric ratios of a 45° angle, a 30° angle, and a 60° angle.

Warm Up

1. Describe the relationship between $\angle P$ and $\angle R$ in right triangle PKR.

   Angle P and angle R are complementary angles.



2. Write a ratio that represents sin $\angle P$.

   $\frac{6}{10} = \frac{3}{5}$

3. Write a ratio that represents cos $\angle R$.

   $\frac{6}{10} = \frac{3}{5}$

4. What do you notice about the ratios representing sin $\angle P$ and cos $\angle R$?

   The ratios are equal.

5. How does the ratio representing cos $\angle P$ compare to the ratio representing the sin $\angle R$?

   $\frac{8}{10} = \frac{4}{5}$

   The ratios are equal.

8

8

We Complement Each Other!

8.5

Complement Angle Relationships

LEARNING GOALS

In this lesson, you will:

- Explore complement angle relationships in a right triangle.
- Solve problems using complement angle relationships.

You've worked with complements before, remember? Two angles whose measures add up to 90 degrees are called complements.

In right triangles, complements are pretty easy to locate. The two angles whose measures are *not* 90 degrees must be complements. In trigonometry, complements are easy to identify by name, too. The prefix "co-" in front of trigonometric ratio names stands for "complement."

8

Problem 1

Students compare trigonometric ratios by organizing them in a table and conclude that, given acute reference angles A and B in a right triangle: $\sin \angle A = \cos \angle B$, $\sin \angle B = \cos \angle A$, $\csc \angle A = \sec \angle B$, $\csc \angle B = \sec \angle A$, $\tan \angle A = \cot \angle B$, and $\tan \angle B = \cot \angle A$.

Grouping

Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

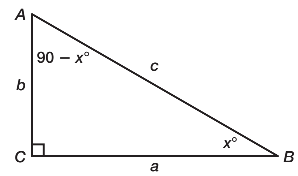
Guiding Questions for Share Phase, Questions 1 through 5

- What is the sine ratio?
- What is the cosine ratio?
- Why do you suppose the ratio representing $\sin \angle A$ is equal to the ratio representing $\cos \angle B$?
- Why do you suppose the ratio representing $\cos \angle A$ is equal to the ratio representing $\sin \angle B$?
- How would you describe the relationship between the side opposite $\angle A$ and the side adjacent to $\angle B$?
- How would you describe the relationship between the side adjacent to $\angle A$ and the side opposite $\angle B$?
- What is the cosecant ratio?
- What is the secant ratio?
- Why do you suppose the ratio representing $\csc \angle A$ is equal to the ratio representing $\sec \angle B$?

PROBLEM 1 Angles Are Very Complementary!

Consider triangle ABC with right angle C. Angles A and B are complementary angles because the sum of their measures is equal to 90 degrees. The trigonometric ratios also have complementary relationships.

1. Use triangle ABC to answer each question.



a. Compare the ratios that represent $\sin \angle A$ and $\cos \angle B$.

$\sin \angle A = \frac{a}{c}$

$\cos \angle B = \frac{a}{c}$

$\sin \angle A$ and $\cos \angle B$ are the same ratio.

b. Compare the ratios that represent $\sin \angle B$ and $\cos \angle A$.

$\sin \angle B = \frac{b}{c}$

$\cos \angle A = \frac{b}{c}$

$\sin \angle B$ and $\cos \angle A$ are the same ratio.

c. Compare the ratios that represent $\csc \angle A$ and $\sec \angle B$.

$\csc \angle A = \frac{c}{a}$

$\sec \angle B = \frac{c}{a}$

$\csc \angle A$ and $\sec \angle B$ are the same ratio.

d. Compare the ratios that represent $\tan \angle A$ and $\cot \angle B$.

$\tan \angle A = \frac{a}{b}$

$\cot \angle B = \frac{a}{b}$

$\tan \angle A$ and $\cot \angle B$ are the same ratio.

- Why do you suppose the ratio representing $\sec \angle A$ is equal to the ratio representing $\csc \angle B$?
- What is the tangent ratio?
- What is the cotangent ratio?
- Why do you suppose the ratio representing $\tan \angle A$ is equal to the ratio representing $\cot \angle B$?
- Why do you suppose the ratio representing $\cot \angle A$ is equal to the ratio representing $\tan \angle B$?

2. Which two functions are described by the ratio $\frac{c}{b}$?

$\sec \angle A = \frac{c}{b}$

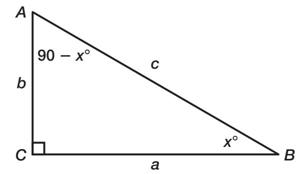
$\csc \angle B = \frac{c}{b}$

3. Which two functions are described by the ratio $\frac{b}{a}$?

$\cot \angle A = \frac{b}{a}$

$\tan \angle B = \frac{b}{a}$

4. Use your answers to Questions 1 through 3 to complete the table.



Reference Angle	sin	cos	tan	csc	sec	cot
A	$\frac{a}{c}$	$\frac{b}{c}$	$\frac{a}{b}$	$\frac{c}{a}$	$\frac{c}{b}$	$\frac{b}{a}$
B	$\frac{b}{c}$	$\frac{a}{c}$	$\frac{b}{a}$	$\frac{c}{b}$	$\frac{c}{a}$	$\frac{a}{b}$

5. Summarize the relationship between the trigonometric functions of complementary angles.

The sine of an angle is equal to the cosine of its complement.

The tangent of an angle is equal to the cotangent of its complement.

The secant of an angle is equal to the cosecant of its complement.

Problem 2

Students use complementary relationships in conjunction with the Pythagorean Theorem to determine the values of the six trigonometric ratios for angle measurements of 30°, 60°, and 45°. They use the information in the table to solve an application problem.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

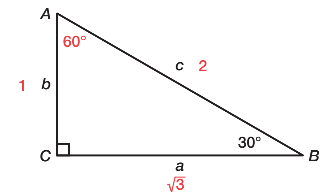
Guiding Questions for Share Phase, Questions 1 and 2

- If sin 30° = 0.5, how can sin 30° be rewritten as a fraction?
- If sin 30° = $\frac{1}{2}$, what is the length of the leg opposite the 30° angle (b), and the length of the hypotenuse (c)?
- How is the Pythagorean Theorem used to solve for the length of the leg adjacent to $\angle A$ (a)?
- What is the relationship between the length of the side opposite the 30° angle (b), and the length of the hypotenuse?
- What is the relationship between the length of the side opposite the 60° angle (b), and the length of the hypotenuse?

PROBLEM 2 Using Complements

1. Given: sin 30° = 0.5

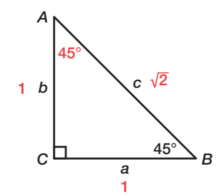
   Use the Pythagorean Theorem and your knowledge of complementary functions to complete the chart. Include both a ratio and its decimal equivalent for each function.



