Properties of Quadrilaterals

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Chapter 7 Overview

This chapter focuses on properties of squares, rectangles, parallelograms, rhombi, kites, and trapezoids. The sum of interior and exterior angles of polygons is also included.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
7.1	Properties of Squares and Rectangles	G.CO.11 G.CO.12 G.SRT.8 G.GPE.5 G.MG.1	2	This lesson begins with proving the Perpendicular/Parallel Line Theorem. Then, students explore properties of squares and rectangles. Questions ask students to construct squares and rectangles, identify and prove their properties, and solve problems using the properties of squares and rectangles.	X				
7.2	Properties of Parallelograms and Rhombi	G.CO.11 G.CO.12 G.GPE.5 G.MG.1	2	This lesson explores the properties of parallelograms and rhombi. Questions ask students to construct parallelograms and rhombi, identify and prove their properties, and solve problems using the properties of parallelograms and rhombi.	X		X		
7.3	Properties of Kites and Trapezoids	G.CO.11 G.SRT.8 G.GPE.5 G.CO.12 G.MG.1	2	This lesson guides students through exploring the properties of kites and trapezoids. Questions ask students to construct kites and trapezoids, identify and prove their properties, and solve problems using the properties of kites and trapezoids.	X	X	X		
7.4	Sum of the Interior Angle Measures of a Polygon	G.CO.9 G.SRT.8 G.MG.1	1	This lesson uses the sum of the interior angle measures of a triangle to derive the sum of the interior angle measures of a polygon. Questions ask students to determine the sum of the interior angles of the polygon by drawing triangles, then generalizing the pattern to a formula. Students also determine the number of sides of a polygon given the sum of the interior angles and the measure of individual interior angles of regular polygons. Lastly, students solve problems involving the interior angles of polygons.	X		X		

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Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
7.5	Sum of the Exterior Angle Measures of a Polygon	G.CO.9 G.CO.12 G.SRT.8 G.MG.1	1	This lesson uses the Linear Pair Postulate to derive the sum of the exterior angle measures of a polygon. Questions ask students to determine the sum of the exterior angles of the polygon based on the Linear Pair Postulate, and then generalize their findings. For regular polygons, students calculate the measure of individual exterior angles, and the number of sides of the polygon given the measure of an exterior angle. Lastly, students solve problems involving the exterior angles of polygons.	X	X			
7.6	Categorizing Quadrilaterals Based on Their Properties	G.CO.12	2	This lesson summarizes properties of quadrilaterals. Questions ask students to list the properties of quadrilaterals, categorize quadrilaterals using a Venn Diagram, and answer true-false questions about quadrilateral properties. Lastly, students describe how to construct quadrilaterals given only one diagonal.	X				

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Skills Practice Correlation for Chapter 7

Lesson		Problem Set	Objectives
7.1	Properties of Squares and Rectangles		Vocabulary
		1 – 6	Identify parallel lines using the Perpendicular/Parallel Line Theorem
		7 – 12	Use properties of squares to complete statements
		13 – 18	Use properties of rectangles to complete statements
		19 – 24	Construct squares and rectangles given descriptions
		25 – 30	Write proofs to prove properties of squares and rectangles
7.2	Properties of Parallelograms and Rhombi		Vocabulary
		1 – 4	Use properties of parallelograms to complete statements
		5 – 8	Use properties of rhombi to complete statements
		9 – 14	Construct parallelograms and rhombi given descriptions
		15 – 20	Determine the statement needed to prove that given quadrilaterals are parallelograms by the Parallelogram/Congruent-Parallel Side Theorem
		21 – 26	Write proofs to prove properties of parallelograms and rhombi
		27 – 32	Use properties of parallelograms and rhombi to answer questions
7.3	Properties of Kites and Trapezoids		Vocabulary
		1 – 4	Use properties of kites to complete statements
		5 – 8	Use properties of trapezoids to complete statements
		9 – 14	Construct kites and trapezoids given descriptions
		15 – 20	Use properties of kites to answer questions
		21 – 26	Use properties of trapezoids to answer questions
		27 – 32	Write proofs to prove properties of kites and trapezoids
		33 – 38	Construct isosceles trapezoids given the perimeter
		39 – 44	Solve problems using properties of kites and trapezoids
7.4	Sum of the Interior Angle Measures of a Polygon		Vocabulary
		1 – 6	Draw all possible diagonals of polygons and write the number of triangles formed
		7 – 12	Calculate the sum of the interior angle measures of polygons using triangles formed by diagonals
		13 – 18	Calculate the sum of the interior angle measures of polygons given the number of sides
		19 – 24	Determine the number of sides of polygons given the sum of the measures of the interior angles
		25 – 30	Calculate the measures of the interior angles of regular polygons
		31 – 36	Calculate the number of sides of regular polygons given the measure of each interior angle

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Lesson		Problem Set	Objectives
7.5	Sum of the Exterior Angle Measures of a Polygon		Vocabulary
		1 – 6	Create exterior angles of polygons
		7 – 12	Calculate the sum of the measures of the exterior angles of polygons
		13 – 18	Calculate the measures of adjacent exterior angles given the measure of an interior angle
		19 – 24	Calculate the measures of exterior angles of regular polygons
		25 – 30	Calculate the number of sides of regular polygons given the measure of each exterior angle
7.6	Categorizing Quadrilaterals Based on Their Properties	1 – 6	List quadrilaterals that match given characteristics
		7 – 12	Use the terms quadrilateral, parallelogram, rectangle, square, trapezoid, rhombus, and kite to describe given figures
		13 – 18	Identify the type of quadrilateral that best describes figures
		19 – 24	Draw Venn diagrams to identify properties of quadrilaterals
		25 – 30	Use quadrilateral properties to determine if statements are true or false
		31 – 36	List the steps to construct given quadrilaterals

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7.1

Squares and Rectangles

Properties of Squares and Rectangles

LEARNING GOALS

In this lesson, you will:

- Prove the Perpendicular/Parallel Line Theorem.
- Construct a square and a rectangle.
- Determine the properties of a square and rectangle.
- Prove the properties of a square and a rectangle.
- Solve problems using the properties of a square and a rectangle.

KEY TERMS

- Perpendicular/Parallel Line Theorem

ESSENTIAL IDEAS

- The Perpendicular/Parallel Line Theorem states: "If two lines are perpendicular to the same line, then the two lines are parallel to each other."
- A square is a quadrilateral with four right angles and all sides congruent. The opposite sides are parallel.
- The diagonals of a square are congruent, bisect each other, bisect the vertex angles, and are perpendicular to each other.
- A rectangle is a quadrilateral with opposite sides congruent and all angles congruent. The opposite sides are parallel.
- The diagonals of a rectangle are congruent and bisect each other.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Prove geometric theorems

11. Prove theorems about parallelograms.

Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

G-SRT Similarity, Right Triangles, and Trigonometry

Define trigonometric ratios and solve problems involving right triangles

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

G-GPE Expressing Geometric Properties with Equations

Use coordinates to prove simple geometric theorems algebraically

5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems.

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G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

Students prove the properties of a square and a rectangle using two-column and paragraph formats. Students apply the theorems to solve problem situations. Construction tools are used in this lesson.

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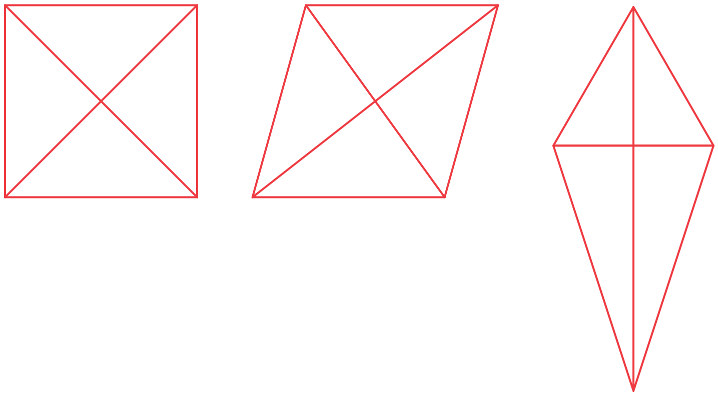
Warm Up

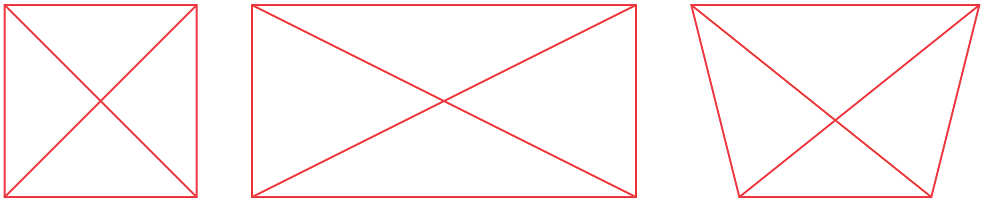
1. Draw three different quadrilaterals with perpendicular diagonals. Use a protractor to verify the diagonals are perpendicular.



2. What are the names of the quadrilaterals?

   The quadrilaterals are a square, a rhombus, and a kite.

3. Draw three different quadrilaterals with two congruent diagonals. Use a ruler to verify the diagonals are congruent.



4. What are the names of the quadrilaterals?

   The quadrilaterals are a square, a rectangle, and an isosceles trapezoid.

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7.1

Squares and Rectangles

Properties of Squares and Rectangles

LEARNING GOALS

In this lesson, you will:

- Prove the Perpendicular/Parallel Line Theorem.
- Construct a square and a rectangle.
- Determine the properties of a square and rectangle.
- Prove the properties of a square and a rectangle.
- Solve problems using the properties of a square and a rectangle.

KEY TERM

- Perpendicular/Parallel Line Theorem

In 2010, a technology company unveiled what was billed at the time as the world's largest plasma TV. This monster of a TV measured $12\frac{2}{3}$ feet across the diagonal (152 inches long) and displayed nearly 9 million pixels in HD and, if necessary, 3D.

Just two years earlier, the same company released a 103-inch version (89.3 inches long) which you could buy for a mere $70,000.

How much would you estimate the 152-inch TV cost if the company set its prices based on the total viewing areas for their TVs?

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Problem 1

A square is a quadrilateral with four right angles and four congruent sides. Students prove the Perpendicular/Parallel Line Theorem: If two lines are perpendicular to the same line, then they are parallel to each other. Next, they sketch a square and listing all known properties. The Perpendicular/Parallel Line Theorem is used to prove other properties of a square. Students prove:

- The diagonals of a square are congruent
- The opposite sides of a square are parallel
- The diagonals of a square bisect each other
- The diagonals of a square bisect the vertex angles
- The diagonals of a square are perpendicular

They also use construction tools to construct a square given one side.

Grouping

- Ask a student to read the information before Question 1. Discuss as a class.
- Have students complete Question 1 with a partner. Then have students share their responses as a class.

PROBLEM 1 Know the Properties or Be Square!

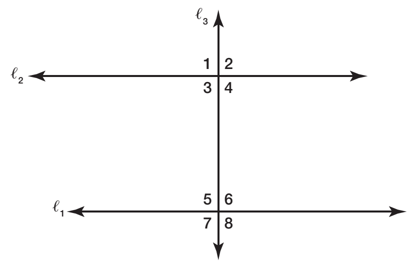
A quadrilateral is a four-sided polygon. A square is a quadrilateral with four right angles and all sides congruent.

Quadrilaterals have different properties that are directly related to the measures of their interior angles and their side lengths. Perpendicular lines and right angles are useful when proving properties of certain quadrilaterals.

1. Ramira says that if two lines are perpendicular to the same line, then the two lines are parallel to each other. Is Ramira correct? If she is correct, complete the proof to justify the reasoning, or state why she is not correct.

Given: $\ell_1 \perp \ell_3$; $\ell_2 \perp \ell_3$

Prove: $\ell_1 \parallel \ell_2$



Statements	Reasons
1. $\ell_1 \perp \ell_3$	1. Given
2. $\ell_2 \perp \ell_3$	2. Given
3. $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$, $\angle 6$, $\angle 7$, and $\angle 8$ are right angles.	3. Definition of perpendicular lines
4. $\angle 1 \cong \angle 2 \cong \angle 3 \cong \angle 4 \cong \angle 5 \cong \angle 6 \cong \angle 7 \cong \angle 8$	4. All right angles are congruent
5. $\ell_1 \parallel \ell_2$	5. Alternate Interior Angle Converse Theorem

Guiding Questions for Share Phase, Question 1

- What is the definition of perpendicular lines?
- Are all right angles congruent? How do you know?
- Did you use the Alternate Interior Angle Converse Theorem to prove this statement?
- Did your classmates use the same theorem to prove the lines are parallel?
- What other theorems can be used to prove the same statement?

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Grouping

Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 2 and 3

- What is true about the length of the four sides of a square?
- What do you know about the measure of the 4 interior angles in a square?
- What is true about the opposite sides of a square?
- Is a square also a parallelogram?
- What do you know about the diagonals of a square?
- Did you duplicate segment *AB* on the starter line?
- What did you do after you duplicated segment *AB* on the starter line?
- How did you locate the side of the square opposite side *AB*?
- What side of the square did you locate second?
- How many perpendicular constructions were necessary to construct the square?
- How did you locate the diagonals of the square?
- What point did you locate after points *A* and *B*?
- Are there any shortcuts that can be taken in this construction?

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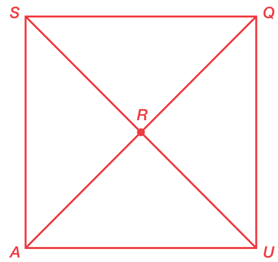
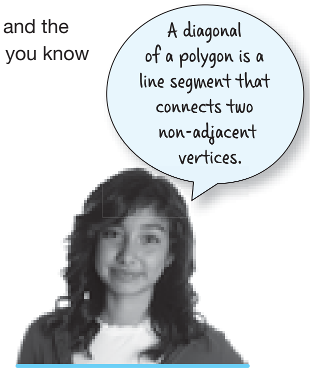
The **Perpendicular/Parallel Line Theorem** states: "If two lines are perpendicular to the same line, then the two lines are parallel to each other."

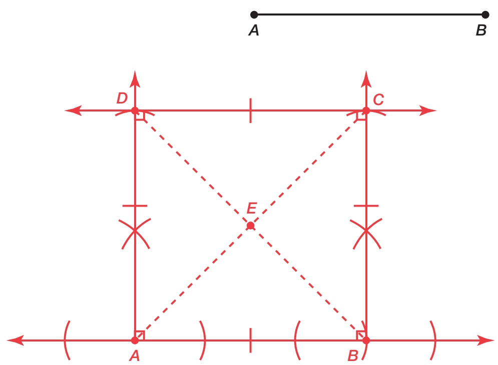
2. Draw a square with two diagonals. Label the vertices and the intersection of the diagonals. List all of the properties you know to be true.





- $\overline{SQ} \cong \overline{QU} \cong \overline{AU} \cong \overline{SA}$
- $\angle ASQ$, $\angle SQU$, $\angle QUA$, and $\angle UAS$ are right angles.
- $\angle ASQ \cong \angle SQU \cong \angle QUA \cong \angle UAS$
- $\overline{SQ} \parallel \overline{AU}$, $\overline{SA} \parallel \overline{QU}$
- $\overline{SU} \cong \overline{QA}$
- $\overline{SR} \cong \overline{UR}$, $\overline{QR} \cong \overline{AR}$
- $\angle RSA \cong \angle RSQ \cong \angle RQS \cong \angle RQU \cong \angle RUQ \cong \angle RUA \cong \angle RAU \cong \angle RAS$
- $\angle SRA$, $\angle SRQ$, $\angle QRU$, and $\angle URA$ are right angles.
- $\angle SRA \cong SRQ \cong \angle QRU \cong \angle URA$

3. Use $\overline{AB}$ to construct square *ABCD* with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at point *E*.



Grouping

Have students complete Questions 4 and 5 with a partner. Then have students share their responses as a class.

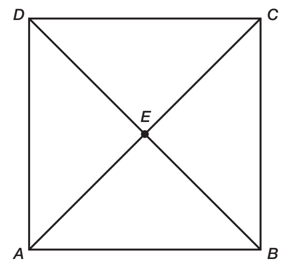
Guiding Questions for Share Phase, Questions 4 and 5

- Are triangle *DAB* and *CBA* overlapping triangles?
- Are triangle *DAB* and *CBA* right triangles? How do you know?
- Do the triangles share a common angle?
- Do the triangles share a common side?
- Are segments *AC* and *BD* corresponding parts of congruent triangles?

4. Prove the statement $\triangle DAB \cong \triangle CBA$.

Given: Square *ABCD* with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at point *E*

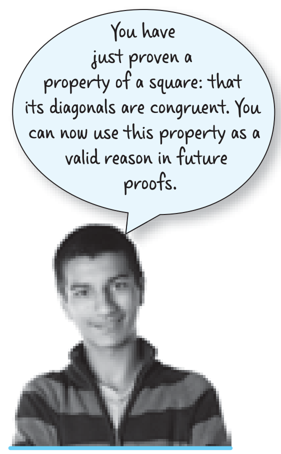
Prove: $\triangle DAB \cong \triangle CBA$



Statements	Reasons
1. Square *ABCD* with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at point *E*	1. Given
2. $\angle DAB$ and $\angle CBA$ are right angles.	2. Definition of square
3. $\angle DAB \cong \angle CBA$	3. All right angles are congruent.
4. $\overline{DA} \cong \overline{CB}$	4. Definition of square
5. $\overline{AB} \cong \overline{AB}$	5. Reflexive Property
6. $\triangle DAB \cong \triangle CBA$	6. SAS Congruence Theorem

5. Do you have enough information to conclude $\overline{AC} \cong \overline{BD}$? Explain your reasoning.

Yes. Because $\overline{AC} \cong \overline{BD}$ by CPCTC.



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Grouping

Have students complete Questions 6 and 7 with a partner. Then have students share their responses as a class.

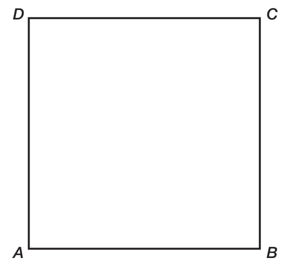
Guiding Questions for Share Phase, Questions 6 and 7

- What is the definition of a square?
- Is this a situation where two lines are perpendicular to the same line?
- What theorem is helpful in proving the lines are parallel?
- Does this prove the opposite sides of a square are parallel?
- Which quadrilaterals have two pairs of opposite sides parallel?

6. Prove the statement $\overline{DA} \parallel \overline{CB}$ and $\overline{DC} \parallel \overline{AB}$.

Given: Square *ABCD*

Prove: $\overline{DA} \parallel \overline{CB}$ and $\overline{DC} \parallel \overline{AB}$



Statements	Reasons
1. Square *ABCD*	1. Given
2. $\angle D$, $\angle A$, $\angle B$, and $\angle C$ are right angles.	2. Definition of square
3. $\overline{DA} \perp \overline{AB}$, $\overline{AB} \perp \overline{BC}$, $\overline{BC} \perp \overline{CD}$, and $\overline{CD} \perp \overline{DA}$	3. Definition of perpendicular lines
4. $\overline{DA} \parallel \overline{CB}$ and $\overline{DC} \parallel \overline{AB}$	4. Perpendicular/Parallel Line Theorem

7. If a parallelogram is a quadrilateral with opposite sides parallel, do you have enough information to conclude square *ABCD* is a parallelogram? Explain your reasoning.

Yes. I have just proven opposite sides of a square are parallel.

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Grouping

Have students complete Questions 8 and 9 with a partner. Then have students share their responses as a class.

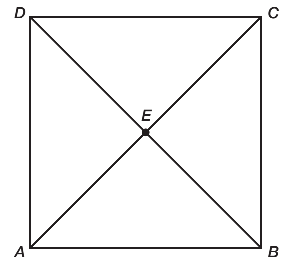
Guiding Questions for Share Phase, Questions 8 and 9

- Are triangle *DEC* and *BEA* overlapping triangles?
- Are triangle *DEC* and *BEA* vertical triangles?
- Are triangle *DAB* and *CBA* right triangles? How do you know?
- Do the triangles share a common angle?
- Do the triangles share a common side?
- Do the triangles contain a pair of vertical angles?
- Will the vertical angles help to prove the triangles congruent? Why not?
- What part of the definition of a square is helpful in this situation?
- Where are the alternate interior angles in these triangles?
- Are segments *DE* and *BE* corresponding parts of congruent triangles?
- Are segments *CE* and *AE* corresponding parts of congruent triangles?
- What is the definition of bisect?

8. Prove the statement $\overline{DE} \cong \overline{BE}$ and $\overline{CE} \cong \overline{AE}$.

Given: Square *ABCD* with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at point *E*

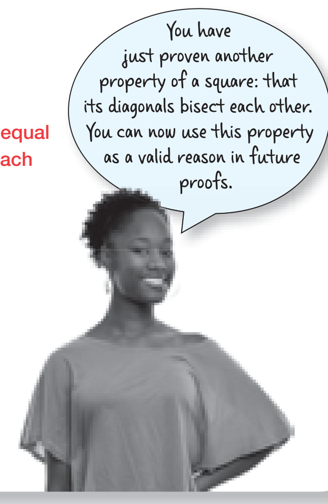
Prove: $\overline{DE} \cong \overline{BE}$ and $\overline{CE} \cong \overline{AE}$



Statements	Reasons
1. Square *ABCD* with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at point *E*	1. Given
2. $\overline{DC} \cong \overline{AB}$	2. Definition of square
3. $\overline{DC} \parallel \overline{AB}$	3. Opposite sides of a square are parallel.
4. $\angle ABD \cong \angle CDB$	4. Alternate Interior Angle Theorem
5. $\angle CAB \cong \angle ACD$	5. Alternate Interior Angle Theorem
6. $\triangle DEC \cong \triangle BEA$	6. ASA Congruence Theorem
7. $\overline{DE} \cong \overline{BE}$	7. CPCTC
8. $\overline{CE} \cong \overline{AE}$	8. CPCTC

9. Do you have enough information to conclude the diagonals of a square bisect each other? Explain your reasoning.

Yes. The definition of bisect is to divide into two equal parts and I have just proven both segments on each diagonal are congruent.



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Grouping

Have students complete Questions 10 and 11 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 10 and 11

- Which two triangles will help to prove diagonal *BD* bisects angle *CDA*?
- Which two triangles will help to prove diagonal *AC* bisects angle *DCB*?
- Which two triangles will help to prove diagonal *BD* bisects angle *ABC*?
- Which two triangles will help to prove diagonal *AC* bisects angle *DAB*?
- Do the two triangles overlap?
- Do the two triangles share a common side?
- What theorem was used to prove the two triangles congruent?
- Can the two triangles be proven congruent using a different triangle congruency theorem?
- What strategy was used to prove the diagonals perpendicular to each other?
- If right angles are formed where the diagonals intersect, is this enough information to conclude the diagonals are perpendicular to each other?
- How did you show right angles were formed at the intersection of the diagonals?