Reference Angle	sin	cos	tan	csc	sec	cot
30°	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} \approx 0.866$	$\frac{1}{\sqrt{3}} \approx 0.577$	2	$\frac{2}{\sqrt{3}} \approx 1.1547$	$\frac{\sqrt{3}}{1} \approx 1.732$
60°	$\frac{\sqrt{3}}{2} \approx 0.866$	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{1} \approx 1.732$	$\frac{2}{\sqrt{3}} \approx 1.1547$	2	$\frac{1}{\sqrt{3}} \approx 0.577$

2. Given: sin 45° = 0.707

   Use the Pythagorean Theorem and your knowledge of complementary functions to complete the chart. Include both a ratio and its decimal equivalence for each function.



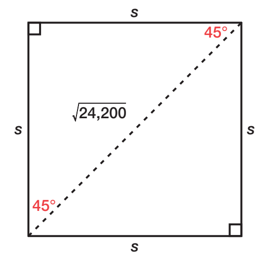
Reference Angle	sin	cos	tan	csc	sec	cot
45°	$\frac{1}{\sqrt{2}} \approx 0.707$	$\frac{1}{\sqrt{2}} \approx 0.707$	$\frac{1}{1} = 1$	$\frac{\sqrt{2}}{1} \approx 1.414$	$\frac{\sqrt{2}}{1} \approx 1.414$	$\frac{1}{1} = 1$

- If the right triangle is a 45°–45°–90°, what is the relationship between the lengths of the legs (a and b)?
- If each leg has a length of 1, what is the length of the hypotenuse?
- How is the Pythagorean Theorem used to solve for the length of the hypotenuse (c)?

Guiding Questions for Share Phase, Question 3

- The diagonal of a square divides the square into what type of triangles?
- If the legs of a right triangle are equal in length, what else is known about the triangle?
- What trigonometric ratio was used to solve for the dimensions of Trafalgar Square?
- Is there another trigonometric ratio that could have been used to solve for the dimensions of Trafalgar Square?
- Did you get the same answer when you used the Pythagorean Theorem to solve for the dimensions of Trafalgar Square?
- Is the answer exact or an approximation?

3. Trafalgar Square is a tourist attraction located in London, England, United Kingdom. The name commemorates the Battle of Trafalgar (1805), a British naval victory of the Napoleonic Wars over France.



a. Use a trigonometric function to solve for the dimensions of Trafalgar Square.

$\cos 45° = \frac{\sqrt{2}}{2}$

$\cos 45° = \frac{s}{\sqrt{24{,}200}}$

$\frac{\sqrt{2}}{2} = \frac{s}{\sqrt{24{,}200}}$

$2s = (\sqrt{2})(\sqrt{24{,}200})$

$2s = 220$

$s = 110$

The dimensions of Trafalgar Square are 110 meters by 110 meters.

b. Use the Pythagorean Theorem to solve for the dimensions of Trafalgar Square.

$a^2 + b^2 = c^2$

$a^2 + a^2 = (\sqrt{24{,}200})^2$

$2a^2 = 24{,}200$

$a^2 = 12{,}100$

$a = \sqrt{12{,}110} \approx 110$

The dimensions of Trafalgar Square are approximately 110 meters by 110 meters.

Problem 3

Students use the trigonometric ratios to solve four application problems.

Grouping

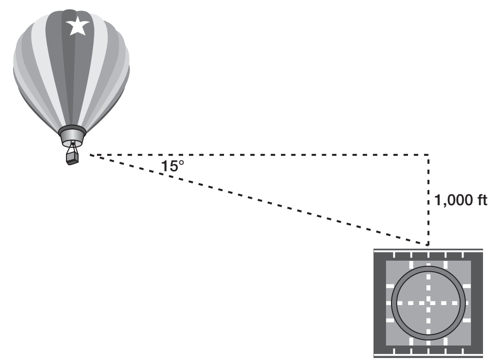
Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Which trigonometric ratio was used to solve this problem?
- Did you use the given angle or the complement of the given angle to solve this problem?
- Is there a different trigonometric ratio which could have been used to solve this problem?
- Which ratio was easier to work with? Why?
- Is this an exact answer or an approximation?

PROBLEM 3 Hot Air Balloons, The Grand Canyon, and Radar

1. At an altitude of 1,000 feet, a balloonist measures the angle of depression from the balloon to the landing zone. The measure of that angle is 15°. How far is the balloon from the landing zone?



$$\sin 15° = \frac{1000}{c}$$

$$c \sin 15° = 1000$$

$$c = \frac{1000}{\sin 15°}$$

$$c \approx 3863.7$$

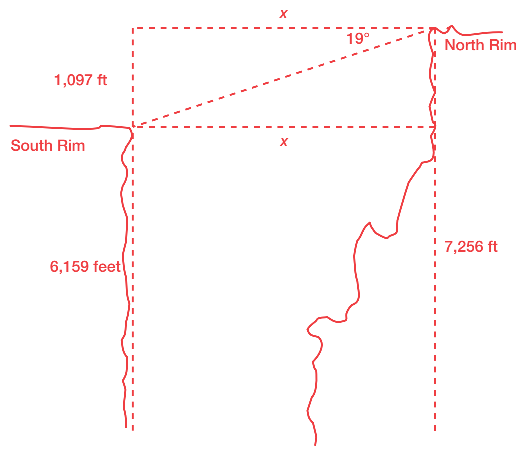
The hot air balloon is approximately 3863.7 feet from the landing zone.

Guiding Questions for Share Phase, Question 2

- Was it difficult to sketch a diagram of this situation? Why?
- Where did you locate the given information?
- Is there a right triangle in your diagram?
- Which parts of the right triangle were given?
- Did you locate the distance of 1,097 feet across from the 19° angle in the right triangle?
- Was the unknown distance the leg adjacent to the 19° angle in the right triangle?

2. To measure the width of the Grand Canyon, a surveyor stands at a point on the North Rim of the canyon and measures the angle of depression to a point directly across on the South Rim of the canyon.

At the surveyor's position on the North Rim, the Grand Canyon is 7,256 feet above sea level. The point on the South Rim, directly across, is 6,159 feet above sea level. Sketch a diagram of the situation and determine the width of the Grand Canyon at the surveyor's position.



$$\tan 19° = \frac{1097}{x}$$

$$x \tan 19° = 1097$$

$$x = \frac{1097}{\tan 19°}$$

$$x \approx 3185.9$$

The distance between the North Rim and the South Rim at the surveyor's position is approximately 3185.9 feet.

8

Guiding Questions for Share Phase, Question 3

- Are there two congruent right triangles in your diagram?
- What side of the right triangle is 8,000 feet?
- Where are the unknown distances in this diagram with respect to the 12° angle?
- Which two trigonometric ratios were used to solve this problem situation?
- When solving for the vertical distance, which trigonometric ratio is easiest to use?

3. An aircraft uses radar to spot another aircraft 8,000 feet away at a 12° angle of depression. Sketch the situation and determine the vertical and horizontal separation between the two aircraft.

$\sin 12° = \frac{y}{8000}$

$8000 \sin 12° = y$

$y \approx 1663.3$

$\cos 12° = \frac{x}{8000}$

$8000 \cos 12° = x$

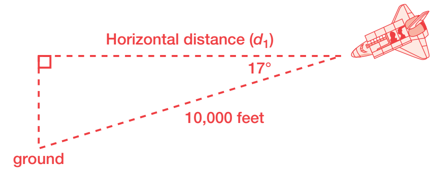
$x \approx 7825.2$

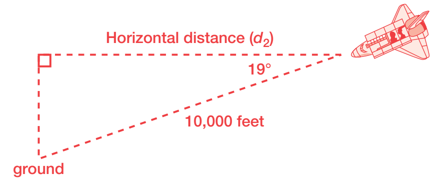
The vertical distance is approximately 1663.3 feet and the horizontal distance is approximately 7825.2 feet.

Guiding Questions for Share Phase, Question 4

- When solving for the horizontal distance, which trigonometric ratio is easiest to use?
- Are the distances exact or approximate?

4. When a space shuttle returns from a mission, the angle of its descent to the ground from the final 10,000 feet above the ground is between 17° and 19° with the horizontal. Sketch a diagram of the situation and determine the maximum and minimum horizontal distances between the landing site and where the descent begins.





$\cos 17° = \frac{d_1}{10{,}000}$

$10{,}000 \cos 17° = d_1$

$d_1 \approx 9563.1$

$\cos 19° = \frac{d_2}{10{,}000}$

$10{,}000 \cos 19° = d_2$

$d_2 \approx 9455.2$

The minimum horizontal distance is approximately 9455.2 feet and the maximum horizontal distance is approximately 9563.1 feet.

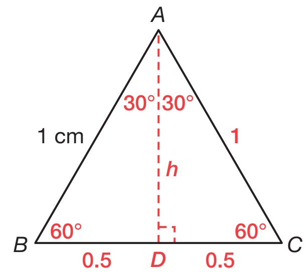
Be prepared to share your solutions and methods.

Check for Students' Understanding

$\triangle ABC$ is an equilateral triangle.

$AB = 1$ cm

Sketch the perpendicular bisector of side BC to form two right triangles.



Determine the height of triangle ABC.

The measure of each interior angle of an equilateral triangle is 60°.

The perpendicular bisector of side BC divides triangle ABC into two congruent triangles. Each triangle is a 30°–60°–90° triangle.

$\sin 60° = \frac{h}{1}$

$0.866 = \frac{h}{1}$

$h \approx 0.866$

The height of triangle ABC is approximately 0.866 centimeters.

8.6

Time to Derive!

Deriving the Triangle Area Formula, the Law of Sines, and the Law of Cosines

LEARNING GOALS

In this lesson, you will:

- Derive the formula for the area of a triangle using the sine function.
- Derive the Law of Sines.
- Derive the Law of Cosines.

KEY TERMS

- Law of Sines
- Law of Cosines

ESSENTIAL IDEAS

- The area formula for any triangle is $A = \frac{1}{2}ab(\sin C)$.
- The Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
- The Law of Cosines:

  $a^2 = b^2 + c^2 - 2bc \cos A$

  $b^2 = a^2 + c^2 - 2ac \cos B$

  $c^2 = a^2 + b^2 - 2ab \cos C$

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-SRT Similarity, Right Triangles, and Trigonometry

Apply trigonometry to general triangles

9. Derive the formula $A = \frac{1}{2}ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. Prove the Laws of Sines and Cosines and use them to solve problems.
11. Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

8

Overview

Students derive the area formula for any triangle, the Law of Sines, and the Law of Cosines algebraically. Students start each activity with a triangle. Altitudes play a major role in forming the right triangles needed to apply the trigonometric ratios. Students apply these concepts in application problems to determine unknown measurements in triangular situations.

Warm Up

Simplify each expression.

1. $(\sin A)^2$

   $\sin^2 A$

2. $(c \sin A)^2$

   $c^2\sin^2 A$

3. $(b - c \cos A)^2$

   $b^2 - 2bc \cos A + c^2 \cos^2 A$

Time to Derive!

Deriving the Triangle Area Formula, the Law of Sines, and the Law of Cosines

8.6

LEARNING GOALS

In this lesson, you will:

- Derive the formula for the area of a triangle using the sine function.
- Derive the Law of Sines.
- Derive the Law of Cosines.

KEY TERMS

- Law of Sines
- Law of Cosines

Suppose you want to measure the height of a tree. You are 100 feet from the tree, and the angle from your feet to the top of the tree is 33 degrees. However, the tree isn't growing straight up from the ground. It leans a little bit toward you. The tree is actually growing out of the ground at an 83 degree angle. How tall is the tree?

Once you finish this lesson, see if you can answer this question.

8

Problem 1

Students derive the area formula for a triangle: $A = \frac{1}{2}ab(\sin C)$, where a and b are lengths of two sides of a triangle and C is the included angle, using the area formula for a triangle $A = \frac{1}{2}bh$ and the sine ratio. Students use the formula to solve for the area of any triangle.

Grouping

- Ask a student to read the information above Question 1 aloud and discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

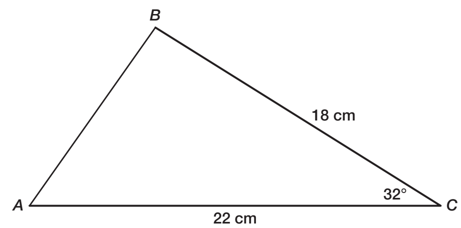
- What does b represent in the formula for the area of a triangle?
- What does h represent in the formula for the area of a triangle?
- What is the ratio representing $\sin C$?
- Is $\angle C$ the included angle with respect to sides a and b?
- What measurements must be given to use this formula to solve for the area of the triangle?
- What is the value of $\sin 32°$?

PROBLEM 1 Deriving Another Version of the Area Formula

Whether you are determining the area of a right triangle, solving for the unknown side lengths of a right triangle, or solving for the unknown angle measurements in a right triangle, the solution paths are fairly straightforward. You can use what you learned previously, such as the area formula for a triangle, the Pythagorean Theorem, and the Triangle Sum Theorem.

Solving for unknown measurements of sides or angles of a triangle becomes more involved if the given triangle is not a right triangle.

Consider triangle *ABC* as shown.



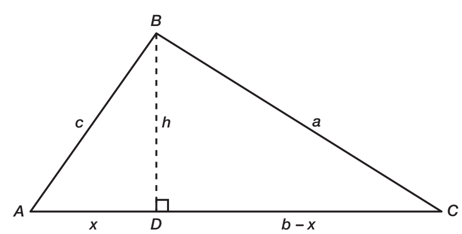
Use of the area formula requires the height of the triangle, which is not given.

Use of the Pythagorean Theorem requires the triangle to be a right triangle, which it is not. Use of the Triangle Sum Theorem requires the measures of two angles of the triangle, which are not given.

In this lesson, you will explore how trigonometric ratios are useful when determining the area of *any* triangle, solving for unknown lengths of sides of *any* triangle, and solving for unknown measures of angles in *any* triangle.