10. Prove that the diagonals of a square bisect the vertex angles. Use square *ABCD* in Question 8.

$\triangle ADE \cong \triangle CDE$ by SSS Congruence Theorem, so $\angle DEC$ to $\angle DEA$ by CPCTC. Diagonal $\overline{BD}$ bisects $\angle CDA$ by the definition of bisector.

$\triangle DCE \cong \triangle BCE$ by SSS Congruence Theorem, so $\angle DCE \cong \angle BCE$ by CPCTC. Diagonal $\overline{AC}$ bisects $\angle DCB$ by the definition of bisector.

$\triangle CBE \cong \triangle ABE$ by SSS Congruence Theorem, so $\angle CBE \cong \angle ABE$ by CPCTC. Diagonal $\overline{BD}$ bisects $\angle ABC$ by the definition of bisector.

$\triangle BAE \cong \triangle DAE$ by SSS Congruence Theorem, so $\angle BAE \cong \angle DAE$ by CPCTC. Diagonal $\overline{AC}$ bisects $\angle DAB$ by the definition of bisector.

11. Prove that the diagonals of a square are perpendicular to each other. Use square *ABCD* in Question 8.

$\triangle ADE \cong \triangle CDE$ by SSS Congruence Theorem, so $\angle AED \cong \angle CED$ by CPCTC. By the Linear Pair Postulate $\angle AED$ and $\angle CED$ are a linear pair. $\angle AED$ and $\angle CED$ are supplementary angles by the definition of a linear pair. If two angles are both congruent and supplementary, then they are right angles. Two lines forming right angles are perpendicular by the definition of perpendicular lines, so the diagonals of a square are perpendicular to each other.

- Do two of the angles at the intersection form a linear pair?
- If the angles form a linear pair, are they also supplementary?
- If the angles are both congruent and supplementary, what else can be concluded about the measure of the angles?

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Problem 2

A rectangle is a quadrilateral with four right angles. Students sketch a rectangle and list all known properties. Next students construct a rectangle using the perpendicular line constructions.

Students prove:

- The opposite sides of a rectangle are congruent
- The opposite sides of a rectangle are parallel
- The diagonals of a rectangle are congruent
- The diagonals of a rectangle bisect each other

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

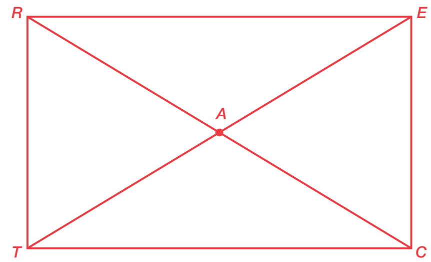
Guiding Questions for Share Phase, Questions 1 and 2

- What is true about the length of the four sides of a rectangle?
- What do you know about the measure of the 4 interior angles in a rectangle?
- What is true about the opposite sides of a rectangle?
- Is a rectangle also a parallelogram?
- What do you know about the diagonals of a rectangle?
- Is a rectangle also a square?
- Is a square also a rectangle?

PROBLEM 2 **The Rectangle**

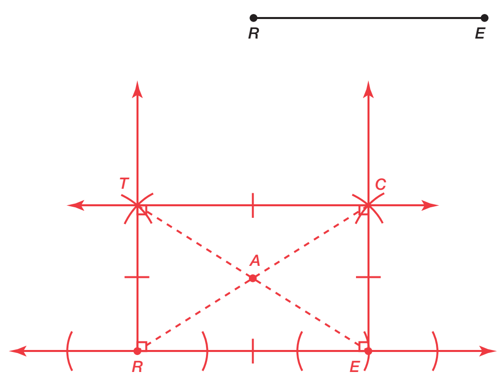
A rectangle is a quadrilateral with opposite sides congruent and all angles congruent.

1. Draw a rectangle with two diagonals. Label the vertices and the intersection of the two diagonals. List all of the properties you know to be true.



- $\overline{RT} \cong \overline{EC}$ and $\overline{RE} \cong \overline{TC}$
- $\angle TRE$, $\angle REC$, $\angle ECT$, and $\angle CTR$ are right angles.
- $\angle TRE \cong \angle REC \cong \angle ECT \cong \angle CTR$
- $\overline{RT} \parallel \overline{EC}$, $\overline{RE} \parallel \overline{TC}$
- $\overline{RC} \cong \overline{ET}$
- $\overline{RA} \cong \overline{CA}$ and $\overline{TA} \cong \overline{EA}$

2. Use $\overline{RE}$ to construct rectangle *RECT* with diagonals $\overline{RC}$ and $\overline{ET}$ intersecting at point *A*. Do not construct a square.



- Did you duplicate segment *RE* on the starter line?
- What did you do after you duplicated segment *RE* on the starter line?
- How did you locate the side of the rectangle opposite side *RE*?
- What side of the rectangle did you locate second?
- How many perpendicular constructions were necessary to construct the rectangle?
- How did you locate the diagonals of the rectangle?

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- What point did you locate after points *R* and *E*?
- Are there any shortcuts that can be taken in this construction?
- How is this construction similar to the construction of a square?
- How is this construction different than the construction of a square?
- Is it easier to construct a square or a rectangle? Why?

Grouping

Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

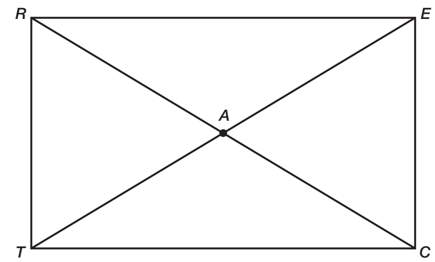
Guiding Questions for Share Phase, Questions 3 and 4

- Are triangle *RCT* and *ETC* overlapping triangles?
- Are triangle *RCT* and *ETC* right triangles? How do you know?
- Do the triangles share a common angle?
- Do the triangles share a common side?
- Are segments *RT* and *EC* corresponding parts of congruent triangles?

3. Prove the statement $\Delta RCT \cong \Delta ETC$.

Given: Rectangle *RECT* with diagonals $\overline{RC}$ and $\overline{ET}$ intersecting at point *A*

Prove: $\Delta RCT \cong \Delta ETC$



Statements	Reasons
1. Rectangle *RECT* with diagonals $\overline{RC}$ and $\overline{ET}$ intersecting at point *A*	1. Given
2. $\angle RTC$, $\angle ECT$, $\angle TRE$, and $\angle CER$ are right angles.	2. Definition of rectangle
3. $\angle RTC \cong \angle ECT \cong \angle TRE \cong \angle CER$	3. All right angles are congruent.
4. $\overline{RT} \perp \overline{RE}$, $\overline{RT} \perp \overline{TC}$	4. Definition of perpendicular lines
5. $\overline{RE} \parallel \overline{TC}$	5. Perpendicular/Parallel Line Theorem
6. $\angle ETC \cong \angle RCT$	6. Alternate Interior Angle Theorem
7. $\overline{TC} \cong \overline{TC}$	7. Reflexive Property
8. $\Delta RCT \cong \Delta ETC$	8. ASA Congruence Theorem

4. Do you have enough information to conclude $\overline{RT} \cong \overline{EC}$? Explain your reasoning.

Yes. Triangle *RCT* is congruent to triangle *ETC*, therefore $\overline{RT} \cong \overline{EC}$ by CPCTC.

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7

Grouping

Have students complete Questions 5 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 8

- Which pair of triangles would you use to prove the second pair of opposite sides congruent?
- What triangle congruency theorem would you use to prove the triangles congruent?
- Are segments *RE* and *TC* corresponding parts of the triangles?
- What theorem would be helpful in proving the rectangle is also a parallelogram?
- Which triangles would be helpful in proving the diagonals of a rectangle are congruent?
- What theorem would be helpful in proving the triangles congruent?
- Are segments *RC* and *ET* corresponding parts of the triangles?
- Do the diagonals divide each other into two equal parts? How do you know?

5. Describe how you could prove the second pair of opposite sides of the rectangle are congruent.

Similarly, I can prove $\triangle TRE \cong \triangle RTC$ by the ASA Congruence Theorem, so $\overline{RE} \cong \overline{TC}$ by CPCTC.

6. Do you have enough information to conclude rectangle *RECT* is a parallelogram? Explain your reasoning.

Yes. I can prove opposite sides parallel the same way we did in a square using the Parallel/Perpendicular Line Theorem.

7. Do you have enough information to conclude the diagonals of a rectangle are congruent? Explain your reasoning.

Yes. Because I can prove $\triangle RTC \cong \triangle ECT$ by the SAS Congruence Theorem, so $\overline{RC} \cong \overline{ET}$ by CPCTC.

8. Do you have enough information to conclude the diagonals of a rectangle bisect each other? Explain your reasoning.

Yes. Because the definition of bisect is to divide into two equal parts, and I can prove both line segments on each diagonal are congruent by CPCTC if I prove $\triangle RAE \cong \triangle CAT$ by the ASA Congruence Theorem.

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Problem 3

Students apply their knowledge of the properties of squares, the properties of rectangles, trigonometric ratios, slopes of parallel and perpendicular lines, and the distance formula to solve problem situations.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- If Ofelia only measures the four sides, is that enough information to conclude the mat is square? Why or why not?
- How can Gretchen use the string to determine the middle point of both support bars?
- If Gretchen crosses the support bars such that the middle most point of each bar overlaps, will that be enough to make certain the bookcase is in a rectangular position?
- How can the string be used to ensure the shelves are parallel to each other?

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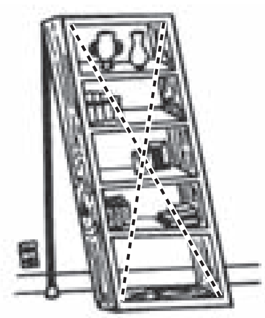
PROBLEM 3 Using Tools

1. Ofelia is making a square mat for a picture frame. How can she make sure the mat is a square using only a ruler?

   Ofelia can measure the four sides and measure the diagonals. If the four sides are congruent and the diagonals are congruent, then it must be a square.

2. Gretchen is putting together a bookcase. It came with diagonal support bars that are to be screwed into the top and bottom on the back of the bookcase. Unfortunately, the instructions were lost and Gretchen does not have the directions or a measuring tool. She has a screwdriver, a marker, and a piece of string. How can Gretchen attach the supports to make certain the bookcase will be a rectangle and the shelves are parallel to the ground?

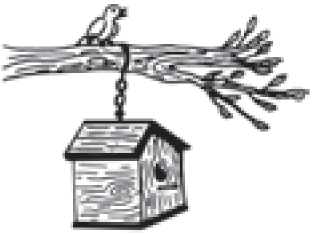


   Gretchen can use the string to locate the middle of both support bars before attaching them by spanning the length of a bar with the string and folding the string in half. Next, she can mark the middle point on both support bars. Then, she can connect the first support to the bookcase, and position the second support such that the support bars cross each other at the middle point of each bar. If the diagonals of a quadrilateral are congruent and bisect each other, the quadrilateral is a rectangle. The string can also be used to make sure the shelves are equidistant, and therefore parallel to the ground.

Guiding Questions for Share Phase, Questions 3 and 4

- What properties does Matsuo know about a rectangle?
- What properties does Matsuo need to know about a square?
- What trigonometric ratio was helpful when determining the width, x, of the rectangular screen?
- What trigonometric ratio was helpful when determining the length, y, of the rectangular screen?
- What formula is helpful in determining the viewing area of the television?

3. Matsuo knows this birdhouse has a rectangular base, but he wonders if it has a square base.



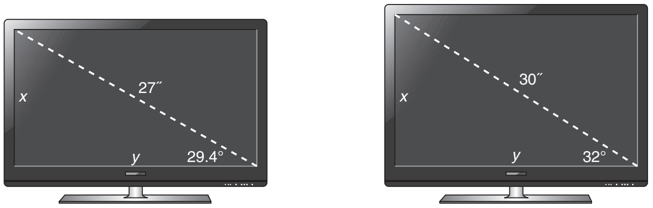
a. What does Matsuo already know to conclude the birdhouse has a rectangular base?

Matsuo knows that the base is a quadrilateral. He also knows the quadrilateral has four right angles with opposite sides congruent.

b. What does Matsuo need to know to conclude the birdhouse has a square base?

Matsuo needs to know if the four sides of the base are congruent.

4. Consider each LED television shown.



a. Determine the dimensions of the 27″ LED TV.

$\sin 29.4° = \frac{x}{27}$

$27\sin 29.4° = x$

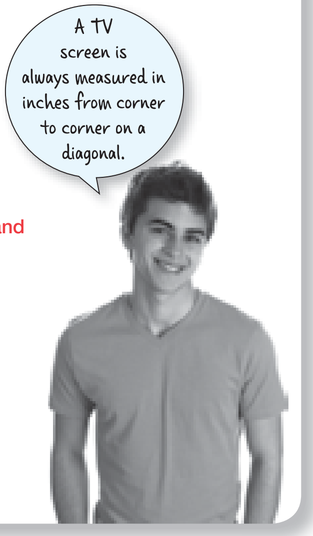
$x \approx 13.25$

$\cos 29.4° = \frac{y}{27}$

$27\cos 29.4° = y$

$y \approx 23.53$

The width of the television screen is 13.25 inches and the length of the television screen is 23.53 inches.



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b. Determine the dimensions of the 30″ LED TV.

$\sin 32° = \frac{x}{30}$

$30\sin 32° = x$

$x \approx 15.9$

$\cos 32° = \frac{y}{30}$

$30\cos 32° = y$

$y \approx 25.44$

The width of the television screen is 15.9 inches and the length of the television screen is 25.44 inches.

c. Compare the viewing area of each size television screen.

The 27″ television screen viewing area is (13.25) (23.53), or 311.77 square inches.

The 30″ television screen viewing area is (15.9) (25.44), or 404.436 square inches.

d. What property of a rectangle would be helpful when locating the center point of the television screen?

The property that diagonals of a rectangle bisect each other would be helpful to locate the center point of the screen.

e. What property of a rectangle would be helpful when determining the perimeter of the television screen?

The property that opposite sides of a rectangle are congruent would be helpful to determine the perimeter of the screen.

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Grouping

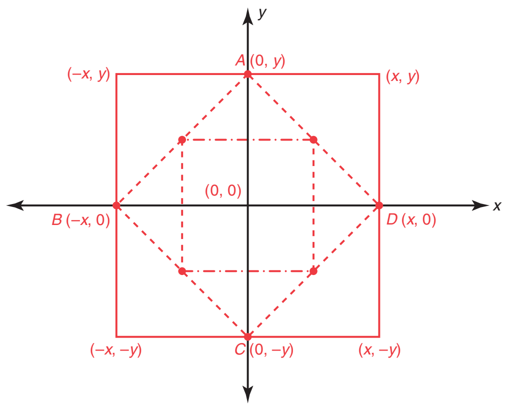
Have students complete Question 5 with a partner. Then have students share their responses as a class.

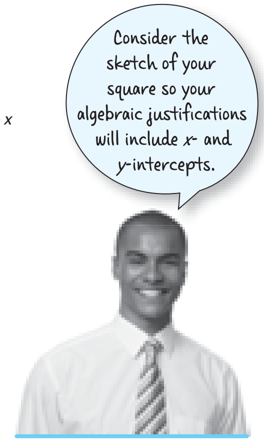
Guiding Questions for Share Phase, Question 5

- Did you sketch the square on a coordinate plane?
- What coordinates represent the four vertices of the square?
- If one vertex of the square is located at (x, y), where are the other three vertices located?
- If one side of the square is determined by the vertices (x, y) and $(-x, y)$, where is the location of the midpoint on this side?
- If one side of the square is determined by the vertices (x, y) and $(x, -y)$, where is the location of the midpoint on this side?
- What formulas were needed to determine the quadrilateral formed by connecting the midpoints of each side of a square?
- How is using the formula for slope important in determining the quadrilateral formed by connecting the midpoints of each side of a square?
- If this entire process was repeated several more times, would the result remain the same each time?

5. Sketch a square. Label the midpoint of each side of the square.

a. Determine the polygon formed by connecting the midpoints of each side of a square and justify your conclusion.





Points A, B, C, and D are midpoints of each side of the given square. Let $x = y$.

$AB = \sqrt{(0 - (-x))^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$

$BC = \sqrt{(-x - 0)^2 + (0 - (-y))^2} = \sqrt{x^2 + y^2}$

$CD = \sqrt{(0 - x)^2 + (-y - 0)^2} = \sqrt{x^2 + y^2}$

$DA = \sqrt{(x - 0)^2 + (0 - y)^2} = \sqrt{x^2 + y^2}$

$AB = BC = CD = DA$

Slope of $\overline{AB}$: $m = \frac{y - 0}{0 - (-x)} = \frac{y}{x}$

Slope of $\overline{BC}$: $m = \frac{-y - 0}{0 - (-x)} = \frac{-y}{x} = -\frac{y}{x}$

Slope of $\overline{CD}$: $m = \frac{-y - 0}{0 - x} = \frac{-y}{-x} = \frac{y}{x}$

Slope of $\overline{DA}$: $m = \frac{0 - y}{x - 0} = \frac{-y}{x} = -\frac{y}{x}$

Remember, $x = y$, so $\frac{y}{x} = 1$ and $-\frac{y}{x} = -1$ have a negative reciprocal relationship.

Line segment AB is perpendicular to line segment BC.

Line segment BC is perpendicular to line segment CD.

Line segment CD is perpendicular to line segment DA.

Line segment DA is perpendicular to line segment AB.

Line segment AB is parallel to line segment CD.

Line segment BC is parallel to line segment DA.

$ABCD$ is a square because all four sides are congruent and all four angles are right angles.

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b. If the same process was repeated one more time by connecting the midpoints of each side of the polygon determined in part (a), describe the polygon that would result.

Another square would be determined. It could be proven a square similar to the method used in part (a). Each vertex would be labeled in a similar fashion.

Be prepared to share your solutions and methods.

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Check for Students' Understanding

The neighbors in a rural community got together for a barn-raising. The first step was to build the rectangular base of the barn. One neighbor was a math teacher and tried to explain to everyone how diagonals could be used to verify the base was rectangular. What did he say?

He told them the diagonals of a rectangle were congruent and bisected each other. All they had to do was check to make sure this was true and that would verify the base was rectangular.

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Parallelograms and Rhombi

7.2

Properties of Parallelograms and Rhombi

LEARNING GOALS

In this lesson, you will:

- Construct a parallelogram.
- Construct a rhombus.
- Prove the properties of a parallelogram.
- Prove the properties of a rhombus.
- Solve problems using the properties of a parallelogram and a rhombus.

KEY TERM

- Parallelogram/Congruent-Parallel Side Theorem

ESSENTIAL IDEAS

- A parallelogram is a quadrilateral with opposite sides parallel. The opposite sides and opposite angles are congruent.
- The diagonals of a parallelogram bisect each other.
- The Parallelogram/Congruent-Parallel Side Theorem states: "If one pair of opposite sides of a quadrilateral are both congruent and parallel, then the quadrilateral is a parallelogram."
- A rhombus is a quadrilateral with all sides congruent. A rhombus is a parallelogram.
- The diagonals of a rhombus are bisect the vertex angles and are perpendicular to each other.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Prove geometric theorems

11. Prove theorems about parallelograms.

Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

G-GPE Expressing Geometric Properties with Equations

Use coordinates to prove simple geometric theorems algebraically

5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

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Overview

Students prove the properties of a parallelogram and a rhombus using two-column and paragraph formats. Students apply the theorems to solve problem situations. Construction tools are used in this lesson.

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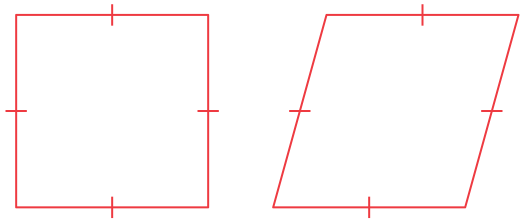
Warm Up

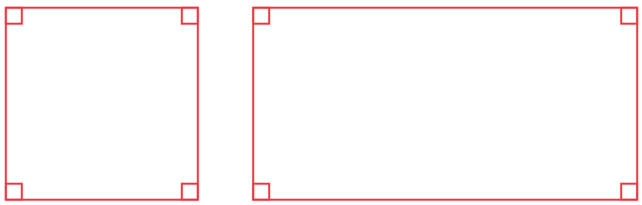
1. Draw two different quadrilaterals with four congruent sides. Use a ruler to verify the sides are congruent.



2. What are the names of the quadrilaterals?

The quadrilaterals are a square and a rhombus.

3. Draw two different quadrilaterals with four congruent angles. Use a protractor to verify the angles are congruent.



4. What are the names of the quadrilaterals?

The quadrilaterals are a square and a rectangle.

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7.2

Parallelograms and Rhombi

Properties of Parallelograms and Rhombi

LEARNING GOALS

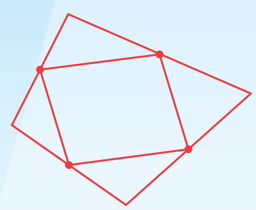
In this lesson, you will:

- Construct a parallelogram.
- Construct a rhombus.
- Prove the properties of a parallelogram.
- Prove the properties of a rhombus.
- Solve problems using the properties of a parallelogram and a rhombus.

KEY TERM

- Parallelogram/Congruent-Parallel Side Theorem

Using a ruler, draw any quadrilateral you want. You're going to locate the midpoints of the sides of the quadrilateral, so you might want to make the sides whole-number lengths. Draw a really wacky quadrilateral. The wackier the better.



Mark the midpoints of each side and then connect these midpoints to form another quadrilateral. What quadrilateral did you create? What quadrilateral did your classmates create? Do you get the same result no matter what quadrilateral you start with?

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Problem 1

A parallelogram is a quadrilateral with opposite sides parallel. The problem begins by students drawing a parallelogram and listing all known properties. Next students construct a parallelogram.

Students prove:

- The opposite sides of a parallelogram are congruent
- The opposite angles of a parallelogram are congruent
- The diagonals of a parallelogram bisect each other
- If one pair of opposite sides of a quadrilateral is both congruent and parallel, then the quadrilateral is a parallelogram.

They also use construction tools to construct a parallelogram given one side.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

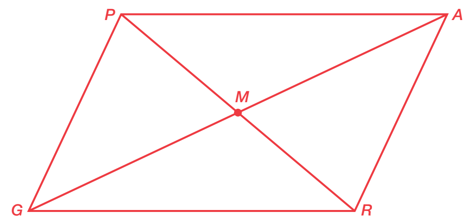
Guiding Questions for Share Phase, Questions 1 and 2

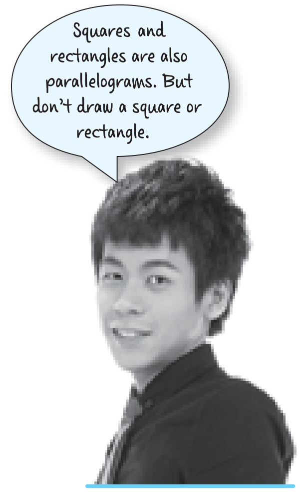
- What is true about the four sides of a parallelogram?
- What do you know about the measure of the 4 interior angles in a parallelogram?

PROBLEM 1 The Parallelogram

A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

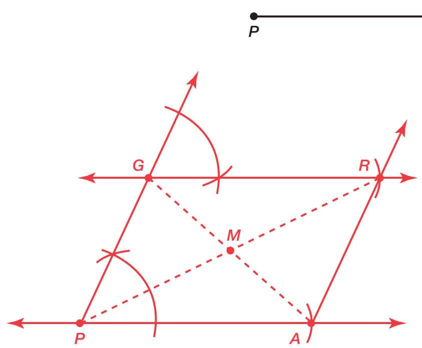
1. Draw a parallelogram with two diagonals. Label the vertices and the intersection of the diagonals. List all of the properties you know to be true.





- $\overline{PG} \cong \overline{AR}$
- $\overline{PA} \cong \overline{GR}$
- $\angle GPA \cong \angle ARG$
- $\angle PAR \cong \angle RGP$
- $\overline{PA} \parallel \overline{GR}$, $\overline{PG} \parallel \overline{AR}$
- $\overline{PM} \cong \overline{RM}$, $\overline{GM} \cong \overline{AM}$

2. Use $\overline{PA}$ to construct parallelogram *PARG* with diagonals $\overline{PR}$ and $\overline{AG}$ intersecting at point *M*.



- What is true about the opposite sides of a parallelogram?
- What do you know about the diagonals of a parallelogram?
- Is a parallelogram also a square or is a square also a parallelogram?
- Is a parallelogram also a rectangle or is a rectangle also a parallelogram?
- Did you duplicate segment *PA* on the starter line?
- What did you do after you duplicated segment *PA* on the starter line?
- How did you locate the side of the parallelogram opposite side *PA*?

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- What side of the parallelogram did you locate second?
- How many parallel line constructions were necessary to construct the parallelogram?
- How did you locate the diagonals of the parallelogram?
- What point did you locate after points *P* and *A*?
- Are there any shortcuts that can be taken in this construction?

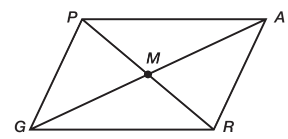
Grouping

Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- Are triangle *PGR* and *RAP* overlapping triangles?
- Are triangle *APG* and *GRA* overlapping triangles?
- Do the triangles share a common angle?
- Do the triangles share a common side?
- Are segments *PG* and *RA* corresponding parts of congruent triangles?
- Are segments *PA* and *RG* corresponding parts of congruent triangles?
- What is the definition of parallelogram?

3. To prove opposite sides of a parallelogram are congruent, which triangles would you prove congruent?



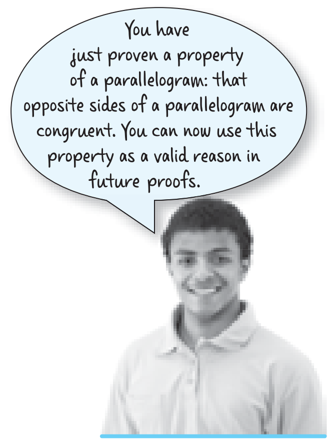
I can prove either $\triangle PGR \cong \triangle RAP$ or $\triangle APG \cong \triangle GRA$.

4. Use $\triangle PGR$ and $\triangle RAP$ in the parallelogram from Question 3 to prove that opposite sides of a parallelogram are congruent. Prove the statement $\overline{PG} \cong \overline{AR}$ and $\overline{GR} \cong \overline{PA}$.

Given: Parallelogram *PARG* with diagonals $\overline{PR}$ and $\overline{AG}$ intersecting at point *M*

Prove: $\overline{PG} \cong \overline{AR}$ and $\overline{GR} \cong \overline{PA}$

Statements	Reasons
1. Parallelogram *PARG* with diagonals $\overline{PR}$ and $\overline{AG}$ intersecting at point *M*	1. Given
2. $\overline{PG} \parallel \overline{AR}$ and $\overline{GR} \parallel \overline{PA}$	2. Definition of parallelogram
3. $\angle GPR \cong \angle ARP$ and $\angle APR \cong \angle GRP$	3. Alternate Interior Angle Theorem
4. $\overline{PR} \cong \overline{PR}$	4. Reflexive Property
5. $\triangle PGR \cong \triangle RAP$	5. ASA Congruence Theorem
6. $\overline{PG} \cong \overline{RA}$ and $\overline{GR} \cong \overline{AP}$	6. CPCTC



- Did you use the Alternate Interior Angle Theorem to prove this statement?
- Did your classmates use the same theorem to prove the corresponding angles congruent?
- What is true about the opposite sides of a parallelogram?
- Which corresponding parts of congruent triangles are used to prove this statement?

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Grouping

Have students complete Questions 5 through 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 9

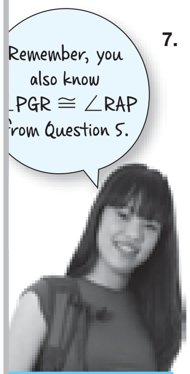
- What else can be stated using CPCTC?
- Which triangles are needed to show angles *PGR* and *RAP* congruent?
- Which triangles are needed to show angles *GPA* and *ARG* congruent?
- What statement can be made using the definition of a parallelogram?
- What is the Alternate Interior Angle Theorem?
- What statements can be made using the Alternate Interior Angle Theorem?
- Do the triangles share a common side?
- Which triangle congruency theorem was used to prove the two triangles congruent?
- Did your classmates use the same triangle congruency theorem?
- Can the triangles be proven congruent using a different theorem?
- When proving the diagonals bisect each other, which pairs of triangles were used?
- Which congruent segments are needed to prove the diagonals bisect each other?

5. Do you have enough information to conclude $\angle PGR \cong \angle RAP$? Explain your reasoning.

Yes. There is enough information because $\angle PGR \cong \angle RAP$ by CPCTC.

6. What additional angles would you need to show congruent to prove that opposite angles of a parallelogram are congruent? What two triangles do you need to prove congruent?

I would also need to show $\angle GPA \cong \angle ARG$. I can prove these angles congruent by CPCTC if I can prove $\triangle APG \cong \triangle GRA$.



7. Use $\triangle APG$ and $\triangle GRA$ in the diagram from Question 3 to prove that opposite angles of a parallelogram are congruent. Create a proof of the statement $\angle GPA \cong \angle ARG$.

Given: Parallelogram *PARG* with diagonals $\overline{PR}$ and $\overline{AG}$ intersecting at point *M*

Prove: $\angle GPA \cong \angle ARG$

Statements	Reasons
1. Parallelogram *PARG* with diagonals $\overline{PR}$ and $\overline{AG}$ intersecting at point *M*	1. Given
2. $\overline{PG} \parallel \overline{AR}$ and $\overline{PA} \parallel \overline{GR}$	2. Definition of parallelogram
3. $\angle PAG \cong \angle RGA$ and $\angle PGA \cong \angle RAG$	3. Alternate Interior Angle Theorem
4. $\overline{GA} \cong \overline{GA}$	4. Reflexive Property
5. $\triangle APG \cong \triangle GRA$	5. ASA Congruence Theorem
6. $\angle GPA \cong \angle ARG$	6. CPCTC

- What is the reason for the segment congruency?
- Which pair of sides was used in proving the Parallelogram/Congruent-Parallel Side Theorem?
- Can either pair of sides be used to prove this theorem?
- If the other pair of sides was used, how would this change the proof strategy?

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8. Prove that the diagonals of a parallelogram bisect each other. Use the parallelogram in Question 3.

I can prove $\triangle PMA \cong \triangle RMG$ by the AAS Congruence Theorem, so $\overline{PM} \cong \overline{RM}$ and $\overline{GM} \cong \overline{AM}$ by CPCTC, proving the diagonals of a parallelogram bisect each other.

9. Ray told his math teacher that he thinks a quadrilateral is a parallelogram if only one pair of opposite sides is known to be both congruent and parallel.

Is Ray correct? Use the diagram from Question 3 to either prove or disprove his conjecture.

Ray is correct.

He can prove $\triangle PAR \cong \triangle RGP$ by the ASA Congruence Theorem, so $\overline{PG} \cong \overline{AR}$ and $\angle PRA \cong \angle RPG$ by CPCTC. Then, he can use the Alternate Interior Angle Converse Theorem to show $\overline{PG} \parallel \overline{AR}$, proving quadrilateral *PARG* is a parallelogram.

The **Parallelogram/Congruent-Parallel Side Theorem** states: "If one pair of opposite sides of a quadrilateral is both congruent and parallel, then the quadrilateral is a parallelogram."

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Problem 2

A rhombus is a quadrilateral with four congruent sides.

The problem begins by students drawing a rhombus and listing all known properties.

Next students construct a rhombus.

Students prove:

- A rhombus is a parallelogram (inheriting all properties of a parallelogram)
- The diagonals of a rhombus are perpendicular
- The diagonals of a rhombus bisect the vertex angles

They also use construction tools to construct a rhombus given one side.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

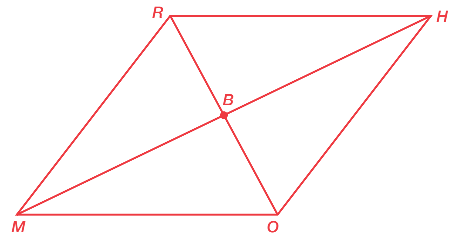
Guiding Questions for Share Phase, Questions 1 and 2

- What is the difference between a rhombus and a square?
- What is true about the length of the four sides of a rhombus?
- What do you know about the measure of the 4 interior angles in a rhombus?
- What is true about the opposite angles of a rhombus?

PROBLEM 2 The Rhombus

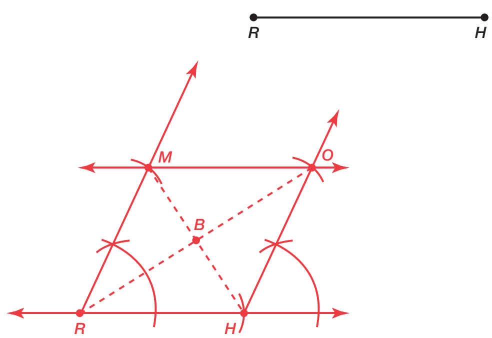
A rhombus is a quadrilateral with all sides congruent.

1. Draw a rhombus with two diagonals. Label the vertices and the intersection of the two diagonals. List all of the properties you know to be true. (Do not draw a square.)



- $\overline{RH} \cong \overline{HO} \cong \overline{OM} \cong \overline{MR}$
- $\angle MRH \cong \angle HOM$ and $\angle RMO \cong \angle OHR$
- $\overline{RM} \parallel \overline{OH}$, $\overline{RH} \parallel \overline{OM}$
- $\overline{RO} \perp \overline{HM}$
- $\overline{RB} \cong \overline{OB}$ and $\overline{MB} \cong \overline{HB}$
- $\angle RHM \cong \angle OHM$, $\angle HOR \cong \angle MOR$, $\angle OMH \cong \angle RMH$, and $\angle MRO \cong \angle HRO$

2. Use $\overline{RH}$ to construct rhombus *RHOM* with diagonals $\overline{RO}$ and $\overline{HM}$ intersecting at point *B*.



- Is a rhombus also a parallelogram?
- What do you know about the diagonals of a rhombus?
- Is a rhombus also a square or is a square also a rhombus?
- Is a rhombus also a rectangle or is a rectangle also a rhombus?
- Did you duplicate segment *RH* on the starter line?
- What did you do after you duplicated segment *RH* on the starter line?
- How did you locate the side of the rhombus opposite side *RH*?

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- What side of the rhombus did you locate second?
- How many parallel line constructions were necessary to construct the rhombus?
- How did you locate the diagonals of the rhombus?
- What point did you locate after points *R* and *H*?
- Are there any shortcuts that can be taken in this construction?

Grouping

Have students complete Questions 3 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- What is the definition of a rhombus?
- Using the definition of rhombus, which pair of triangles can be proven congruent using the SSS triangle congruency theorem?
- Are there other pairs of triangles that could be used? Explain.
- How is the Alternate Interior Angle Theorem used to show the opposite sides of the rhombus are parallel?
- If a rhombus can be proven to be a parallelogram, do all of the properties of a parallelogram hold true for any rhombus?

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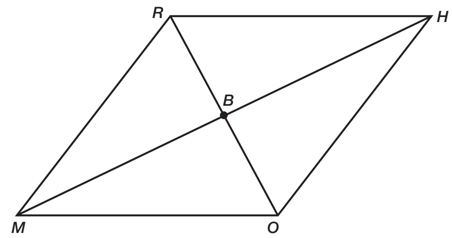
3. Prove that rhombus *RHOM* is a parallelogram.



I can prove $\triangle ROM \cong \triangle ORH$ by SSS by using the definition of a rhombus and the reflexive property of congruency and $\angle HRO \cong \angle MOR$ by CPCTC. This gives us $\overline{RM} \parallel \overline{OH}$ by the Alternate Interior Angle Theorem. Similarly, $\angle HOR \cong \angle MRO$ by CPCTC which shows $\overline{RM} \parallel \overline{OH}$ by the Alternate Interior Angle Theorem.

4. Since a rhombus is a parallelogram, what properties hold true for all rhombi?

Opposite angles are congruent, opposite sides are congruent, and diagonals bisect each other.

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Guiding Questions for Share Phase, Questions 5 and 6

- Using the definition of rhombus, and the reflexive property, which pair of triangles can be proven congruent using the SSS triangle congruency theorem to show angles *HRB* and *MRB* are congruent?
- If angles *HRB* and *MRB* are congruent, is this enough information to conclude segment *OR* bisects angle *HRM*? Explain.
- Using the definition of rhombus, and the reflexive property, which pair of triangles can be proven congruent using the SSS triangle congruency theorem to show angles *HOB* and *MOB* are congruent?
- If angles *HOB* and *MOB* are congruent, is this enough information to conclude segment *OR* bisects angle *HOM*? Explain.
- Using the definition of rhombus, and the reflexive property, which pair of triangles can be proven congruent using the SSS triangle congruency theorem to show angles *RHB* and *OHB* are congruent?
- If angles *RHB* and *OHB* are congruent, is this enough information to conclude segment *MH* bisects angle *RHO*? Explain.

5. Prove that the diagonals of a rhombus are perpendicular. Use the rhombus in Question 3.

I can prove $\triangle RBH \cong \triangle RBM$ by SSS by using the definition of a rhombus and the reflexive property of congruency to show $\angle RBH \cong \angle RBM$ by CPCTC. These angles also form a linear pair using the Linear Pair Postulate. The angles are supplementary using the definition of a linear pair, and two angles that are both congruent and supplementary are right angles. If they are right angles, then the line segments forming the angles must be perpendicular.

6. Prove that the diagonals of a rhombus bisect the vertex angles. Use the rhombus in Question 3.

I can prove $\triangle RBH \cong \triangle RBM$ by SSS by using the definition of a rhombus and the reflexive property of congruency to show $\angle HRB \cong \angle MRB$ by CPCTC. Then $\overline{OR}$ must bisect $\angle HRM$ by the definition of bisect. This can be done with each angle of the rhombus using the two different triangles for each vertex angle of the rhombus.

- Using the definition of rhombus, and the reflexive property, which pair of triangles can be proven congruent using the SSS triangle congruency theorem to show angles *RMB* and *OMB* are congruent?
- If angles *RMB* and *OMB* are congruent, is this enough information to conclude segment *MH* bisects angle *RMO*? Explain.

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Problem 3

Students apply their knowledge of the properties of parallelograms, the properties of rhombi, slopes of parallel and perpendicular lines, and the distance formula to solve problem situations.

Grouping

Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- If Jim tells you the quadrilateral is either a square of a rhombus, but not both, do you even need to ask a question to determine which quad he is thinking about?
- If Jim were thinking about a square, wouldn't it also be a rhombus?
- How could Albert use the diagonal length to show this flag was in the shape of a parallelogram?
- Could Albert use the length of the sides of the flag to show the flag was in the shape of a parallelogram? How?

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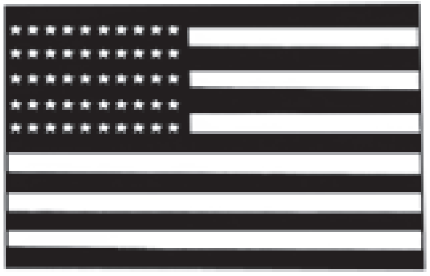
PROBLEM 3 Flags and Whatnot

1. Jim tells you he is thinking of a quadrilateral that is either a square or a rhombus, but not both. He wants you to guess which quadrilateral he is thinking of and allows you to ask one question about the quadrilateral. Which question should you ask?

   You don't need to ask Jim any questions to determine the identity of the quadrilateral. If it is a square, then it is also considered a rhombus. Since Jim said that it can't be a rhombus and a square, then it must be a rhombus, which is not a square.

2. Ms. Baker told her geometry students to bring in a picture of a parallelogram for extra credit. Albert brought in a picture of the flag shown. The teacher handed Albert a ruler and told him to prove it was a parallelogram. What are two ways Albert could prove the picture is a parallelogram?



   Albert could measure the opposite sides and show they are congruent, or he could measure the diagonals and show they bisect each other.

Guiding Questions for Share Phase, Question 3

- Why did Rena fold each piece of rope in half first?
- How did Rena use a perpendicular bisector to accomplish this task?

3. Ms. Baker held up two different lengths of rope shown and a piece of chalk. She asked her students if they could use this rope and chalk to construct a rhombus on the blackboard. Rena raised her hand and said she could construct a rhombus with the materials. Ms. Baker handed Rena the chalk and rope. What did Rena do?

Rena folded each piece of rope in half to locate the middle and marked it. Then, she crossed the two ropes such that the middle points overlapped and created a perpendicular bisector. Next, she used the chalk to connect the ends of the ropes to form the rhombus. To be sure the ropes were perpendicular, after drawing the rhombus she removed one rope and used it to compare the length of each side to verify all four sides were the same length.

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Guiding Questions for Share Phase, Question 4

- Do the diagonals of all parallelograms bisect each other?
- Are all of the sides of all rhombi congruent?
- Are all of the sides congruent and all of the angles congruent in all squares?
- Do the diagonals bisect the vertex angles in all rhombi?
- Are the four angles congruent in all rectangles?
- Are opposite sides both congruent and parallel in all rhombi?
- Are opposite angles congruent in all parallelograms?
- Are the diagonals perpendicular to each other in all rhombi?
- Are the diagonals congruent in all rectangles?

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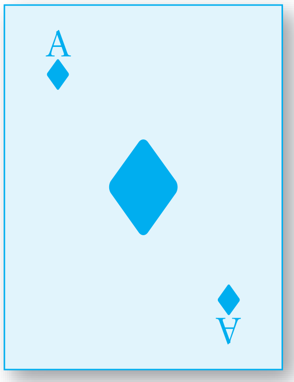
4. Consider the Ace of Diamonds playing card shown. The large diamond in the center of the playing card is a quadrilateral. Classify the quadrilateral based only on each piece of given information.



a. The diagonals of the quadrilateral bisect each other.

The quadrilateral is a parallelogram if the diagonals bisect each other.

b. The four sides of the quadrilateral are congruent.

The quadrilateral is a rhombus if the four sides are congruent.

c. The four angles and the four sides of the quadrilateral are congruent.

The quadrilateral is a square if the four angles and the four sides are congruent.

d. The diagonals of the quadrilateral bisect the vertex angles.

The quadrilateral is a rhombus if the diagonals bisect the vertex angles.

e. The four angles of the quadrilateral are congruent.

The quadrilateral is a rectangle if the four angles are congruent.

f. The opposite sides of the quadrilateral are both congruent and parallel.

The quadrilateral is a rhombus if the opposite sides are both congruent and parallel.

g. The opposite angles of the quadrilateral are congruent.

The quadrilateral is a parallelogram if the opposite angles are congruent.

h. The diagonals of the quadrilateral are perpendicular to each other.

The quadrilateral is a rhombus if the diagonals are perpendicular to each other.

i. The diagonals of the quadrilateral are congruent.

The quadrilateral is a rectangle if the diagonals are congruent.

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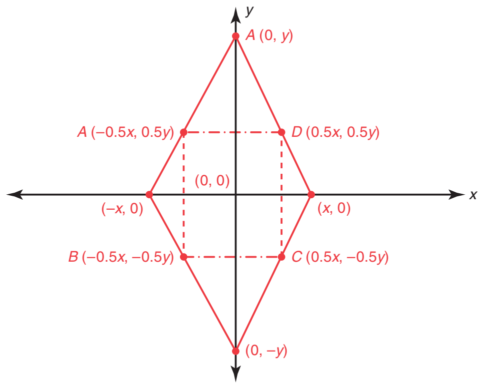
Guiding Questions for Share Phase, Question 5

- What coordinates represent the four vertices of the rhombus?
- If one vertex of the rhombus is located at $(x, 0)$, where are the other three vertices located?
- If one side of the rhombus is determined by the vertices $(x, 0)$ and $(0, y)$, where is the location of the midpoint on this side?
- If one side of the rhombus is determined by the vertices $(0, y)$ and $(-x, 0)$, where is the location of the midpoint on this side?
- What formulas were needed to determine the quadrilateral formed by connecting the midpoints of each side of a rhombus?
- How is using the formula for slope important in determining the quadrilateral formed by connecting the midpoints of each side of a rhombus?
- If the slope is equal to 0, what can you conclude?
- If the slope is undefined, what can you conclude?
- What can you conclude about the measures of the angles formed, when a horizontal line intersects a vertical line?

5. Sketch a rhombus that is not a square. Label the midpoint of each side of the rhombus.

a. Determine the polygon formed by connecting the midpoints of each side of a rhombus and justify your conclusion.



Points A, B, C, and D are midpoints of each side of the given rhombus. Let $x \neq y$.

$AB = \sqrt{(-0.5x - (-0.5x))^2 + (0.5y - (-0.5y))^2} = \sqrt{y^2} = y$

$BC = \sqrt{(-0.5x - 0.5x)^2 + (-0.5y - (-0.5y))^2} = \sqrt{(-x)^2} = \sqrt{x^2} = x$

$CD = \sqrt{(0.5x - 0.5x)^2 + (-0.5y - 0.5y)^2} = \sqrt{(-y)^2} = \sqrt{y^2} = y$

$DA = \sqrt{(0.5x - (-0.5x))^2 + (0.5y - 0.5y)^2} = \sqrt{x^2} = x$

$AB = CD$ and $BC = DA$

Slope of $\overline{AB}$: $m = \dfrac{0.5y - (-0.5y)}{-0.5x - (-0.5x)} = \dfrac{y}{0}$ (undefined)

Slope of $\overline{BC}$: $m = \dfrac{-0.5y - (-0.5y)}{-0.5x - 0.5x} = \dfrac{0}{-x} = 0$

Slope of $\overline{CD}$: $m = \dfrac{-0.5y - 0.5y}{0.5x - 0.5x} = \dfrac{y}{0}$ (undefined)

Slope of $\overline{DA}$: $m = \dfrac{0.5y - 0.5y}{0.5x - (-0.5x)} = \dfrac{0}{x} = 0$

Remember undefined slopes describe vertical lines and 0 slopes describe horizontal lines. Vertical lines are perpendicular to horizontal lines.