1. Analyze triangle *ABC*.

   a. Write the formula for the area of triangle *ABC* in terms of b and h.



$$A = \frac{1}{2}(\text{base})(\text{height})$$
$$= \frac{1}{2}(b - x + x)(h)$$
$$= \frac{1}{2}bh$$

- Is the sine of an interior angle of a triangle always less than 1, greater than 1, or equal to 1?
- Is the answer exact or an approximation?
- When is it appropriate to use this area formula?

b. Write the ratio that represents sin C and solve for the height, h.

$\sin C = \frac{h}{a}$

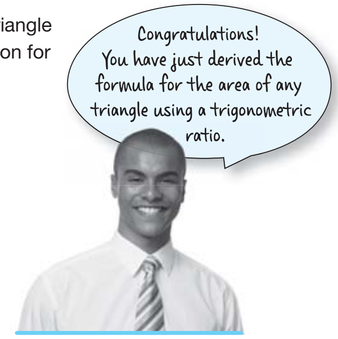
$h = a \sin C$

c. Rewrite the formula you wrote for the area of triangle ABC in Question 1 by substituting the expression for the value of h in Question 2.

$A = \frac{1}{2}bh$

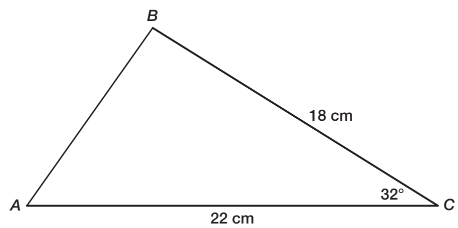
$= \frac{1}{2}b(a \sin C)$

$= \frac{1}{2}ab(\sin C)$



The area formula, $A = \frac{1}{2}ab \sin C$, can be used to determine the area of any triangle if you know the lengths of two sides and the measure of the included angle.

2. Use a trigonometric ratio to determine the area of the triangle.



$A = \frac{1}{2}ab(\sin C)$

$= \frac{1}{2}(18)(22)(\sin 32°)$

≈ 104.9

The area of the triangle is approximately 104.9 square centimeters.

Problem 2

Students derive the Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, where a, b, and c are lengths of the sides of a triangle and A, B, and C are the interior angles of a triangle by writing equivalent trigonometric ratios. Students use the Law of Sines to determine the lengths of sides and the measures of angles in any triangle.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

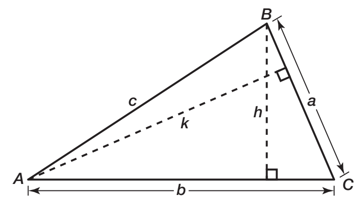
- If both ratios are equal to h, what can you conclude about the ratios using the transitive property?
- What is a proportion?
- How is a proportion written?
- When writing the proportion, what was written in the numerators and what was written in the denominators of each ratio?

PROBLEM 2 Deriving the Law of Sines

You have used the trigonometric ratios to solve for unknown side length and angle measures in right triangles. Let's explore relationships between side lengths and angle measures in any triangle.

1. Analyze triangle *ABC*.

   a. Write a ratio to represent sin *A*, and then solve for the height, *h*.



$\sin A = \frac{h}{c}$

$h = c \cdot \sin A$

   b. Write a ratio that represents sin *C*, and then solve for the height, *h*.

$\sin C = \frac{h}{a}$

$h = a \cdot \sin C$

   c. What can you conclude about the relationship between $c \cdot \sin A$ and $a \cdot \sin C$?

Both expressions are equal to h, the height of the triangle, so $c \cdot \sin A = a \cdot \sin C$.

   d. Express $c \cdot \sin A = a \cdot \sin C$ as a proportion.

$c \cdot \sin A = a \cdot \sin C$

$\frac{\sin A}{a} = \frac{\sin C}{c}$

   e. Write a ratio that represents sin *B*, and then solve for the height, *k*.

$\sin B = \frac{k}{c}$

$k = c \cdot \sin B$

   f. Write a ratio that represents sin *C*, and then solve for the height, *k*.

$\sin C = \frac{k}{b}$

$k = b \cdot \sin C$

Grouping

- Ask a student to read the information above Question 2 and discuss as a class.
- Have students complete Question 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- Did your classmates write the same proportion?
- Is there a different way the proportion can be written?
- When is it appropriate to use the Law of Sines?

g. What can you conclude about the relationship between $c \cdot \sin B$ and $b \cdot \sin C$?

Both expressions are equal to k, the height of the triangle, so $c \cdot \sin B = b \cdot \sin C$.

h. Express $c \cdot \sin B = b \cdot \sin C$ as a proportion.

$c \cdot \sin B = b \cdot \sin C$

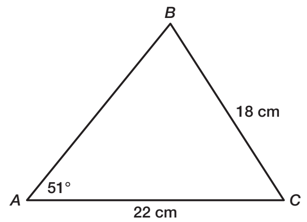
$\frac{\sin B}{b} = \frac{\sin C}{c}$

i. Derive the *Law of Sines* by combining the proportions formed in parts (d) and (h).

If $\frac{\sin A}{a} = \frac{\sin C}{c}$ and $\frac{\sin B}{b} = \frac{\sin C}{c}$, then $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

The **Law of Sines**, or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, can be used to determine the unknown side lengths or the unknown angle measures in *any* triangle.

2. Use the Law of Sines to determine the measure of $\angle B$.



$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$\frac{\sin A}{a} = \frac{\sin B}{b}$

$\frac{\sin 51}{18} = \frac{\sin B}{22}$

$22\sin 51 = 18\sin B$

$17.1 = 18 \sin B$

$\sin B = \frac{17.1}{18} \approx 0.95$

$\sin^{-1} B \approx 71.8°$

Problem 3

Students derive the Law of Cosines

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

algebraically using trigonometric ratios, the Pythagorean Theorem, multiplying binomials, combining like terms, and the Substitution Property.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

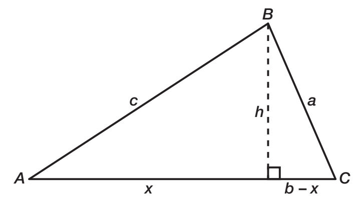
Guiding Questions for Share Phase, Question 1

- Which triangle is used to solve for the value of a^2?
- How do you square $(c \sin A)$?
- How do you square $(b - c \cos A)$?
- Did you group together the two terms containing the factor c^2?
- Did you factor out c^2 of two terms?
- Did you substitute 1 for $(\sin^2 + \cos^2)$?

PROBLEM 3 Deriving the Law of Cosines

The Law of Sines is one relationship between the side lengths and angle measures of any triangle. Another relationship is called the *Law of Cosines*.

1. Analyze triangle *ABC*.



a. Write a ratio that represents sin *A*, and then solve for the height, *h*.

$\sin A = \frac{h}{c}$

$h = c \cdot \sin A$

b. Write a ratio that represents cos *A*, and then solve for *x*.