Line segment AB is perpendicular to line segment BC.

Line segment BC is perpendicular to line segment CD.

Line segment CD is perpendicular to line segment DA.

Line segment DA is perpendicular to line segment AB.

Line segment AB is parallel to line segment CD.

Line segment BC is parallel to line segment DA.

ABCD is a rectangle because opposite sides are congruent and all four angles are right angles.

- If opposite sides are congruent, and all angles are congruent, what can you conclude about the quadrilateral?
- If this entire process was repeated several more times, would the result remain the same each time? Explain.

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b. If the same process was repeated one more time by connecting the midpoints of each side of the polygon determined in part (a), describe the polygon that would result.

A rhombus would be determined. It could be proven a rhombus similar to the method used in part (a). Each vertex would be labeled in a similar fashion.

Be prepared to share your solutions and methods.

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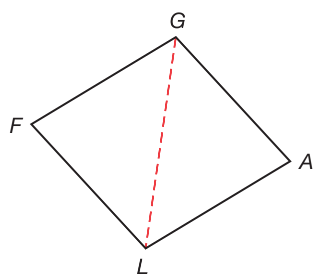
Check for Students' Understanding

John's math teacher, Ms. Diamond announced to the class that anyone bringing in a picture of a flag with a parallelogram would get extra credit. John brought in a picture of the Arkansas state flag.

John told Ms. Diamond that he measured each side of the blue quadrilateral and opposite sides were congruent, so he concluded it is a parallelogram. Ms. Diamond explained to John that the class had proven that if a quadrilateral is a parallelogram then opposite sides are congruent. To say if a quadrilateral has opposite sides congruent then it is a parallelogram is technically the converse and it would need to be proven. Ms. Diamond wrote some given information on the blackboard and drew this figure:



Given: Quadrilateral *FLAG*

$\overline{FL} \cong \overline{LA} \cong \overline{AG} \cong \overline{GF}$

Ms. Diamond told John he needed to prove that if a quadrilateral has opposite sides congruent, then the quadrilateral is a parallelogram to get the extra credit. Can you help John?

Statements	Reasons
1. $\overline{GL}$ is a diagonal of quadrilateral *RHOM*	1. Construction
2. Quadrilateral *FLAG*	2. Given
3. $\overline{FL} \cong \overline{LA} \cong \overline{AG} \cong \overline{GF}$	3. Given
4. $\overline{GL} \cong \overline{GL}$	4. Reflexive property
5. $\angle FLG \cong \angle AGL$	5. SSS Congruence Theorem
6. $\angle FGL \cong \angle ALG$, $\angle FLG \cong \angle AGL$	6. CPCTC
7. $\overline{FL} \parallel \overline{AG}$ and $\overline{FG} \parallel \overline{AL}$	7. Alternate Interior Angle Converse Theorem
8. *FLAG* is a parallelogram	8. Definition of parallelogram

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7.3

Kites and Trapezoids

Properties of Kites and Trapezoids

LEARNING GOALS

In this lesson, you will:

- Construct a kite and a trapezoid.
- Determine the properties of a kite and a trapezoid.
- Prove the properties of kites and trapezoids.
- Solve problems using the properties of kites and trapezoids.

KEY TERMS

- base angles of a trapezoid
- isosceles trapezoid
- biconditional statement
- midsegment
- Trapezoid Midsegment Theorem

ESSENTIAL IDEAS

- A kite is a quadrilateral with two pairs of consecutive congruent sides with opposite sides that are not congruent.
- One pair of opposite angles of a kite is congruent.
- One diagonal of a kite bisects the other diagonal.
- One diagonal of a kite bisects its vertex angles.
- The diagonals of a kite are perpendicular to each other.
- An isosceles trapezoid is a trapezoid with congruent non-parallel sides.
- The base angles of an isosceles trapezoid are congruent.
- An *if and only if* statement is called a bi-conditional statement because it is two separate statements rewritten as one statement. It is a combination of both the conditional statement and the converse of the conditional statement.
- A trapezoid is isosceles if and only if its diagonals are congruent.
- The Trapezoid Midsegment Theorem states: "The midsegment of a trapezoid is parallel to each of the bases and its length is one half the sum of the lengths of the bases."

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Prove geometric theorems

11. Prove theorems about parallelograms.

G-SRT Similarity, Right Triangles, and Trigonometry

Define trigonometric ratios and solve problems involving right triangles

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

G-GPE Expressing Geometric Properties with Equations

Use coordinates to prove simple geometric theorems algebraically

5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems.

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Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

Students prove the properties of a kite and a trapezoid using two-column and paragraph formats. Students apply the theorems to solve problem situations. Construction tools are used in this lesson.

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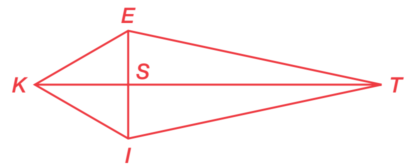
Warm Up



1. If $\overline{KT} \perp \overline{EI}$, is that enough information to prove two triangles congruent? If so, name the triangles.
   No, it is not enough information to prove two triangles congruent.

2. If $\overline{KT}$ bisects $\overline{EI}$, is that enough information to prove two triangles congruent? If so, name the triangles.
   No, it is not enough information to prove two triangles congruent.

3. If $\overline{KT}$ is the perpendicular bisector of $\overline{EI}$, is that enough information to prove two triangles congruent? If so, name the triangles.
   Yes, we could prove $\triangle KES \cong \triangle KIS$ and $\triangle TES \cong \triangle TIS$ by SAS Congruence Theorem.

4. If $\overline{KE} \cong \overline{KI}$, is that enough information to prove two triangles congruent? If so, name the triangles.
   No, it is not enough information to prove two triangles congruent.

5. If $\overline{TE} \cong \overline{TI}$, is that enough information to prove two triangles congruent? If so, name the triangles.
   No, it is not enough information to prove two triangles congruent.

6. If $\overline{KE} \cong \overline{KI}$ and $\overline{TE} \cong \overline{TI}$, is that enough information to prove two triangles congruent? If so, name the triangles.
   Yes, we could prove $\triangle KIT \cong \triangle KET$ by SSS Congruence Theorem.

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7.3

Kites and Trapezoids

Properties of Kites and Trapezoids

LEARNING GOALS

In this lesson, you will:

- Construct a kite and a trapezoid.
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- Solve problems using the properties of kites and trapezoids.

KEY TERMS

- base angles of a trapezoid
- isosceles trapezoid
- biconditional statement
- midsegment
- Trapezoid Midsegment Theorem

Your trapezius muscles are trapezoid-shaped muscles that extend down from the base of your head to the middle of your back and out to your shoulders.

When you lift your shoulders or try to squeeze your shoulder blades together, you are using your trapezius muscles.

Competitive weightlifters make heavy use of their trapezius muscles. When lifting the barbell from the floor to their collarbones—called the "clean" phase—weightlifters develop the upper portion of their trapezius muscles, which often gives them the appearance of having no neck.

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Problem 1

A kite is a quadrilateral with two pairs of consecutive congruent sides with opposite sides that are not congruent. The problem begins by students drawing a kite and listing all known properties. Next students construct a kite.

Students prove:

- One pair of opposite angles of a kite are congruent
- One diagonal of a kite bisects the second diagonal of a kite
- The diagonals of a kite are perpendicular to each other

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

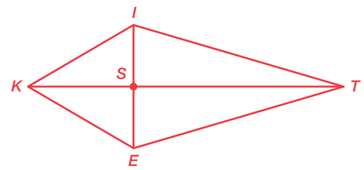
Guiding Questions for Share Phase, Questions 1 and 2

- What is true about the sides of a kite?
- What do you know about the measure of the 4 interior angles in a kite?
- What do you know about the diagonals of a kite?
- Is a kite also a parallelogram or is a parallelogram also a kite?
- Did you place a diagonal or a side of the kite on the starter line?
- What part of the kite did you locate second?

PROBLEM 1 **Let's Go Fly a Kite!**

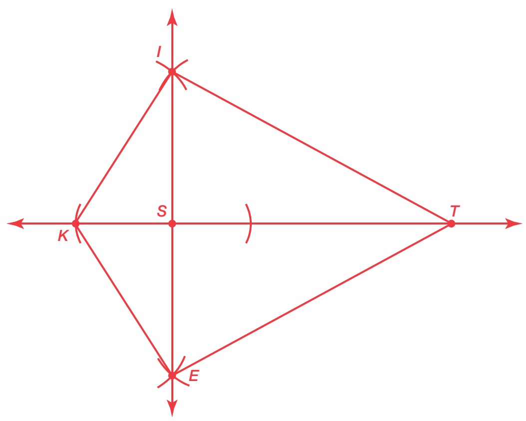
A kite is a quadrilateral with two pairs of consecutive congruent sides with opposite sides that are not congruent.

1. Draw a kite with two diagonals. Label the vertices and the intersection of the two diagonals. List all of the properties you know to be true.



- $\overline{KI} \cong \overline{KE}$
- $\overline{TI} \cong \overline{TE}$
- $\angle KIT \cong \angle KET$
- $\overline{IS} \cong \overline{ES}$
- $\angle IKS \cong \angle EKS$
- $\angle ITS \cong \angle ETS$

2. Construct kite *KITE* with diagonals $\overline{IE}$ and $\overline{KT}$ intersecting at point *S*.



- How many perpendicular line constructions were necessary to construct the kite?
- How did you locate the diagonals of the kite?
- What point did you locate first? Second? Third?
- Are there any shortcuts that can be taken in this construction?

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Grouping

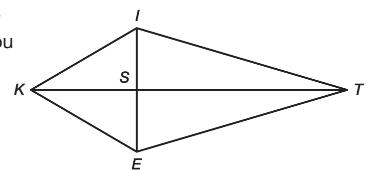
Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- Which two angles of the kite appear to be congruent?
- Are the two triangles used to show one pair of opposite angles in a kite are congruent overlapping triangles?
- Do the triangles share a common angle?
- Do the triangles share a common side?
- What is the definition of a kite?
- Which triangle congruency theorem was used to prove the two triangles congruent?
- Did your classmates use the same triangle congruency theorem?
- Can the triangles be proven congruent using a different theorem?
- Which corresponding parts of congruent triangles are used to prove this statement?

3. To prove that two opposite angles of a kite are congruent, which triangles in the kite would you prove congruent? Explain your reasoning.

I can prove $\triangle KIT \cong \triangle KET$ to show $\angle KIT \cong \angle KET$ by CPCTC.



4. Prove that two opposite angles of a kite are congruent.

Given: Kite *KITE* with diagonals $\overline{KT}$ and $\overline{IE}$ intersecting at point *S*.

Prove: $\angle KIT \cong \angle KET$

Statements	Reasons
1. Kite *KITE* with diagonals $\overline{KT}$ and $\overline{IE}$ intersecting at point *S*	1. Given
2. $\overline{KI} \cong \overline{KE}$	2. Definition of kite
3. $\overline{TI} \cong \overline{TE}$	3. Definition of kite
4. $\overline{KT} \cong \overline{KT}$	4. Reflexive Property
5. $\triangle KIT \cong \triangle KET$	5. SSS Congruence Theorem
6. $\angle KIT \cong \angle KET$	6. CPCTC

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Grouping

Have students complete Questions 5 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 8

- What else can be stated using CPCTC?
- Which triangles are needed to show angles *IKT* and *EKT* congruent?
- Which triangles are needed to show angles *ITK* and *ETK* congruent?
- What statement can be made using the definition of a kite?
- If segments *IS* and *ES* are congruent, does diagonal *KT* bisect diagonal *IE* or does diagonal *IE* bisect diagonal *KT*?
- Which congruency theorem can be used to show triangles *KIS* and *KES* are congruent?
- How is the Linear Pair Postulate helpful when proving the diagonals of a kite are perpendicular to each other?
- If two angles are both congruent and supplementary, what can you conclude about the measures of the angles?

5. Do you have enough information to conclude $\overline{KT}$ bisects $\angle IKE$ and $\angle ITE$? Explain your reasoning.

Yes. There is enough information to conclude because $\angle IKT \cong \angle EKT$ and $\angle ITK \cong \angle ETK$ by CPCTC.

6. What two triangles could you use to prove $\overline{IS} \cong \overline{ES}$?

I can first prove $\triangle KIS \cong \triangle KES$, or $\triangle ITS \cong \triangle ETS$ by the SAS Congruence Theorem, and then $\overline{IS} \cong \overline{ES}$ by CPCTC.

7. If $\overline{IS} \cong \overline{ES}$, is that enough information to determine that one diagonal of a kite bisects the other diagonal? Explain your reasoning.

Yes. If $\overline{IS} \cong \overline{ES}$, then by the definition of bisect, diagonal $\overline{KT}$ bisects diagonal $\overline{IE}$.

8. Prove that the diagonals of a kite are perpendicular to each other.

I can first prove $\triangle KIS \cong \triangle KES$ by the SAS Congruence Theorem, and then $\angle KIS \cong \angle KES$ by CPCTC. These angles also form a linear pair by the Linear Pair Postulate. The angles are supplementary by the definition of a linear pair, and two angles that are both congruent and supplementary are right angles. If they are right angles, then the lines forming the angles must be perpendicular.

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Problem 2

A trapezoid is a quadrilateral with exactly one pair of parallel sides. An isosceles trapezoid is a trapezoid with congruent non-parallel sides. The problem begins by students identifying the vertices, bases, base angles and legs of a trapezoid. Next students construct a trapezoid.

Students prove:

- The base angles of an isosceles trapezoid are congruent.

An *if and only if* statement or a bi-conditional statement is introduced and students prove a bi-conditional statement: A trapezoid is isosceles if and only if its diagonals are congruent.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

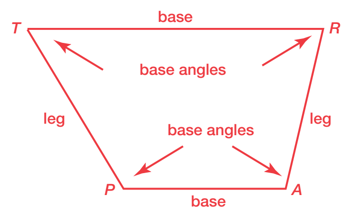
- What is the difference between a trapezoid and a parallelogram?
- What is the difference between the bases and the legs of a trapezoid?
- Did you duplicate segment *TR* on the starter line?
- What did you do after you duplicated segment *TR* on the starter line?
- How did you locate the side of the trapezoid opposite side *TR*?
- What side of the trapezoid did you locate second?
- How many parallel line constructions were necessary to construct the trapezoid?
- Do trapezoids have diagonals?
- What point did you locate after points *T* and *R*?
- Are there any shortcuts that can be taken in this construction?

PROBLEM 2 The Trapezoid

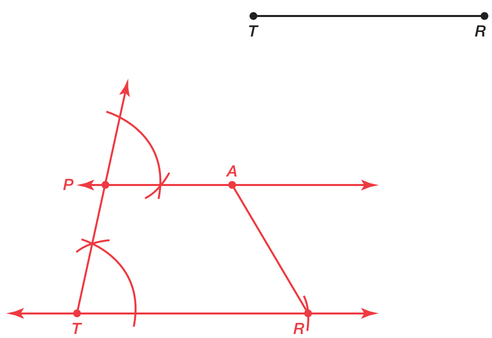
A trapezoid is a quadrilateral with exactly one pair of parallel sides.

The bases of a trapezoid are its parallel sides. The **base angles of a trapezoid** are either pair of angles that share a base as a common side. The legs of a trapezoid are its non-parallel sides.

1. Draw a trapezoid. Identify the vertices, bases, base angles, and legs.



2. Use $\overline{TR}$ to construct trapezoid *TRAP*.



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Grouping

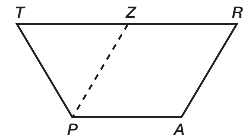
Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- How is an isosceles trapezoid different from other trapezoids?
- What are the similarities and differences between the two proofs?
- How do you know lines segments *PZ* and *AR* are parallel to each other?
- If *PZ* and *AR* are drawn parallel to each other, is quadrilateral *PZRA* a parallelogram?
- How is the transitive property used to determine triangle *TZP* is isosceles?
- How is the transitive property used to determine angles *T* and *R* congruent?
- How is the Same Side Interior Angle Theorem used to determine angles *P* and *A* congruent?
- How is the Isosceles Triangle Converse Theorem used to determine the trapezoid is isosceles?

An **isosceles trapezoid** is a trapezoid with congruent non-parallel sides.

3. Prove that the base angles of an isosceles trapezoid are congruent.



a. Given: Isosceles Trapezoid *TRAP* with $\overline{TR} \parallel \overline{PA}$, $\overline{TP} \cong \overline{RA}$

Prove: $\angle T \cong \angle R$

Statements	Reasons
1. Isosceles trapezoid *TRAP* with $\overline{TR} \parallel \overline{PA}$, $\overline{TP} \cong \overline{RA}$, and $\overline{ZP} \parallel \overline{RA}$	1. Given
2. $\overline{ZP} \parallel \overline{RA}$	2. Construction
3. Quadrilateral *ZRAP* is a parallelogram.	3. Definition of parallelogram
4. $\overline{ZP} \cong \overline{RA}$	4. Opposite sides of a parallelogram are congruent.
5. $\overline{TP} \cong \overline{ZP}$	5. Transitive Property
6. $\triangle TPZ$ is an isosceles triangle	6. Definition of isosceles triangle
7. $\angle T \cong \angle TZP$	7. Isosceles Triangle Theorem
8. $\angle TZP \cong \angle R$	8. Corresponding Angle Theorem
9. $\angle T \cong \angle R$	9. Transitive Property

b. You must also prove $\angle A \cong \angle TPA$. Prove $\angle A \cong \angle TPA$.

Angle *T* and $\angle P$ as well as $\angle R$ and $\angle A$ are pairs of supplementary angles using the Same Side Interior Angle Theorem.

Since I have already proven $\angle T \cong \angle R$, then $\angle P \cong \angle A$ by supplements of congruent angles are congruent.

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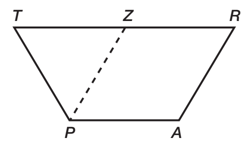
4. Kala insists that if a trapezoid has only one pair of congruent base angles, then the trapezoid must be isosceles. She thinks proving two pairs of base angles are congruent is not necessary. Prove the given statement to show that Kala is correct.

Given: Isosceles trapezoid *TRAP* with $\overline{TR} \parallel \overline{PA}$, $\angle T \cong \angle R$

Prove: $\overline{TP} \cong \overline{RA}$



Statements	Reasons
1. Isosceles trapezoid *TRAP* with $\overline{TR} \parallel \overline{PA}$, $\angle T \cong \angle R$	1. Given
2. $\overline{ZP} \parallel \overline{RA}$	2. Construction
3. Quadrilateral *ZRAP* is a parallelogram.	3. Definition of parallelogram
4. $\overline{ZP} \cong \overline{RA}$	4. Opposite sides of a parallelogram are congruent.
5. $\angle R \cong \angle TZP$	5. Corresponding Angle Theorem
6. $\angle T \cong \angle TZP$	6. Transitive Property
7. $\triangle TPZ$ is an isosceles triangle	7. Isosceles Triangle Theorem
8. $\overline{ZP} \cong \overline{TP}$	8. Isosceles Triangle Converse Theorem
9. $\overline{TP} \cong \overline{RA}$	9. Transitive Property
10. Trapezoid *TRAP* is isosceles	10. Definition of Isosceles Trapezoid

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Grouping

- Ask a student to read the information and definition before Question 5. Discuss as a class.
- Have students complete Questions 5 and 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Discuss Phase

- What is a conditional statement?
- Provide an example of a real-life conditional statement.
- What is a biconditional statement?
- Provide an example of a real-life biconditional statement.
- What is the difference between a conditional statement and a biconditional statement?

Guiding Questions for Share Phase, Questions 5 and 6

- Which triangles are used to show the diagonals of the trapezoid are congruent?
- Are the triangles overlapping triangles?
- Do the triangles share a common side?
- Do the triangles share a common angle?
- Which triangle congruency theorem is used to prove the two triangles congruent?
- Are segments *TA* and *PR* corresponding parts of congruent triangles?

An *if and only if* statement is called a **biconditional statement** because it consists of two separate conditional statements rewritten as one statement. It is a combination of both a conditional statement and the converse of that conditional statement. A biconditional statement is true only when the conditional statement and the converse of the statement are both true.

Consider the following property of an isosceles trapezoid:

The diagonals of an isosceles trapezoid are congruent.

The property clearly states that if a trapezoid is isosceles, then the diagonals are congruent. Is the converse of this statement true? If so, then this property can be written as a biconditional statement. Rewording the property as a biconditional statement becomes:

A trapezoid is isosceles *if and only if* its diagonals are congruent.

To prove this biconditional statement is true, rewrite it as two conditional statements and prove each statement.

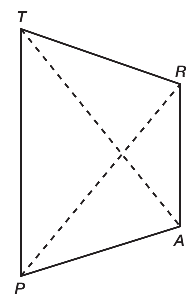
Statement 1: If a trapezoid is an isosceles trapezoid, then the diagonals of the trapezoid are congruent. (Original statement)

Statement 2: If the diagonals of a trapezoid are congruent, then the trapezoid is an isosceles trapezoid. (Converse of original statement)

5. Prove that the diagonals of an isosceles trapezoid are congruent.

Given: Isosceles trapezoid *TRAP* with $\overline{TP} \parallel \overline{RA}$, $\overline{TR} \cong \overline{PA}$, and diagonals $\overline{TA}$ and $\overline{PR}$.

Prove: $\overline{TA} \cong \overline{PR}$



Statements	Reasons
1. Isosceles trapezoid *TRAP* with $\overline{TP} \parallel \overline{RA}$, $\overline{TR} \cong \overline{PA}$, and diagonals $\overline{TA}$ and $\overline{PR}$	1. Given
2. $\angle RTP \cong \angle APT$	2. Base angles of an isosceles trapezoid are congruent.
3. $\overline{TP} \cong \overline{TP}$	3. Reflexive Property
4. $\triangle RTP \cong \triangle APT$	4. SAS Congruence Theorem
5. $\overline{TA} \cong \overline{PR}$	5. CPCTC

- What is true about the opposite sides of a rectangle?
- What is true about the angles of a rectangle?
- Are triangles *TEA* and *PZR* right triangles? How do you know?
- Which congruency theorem is used to show triangles *TEA* and *PZR* are congruent?
- Which congruency theorem is used to show triangles *RTA* and *APR* are congruent?

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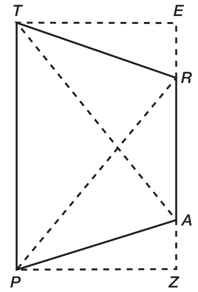
6. Prove the converse, that the trapezoid is isosceles if the diagonals are congruent.

Given: Trapezoid *TRAP* with $\overline{TP} \parallel \overline{RA}$, and diagonals $\overline{TA} \cong \overline{PR}$

Prove: Trapezoid *TRAP* is isosceles

To prove the converse, auxiliary lines must be drawn such that $\overline{RA}$ is extended to intersect a perpendicular line passing through point *T* perpendicular to $\overline{RA}$ ($\overline{TE}$) and intersect a second perpendicular line passing through point *P* perpendicular to $\overline{RA}$ ($\overline{PZ}$).