$\cos A = \frac{x}{c}$

$x = c \cdot \cos A$

c. Solve for a^2 using the Pythagorean Theorem.

$h^2 + (b - x)^2 = a^2$

d. Substitute the expressions for *h* and *x* into the equation in part (c).

$h^2 + (b - x)^2 = a^2$

$(c \cdot \sin A)^2 + (b - c \cdot \cos A)^2 = a^2$

e. Use the equation you wrote in part (d) to solve for a^2.

$a^2 = (c \cdot \sin A)^2 + (b - c \cdot \cos A)^2$

$a^2 = c^2 \cdot \sin^2 A + b^2 - 2bc \cdot \cos A + c^2 \cdot \cos^2 A$

$a^2 = c^2 \cdot \sin^2 A + b^2 - 2bc \cdot \cos A + c^2 \cdot \cos^2 A$

$a^2 = c^2(\sin^2 A + \cos^2 A) + b^2 - 2bc \cdot \cos A$

$a^2 = c^2(1) + b^2 - 2bc \cdot \cos A$

$a^2 = c^2 + b^2 - 2bc \cdot \cos A$

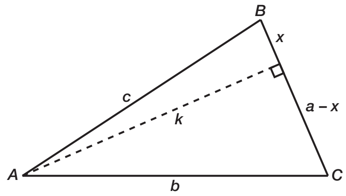
Grouping

Have students complete Question 2 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 2 through 4

- Looking at the three conclusions, do you see a pattern?
- When is it appropriate to use the Law of Cosines?
- What is the value of the cosine of a right angle?
- If the value of the cosine of a right angle is 0, what happens to the last term in the equation?
- Do you recognize the final equation?

2. Repeat the steps in Question 1 to solve for b^2.



$\sin B = \frac{k}{c}$

$k = c \cdot \sin B$

$\cos B = \frac{x}{c}$

$x = c \cdot \cos B$

$b^2 = k^2 + (a - x)^2$

$b^2 = (c \cdot \sin B)^2 + (a - c \cdot \cos B)^2$

$b^2 = c^2 \cdot \sin^2 B + a^2 - 2ac \cdot \cos B + c^2 \cdot \cos^2 B$

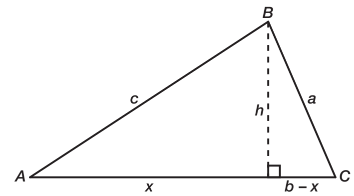
$b^2 = c^2 \cdot \sin^2 B + a^2 - 2ac \cdot \cos B + c^2 \cdot \cos^2 B$

$b^2 = c^2(\sin^2 B + \cos^2 B) + a^2 - 2ac \cdot \cos B$

$b^2 = c^2(1) + a^2 - 2ac \cdot \cos B$

$b^2 = a^2 + c^2 - 2ac \cdot \cos B$

3. Repeat the steps in Question 1 to solve for c^2.



$\sin C = \frac{h}{a}$

$h = a \cdot \sin C$

$\cos C = \frac{b - x}{a}$

$b - x = a \cdot \cos C$

$x = b - a \cdot \cos C$

$c^2 = h^2 + x^2$

$c^2 = (a \cdot \sin C)^2 + (b - a \cdot \cos C)^2$

$c^2 = a^2 \cdot \sin^2 C + b^2 - 2ab \cdot \cos C + a^2 \cdot \cos^2 C$

$c^2 = a^2 \cdot \sin^2 C + b^2 - 2ab \cdot \cos C + a^2 \cdot \cos^2 C$

$c^2 = a^2(\sin^2 C + \cos^2 C) + b^2 - 2ab \cdot \cos C$

$c^2 = a^2(1) + b^2 - 2ab \cdot \cos C$

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$

The **Law of Cosines**, or

$$a^2 = c^2 + b^2 - 2bc \cdot \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cdot \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$

can be used to determine the unknown lengths of sides or the unknown measures of angles in *any* triangle.

4. Why is the Pythagorean Theorem considered to be a special case of the Law of Cosines?

The Pythagorean Theorem is considered to be a special case of the Law of Cosines because when a triangle is a right triangle, the cosine of the right angle has a value of zero and the Law of Cosines simplifies to the Pythagorean Theorem.

Suppose $\angle C$ is the right angle in triangle ABC.

$c^2 = a^2 + b^2 - 2ab(\cos C)$

$c^2 = a^2 + b^2 - 2ab(0)$

$c^2 = a^2 + b^2$

Problem 4

The Law of Cosines or the Law of Sines is used to solve application problems.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

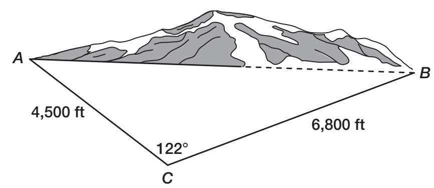
Guiding Questions for Share Phase, Question 1

- Will the Law of Sines or the Law of Cosines be useful in determining the length of the proposed tunnel? How do you know?
- How did you determine which Law of Cosines was needed to determine the length of the proposed tunnel?
- Is the distance exact or approximate?

PROBLEM 4 Applying Yourself!

A surveyor was hired to determine the approximate length of a proposed tunnel which, will be necessary to complete a new highway. A mountain stretches from point A to point B as shown. The surveyor stands at point C and measures the distance from where she is standing to both points A and B, then measures the angle formed between these two distances.

1. Use the surveyor's measurements to determine the length of the proposed tunnel.



$c^2 = a^2 + b^2 - 2ab \cdot \cos C$

$c^2 = (6800)^2 + (4500)^2 - 2(6800)(4500) \cos 122°$

$c^2 = 46{,}240{,}000 + 20{,}250{,}000 - 61{,}200{,}000 \cos 122°$

$c^2 = 46{,}240{,}000 + 20{,}250{,}000 - (-32{,}431{,}058.97)$

$c^2 = 46{,}240{,}000 + 20{,}250{,}000 - (-32{,}431{,}058.97)$

$c^2 = 98{,}921{,}058.97$

$c = \sqrt{98{,}921{,}058.97} \approx 9945.9$

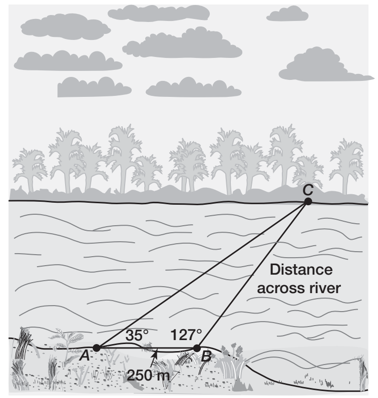
The length of the proposed tunnel is 9945.9 feet or approximately $\frac{9945.9}{5280} = 1.88$ miles.

8

Guiding Questions for Share Phase, Question 2

- Will the Law of Sines or the Law of Cosines be useful in determining the width of the river? How do you know?
- How did you determine which ratios to use in the proportion?
- Is the width of the river exact or approximate?