Notice that quadrilateral *TEZP* is a rectangle.

I know $\overline{TE} \cong \overline{PZ}$ because opposite sides of a rectangle are congruent. Angle *E* and $\angle Z$ are right angles by the definition of a rectangle, so $\triangle TEA$ and $\triangle PZR$ are right triangles. It is given that $\overline{TA} \cong \overline{PR}$, so $\triangle TEA$ and $\triangle PZR$ by *HL*. I get $\angle EAT \cong \angle ZRP$ by CPCTC. I know $\overline{AR} \cong \overline{AR}$ by the Reflexive Property, so $\triangle RTA \cong \triangle APR$ by the SAS Congruence Theorem. We get $\overline{TR} \cong \overline{PA}$ by CPCTC, proving trapezoid *TRAP* is an isosceles trapezoid by the definition of an isosceles trapezoid.

The property of an isosceles trapezoid can now be written as a biconditional statement because the conditional statement and its converse have both been proven to be true: A trapezoid is isosceles if and only if its diagonals are congruent.

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Problem 3

Students construct an isosceles trapezoid given the perimeter. Steps to guide the students through the construction are provided.

Grouping

Have students complete Questions 1 through 10 with a partner. Then have students share their responses as a class.

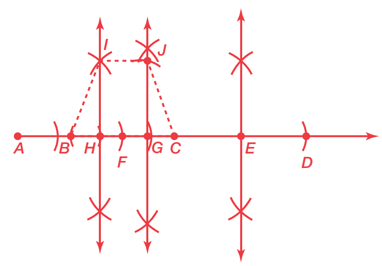
Guiding Questions for Share Phase, Questions 1 through 10

- If *AD* is the perimeter of an isosceles trapezoid, do you suppose this length determines a unique isosceles trapezoid, or could this length determine different isosceles trapezoids? Explain.
- What should determine or limit the length of the shorter base, *AB*?
- What should determine or limit the length of the longer base, *BC*?
- If segment *CD* represents the sum of the length of the two legs, does that mean it is twice as long as one leg?
- Are both legs of the trapezoid the same length?
- Why is it necessary to determine the length of the difference (*BF*) between the bases?
- What is the importance of bisecting segment *FC* to locate point *G*?

PROBLEM 3 **Construction**

Segment *AD* is the perimeter of an isosceles trapezoid. Follow the steps to construct the isosceles trapezoid.

1. Choose a line segment on $\overline{AD}$ for the shorter base ($\overline{AB}$).
2. Choose a line segment on $\overline{AD}$ for the longer base ($\overline{BC}$).
3. Line segment *CD* represents the sum of the length of the two legs.
4. Bisect $\overline{CD}$ to determine the length of each congruent leg. Label the midpoint *E*.
5. Copy $\overline{AB}$ onto $\overline{BC}$ (creating $\overline{BF}$) to determine the difference between the bases ($\overline{FC}$).
6. Bisect $\overline{FC}$ to determine point *G* ($FG = CG$).
7. Take $\overline{FG}$ (half the difference of the base lengths) and copy it onto the left end of the long base $\overline{BC}$ (creating distance $\overline{BH}$). Notice that it is already marked off on the right end of the long base ($\overline{GC}$).
8. Construct the perpendicular through point *H*. Note the distance between the two most left perpendiculars is the length of the short base.
9. Place the compass point on *B* and mark off the distance of one leg ($\overline{CE}$) on the left most perpendicular, naming the new point *I*. This forms one leg of the isosceles trapezoid.
10. Place the compass point on *C* and mark off the length of one leg ($\overline{CE}$) on the other perpendicular, naming the new point *J*. This is one leg of the isosceles trapezoid. Note that $IJ = AB$. *BIJC* is an isosceles trapezoid!



- How many perpendicular constructions were necessary to complete this construction?
- Does this construction result in a unique construction?
- How does your isosceles trapezoid compare to your classmates' constructions?
- Can this construction be completed in fewer steps?

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Problem 4

Students create a trapezoid on the coordinate plane and use it to determine characteristics of its midsegment. Then, students prove the Trapezoid Midsegment Theorem.

Grouping

Have students complete Questions 1 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- How does your trapezoid compare to your classmates' trapezoids?
- What do all of the trapezoids have in common?
- How are some of the trapezoids different?
- What formula was used to determine the coordinates of the midpoints of the legs of the trapezoid?
- Was a formula needed to determine the length of the line segment connecting the midpoints of the legs of the trapezoid? Explain.

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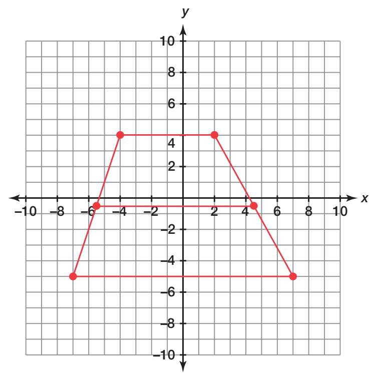
PROBLEM 4 Midsegment of a Trapezoid

1. Locate four points on the coordinate plane to form a trapezoid.



2. Identify the coordinates of the four points you chose.

   Answers will vary.

   (−4, 4), (2, 4), (−7, −5) and (7, −5)

3. Determine the coordinates of the midpoints of the legs of your trapezoid. Use the midpoint formula.

   Answers will vary.

   $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-4 - 7}{2}, \frac{4 - 5}{2}\right) = \left(\frac{-11}{2}, \frac{-1}{2}\right) = (-5.5, -0.5)$

   $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2 + 7}{2}, \frac{4 - 5}{2}\right) = \left(\frac{9}{2}, \frac{-1}{2}\right) = (4.5, -0.5)$

4. Plot and connect the midpoints of the legs. Determine the distance between the two midpoints.

   Answers will vary.

   The distance between the two midpoints of the legs of the trapezoid is 10 units.

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Guiding Questions for Share Phase, Questions 5 through 8

- What is the sum of the lengths of the two bases of the trapezoid?
- What slope is associated with the midsegment of the trapezoid?
- What slope is associated with the bases of the trapezoid?
- If the slopes associated with the midsegment and the two bases are the same, what can you conclude?

The **midsegment** of a trapezoid is a segment formed by connecting the midpoints of the legs of the trapezoid.

5. Determine the lengths of the two bases of your trapezoid.

Answers will vary.

$d = \sqrt{(-4 - 2)^2 + (4 - 4)^2}$

$= \sqrt{36 + 0}$

$= 6$

$d = \sqrt{(-7 - 7)^2 + (-5 + 5)^2}$

$= \sqrt{196 + 0}$

$= 14$

The lengths of the two bases of the trapezoid are 6 units and 14 units.

6. Determine the length of the midsegment of your trapezoid.

Answers will vary.

$d = \sqrt{(-5.5 - 4.5)^2 + (-0.5 - (-0.5))^2}$

$= \sqrt{100 + 0} = 10$

The length of the midsegment of the trapezoid is 10 units.

7. Compare the length of the midsegment to the sum of the lengths of the bases.

Answers will vary.

The length of the midsegment of a trapezoid is one half (10) the sum of the lengths of the bases of the trapezoid (20).

8. Is the midsegment of the trapezoid parallel to the bases of the trapezoid? Explain your reasoning.

Yes. The bases of the trapezoid are horizontal line segments with a slope equal to zero. The midsegment of the trapezoid is also a horizontal line segment with a slope equal to zero. Lines with the same slope are parallel, so the midsegment is parallel to the bases.

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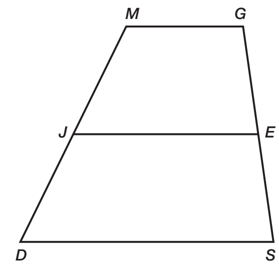
Grouping

Have students complete Question 9 with a partner. Then have students share their responses as a class.

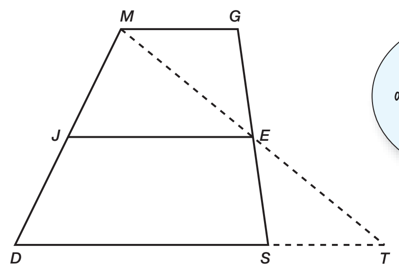
Guiding Questions for Share Phase, Question 9

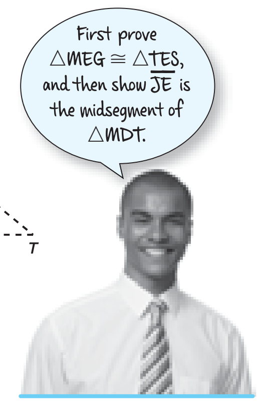
- How many prove statements are needed to prove this theorem?
- Are triangles *MEG* and *TES* overlapping triangles?
- Are triangles *MEG* and *TES* vertical triangles?
- What is the definition of trapezoid?
- Which line segments are parallel?
- How is the Alternate Interior Angle Theorem used to show the triangles are congruent?
- Which triangle congruency theorem is used to show the triangles are congruent?
- What is the Midsegment of a Triangle Theorem?
- How is the Midsegment of a Triangle Theorem used in this situation?
- If two lines are parallel to the same line, what can you conclude?
- How is segment addition used to prove this theorem?
- How is substitution used in the proof of this theorem?

The **Trapezoid Midsegment Theorem** states: "The midsegment of a trapezoid is parallel to each of the bases and its length is one half the sum of the lengths of the bases."



9. Prove the Trapezoid Midsegment Theorem. It will be necessary to connect points *M* and *E* to form $\overline{ME}$, and then extend $\overline{ME}$ until it intersects the extension of $\overline{DS}$ at point *T*.





a. Complete the "Prove" statement.

Given: *MDSG* is a trapezoid

J is the midpoint of $\overline{MD}$

E is the midpoint of $\overline{GS}$

Prove: $\overline{JE} \parallel \overline{MG}$

$\overline{JE} \parallel \overline{DS}$

$JE = \frac{1}{2}(MG + DS)$

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b. Create a proof of the Trapezoid Midsegment Theorem.

Statements	Reasons
1. *MDSG* is a trapezoid, *J* is the midpoint of $\overline{MD}$, and *E* is the midpoint of $\overline{GS}$.	**1.** Given
2. Connect points *M* and *E* to form $\triangle MEG$ and extend $\overline{ME}$ until it intersects the extension of $\overline{DS}$ at point *T*.	**2.** Construction
3. $\overline{GE} \cong \overline{ES}$	**3.** Definition of midpoint
4. $\overline{MG} \parallel \overline{DS}$	**4.** Definition of trapezoid
5. $\angle MGE \cong \angle TSE$	**5.** Alternate Interior Angle Theorem
6. $\angle MEG \cong \angle TES$	**6.** Vertical angles are congruent.
7. $\triangle MEG \cong \triangle TES$	**7.** ASA Congruence Theorem
8. $\overline{ME} \cong \overline{TE}$	**8.** CPCTC
9. *E* is the midpoint of $\overline{MT}$.	**9.** Definition of midpoint
10. $\overline{JE}$ is the midsegment of $\triangle MDT$.	**10.** Definition of midsegment of a triangle
11. $\overline{JE} \parallel \overline{DT}$, so $\overline{JE} \parallel \overline{DS}$	**11.** Midsegment of a Triangle Theorem
12. $\overline{JE} \parallel \overline{MG}$	**12.** Two lines parallel to the same line are parallel to each other.
13. $JE = \frac{1}{2} DT$	**13.** Midsegment of a Triangle Theorem
14. $DT = DS + ST$	**14.** Segment addition
15. $ST \cong MG$	**15.** CPCTC
16. $\overline{ST} = \overline{MG}$	**16.** Definition of congruent segments
17. $DT = DS + MG$	**17.** Substitution steps 14 and 16
18. $JE = \frac{1}{2}(DS + MG)$	**18.** Substitution steps 13 and 17

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Problem 5

Students solve problem situations involving the properties of kites and trapezoids.

Grouping

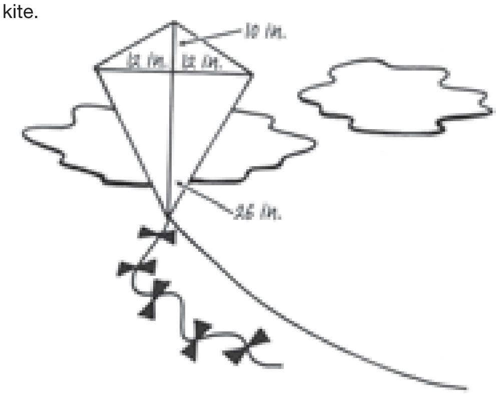
Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- How is the Pythagorean Theorem used to determine the perimeter of the kite?
- Do you need to determine the length of the four sides of the kite to determine the perimeter?
- Are regions 4 and 5 right triangles? Explain.
- Does Trevor know anything about the length of the bases of the three trapezoids?
- If trapezoids are congruent, are the length of their corresponding bases also congruent?

PROBLEM 5 Kites for Real

1. Determine the perimeter of the kite.



$c^2 = 10^2 + 12^2$

$c^2 = 100 + 144$

$c^2 = 244$

$c = \sqrt{244} \approx 15.6$

$c^2 = 12^2 + 26^2$

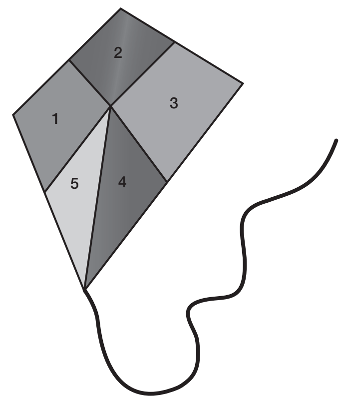
$c^2 = 144 + 676$

$c^2 = 820$

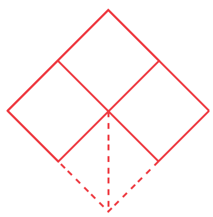
$c = \sqrt{820} \approx 28.6$

The perimeter of the kite is 2(15.6) + 2(28.6), or approximately 88.4 inches.

2. Could quadrilaterals 1, 2, and 3 on the kite shown be squares? Explain your reasoning.



No. Quadrilaterals 1, 2, and 3 could not be squares. If they were squares, it would not be possible to form obtuse triangles 4 and 5. The triangles would be congruent right triangles, with each triangle half the area of one square as shown. The entire kite would be shaped like a square.



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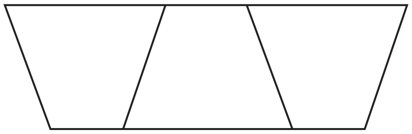
3. Trevor used a ruler to measure the height of each trapezoid and the length of each leg. He tells Carmen the three trapezoids must be congruent because they are all the same height and have congruent legs. What does Carmen need to do to convince Trevor that he is incorrect?



Carmen needs to extend the length of the base lines on one of the trapezoids, which will not affect the height or the length of the legs. Trevor will then see the extended trapezoid is a different shape, as shown.

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4. Mr. King said he was thinking of a quadrilateral and wanted his students to name the quadrilateral. He said he would answer a few yes-no questions to give them a hint. What questions would you ask Mr. King?

What is the fewest number of questions you could ask Mr. King to be sure you knew the correct answer?

Answers will vary.

Does the quadrilateral contain 4 right angles?
(Tells you it is a rectangle or square if it has 4 right angles)

Are there 2 pairs of parallel sides in the quadrilateral?
(Tells you it is a parallelogram if it has 2 pairs of parallel sides)

Is there exactly 1 pair of parallel sides in the quadrilateral?
(Tells you it is a trapezoid if it has 1 pair of parallel sides)

Are all 4 sides congruent in the quadrilateral?
(Tells you it is a rhombus or square if it has 4 congruent sides)

Are the 2 diagonals congruent to each other?
(Tells you it is a parallelogram if it has congruent diagonals)

Are the 2 diagonals perpendicular to each other?
(Tells you it is a kite, rhombus, or square if it has perpendicular diagonals)

Is exactly 1 pair of opposite angles congruent in the quadrilateral?
(Tells you it is a kite if exactly 1 pair of opposite angles are congruent)

The fewest number of questions would depend on Mr. King's answers. If he says it has 4 right angles, then I know it is a square or a rectangle and I would only have to ask if the diagonals are perpendicular to each other to know the correct answer. In this case, I would only have to ask 2 questions.

Be prepared to share your solutions and methods.

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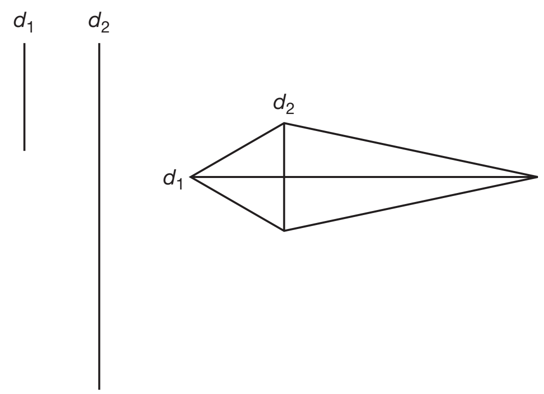
Check for Students' Understanding

Nick found an easy way to determine the area of a kite using only the length of the diagonals. Can you? Write a formula for the area of a kite in terms of d_1 and d_2.



The kite can be divided into two triangles where d_1 is the base and $\frac{1}{2}d_2$ is the height

In the upper and lower triangles in which d_1 is the base and $\frac{1}{2}d_2$ is the height, the area in each triangle is:

$A = \frac{1}{2}(d_1)\left(\frac{1}{2}d_2\right)$

$A = \frac{1}{4}(d_1)(d_2)$

The area of the kite is $(2)\frac{1}{4}(d_1)(d_2) = \frac{1}{2}d_1d_2$.

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Interior Angles of a Polygon

7.4

Sum of the Interior Angle Measures of a Polygon

LEARNING GOALS

In this lesson, you will:

- Write the formula for the sum of the measures of the interior angles of any polygon.
- Calculate the sum of the measures of the interior angles of any polygon, given the number of sides.
- Calculate the number of sides of a polygon, given the sum of the measures of the interior angles.
- Write a formula for the measure of each interior angle of any regular polygon.
- Calculate the measure of an interior angle of a regular polygon, given the number of sides.
- Calculate the number of sides of a regular polygon, given the sum of the measures of the interior angles.

KEY TERM

- interior angle of a polygon

ESSENTIAL IDEAS

- An interior angle of a polygon is an angle which faces the inside of a polygon and is formed by consecutive sides of the polygon.
- The sum of the measures of the interior angles of an n-sided polygon is $180°(n - 2)$.
- The measure of each interior angle of a regular n-sided polygon is $\frac{180°(n - 2)}{n}$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Prove geometric theorems

9. Prove theorems about lines and angles.

G-SRT Similarity, Right Triangles, and Trigonometry

Define trigonometric ratios and solve problems involving right triangles

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

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G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

Students derive a formula for calculating the sum of the measures of the interior angles of an *n*-sided polygon and the formula for determining the measure of each interior angle of a regular *n*-sided polygon. The formulas are also used to solve problem situations in which they determine the number of sides of the polygon given information about the measures of the interior angles or the measure of the interior angle sum.

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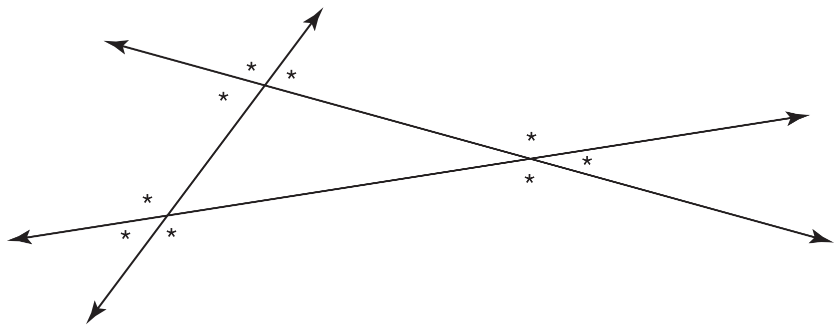
Warm Up

John announced to the class that he could calculate the sum of the measures of the starred angles in this diagram without knowing the measure of any specific angle. How is this possible? Using theorems or postulates, explain what John is thinking.



Using the Linear Pair Postulate, John first determined the sum of the measures of all 12 angles formed by the three intersecting lines is 180(6) = 1080°. John then used the Triangle Sum Theorem to determine the sum of the interior angles of the triangle is 180°. By subtracting the sum of the interior angles of the triangle from the sum of the 12 angles, 1080–180, John determined that the sum of the starred angles is 900 degrees.

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Interior Angles of a Polygon

7.4

Sum of the Interior Angle Measures of a Polygon

LEARNING GOALS

In this lesson, you will:

- Write the formula for the sum of the measures of the interior angles of any polygon.
- Calculate the sum of the measures of the interior angles of any polygon, given the number of sides.
- Calculate the number of sides of a polygon, given the sum of the measures of the interior angles.
- Write a formula for the measure of each interior angle of any regular polygon.
- Calculate the measure of an interior angle of a regular polygon, given the number of sides.
- Calculate the number of sides of a regular polygon, given the sum of the measures of the interior angles.

KEY TERM

- interior angle of a polygon

The Susan B. Anthony dollar coin was minted from 1979 to 1981 and then again in 1999. This coin was the first to show the image of a real woman.

The coin also had an unusual shape inscribed inside of it—a regular undecagon, or hendecagon, which is an 11-sided polygon. The shape was no accident, as you can see from the back of the coin which commemorates the Apollo 11 moon landing.

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Problem 1

Students explore the sum of the measures of the interior angles of a quadrilateral using diagonals.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

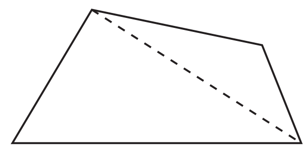
- What is the sum of the measures of the interior angles of a triangle?
- A quadrilateral containing one diagonal is divided into how many separate triangles?
- A quadrilateral containing two intersecting diagonals is divided into how many separate triangles?

PROBLEM 1 Inside Job

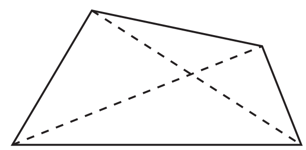
An **interior angle of a polygon** faces the inside of a polygon and is formed by consecutive sides of the polygon.

1. Ms. Lambert asked her class to determine the sum of the interior angle measures of a quadrilateral.

Carson drew a quadrilateral and added one diagonal as shown. He concluded that the sum of the measures of the interior angles of a quadrilateral must be equal to 360°.



Juno drew a quadrilateral and added two diagonals as shown. She concluded that the sum of the measures of the interior angles of a quadrilateral must be equal to 720°.



Who is correct? Who is incorrect? Explain your reasoning.

Carson is correct. Carson concluded the sum of the interior angle measures of a quadrilateral is equal to 360° because the diagonal formed two distinct triangles within the quadrilateral. Thus, the sum of the measures of the interior angles of each triangle is 180°. Therefore, $2(180°) = 360°$.

When drawing the two intersecting diagonals, Juno created extra angles that are not considered interior angles of the original quadrilateral. Therefore, her answer has an extra 360° because the additional angles form a circle.

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Problem 2

This activity provides students the opportunity to develop the formula used to determine the sum of the measures of the interior angles of a polygon: $(n - 2)180$ where n is the number of sides. Students will use prior knowledge, specifically the Triangle Sum Theorem to gather data related to number of sides of the polygon, number of diagonals drawn from one vertex, number of triangles formed and sum of the measures of the interior angles of the total number of triangles formed by the diagonals. Viewing this data in a chart format allows students to identify a pattern and eventually generalize. This generalization enables the students to create a formula for polygons with n sides.

Allowing students the time to gather data and develop their own formula will empower them to reconstruct their knowledge when needed, rather than memorize the formula and not understand why we subtract 2 from the number of sides or error in trying to recall what number was subtracted.

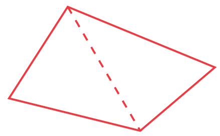
The questions in this activity require students to work arithmetically both forward and backward. Students will be given the number of sides of the polygon and asked to solve for the sum of the measures of the interior angles of the polygon, as well as solve for the number of sides in the polygon given the sum of the measures of the interior angles.

PROBLEM 2 How Many Triangles?

The Triangle Sum Theorem states that the sum of the three interior angles of any triangle is equal to 180°. You can use this information to calculate the sum of the interior angles of other polygons.

1. Calculate the sum of the interior angle measures of a quadrilateral by completing each step.

   a. Draw a quadrilateral. Draw a diagonal using only one vertex of the quadrilateral.



   b. How many triangles are formed when the diagonal divides the quadrilateral?

   Two triangles are formed when the diagonal divides the quadrilateral.

   c. If the sum of the interior angle measures of each triangle is 180°, what is the sum of all the interior angle measures of the triangles formed by the diagonal?

   The sum of all the interior angle measures of the two triangles is 360°.

Grouping

Have students complete Questions 1 through 10 with a partner. Then have students share their responses as a class.

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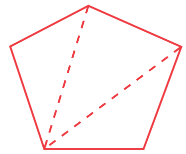
Guiding Questions for Share Phase, Questions 1 through 10

- Why is it important to draw all possible diagonals from a single vertex to form the triangles in the polygon?
- If the diagonals were drawn from more than one diagonal, how would this affect the total number of triangles formed in each polygon?
- If the diagonals are drawn from a single vertex, will the diagonals ever intersect each other?
- Why is there always one more triangle than the number of diagonals drawn from one vertex in the polygon?
- Why is the number of diagonals drawn always three less than the number of sides of the polygon?
- Why is the number of triangles drawn always two less than the number of sides of the polygon?
- As the number of sides of each polygon increases by 1, how much does the sum of the interior angle measures of the polygon increase by?
- Four diagonals drawn inside a polygon forms how many triangles?
- If the sum of the measures of the interior angles of a polygon is 900°, how many sides does the polygon have?

2. Calculate the sum of the interior angle measures of a pentagon by completing each step.

a. Draw a pentagon. Draw all possible diagonals using only one vertex of the pentagon.



b. How many triangles are formed when the diagonal(s) divide the pentagon?

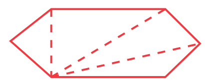
Three triangles are formed when the diagonals divide the pentagon.

c. If the sum of the interior angle measures of each triangle is 180°, what is the sum of all the interior angle measures of the triangles formed by the diagonal(s)?

The sum of all the interior angle measures of the three triangles is 540°.

3. Calculate the sum of the interior angle measures of a hexagon by completing each step.

a. Draw a hexagon. Draw all possible diagonals using one vertex of the hexagon.



b. How many triangles are formed when the diagonal(s) divide the hexagon?

Four triangles are formed when the diagonals divide the hexagon.

c. If the sum of the interior angle measures of each triangle is 180°, what is the sum of all the interior angle measures of the triangles formed by the diagonal(s)?

The sum of all the interior angle measures of the four triangles is 720°.

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4. Complete the table shown.

Number of Sides of the Polygon	3	4	5	6
Number of Diagonals Drawn	0	1	2	3
Number of Triangles Formed	1	2	3	4
Sum of the Measures of the Interior Angles	180°	360°	540°	720°

5. What pattern do you notice between the number of possible diagonals drawn from one vertex of the polygon, and the number of triangles formed by those diagonals?

One diagonal forms two triangles, two diagonals form three triangles, and so on. There is always one more triangle than the number of diagonals.

6. Compare the number of sides of the polygon to the number of possible diagonals drawn from one vertex. What do you notice?

The number of diagonals drawn is always three less than the number of sides of the polygon.

7. Compare the number of sides of the polygon to the number of triangles formed by drawing all possible diagonals from one vertex. What do you notice?

The number of triangles formed is always two less than the number of sides of the polygon.

8. What pattern do you notice about the sum of the interior angle measures of a polygon as the number of sides of each polygon increases by 1?

As the number of sides of each polygon increases by 1, the sum of the interior angle measures of the polygon increases by 180°.

9. Predict the number of possible diagonals drawn from one vertex and the number of triangles formed for a seven-sided polygon using the table you completed.

Using the table, I predict that a seven-sided polygon will have four diagonals drawn from one vertex and it will form five triangles.

10. Predict the sum of all the interior angle measures of a seven-sided polygon using the table your completed.

I predict the sum of the measures of the interior angles of a seven-sided polygon will be 900°.

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Grouping

Have students complete Questions 11 through 18 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 11 through 18

- If the polygon has 10 sides, how many diagonals can be drawn from one vertex?
- If the polygon has 10 sides, how many triangles are formed from the diagonals drawn from one vertex?
- If the polygon has 10 sides, what is the sum of the measures of the interior angles?
- If the polygon has 11 sides, how many diagonals can be drawn from one vertex?
- If the polygon has 11 sides, how many triangles are formed from the diagonals drawn from one vertex?
- If the polygon has 11 sides, what is the sum of the measures of the interior angles?
- If the polygon has 20 sides, how many diagonals can be drawn from one vertex?
- If the polygon has 20 sides, how many triangles are formed from the diagonals drawn from one vertex?
- If the polygon has 20 sides, what is the sum of the measures of the interior angles?

11. Continue the pattern to complete the table.

Number of Sides of the Polygon	7	8	9	16
Number of Diagonals Drawn	4	5	6	13
Number of Triangles Formed	5	6	7	14
Sum of the Measures of the Interior Angles	900°	1080°	1260°	2520°

12. When you calculated the number of triangles formed in the 16-sided polygon, did you need to know how many triangles were formed in a 15-sided polygon first? Explain your reasoning.

No. I knew there were two fewer triangles formed than the number of sides, so I just subtracted 2 from 16 to calculate the number of triangles formed.

13. If a polygon has 100 sides, how many triangles are formed by drawing all possible diagonals from one vertex? Explain your reasoning.

A 100-sided polygon will be divided into 98 triangles. There are always two fewer triangles than the number of sides.

14. What is the sum of all the interior angle measures of a 100-sided polygon? Explain your reasoning.

The sum of all the interior angle measures of 100-sided polygon is 17,640°. I can multiply the number of triangles by 180°.

15. If a polygon has n sides, how many triangles are formed by drawing all diagonals from one vertex? Explain your reasoning.

An n-sided polygon will be divided into $(n - 2)$ triangles. There are always two fewer triangles than the number of sides.

16. What is the sum of all the interior angle measures of an n-sided polygon? Explain your reasoning.

The sum of all the interior angle measures of an n-sided polygon is $180°(n - 2)$. I can multiply the number of triangles by 180°.

17. Use the formula to calculate the sum of all the interior angle measures of a polygon with 32 sides.

The sum of all the interior angle measures of a polygon with 32 sides is $180(30) = 5400°$.

18. If the sum of all the interior angle measures of a polygon is 9540°, how many sides does the polygon have? Explain your reasoning.

The polygon has 55 sides. I divided 9540 by 180, and then added 2 to the quotient.

- What is the sum of the measures of the interior angles of a polygon with 50 sides?
- If the sum of the interior angle measure of a polygon is 9000°, how many sides does the polygon have?

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Problem 3

Students use their new knowledge from problem 1 to derive a formula that determines the measure of each interior angle of a regular polygon: $\frac{(n - 2)180}{n}$ where n is the number of sides. A chart is used to organize their data and pattern recognition guides the students to a generalization.

Students will be given the number of sides of the regular polygon and asked to solve for the measure of each interior angle as well as solve for the number of sides in the regular polygon given the measure of each interior angle.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

- How many sides are on a decagon?
- How many angles does a decagon have?
- What operation is used to determine the measure of each interior angle of a regular decagon?
- What is another name for a regular triangle?
- What is the measure of each interior angle of an equilateral triangle?

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PROBLEM 3 Regular Stuff

1. Use the formula developed in Problem 2, Question 16 to calculate the sum of all the interior angle measures of a decagon.

   $180°(n - 2) = 180°(10 - 2) = 1440°$

   The sum of the interior angle measures of a decagon is 1440°.

2. Calculate each interior angle measure of a decagon if each interior angle is congruent. How did you calculate your answer?

   The measure of each interior angle of the decagon is equal to 144°. I divided the sum of all the interior angle measures by the number of sides, 10.

3. Complete the table.

Number of Sides of Regular Polygon	3	4	5	6	7	8
Sum of Measures of Interior Angles	180°	360°	540°	720°	900°	1080°
Measure of Each Interior Angle	60°	90°	108°	120°	128.57°	135°

4. If a regular polygon has n sides, write a formula to calculate the measure of each interior angle.

   $\frac{180(n - 2)}{n}$

5. Use the formula to calculate each interior angle measure of a regular 100-sided polygon.

   $\frac{180(n - 2)}{n} = \frac{180(100 - 2)}{100} = 176.4°$

6. If each interior angle measure of a regular polygon is equal to 150°, determine the number of sides. How did you calculate your answer?

   The regular polygon has 12 sides.

   I worked the formula backwards to calculate my answer.

   $\frac{180(n - 2)}{n} = 150$

   $180n - 360 = 150n$

   $30n = 360$

   $n = 12$

- What is another name for a regular quadrilateral?
- What is the measure of each interior angle of square?
- Will this strategy for determining the measure of each interior angle work if the polygon is not regular? Explain.
- How would you determine the measure of each interior angle of a regular polygon with 1,000 sides?
- If the measure of each interior angle of a regular polygon is 156°, how many sides are on the polygon?

Problem 4

Students use interior angle measures of a polygon to solve problems. Questions 3 and 4 ask students to apply their knowledge in other areas of geometry, including trigonometry, in a problem situation.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- How is each interior angle in the pentagon related to a base angle in the triangle?
- Is each triangle isosceles? How do you know?
- What do you know about the measures of the base angles of an isosceles triangle?
- How is the Triangle Sum Theorem used to determine the value of x?
- What's the difference between a regular polygon and a non-regular polygon?
- How can you determine the length from the center of a regular hexagon to a vertex?
- What are the side ratios for a 30°–60°–90° triangle?
- Which side lengths are used for the sine ratio?
- Which side lengths are used for the cosine ratio?

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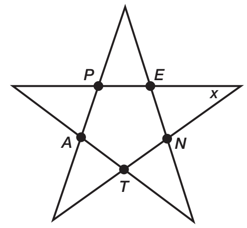
PROBLEM 4 The Angle Inside

1. *PENTA* is a regular pentagon. Solve for x.



By adding two diagonals in the pentagon, I can determine that the sum of all the interior angle measures is equal to 540°. Each interior angle must be equal because the pentagon is regular, so each interior angle is 108°. Each interior angle of the pentagon is an exterior angle of a triangle, forming a linear pair with one interior angle of a triangle. So, each interior angle in the linear pair must be 72°. If two angles in each triangle are 72°, that leaves 36° for the third angle in the triangle. Therefore, $x = 36°$.

2. The Susan B. Anthony dollar coin minted in 1999 features a regular 11-gon, or hendecagon, inside a circle on both sides of the coin.

What is the measure of each interior angle of the regular hendecagon?

$$\frac{180°(11-2)}{11} = \frac{180(9)}{11} = \frac{1620}{11} = 147.\overline{27}°$$

Each interior angle measure of a regular hendecagon is $147.\overline{27}°$.

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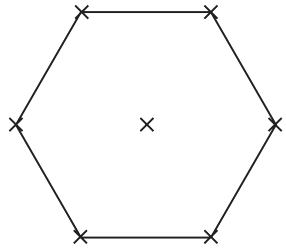
3. The high school pep squad is preparing a halftime performance for the next basketball game. Six students will hold banners to form a regular hexagon as shown with the school mascot in the very center. Each of the banners is exactly 4 feet long.



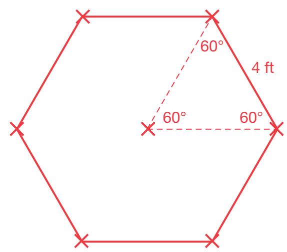
a. What angle does each student form with his or her banners? Explain your reasoning.

The banners each student holds form an interior angle of the hexagon.

$\frac{180(6-2)}{6} = 120$

Each student forms a 120° angle with his or her banners.

b. What is the distance from each student on the regular hexagon to the school mascot in the center? Show your work and explain your reasoning.



I can draw two line segments from the center of the hexagon to two adjacent vertices, bisecting two 120° angles. This creates an equilateral triangle, because the measures of all three angles are 60° (180 − 120 = 60).

So, every side of the triangle is 4 feet long. Thus, the distance from the school mascot at the center to two of the students on the hexagon is 4 feet. This will be the same for all the other vertices of the hexagon, so the distance from the school mascot to any student on the hexagon is 4 feet.

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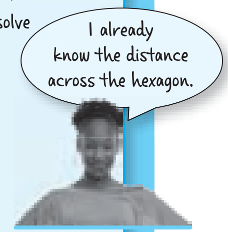
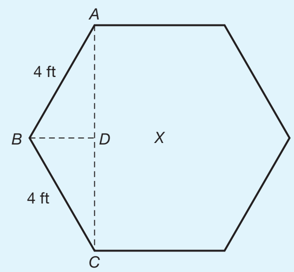
4. Yael and Brynn wanted to calculate how much space the halftime show would take up.

Yael

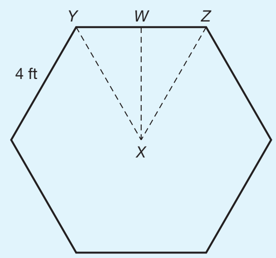
To determine the height of the hexagon, I first drew $\overline{AC}$. Then, I drew $\overline{BD}$ to create two right triangles. Finally, I used trigonometry to solve for the lengths AD and DC.





Brynn

To determine the height of the hexagon, I first drew line segments $\overline{XY}$ and $\overline{XZ}$ to create triangle XYZ. Then, I drew $\overline{XW}$ perpendicular to $\overline{YZ}$ to create two right triangles. Finally, I used trigonometry to solve for the height.



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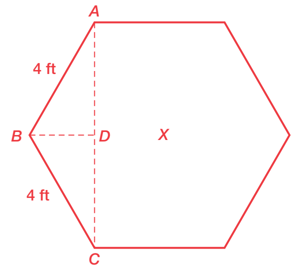
a. Verify that both Yael's method and Brynn's method result in the same height.

Yael's Method

Each of the interior angles of the regular hexagon has a measure of 120°, so $m\angle B = 120°$. This means that angles *A* and *C* must measure 30° each, because triangle *ABC* is an isosceles triangle with two sides measuring 4 feet each.

Triangle *ABD* is a 30°-60°-90° right triangle with a hypotenuse of 4 feet. I can set the cosine of 30° equal to the ratio $\frac{AD}{4}$, or I can set the sine of 60° equal to the ratio $\frac{AD}{4}$. Then, I can solve for *AD*.



$$\cos(30°) = \sin(60°) = \frac{\sqrt{3}}{2}$$

$$\frac{AD}{4} = \frac{\sqrt{3}}{2}$$

$$2(AD) = 4\sqrt{3}$$

$$AD = 2\sqrt{3}$$

The length of line segment *CD* must be equal to the length of line segment *AD*, because the ratio $\frac{CD}{4}$ is also equal to the cosine of 30°, or the sine of 60°.

So, $CD = 2\sqrt{3}$.

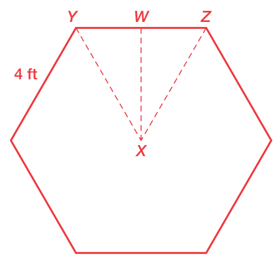
The height of the hexagon, *AC*, is $2 \times 2\sqrt{3}$, or $4\sqrt{3}$ feet.

Brynn's Method

I know from Question 3, part (b), that the distance from the center of the hexagon to each of its vertices is equal to the side length of 4 feet. So line segments *XY* and *XZ* both measure 4 feet.

Line segments *XY* and *XZ* bisect the interior angles *Y* and *Z*, so angles *Y* and *Z* each measure 60°. This means that triangles *XYW* and *XWZ* are both 30°-60°-90° right triangles, because angles *XWZ* and *XWY* are both right angles.

Triangle *XYW* has a hypotenuse of 4 feet. I can set the cosine of 30° equal to the ratio $\frac{XW}{4}$, or I can set the sine of 60° equal to the ratio $\frac{XW}{4}$. Then, I can solve for *XW*.



$$\cos(30°) = \sin(60°) = \frac{\sqrt{3}}{2}$$

$$\frac{XW}{4} = \frac{\sqrt{3}}{2}$$

$$2(XW) = 4\sqrt{3}$$

$$XW = 2\sqrt{3}$$

The length of line segment *XW* must be equal to exactly half the height of the hexagon because point *X* is at the center of the hexagon.

So, the height of the hexagon is $2 \times 2\sqrt{3}$, or $4\sqrt{3}$ feet.

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b. Calculate how much rectangular space the students need for the halftime show. Show your work and explain your reasoning.

At its widest, the width of the hexagon is 8 feet, because the length of each line segment from a vertex of the hexagon to the center is 4 feet. The height of the hexagon is $4\sqrt{3}$ feet.

So, the rectangular area needed for the halftime show is $8 \times 4\sqrt{3}$ ft². This is $32\sqrt{3}$ ft², or approximately 55.43 ft².

c. About how much space would the halftime show take up if 8 students formed a regular octagon with 4-foot banners? Show your work.

The interior angles of a regular octagon each measure $\frac{180(8-2)}{8}$, or 135°. Using Brynn's method, I can draw three line segments from the center to create two right triangles which would be 27.5°-62.5°-90° right triangles.

$\tan(27.5°) \approx 0.521$

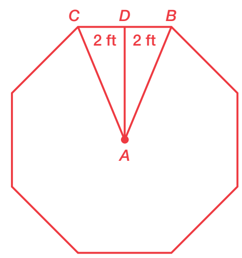
$\frac{2}{AD} \approx 0.521$

$\frac{2}{0.521} \approx AD \approx 3.84$ ft

width of octagon ≈ 3.84 × 2, or 7.68 ft

height of octagon ≈ 3.84 × 2, or 7.68 ft

The rectangular area needed would be 7.68^2, or approximately 58.9, square feet.



Be prepared to share your methods and solutions.

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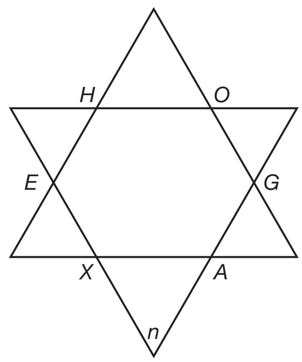
Check for Students' Understanding

Use the six pointed star and the regular hexagon HEXAGO to solve for n.



$$\frac{180(6 - 2)}{6} = x$$

$$\frac{180(4)}{6} = x$$

$$\frac{720}{6} = x$$

$$x = 120°$$

Each interior angle in the regular hexagon is 120°. Using the Linear Pair Postulate, it can be determined that two of the interior angles in the bottom triangle each have a measure 60°. Using the Triangle Sum Theorem, the third angle must be 60°, so $n = 60°$.

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Exterior and Interior Angle Measurement Interactions

7.5

Sum of the Exterior Angle Measures of a Polygon

LEARNING GOALS

In this lesson, you will:

- Write a formula for the sum of the exterior angles of any polygon.
- Calculate the sum of the exterior angles of any polygon, given the number of sides.
- Write a formula for the measure of each exterior angle of any regular polygon.
- Calculate the measure of an exterior angle of a regular polygon, given the number of sides.
- Calculate the number of sides of a regular polygon, given the measure of each exterior angle.

KEY TERM

- exterior angle of a polygon

ESSENTIAL IDEAS

- An exterior angle of a polygon is formed adjacent to each interior angle by extending one side of each vertex of the polygon.
- The sum of the measures of the exterior angles of a polygon is 360°.
- The measure of each exterior angle of a regular polygon is $\frac{360}{n}$ where n is the number of sides.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Prove geometric theorems

9. Prove theorems about lines and angles.

Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

G-SRT Similarity, Right Triangles, and Trigonometry

Define trigonometric ratios and solve problems involving right triangles

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

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G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

Students discover the sum of the measures of the exterior angles of an *n*-sided polygon is always 360° and the formula for determining the measure of each exterior angle of a regular *n*-sided polygon is $\frac{360}{n}$. The formulas are also used to solve problem situations in which they determine the number of sides of the polygon given information about the measures of the exterior angles or the measure of the interior angle sum.

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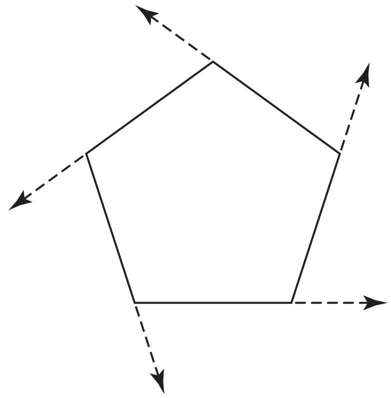
Warm Up



1. Calculate the sum of the measures of the interior angles of this pentagon.

   $180(5 - 2) = 540°$

2. How many linear pairs are in the diagram?

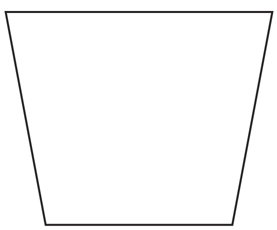
   There are 5 pairs of linear pairs in the diagram.

3. Calculate the sum of the measures of all of the linear pairs of the pentagon.

   $180(5) = 900°$

4. Calculate the sum of the measures of the exterior angles of the pentagon.

   $900 - 540 = 360°$



5. Calculate the sum of the measures of the interior angles of this trapezoid.

   $180(4 - 2) = 360°$

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6. Are of the interior angles of the trapezoid congruent? Explain.

The interior angles of the trapezoid are not all congruent because the same side interior angles are supplementary and each pair of supplementary angles is composed of one obtuse and one acute angle. However, for an isosceles trapezoid each pair of base angles is congruent.

7. Can a trapezoid contain four congruent interior angles? Explain.

No, it is impossible. The sum of the interior angles of a quadrilateral is 360° and if the four interior angles were congruent, the measure of each angle would be equal to 90°.The definition of trapezoid is a quadrilateral with exactly one pair of parallel sides. If the four interior angles were congruent, the quadrilateral would have two pair of parallel sides and it would not be a trapezoid.