2. A nature lover decides to use geometry to determine if she can swim across a river. She locates two points, *A* and *B*, along one side of the river and determines the distance between these points is 250 meters. She then spots a point *C* on the other side of the river and measures the angles formed using point *C* to point *A* and then point *C* to point *B*. She determines the measure of the angle whose vertex is located at point *A* to be 35° and the angle whose vertex is located at point *B* to be 127° as shown.



How did she determine the distance across the river from point *B* to point *C* and what is that distance?

$m\angle C = 180° - 127° - 35° = 18°$

$\frac{\sin A}{a} = \frac{\sin C}{c}$

$a \sin C = c \sin A$

$a = \frac{c \sin A}{\sin C}$

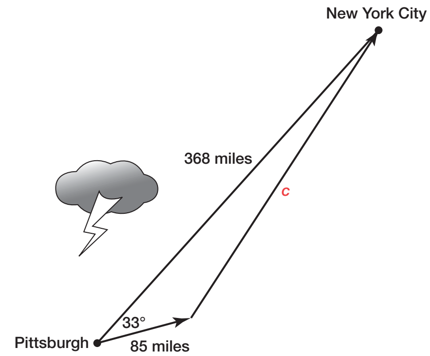
$= \frac{250 \sin 35°}{\sin 18°} \approx 464.03$

The distance across the river from point *B* to point *C* is approximately 464.03 meters.

Guiding Questions for Share Phase, Question 3

- Will the Law of Sines or the Law of Cosines be useful in determining the additional miles the aircraft traveled to avoid the storm? How do you know?
- How did you determine which Law of Cosines was needed to determine the additional miles the aircraft traveled to avoid the storm?
- Approximately how many gallons per mile is 11.875 liters per kilometer?

3. A typical direct flight from Pittsburgh, Pennsylvania, to New York City is approximately 368 miles. A pilot alters the course of his aircraft 33° for 85 miles to avoid a storm and then turns the aircraft heading straight for New York City, as shown.



a. How many additional miles did the aircraft travel to avoid the storm?

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$

$c^2 = (368)^2 + (85)^2 - 2(368)(85) \cos 33°$

$c^2 = 135{,}424 + 7225 - 62{,}560 \cos 33°$

$c^2 = 135{,}424 + 7225 - (52{,}467.23)$

$c^2 = 90{,}181.77$

$c = \sqrt{90{,}181.77} \approx 300.3$

$300.3 + 85 = 385.3$ miles

$385.3 - 368 = 17.3$ miles

The aircraft traveled an additional 17.3 miles to avoid the storm.

b. If a commercial jet burns an average of 11.875 liters per kilometer, and the cost of jet fuel is \$3.16 per gallon, this alteration in route due to the storm cost the airline company how much money?

11.875 liters per kilometer is approximately 5 gallons per mile. If the aircraft traveled an additional 17.3 miles to avoid the storm, then avoiding the storm cost the airline company an additional 17.3(5)(\$3.16) = \$273.34.

8

Talk the Talk

Students discuss when it is appropriate to use the Law of Sines and the Law of Cosines.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Talk the Talk

1. When is it appropriate to use the Law of Sines?

   The Law of Sines should be used when you know the length of two sides of a triangle and an angle opposite one of those sides, or when you know the measure of two angles of a triangle and the length of one side opposite one of the angles of known measure.

2. When is it appropriate to use the Law of Cosines?

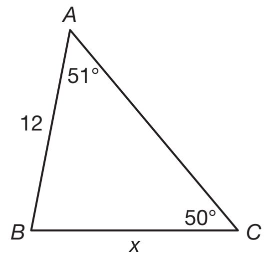
   The Law of Cosines should be used when you know the length of three sides of a triangle and want to solve for the measure of an interior angle or when you know the lengths of two sides and the measure of the included angle and you want to solve for the measure of either of the other two interior angles or the length of the third side.

Be prepared to share your solutions and methods.

Check for Students' Understanding

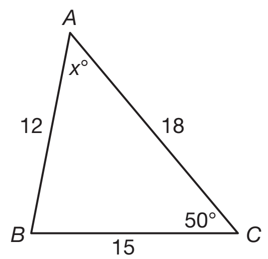
State your strategy for solving each situation: the **Law of Sines** or the **Law of Cosines**. (Do not solve for the unknown measurement.)

1.



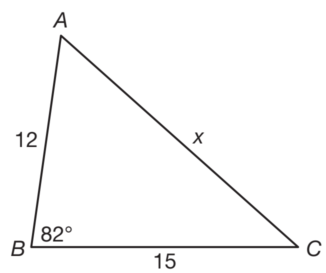
The Law of Sines

2.



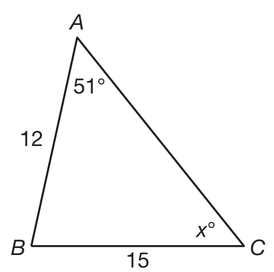
The Law of Cosines

3.



The Law of Cosines

4.



The Law of Sines

Chapter 8 Summary

KEY TERMS

- reference angle (8.1)
- opposite side (8.1)
- adjacent side (8.1)
- rationalizing the denominator (8.2)
- tangent (tan) (8.2)
- cotangent (cot) (8.2)
- inverse tangent (8.2)
- sine (sin) (8.3)
- cosecant (csc) (8.3)
- inverse sine (8.3)
- cosine (cos) (8.4)
- secant (sec) (8.4)
- inverse cosine (8.4)
- Law of Sines (8.6)
- Law of Cosines (8.6)

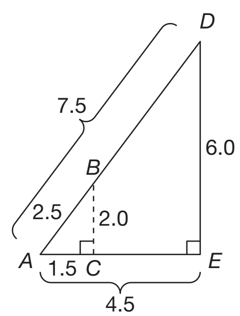
8.1 Analyzing the Properties of Similar Right Triangles

In similar triangles, the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ is equal for corresponding reference angles.

In similar triangles, the ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$ is equal for corresponding reference angles.

In similar triangles, the ratio $\frac{\text{opposite}}{\text{adjacent}}$ is equal for corresponding reference angles.

Example



Right triangles *ABC* and *ADE* are similar. Consider angle *A* as the reference angle.

$\frac{\text{opposite}}{\text{hypotenuse}}$ ratios:

triangle *ABC*: $\frac{2.0}{2.5} = 0.8$

triangle *ADE*: $\frac{6.0}{7.5} = 0.8$

$\frac{\text{adjacent}}{\text{hypotenuse}}$ ratios:

triangle *ABC*: $\frac{1.5}{2.5} = 0.6$

triangle *ADE*: $\frac{4.5}{7.5} = 0.6$

$\frac{\text{opposite}}{\text{adjacent}}$ ratios:

triangle *ABC*: $\frac{0.8}{0.6} \approx 1.33$

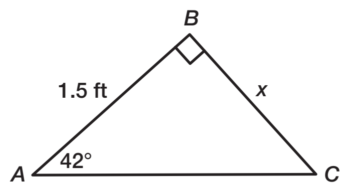
triangle *ADE*: $\frac{0.8}{0.6} \approx 1.33$

8

8.2 Using the Tangent Ratio

The tangent (tan) of an acute angle in a right triangle is the ratio of the length of the side that is opposite the angle to the length of the side that is adjacent to the angle. You can use the tangent of an angle to determine the length of a leg in a right triangle when you know the measure of an acute angle and the length of the other leg.