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Exterior and Interior Angle Measurement Interactions

7.5

Sum of the Exterior Angle Measures of a Polygon

LEARNING GOALS

In this lesson, you will:

- Write a formula for the sum of the exterior angles of any polygon.
- Calculate the sum of the exterior angles of any polygon, given the number of sides.
- Write a formula for the measure of each exterior angle of any regular polygon.
- Calculate the measure of an exterior angle of a regular polygon, given the number of sides.
- Calculate the number of sides of a regular polygon, given the measure of each exterior angle.

KEY TERM

- exterior angle of a polygon

On April 5, 1968, Jane Elliott decided to try what is now a famous "experiment" with her third grade class in Riceville, Iowa. The purpose of her experiment was to allow her young students to feel what it was like to be in the "in group" and to be in the "out group."

She divided the class into blue-eyed students and brown-eyed students and informed the class that the blue-eyed students were the "superior group," who received special privileges at school. Elliott also encouraged students to only talk and play with other members of their in group or out group.

The results were shocking and immediate. Students in the out group began to do more poorly academically, while those in the in group did better. The in group students also began acting arrogant and bossy, while those in the out group became quiet and timid.

All of this happened, even though the students knew they were part of an experiment.

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Problem 1

This activity provides students with an opportunity to understand why the sum of the measures of the exterior angles of any polygon is always equal to 360°. Students will use prior knowledge, specifically the number of sides of the polygon, the number of linear pairs formed, and the sum of the measures of the interior angles of each polygon. Viewing this data in a chart format allows students to identify the pattern and quickly generalize.

Allowing students the time to gather data and notice the result is the same for several polygons will help them to understand why this happens, rather than memorize an obscure fact.

Grouping

Ask students to read the introduction, worked example, and Question 1. Discuss as a class.

Guiding Questions for Discuss Phase, Question 1

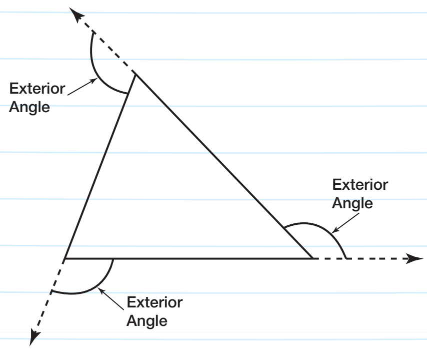
- What is the sum of the measures of the interior angles of a triangle?
- How many linear pairs are formed using the interior and exterior angles of the triangle?
- What is the sum of the measures of the three linear pairs?

PROBLEM 1 Is There a Formula?

Previously, you wrote a formula for the sum of all the interior angle measures of a polygon. In this lesson, you will write a formula for the sum of all the exterior angle measures of a polygon.

Each interior angle of a polygon can be paired with an exterior angle. An **exterior angle of a polygon** is formed adjacent to each interior angle by extending one side of each vertex of the polygon as shown in the triangle.

Each exterior angle and the adjacent interior angle form a linear pair.



1. Use the formula for the sum of interior angle measures of a polygon and the Linear Pair Postulate to calculate the sum of the exterior angle measures of a triangle.

   The sum of the exterior angle measures of a triangle is equal to 360°.

   Using the formula for the sum of the interior angle measures of a polygon, the sum of the interior angle measures of a triangle is equal to 180°. The Linear Pair Postulate helps to determine that the sum of the angle measures formed by three linear pairs is equal to 3(180°) = 540°. By subtracting the sum of the interior angle measures (180°) from the sum of the linear pair measures (540°), I calculated the sum of the exterior angle measures (360°).

- What is the difference between the sum of the measures of the linear pairs of angles and the sum of the measures of the interior angles?

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Grouping

Have students complete Questions 2 through 10 with a partner. Then have students share their responses as a class.

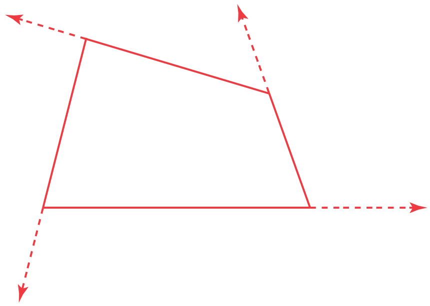
Guiding Questions for Share Phase, Questions 2 through 10

- What is the sum of the measures of the interior angles of a quadrilateral?
- How many linear pairs are formed using the interior and exterior angles of the quadrilateral?
- What is the sum of the measures of the four linear pairs?
- What is the difference between the sum of the measures of the linear pairs of angles and the sum of the measures of the interior angles?
- What is the sum of the measures of the interior angles of a pentagon?
- How many linear pairs are formed using the interior and exterior angles of the pentagon?
- What is the sum of the measures of the five linear pairs?
- What is the difference between the sum of the measures of the linear pairs of angles and the sum of the measures of the interior angles?

Let's explore the sum of the exterior angle measures of other polygons.

2. Calculate the sum of the exterior angle measures of a quadrilateral by completing each step.

   a. Draw a quadrilateral and extend each side to locate an exterior angle at each vertex.



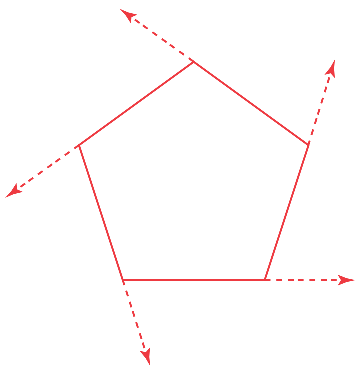
   b. Use the formula for the sum of interior angle measures of a polygon and the Linear Pair Postulate to calculate the sum of the exterior angle measures of a quadrilateral.

   The sum of the exterior angle measures of a quadrilateral is equal to 360°.

   Using the formula for the sum of the interior angle measures of a polygon, the sum of the interior angle measures of a quadrilateral is equal to 360°. The Linear Pair Postulate helps to determine that the sum of the angle measures formed by four linear pairs is equal to 4(180°) = 720°. By subtracting the sum of the interior angle measures (360°) from the sum of the linear pair measures (720°), I calculated the sum of the exterior angle measures (360°).

3. Calculate the sum of the exterior angle measures of a pentagon by completing each step.

   a. Draw a pentagon and extend each side to locate an exterior angle at each vertex.



- What determines the number of linear pairs formed by each interior and adjacent exterior angle of the polygon?
- Why will the number of linear pairs always be two more than the number of triangles formed by the diagonals drawn from one vertex of the polygon?

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b. Use the formula for the sum of the interior angle measures of a polygon and the Linear Pair Postulate to calculate the sum of the exterior angle measures of a pentagon.

The sum of the exterior angle measures of a pentagon is equal to 360°.

Using the formula for the sum of interior angle measures of a polygon, the sum of the interior angle measures of a pentagon is equal to 540°. The Linear Pair Postulate helps to determine that the sum of the angle measures formed by five linear pairs is equal to 5(180°) = 900°. By subtracting the sum of the interior angle measures (540°) from the sum of the linear pair measures (900°), I calculated the sum of the exterior angle measures (360°).

4. Calculate the sum of the exterior angle measures of a hexagon by completing each step.

a. Without drawing a hexagon, how many linear pairs are formed by each interior and adjacent exterior angle? How do you know?

There are six linear pairs formed in a hexagon. A hexagon has six sides and six vertices, so if each side is extended to form an exterior angle, there will be six linear pairs formed.

b. What is the relationship between the number of sides of a polygon and the number of linear pairs formed by each interior angle and its adjacent exterior angle?

The number of linear pairs formed by each interior angle and its adjacent exterior angle is equal to the number of sides of the polygon.

c. Use the formula for the sum of the interior angle measures of a polygon and the Linear Pair Postulate to calculate the sum of the measures of the exterior angles of a hexagon.

The sum of the exterior angle measures of a hexagon is equal to 360°.

Using the formula for the sum of the interior angle measures of a polygon, the sum of the interior angle measures of a hexagon is equal to 720°. The Linear Pair Postulate helps to determine that the sum of the angle measures formed by six linear pairs is equal to 6(180°) = 1080°. By subtracting the sum of the interior angle measures (720°) from the sum of the linear pair measures (1080°), I calculated the sum of the exterior angle measures (360°).

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5. Complete the table.

Number of Sides of the Polygon	3	4	5	6	7	15
Number of Linear Pairs Formed	3	4	5	6	7	15
Sum of Measures of Linear Pairs	540°	720°	900°	1080°	1260°	2700°
Sum of Measures of Interior Angles	180°	360°	540°	720°	900°	2340°
Sum of Measures of Exterior Angles	360°	360°	360°	360°	360°	360°

6. When you calculated the sum of the exterior angle measures in the 15-sided polygon, did you need to know anything about the number of linear pairs, the sum of the linear pair measures, or the sum of the interior angle measures of the 15-sided polygon? Explain your reasoning.

No. I knew the sum of the exterior angle measures was equal to 360° because the difference between the sum of the angle measures in the linear pairs of any polygon and the sum of the interior angle measures of the polygon will always be 360°.

7. If a polygon has 100 sides, calculate the sum of the exterior angle measures. Explain how you calculated your answer.

The sum of the exterior angle measures of a 100-sided polygon is equal to 360°. The sum of the exterior angle measures of all polygons is always equal to 360°.

8. What is the sum of the exterior angle measures of an *n*-sided polygon?

The sum of the exterior angle measures of an *n*-sided polygon is 360°.

9. If the sum of the exterior angle measures of a polygon is 360°, how many sides does the polygon have? Explain your reasoning.

I have no way of knowing how many sides the polygon has because 360° describes the sum of the exterior angle measures of all polygons.

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Problem 2

Students use their new knowledge from Problem 1 to derive a formula that determines the measure of each exterior angle of a regular polygon: $\frac{360}{n}$ where n is the number of sides. A chart is used to organize their data and pattern recognition guides the students to a generalization.

Students will be given the number of sides of the regular polygon and asked to solve for the measure of each exterior angle as well as solve for the number of sides in the regular polygon given the measure of each exterior angle.

Grouping

Have students complete Questions 1 through 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 9

- What is the measure of each interior angle of an equilateral triangle?
- What is the measure of each interior angle of a square?
- What is the measure of each interior angle of a regular pentagon?
- What is the measure of each interior angle of a regular hexagon?
- If the sum of the measures of the exterior angles (360°) is divided by the number of sides of the polygon, what information does the result provide?
- If the measure of each exterior angle of a regular polygon is 14.4°, how many sides does the polygon have?

10. Explain why the sum of the exterior angle measures of any polygon is always equal to 360°.

To calculate the sum of the interior angle measures, I subtract 2 from the number of sides and multiply by 180. The diagonals drawn from one vertex always form two less triangles than the number of sides. The number of sides determines the number of linear pairs formed by each interior and adjacent exterior angle of the polygon. The number of linear pairs (interior plus exterior angle measures) will always be 2 more than the number of triangles formed (interior angles), which is a difference of 360°.

PROBLEM 2 Regular Polygons

1. Calculate the measure of each exterior angle of an equilateral triangle. Explain your reasoning.

The measure of each exterior angle of an equilateral triangle is equal to 120° because the measure of each interior angle is equal to 60° and the interior and exterior angles form a linear pair.

2. Calculate the measure of each exterior angle of a square. Explain your reasoning.

The measure of each exterior angle of a square is equal to 90° because the measure of each interior angle is equal to 90° and the interior and exterior angles form a linear pair.

3. Calculate the measure of each exterior angle of a regular pentagon. Explain your reasoning.

The measure of each exterior angle of a regular pentagon is equal to 72° because each interior angle is equal to 108° and the interior and exterior angles form a linear pair.

4. Calculate the measure of each exterior angle of a regular hexagon. Explain your reasoning.

The measure of each exterior angle of a regular hexagon is equal to 60° because each interior angle is equal to 120° and the interior and exterior angles form a linear pair.

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5. Complete the table shown to look for a pattern.

Number of Sides of a Regular Polygon	3	4	5	6	7	15
Sum of Measures of Exterior Angles	360°	360°	360°	360°	360°	360°
Measure of Each Interior Angle	60°	90°	108°	120°	128.57°	156°
Measure of Each Exterior Angle	120°	90°	72°	60°	51.43°	24°

6. When you calculated the measure of each exterior angle in the 15-sided regular polygon, did you need to know anything about the measure of each interior angle? Explain your reasoning.

No. I knew the sum of the exterior angle measures was equal to 360° and just divided it by the number of sides of the polygon.

7. If a regular polygon has 100 sides, calculate the measure of each exterior angle. Explain how you calculated your answer.

The measure of each exterior angle of a 100-sided regular polygon is equal to 3.6°. I divided 360° by 100.

8. What is the measure of each exterior angle of an n-sided regular polygon?

The measure of each exterior angle of an n-sided regular polygon is $\frac{360}{n}$, where n is the number of sides.

9. If the measure of each exterior angle of a regular polygon is 18°, how many sides does the polygon have? Explain how you calculated your answer.

The regular polygon has 20 sides.

$\frac{360}{n} = 18$

$360 = 18n$

$20 = n$

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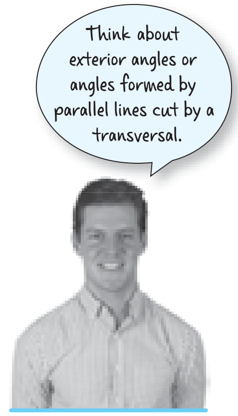
Grouping

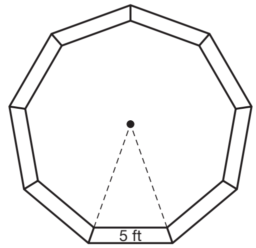
Have students complete Question 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 10

- What angles are formed by the lines shown in the diagram?
- What angle measures of the triangle do you know?
- How can you use exterior angle measures to help you get more information?
- Which line segments in the diagram are parallel?
- Can you use the Pythagorean Theorem to help you get more information?
- What equation describes the area of a trapezoid?

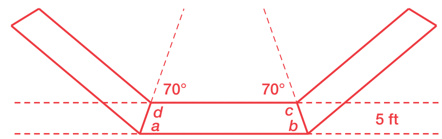
10. Simon installed a custom-built pool in his backyard in the shape of a regular nonagon. The pool has a perimeter of 180 feet. Simon wants to install decking around the pool as shown. Each trapezoid section of deck will be congruent and have a width of 5 feet.





a. What are the interior angle measures of each section of deck? Show your work and explain your reasoning.

I know that angles *c* and *d* measure 180° − 70°, or 110° each because they each form a linear pair with a 70° angle. Angles *a* and *b* are each corresponding angles with a 70° angle so angles *a* and *b* each measure 70°.



I can extend the bases of each trapezoidal bench and the line segments from the center to create parallel lines cut by transversals.

The line segments drawn from the center bisect the interior angles of the regular nonagon. These angles each have a measure of $\frac{180(9-2)}{9 \times 2}$, or 70°. Angles *a* and *b* are corresponding angles to these interior angles of the nonagon, so they each measure 70° also.

Angles *a* and *d* are same-side interior angles, so they are supplementary. So, angle *d* measures 110°. Angles *b* and *c* are also same-side interior angles, so angle *c* also measures 110°.

These interior angle measures are the same for each section of deck.

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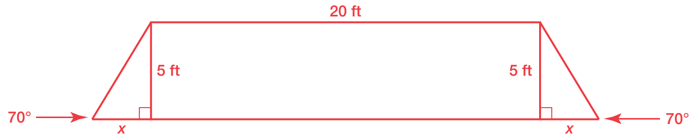
b. What are the dimensions of each section of deck? Show your work and explain your reasoning.

Because the pool has a perimeter of 180 feet, the side of each deck section that is against the pool is 180 ÷ 9, or 20, feet long. Each trapezoidal section of deck has a height of 5 feet. I can draw line segments to divide one of the benches into two right triangles and a rectangle.



I know that each of the right triangles has a 70° angle. I can set the tangent of 70° equal to the ratio $\frac{5}{x}$ to determine the length of each side of the deck farthest from the pool.

$\tan(70°) \approx \frac{5}{x}$

$2.747 \approx \frac{5}{x}$

$2.747x \approx 5$

$x \approx 1.82$ ft

$1.82(2) + 20 = 23.64$ ft

Each deck section is 20 feet long on the side closest to the pool, approximately 23.64 feet long on the side farthest from the pool, and each section has a height of 5 feet.

c. What is the total sitting area of the entire deck?

The area of each trapezoid deck section is approximately $\frac{1}{2}(20 + 23.64)(5)$, or 109.1 ft^2. So, the total sitting area of the entire deck is approximately 109.1×9, or 981.9, square feet.

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Problem 3

A scenario requires students to construct a regular hexagon, a square and a regular pentagon by cutting arcs around a circle. The pentagon is most challenging and a few hints are provided to begin this construction. Give the students about five minutes to try this construction. If no one has completed it, ask for a student volunteer to do the construction on the board and guide the class through the steps.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

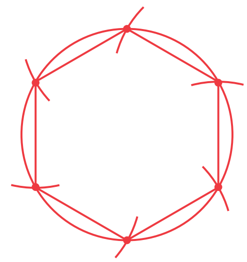
- If Lily used her compass to draw a circle and didn't change the radius of the compass, how many times can she mark off the radius distance around the circumference of the circle?
- After drawing a circle, how did Molly use a diameter and a perpendicular bisector construction to construct a square?
- Four points are needed for the construction of the regular pentagon, three of which are located on the starter line, where is the location of the other point?
- After constructing a line perpendicular to the starter line, where can a circle be drawn and which point is used as the center of the circle?
- How was the radius on the compass determined to divide the circumference of the circle into exactly five equal parts?

PROBLEM 3 Old Sibling Rivalry

Two sisters, Molly and Lily, were arguing about who was better at using a compass and a straightedge.

1. Molly challenged Lily to construct a regular hexagon. Undaunted by the challenge, Lily took the compass and went to work. What did Lily do?

   Lily drew a circle. Then, using the same radius, she went around the entire circle marking exactly six arcs, placing her compass on each arc marker to create the next arc. Finally, she connected the consecutive arcs to form the regular hexagon.



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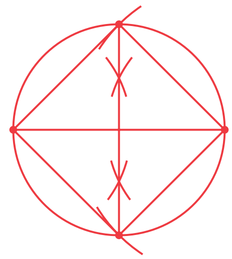
2. Lily then challenged Molly to construct a square. Molly grabbed her compass with gusto and began the construction. What did Molly do?

Molly drew a circle. Then, she drew a diameter. Next, she constructed the diameter's perpendicular bisector and marked arcs where the bisector intersected the circle. Finally, she connected the arc intersections with the ends of the diameter to form the square.



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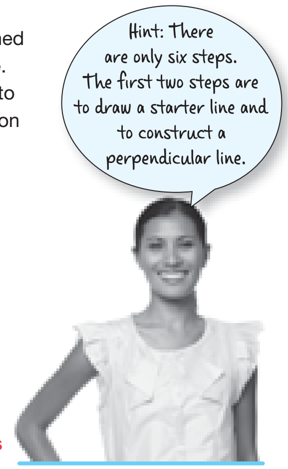
3. Both sisters were now glaring at each other and their mother, a math teacher, walked into the room. Determined to end this dispute, she gave her daughters a challenge. She told them the only way to settle the argument was to see who could be the first to come up with a construction for a regular pentagon. Give it a try!



a. Draw a starter line.

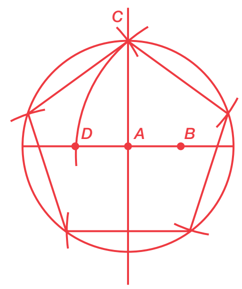
b. Construct a line perpendicular to the starter line. Label the point of intersection point *A*.

c. Locate point *B* on the starter line.

d. Use point *A* as the center and draw a circle with a radius of twice *AB*. Label the point at which the perpendicular line intersects the circle point *C*.

e. Use point *B* as the center and draw an arc through point *C*. Label the point at which this arc intersects the starter line point *D*.

f. *CD* is the length needed to strike arcs around the circle to create a regular pentagon. Go around the circle and mark off arcs using *CD* as a radius.



Be prepared to share your solutions and methods.

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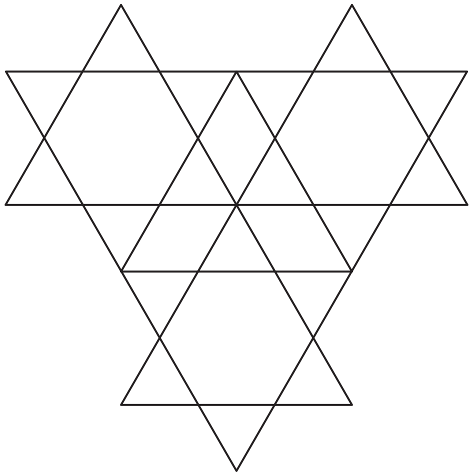
Check for Students' Understanding

Each triangle in the design is equilateral. Are the hexagons formed regular hexagons? Use exterior angles to justify your conclusion.



$$\frac{360}{6} = x$$

$$60° = x$$

No. Additional information is needed to determine that the hexagons are regular. I determined that the measure of each exterior angle is 60 degrees. Every exterior angle of the hexagon is also an interior angle of an equilateral triangle. All equilateral triangles are similar, but additional information is needed to show that the equilateral triangles are congruent. As a result, I cannot show that the hexagons are regular.

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7.6

Quadrilateral Family

Categorizing Quadrilaterals Based on Their Properties

LEARNING GOALS

In this lesson, you will:

- List the properties of quadrilaterals.
- Categorize quadrilaterals based upon their properties.
- Construct quadrilaterals given a diagonal.

ESSENTIAL IDEA

- Quadrilaterals, trapezoids, parallelograms, kites, rhombi, rectangles, and squares can be organized based on properties of their sides, angles, and diagonals.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

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Overview

Students review the properties associated with quadrilaterals, parallelograms, trapezoids, kites, rhombi, rectangles, and squares. Charts, Venn diagrams, and flow charts are used to visualize the relationships that exist among these figures.

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Warm Up

List two properties of each quadrilateral.

1. A kite
   A kite has two pair of consecutive sides congruent and the diagonals are perpendicular to each other.

2. A trapezoid
   A trapezoid has one pair of parallel sides and two pair of consecutive angles are supplementary.

3. A rhombus.
   A rhombus has four sides congruent and the diagonals are perpendicular to each other.

4. A parallelogram.
   A parallelogram has opposite sides congruent and opposite angles congruent.

5. Which quadrilaterals have two pair of opposite angles congruent?
   A square, rectangle, rhombus, parallelogram and isosceles trapezoid have two pair of opposite angles congruent.

6. Which quadrilaterals have two pair of consecutive angles congruent?
   A square and a rectangle have two pair of consecutive angles congruent.

7. Which quadrilaterals have two pair of opposite sides congruent?
   A square, rectangle, rhombus, and a parallelogram have two pair of opposite sides congruent.

8. Which quadrilaterals have two pair of consecutive sides congruent?
   A square, rhombus, and kite have two pair of consecutive sides congruent.