Example



$$\tan 42^\circ = \frac{x}{1.5}$$

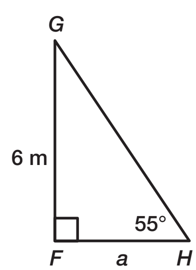
$$1.5(\tan 42^\circ) = x$$

$$x \approx 1.35 \text{ ft}$$

8.2 Using the Cotangent Ratio

The cotangent (cot) of an acute angle in a right triangle is the ratio of the length of the side that is adjacent to the angle to the length of the side that is opposite the angle. You can use the cotangent of an angle to determine the length of a leg in a right triangle when you know the measure of an acute angle and the length of the other leg.

Example



$$\cot 55^\circ = \frac{a}{6}$$

$$6(\cot 55^\circ) = a$$

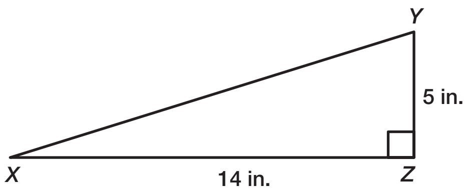
$$6\left(\frac{1}{\tan 55^\circ}\right) = a$$

$$a \approx 4.20 \text{ m}$$

8.2 Using the Inverse Tangent

The inverse tangent, or arc tangent, of x is defined as the measure of an acute angle whose tangent is x. You can use the inverse tangent to calculate the measure of either acute angle in a right triangle when you know the lengths of both legs.

Example



$m\angle X = \tan^{-1}\left(\frac{5}{14}\right) \approx 19.65°$

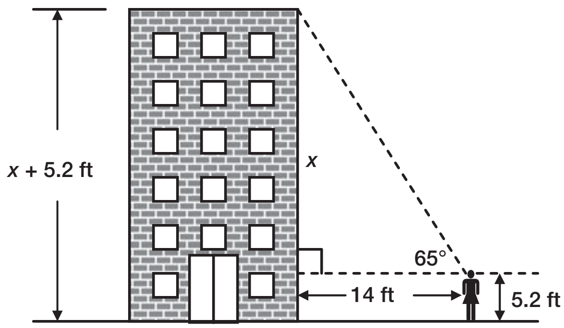
$m\angle Y = \tan^{-1}\left(\frac{14}{5}\right) \approx 70.35°$

8.2 Solving Problems Using the Tangent Ratio

An angle of elevation is the angle above a horizontal. You can use trigonometric ratios to solve problems involving angles of elevation.

Example

Mitchell is standing on the ground 14 feet from a building and he is looking up at the top of the building. The angle of elevation that his line of sight makes with the horizontal is 65°. His eyes are 5.2 feet from the ground. To calculate the height of the building, first draw a diagram of the situation. Then write and solve an equation involving a trigonometric ratio.



$$\tan 65° = \frac{x}{14}$$

$$14(\tan 65°) = x$$

$$x \approx 30 \text{ ft}$$

$$x + 5.2 = 30 + 5.2 = 35.2 \text{ ft}$$

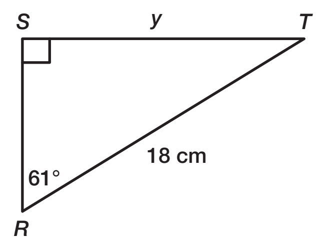
The building is about 35.2 feet tall.

8

8.3 Using the Sine Ratio

The sine (sin) of an acute angle in a right triangle is the ratio of the length of the side that is opposite the angle to the length of the hypotenuse. You can use the sine of an angle to determine the length of a leg in a right triangle when you know the measure of the angle opposite the leg and the length of the hypotenuse. You can also use the sine of an angle to determine the length of the hypotenuse when you know the measure of an acute angle and the length of the leg opposite the angle.

Example



$$\sin 61^\circ = \frac{y}{18}$$

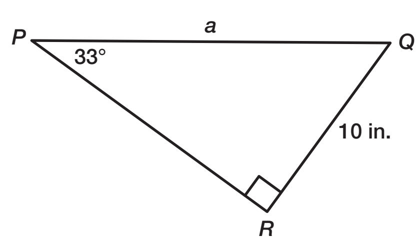
$$18(\sin 61^\circ) = y$$

$$x \approx 15.74 \text{ cm}$$

8.3 Using the Cosecant Ratio

The cosecant (csc) of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is opposite the angle. You can use the cosecant of an angle to determine the length of a leg in a right triangle when you know the measure of the angle opposite the leg and the length of the hypotenuse. You can also use the cosecant of an angle to determine the length of the hypotenuse when you know the measure of an acute angle and the length of the leg opposite the angle.

Example



$$\csc 33^\circ = \frac{a}{10}$$

$$10(\csc 33^\circ) = a$$

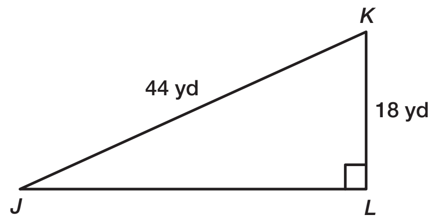
$$10\left(\frac{1}{\sin 33^\circ}\right) = a$$

$$a \approx 18.36 \text{ in.}$$

8.3 Using the Inverse Sine

The inverse sine, or arc sine, of *x* is defined as the measure of an acute angle whose sine is *x*. You can use the inverse sine to calculate the measure of an acute angle in a right triangle when you know the length of the leg opposite the angle and the length of the hypotenuse.

Example

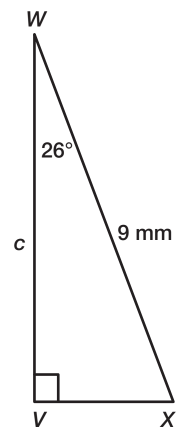


$m\angle J = \sin^{-1}\left(\frac{18}{44}\right) \approx 24.15°$

8.4 Using the Cosine Ratio

The cosine (cos) of an acute angle in a right triangle is the ratio of the length of the side that is adjacent to the angle to the length of the hypotenuse. You can use the cosine of an angle to determine the length of a leg in a right triangle when you know the measure of the angle adjacent to the leg and the length of the hypotenuse. You can also use the cosine of an angle to determine the length of the hypotenuse when you know the measure of an acute angle and the length of the leg adjacent to the angle.

Example



$\cos 26° = \frac{c}{9}$

$9(\cos 26°) = c$

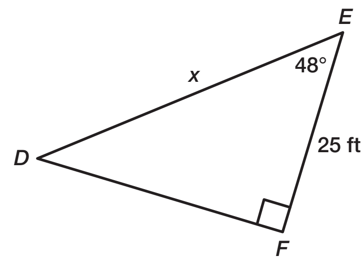
$c \approx 8.09 \text{ mm}$

8

8.4 Using the Secant Ratio

The secant (sec) of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is adjacent to the angle. You can use the secant of an angle to determine the length of a leg in a right triangle when you know the measure of the angle adjacent to the leg and the length of the hypotenuse. You can also use the secant of an angle to determine the length of the hypotenuse when you know the measure of an acute angle and the length of the leg adjacent to the angle.

Example



$$\sec 48° = \frac{x}{25}$$

$$25(\sec 48°) = x$$

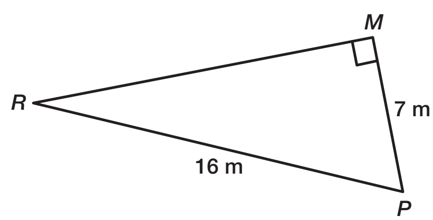
$$25\left(\frac{1}{\cos 48°}\right) = x$$

$$x \approx 37.36 \text{ ft}$$

8.4 Using the Inverse Cosine

The inverse cosine, or arc cosine, of *x* is defined as the measure of an acute angle whose cosine is *x*. You can use the inverse cosine to calculate the measure of an acute angle in a right triangle when you know the length of the leg adjacent to the angle and the length of the hypotenuse.

Example



$$m\angle P = \cos^{-1}\left(\frac{7}{16}\right) \approx 64.06°$$

8.5 Exploring Complementary Angle Relationships in a Right Triangle

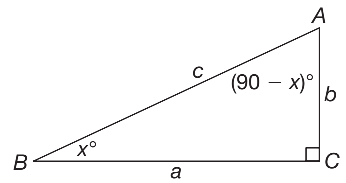
8

The two acute angles of a right triangle are complementary angles because the sum of their measures is 90 degrees. The trigonometric ratios also have complementary relationships:

The sine of an acute angle is equal to the cosine of its complement.

The tangent of an acute angle is equal to the cotangent of its complement.

The secant of an acute angle is equal to the cosecant of its complement.



Example

Angle *A* and angle *B* are complementary angles.

$\sin \angle A = \frac{a}{c}$

$\cos \angle B = \frac{a}{c}$

$\sin \angle A$ and $\cos \angle B$ are the same ratio.

$\sin \angle B = \frac{b}{c}$

$\cos \angle A = \frac{b}{c}$

$\sin \angle B$ and $\cos \angle A$ are the same ratio.

$\tan \angle A = \frac{a}{b}$

$\cot \angle B = \frac{a}{b}$

$\tan \angle A$ and $\cot \angle B$ are the same ratio.

$\tan \angle B = \frac{b}{a}$

$\cot \angle A = \frac{b}{a}$

$\tan \angle B$ and $\cot \angle A$ are the same ratio.

$\sec \angle A = \frac{c}{b}$

$\csc \angle B = \frac{c}{b}$

$\sec \angle A$ and $\csc \angle B$ are the same ratio.

$\sec \angle B = \frac{c}{a}$

$\csc \angle A = \frac{c}{a}$

$\sec \angle B$ and $\csc \angle A$ are the same ratio.

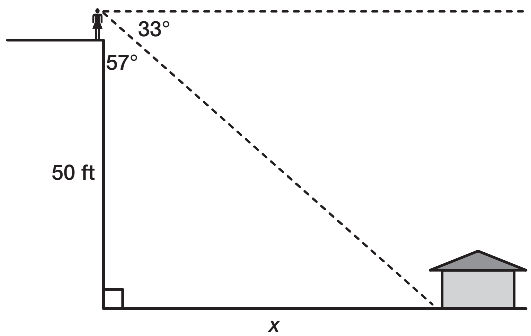
8

8.5 Solving Problems Using Complementary Angle Relationships

An angle of elevation is the angle below a horizontal. You can use trigonometric ratios to solve problems involving angles of depression.

Example

You are standing on a cliff and you see a house below you. You are 50 feet above the house. The angle of depression that your line of sight makes with the horizontal is 33°. To calculate the horizontal distance x you are from the house, first draw a diagram of the situation. Then write and solve an equation involving a trigonometric ratio.



$$\tan 57° = \frac{x}{50}$$

$$50(\tan 57°) = x$$

$$x \approx 77 \text{ ft}$$

You are about 77 feet from the house.

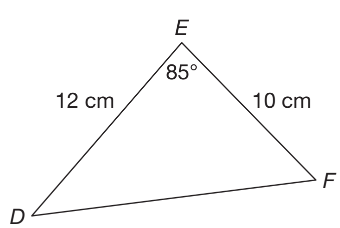
8.6 Deriving and Using a Formula for the Area of a Triangle

You can calculate the area of any triangle if you know the lengths of two sides and the measure of the included angle.

Area formula:

For any triangle ABC, $A = \frac{1}{2}ab(\sin C)$.

Example



$$A = \frac{1}{2}df(\sin E)$$

$$A = \frac{1}{2}(10)(12)(\sin 85°)$$

$$A \approx 59.8 \text{ square centimeters}$$

8.6 Deriving and Using the Law of Sines

You can use the Law of Sines when

- you know the lengths of two sides of a triangle and the measure of an angle opposite one of those sides, and you want to know the measure of the angle opposite the other known side.

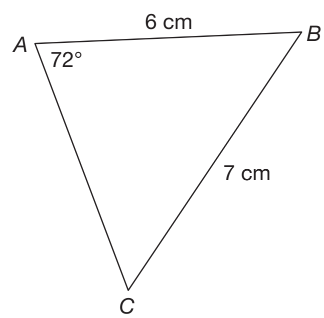
or

- you know the measures of two angles of a triangle and the length of a side opposite one of those angles, and you want to know the length of the side opposite the other known angle.

Law of Sines:

For any triangle ABC, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Example



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin 72^\circ}{7} = \frac{\sin C}{6}$$

$$6 \sin 72^\circ = 7 \sin C$$

$$\sin C = \frac{6 \sin 72^\circ}{7} \approx 0.815$$

$$C \approx 54.6^\circ$$

8.6 Deriving and Using the Law of Cosines

You can use the Law of Cosines when

- You know the lengths of all three sides of a triangle and you want to solve for the measure of any of the angles

or

- You know the lengths of two sides of a triangle and the measure of the included angle, and you want to solve for the length of the third side.

Law of Cosines

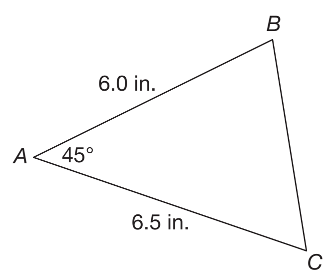
For any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Example



$a^2 = b^2 + c^2 - 2bc \cos A$

$a^2 = 6.5^2 + 6.0^2 - 2(6.5)(6.0)(\cos 45^\circ)$

$a^2 = 42.25 + 36 - 78(\cos 45^\circ)$

$a^2 \approx 23.10$

$a \approx 4.8$ inches