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7.6

Quadrilateral Family

Categorizing Quadrilaterals Based on Their Properties

LEARNING GOALS

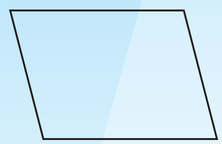
In this lesson, you will:

- List the properties of quadrilaterals.
- Categorize quadrilaterals based upon their properties.
- Construct quadrilaterals given a diagonal.

Okay, maybe the trapezoid is a kind of oddball of quadrilaterals, but did you know that its area formula, $\frac{1}{2}(b_1 + b_2)h$, can be used to determine the area of other polygons?

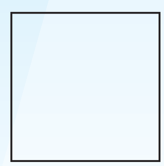
Take a parallelogram, for example. Since its bases are the same length, you can just use b to describe the length of each base.

$$\begin{aligned} A &= \frac{1}{2}(b + b)h \\ &= \frac{1}{2}(2b)h \\ &= bh \end{aligned}$$



The square, too. All three measurements (b_1, b_2, and h) are the same. So, change all the variables to, say, s.

$$\begin{aligned} A &= \frac{1}{2}(s + s)s \\ &= \frac{1}{2}(2s)s \\ &= \frac{1}{2} \cdot 2s^2 \\ &= s^2 \end{aligned}$$



Even the triangle's area can be represented using the trapezoid area formula. Can you figure out how?

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Problem 1

Students complete a chart by identifying the properties of quadrilaterals, trapezoids, parallelograms, kites, rhombi, rectangles, and squares. The vertical column of a chart lists properties of quadrilaterals and the top horizontal row contains the names of quadrilaterals. Students complete the chart by placing checkmarks indicating the appropriate properties for each quadrilateral.

Grouping

Have students complete the chart with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase

- Does a kite have any pairs of parallel sides?
- Do all quadrilaterals have at least one pair of parallel sides?
- If a quadrilateral has two pairs of parallel sides, are the opposite sides congruent?
- If a quadrilateral has two pairs of parallel sides, are the opposite angles congruent?
- If the consecutive angles of a quadrilateral are supplementary, are the opposite angles always congruent?

PROBLEM 1 **Characteristics of Quadrilaterals**

Complete the table by placing a checkmark in the appropriate row and column to associate each figure with its properties.

Characteristic	Quadrilateral	Trapezoid	Parallelogram	Kite	Rhombus	Rectangle	Square
No parallel sides	✓			✓			
Exactly one pair of parallel sides		✓					
Two pairs of parallel sides			✓		✓	✓	✓
One pair of sides are both congruent and parallel			✓		✓	✓	✓
Two pairs of opposite sides are congruent			✓		✓	✓	✓
Exactly one pair of opposite angles are congruent				✓			
Two pairs of opposite angles are congruent			✓		✓	✓	✓
Consecutive angles are supplementary			✓		✓	✓	✓
Diagonals bisect each other			✓		✓	✓	✓
All sides are congruent					✓		✓
Diagonals are perpendicular to each other				✓	✓		✓
Diagonals bisect the vertex angles					✓		✓
All angles are congruent						✓	✓
Diagonals are congruent						✓	✓

- Which quadrilateral satisfies the most characteristics of the table?
- Which quadrilateral satisfies the fewest characteristics of the table?

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Problem 2

Students create a Venn diagram describing the relationships between quadrilaterals, trapezoids, parallelograms, kites, rhombi, rectangles, and squares.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Which word(s) describe(s) the region that contains the highest number of different figures?
- Which word(s) describe(s) the region that contains the fewest number of different figures?
- How many separate regions does the Venn diagram contain?
- Which region is associated with the greatest number of properties?
- Which region is associated with the fewest number of properties?
- Which regions do not intersect each other?
- Which regions intersect each other?
- Is there more than one way to draw the Venn diagram?

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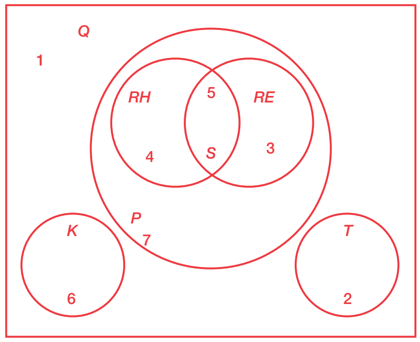
PROBLEM 2 Now I Can See the Relationships!

1. Create a Venn diagram that describes the relationships between all of the quadrilaterals listed. Number each region and name the figure located in each region.



2. Write a description for each region.

Trapezoids	Kites
Rhombi	Rectangles
Parallelograms	Squares
Quadrilaterals	

Region 1: All four-sided polygons that are not parallelograms, trapezoids, or kites.

Region 2: All quadrilaterals that are trapezoids.

Region 3: All rectangles that are not squares.

Region 4: All rhombi that are not squares.

Region 5: All rhombi and rectangles that are squares.

Region 6: All quadrilaterals that are kites.

Region 7: All parallelograms that are not rectangles or rhombi.

Problem 3

Statements describing quadrilaterals and their properties are given. Students determine which statements are true and which statements are false.

Grouping

Have students complete Questions 1 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 8

- Which properties are shared by both a square and a rectangle?
- Which properties are shared by both a parallelogram and a trapezoid?
- Which properties are shared by both a parallelogram and a kite?
- Which properties are shared by both a rhombus and a rectangle?
- Which properties are shared by both a square and a rhombus?

PROBLEM 3 True or False

Determine whether each statement is true or false. If it is false, explain why.

1. A square is also a rectangle.

   True.

   If a quadrilateral is a square, it has all of the properties of a rectangle.

2. A rectangle is also a square.

   False.

   If a quadrilateral is a rectangle, that does not imply it has four congruent sides.

3. The base angles of a trapezoid are congruent.

   False.

   The base angles of a trapezoid are congruent only if the trapezoid is isosceles.

4. A parallelogram is also a trapezoid.

   False.

   A parallelogram has two pairs of opposite sides parallel and a trapezoid has exactly one pair of opposite sides parallel.

5. A square is a rectangle with all sides congruent.

   True.

   If a rectangle has four congruent sides, then it has all of the properties of a square.

6. The diagonals of a trapezoid are congruent.

   False.

   The diagonals of a trapezoid are congruent only if the trapezoid is isosceles.

7. A kite is also a parallelogram.

   False.

   Opposite sides of a kite are not parallel.

8. The diagonals of a rhombus bisect each other.

   True.

   Rhombi are parallelograms and their diagonals bisect each other.

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Problem 4

Students use deductive reasoning skills to determine the identity of a quadrilateral, given six hints. Progressively, each hint narrows the possibilities.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

- Does a pentagon have four sides?
- What is the meaning of "at least one pair of parallel sides?"
- Which quadrilaterals have diagonals that bisect each other?
- Which quadrilaterals have congruent opposite sides?
- Which quadrilaterals have diagonals that are perpendicular to each other?
- Which quadrilaterals do not have four congruent angles?
- If a quadrilateral has four congruent angles, must it have four congruent sides?
- If a quadrilateral has four congruent sides, must it have four congruent angles?

PROBLEM 4 Can You Read Joe's Mind?

Joe is thinking of a specific polygon. He has listed six hints. As you read each hint, use deductive reasoning to try and guess Joe's polygon. By the last hint you should be able to read Joe's mind.

1. The polygon has four sides.

   The polygon is a quadrilateral.

2. The polygon has at least one pair of parallel sides.

   The polygon could be a trapezoid, a parallelogram, a rhombus, a rectangle, or a square.

3. The diagonals of the polygon bisect each other.

   The polygon is a parallelogram, so it could also be a rhombus, a rectangle, or a square.

4. The polygon has opposite sides congruent.

   The polygon is a parallelogram, so it could also be a rhombus, a rectangle, or a square.

5. The diagonals of the polygon are perpendicular to each other.

   The polygon could be a rhombus or a square.

6. The polygon does not have four congruent angles.

   The polygon is a rhombus.

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Problem 5

Students are asked to describe how a parallelogram, a kite and a rhombus can be constructed, given the length of one diagonal. It is not important that everyone constructs the same quadrilateral, it is important that students apply the appropriate properties of each quadrilateral to perform the construction. A compass and straightedge are needed.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What is true about the diagonals of a parallelogram?
- Is locating the midpoint of each of the diagonals enough to form a parallelogram? Explain.
- What is true about the diagonals of a rhombus?
- Is locating the midpoint of each of the diagonals enough to form a rhombus? Explain.
- What is true about the diagonals of a kite?
- Is locating the midpoint of each of the diagonals enough to form a kite? Explain.

PROBLEM 5 Using Diagonals

Knowing certain properties of each quadrilateral makes it possible to construct the quadrilateral given only a single diagonal.

1. Describe how you could construct parallelogram *WXYZ* given only diagonal $\overline{WY}$.

   I would begin by duplicating line segment *WY*. Then, I would bisect line segment *WY* to locate the midpoint of the diagonal. Next, I would duplicate line segment *WY* a second time, labeling it $\overline{YZ}$, and locate the midpoint. I would then draw line segment *YZ* intersecting line segment *WY* at both their midpoints. Finally, I would connect the endpoints of line segment *WY* and line segment *YZ* to form parallelogram *WXYZ*. This will create a rectangle, which is also a parallelogram.

2. Describe how you could construct rhombus *RHOM* given only diagonal $\overline{RO}$.

   First, I would begin by duplicating line segment *RO*. Then, I would construct a new line segment on the perpendicular bisector of line segment *RO* such that line segment *RO* also bisects this line segment. Next, I would label the new line segment *HM*. Finally, I would connect the endpoints of line segment *RO* and line segment *HM* to form rhombus *RHOM*.

3. Describe how you could construct kite *KITE* given only diagonal $\overline{KT}$.

   First, I would begin by duplicating line segment *KT*. Then, I would construct a perpendicular line segment to *KT* such that *KT* bisects the perpendicular line segment. Next, I would label the perpendicular line segment *IE*. Finally, I would connect the endpoints of line segment *KT* and line segment *IE* to form kite *KITE*.

Be prepared to share your solutions and methods.

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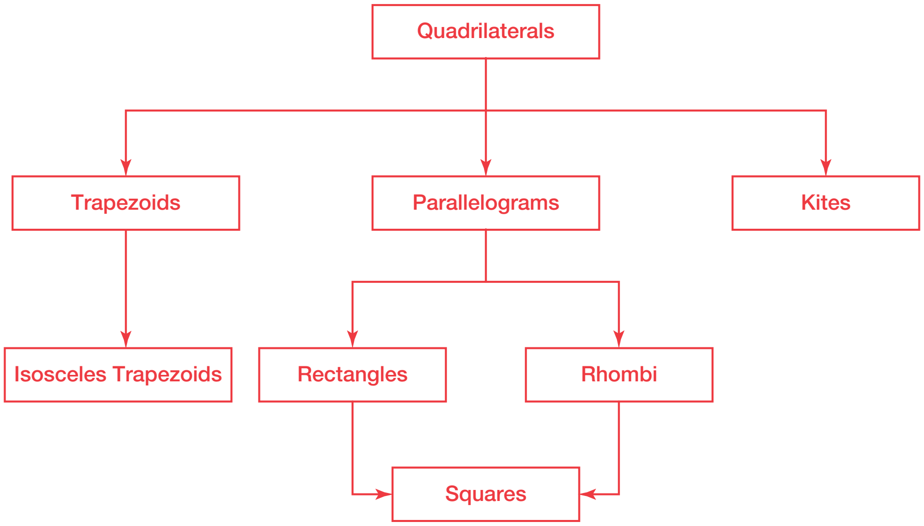
Check for Students' Understanding

Design a flow chart that relates the family of quadrilaterals.



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Chapter 7 Summary

KEY TERMS

- base angles of a trapezoid (7.3)
- isosceles trapezoid (7.3)
- biconditional statement (7.3)
- midsegment (7.3)
- interior angle of a polygon (7.4)
- exterior angle of a polygon (7.5)

THEOREMS

- Perpendicular/Parallel Line Theorem (7.1)
- Parallelogram/Congruent-Parallel Side Theorem (7.2)
- Trapezoid Midsegment Theorem (7.3)

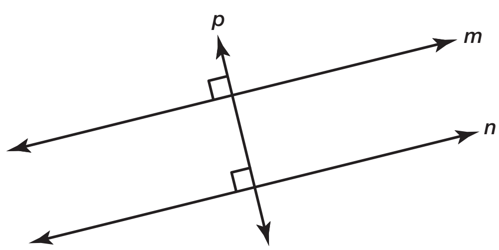
CONSTRUCTIONS

- square (7.1)
- rectangle (7.1)
- parallelogram (7.2)
- rhombus (7.2)
- kite (7.3)
- trapezoid (7.3)
- isosceles trapezoid (7.3)

7.1 Using the Perpendicular/Parallel Line Theorem

The Perpendicular/Parallel Line Theorem states: "If two lines are perpendicular to the same line, then the two lines are parallel to each other."

Example



Because line p is perpendicular to line m and line p is perpendicular to line n, lines m and n are parallel.

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7

7.1 Determining Properties of Squares

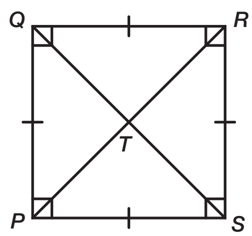
A square is a quadrilateral with four right angles and all sides congruent. You can use the Perpendicular/Parallel Line Theorem and congruent triangles to determine the following properties of squares.

- The diagonals of a square are congruent.
- Opposite sides of a square are parallel.
- The diagonals of a square bisect each other.
- The diagonals of a square bisect the vertex angles.
- The diagonals of a square are perpendicular to each other.

Example

For square *PQRS*, the following statements are true:

- $\overline{PR} \cong \overline{QS}$
- $\overline{PQ} \parallel \overline{RS}$ and $\overline{PS} \parallel \overline{QR}$
- $\overline{PT} \cong \overline{RT}$ and $\overline{QT} \cong \overline{ST}$
- $\angle PQS \cong \angle RQS$, $\angle QRP \cong \angle SRP$, $\angle RSQ \cong \angle PSQ$, and $\angle SPR \cong \angle QPR$
- $\overline{PR} \perp \overline{QS}$



7.1 Determining Properties of Rectangles

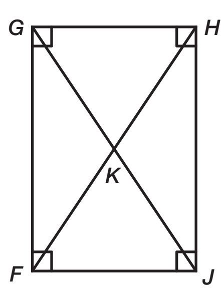
A rectangle is a quadrilateral with opposite sides congruent and with four right angles. You can use the Perpendicular/Parallel Line Theorem and congruent triangles to determine the following properties of rectangles.

- Opposite sides of a rectangle are congruent.
- Opposite sides of a rectangle are parallel.
- The diagonals of a rectangle are congruent.
- The diagonals of a rectangle bisect each other.

Example

For rectangle *FGHJ*, the following statements are true.

- $\overline{FG} \cong \overline{HJ}$ and $\overline{FJ} \cong \overline{GH}$
- $\overline{FG} \parallel \overline{HJ}$ and $\overline{FJ} \parallel \overline{GH}$
- $\overline{FH} \cong \overline{GJ}$
- $\overline{GK} \cong \overline{JK}$ and $\overline{FK} \cong \overline{HK}$



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7.2 Determining Properties of Parallelograms

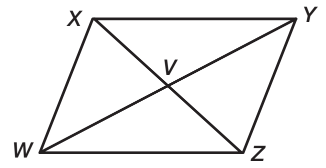
A parallelogram is a quadrilateral with both pairs of opposite sides parallel. You can use congruent triangles to determine the following properties of parallelograms.

- Opposite sides of a parallelogram are congruent.
- Opposite angles of a parallelogram are congruent.
- The diagonals of a parallelogram bisect each other.

Example

For parallelogram *WXYZ*, the following statements are true.

- $\overline{WX} \cong \overline{YZ}$ and $\overline{WZ} \cong \overline{XY}$
- $\angle WXY \cong \angle WZY$ and $\angle XYZ \cong \angle XWZ$
- $\overline{WV} \cong \overline{YV}$ and $\overline{XV} \cong \overline{ZV}$

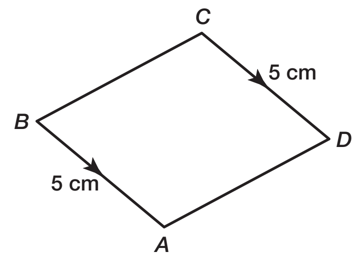


7.2 Using the Parallelogram/Congruent-Parallel Side Theorem

The Parallelogram/Congruent-Parallel Side Theorem states: "If one pair of opposite sides of a quadrilateral is both congruent and parallel, then the quadrilateral is a parallelogram."

Example

In quadrilateral *ABCD*, $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$. So, quadrilateral *ABCD* is a parallelogram, and thus has all of the properties of a parallelogram.



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7.2 Determining Properties of Rhombi

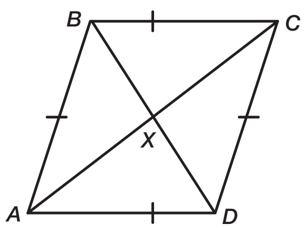
A rhombus is a quadrilateral with all sides congruent. You can use congruent triangles to determine the following properties of rhombi.

- Opposite angles of a rhombus are congruent.
- Opposite sides of a rhombus are parallel.
- The diagonals of a rhombus are perpendicular to each other.
- The diagonals of a rhombus bisect each other.
- The diagonals of a rhombus bisect the vertex angles.

Example

For rhombus *ABCD*, the following statements are true:

- $\angle ABC \cong \angle CDA$ and $\angle BCD \cong \angle DAB$
- $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{DA}$
- $\overline{AC} \perp \overline{BD}$
- $\overline{AX} \cong \overline{CX}$ and $\overline{BX} \cong \overline{DX}$
- $\angle BAC \cong \angle DAC$, $\angle ABD \cong \angle CBD$, $\angle BCA \cong \angle DCA$, and $\angle CDB \cong \angle ADB$



7.3 Determining Properties of Kites

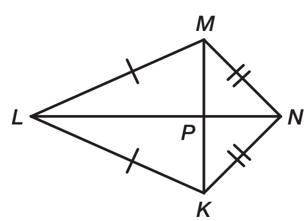
A kite is a quadrilateral with two pairs of consecutive congruent sides with opposite sides that are not congruent. You can use congruent triangles to determine the following properties of kites.

- One pair of opposite angles of a kite is congruent.
- The diagonals of a kite are perpendicular to each other.
- The diagonal that connects the opposite vertex angles that are not congruent bisects the diagonal that connects the opposite vertex angles that are congruent.
- The diagonal that connects the opposite vertex angles that are not congruent bisects the vertex angles.

Example

For kite *KLMN*, the following statements are true:

- $\angle LMN \cong \angle LKN$
- $\overline{KM} \perp \overline{LN}$
- $\overline{KP} \cong \overline{MP}$
- $\angle KLN \cong \angle MLN$ and $\angle KNL \cong \angle MNL$



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7.3 Determining Properties of Trapezoids

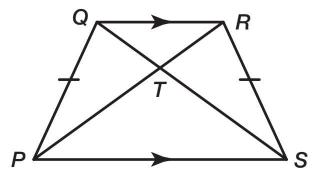
A trapezoid is a quadrilateral with exactly one pair of parallel sides. An isosceles trapezoid is a trapezoid with congruent non-parallel sides. You can use congruent triangles to determine the following properties of isosceles trapezoids:

- The base angles of a trapezoid are congruent.
- The diagonals of a trapezoid are congruent.

Example

For isosceles trapezoid *PQRS*, the following statements are true:

- $\angle QPS \cong \angle RSP$
- $\overline{PR} \cong \overline{QS}$



7.4 Determining the Sum of the Interior Angle Measures of Polygons

You can calculate the sum of the interior angle measures of a polygon by using the formula $180°(n - 2)$, where n is the number of sides of the polygon. You can calculate the measure of each interior angle of a regular polygon by dividing the formula by n, the number of sides of the regular polygon.

Examples

The sum of the interior angle measures of a pentagon is $180°(5 - 2) = 540°$.

Each interior angle of a regular pentagon measures $\frac{540°}{5} = 108°$.

The sum of the interior angle measures of a hexagon is $180°(6 - 2) = 720°$.

Each interior angle of a regular hexagon measures $\frac{720°}{6} = 120°$.

The sum of the interior angle measures of a decagon is $180°(10 - 2) = 1440°$.

Each interior angle of a regular decagon measures $\frac{1440°}{10} = 144°$.

The sum of the interior angle measures of an 18-gon is $180°(18 - 2) = 2880°$.

Each interior angle of a regular 18-gon measures $\frac{2880°}{18} = 160°$.

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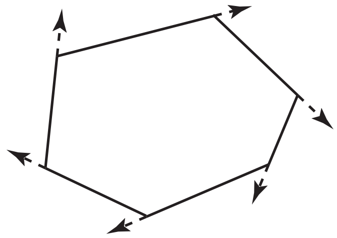
7.5 Determining the Sum of the Exterior Angle Measures of Polygons

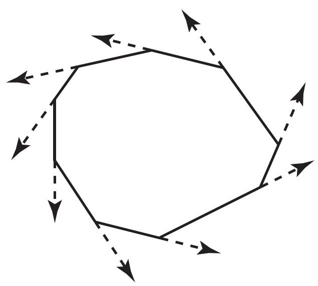
You can use the formula for the sum of the interior angle measures of a polygon and the Linear Pair Postulate to determine that the sum of the exterior angle measures of any polygon is 360°.

Examples

You can use the formula for the sum of the interior angle measures of a polygon to determine that the interior angle measures of the hexagon is 720°. Then, you can use the Linear Pair Postulate to determine that the sum of the angle measures formed by six linear pairs is $6(180°) = 1080°$. Next, subtract the sum of the interior angle measures from the sum of the linear pair measures to get the sum of the exterior angle measures: $1080° - 720° = 360°$.



You can use the formula for the sum of the interior angle measures of a polygon to determine that the interior angle measures of the nonagon is 1260°. Then, you can use the Linear Pair Postulate to determine that the sum of the angle measures formed by nine linear pairs is $9(180°) = 1620°$. Next, subtract the sum of the interior angle measures from the sum of the linear pair measures to get the sum of the exterior angle measures: $1620° - 1260° = 360°$.



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7.6 Identifying Characteristics of Quadrilaterals

The table shows the characteristics of special types of quadrilaterals.

	Trapezoid	Parallelogram	Kite	Rhombus	Rectangle	Square
No parallel sides			•			
Exactly one pair of parallel sides	•					
Two pairs of parallel sides		•		•	•	•
One pair of sides are both congruent and parallel		•		•	•	•
Two pairs of opposite sides are congruent		•		•	•	•
Exactly one pair of opposite angles are congruent			•			
Two pairs of opposite angles are congruent		•		•	•	•
Consecutive angles are supplementary		•		•	•	•
Diagonals bisect each other		•		•	•	•
All sides are congruent				•		•
Diagonals are perpendicular to each other			•	•		•
Diagonals bisect the vertex angles				•		•
All angles are congruent					•	•
Diagonals are congruent					•	•

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