Using Congruence Theorems

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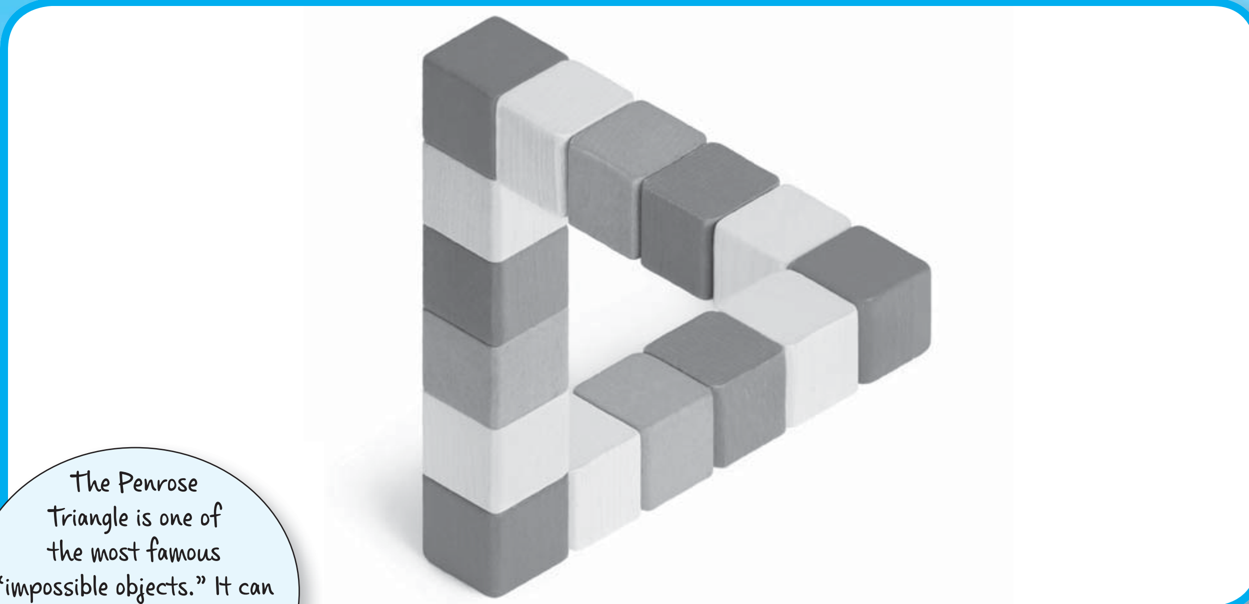


Chapter 6 Overview

This chapter covers triangle congruence, including right triangle and isosceles triangle congruence theorems. Lessons provide opportunities for students to explore the congruence of corresponding parts of congruent triangles as well as continuing work with proof, introducing indirect proof, or proof by contradiction. Throughout, students apply congruence theorems to solve problems.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
6.1	Right Triangle Congruence Theorems	G.CO.6 G.CO.7 G.CO.8 G.CO.10 G.CO.12 G.MG.1	1	This lesson explores four right triangle congruence theorems: The Hypotenuse-Leg, Leg-Leg, Hypotenuse-Angle, and Leg-Angle Congruence Theorems. Questions ask students to explore each theorem using algebra and constructions and prove the theorems. Students also apply the theorems to solve problems.	X			X	
6.2	Corresponding Parts of Congruent Triangles are Congruent	G.CO.10 G.MG.1	1	In this lesson, students investigate that the corresponding parts of congruent triangles are congruent (CPCTC). Questions ask students to prove the Isosceles Triangle Base Theorem and its converse as well as apply CPCTC.	X	X			
6.3	Isosceles Triangle Theorems	G.CO.10 G.MG.1	1	In this lesson, students prove several theorems related to isosceles triangles using drawings and two-column proofs.	X			X	
6.4	Inverse, Contrapositive, Direct Proof, and Indirect Proof	G.CO.10 G.MG.1	1	This lesson introduces inverses and contrapositives of conditional statements as well as proof by contradiction, or indirect proof. Students prove the Hinge Theorem along with its converse.	X	X			

Skills Practice Correlation for Chapter 6

Lesson		Problem Set	Objectives
6.1	Right Triangle Congruence Theorems		Vocabulary
		1 – 4	Mark triangle sides to make congruence statements true by the HL Congruence Theorem
		5 – 8	Mark triangle sides to make congruence statements true by the LL Congruence Theorem
		9 – 12	Mark triangle sides to make congruence statements true by the HA Congruence Theorem
		13 – 16	Mark triangle sides to make congruence statements true by the LA Congruence Theorem
		17 – 22	Determine if there is enough information to prove triangles congruent
		23 – 26	Use right triangle congruence theorems to solve problems
		27 – 30	Write two-column proofs using right triangle congruence theorems
6.2	Corresponding Parts of Congruent Triangles are Congruent		Vocabulary
		1 – 12	Write two-column proofs using CPCTC
		13 – 18	Use CPCTC to solve problems
6.3	Isosceles Triangle Theorems		Vocabulary
		1 – 6	Write isosceles triangle theorems to justify statements
		7 – 12	Determine unknown values given isosceles triangle diagrams
		13 – 16	Use isosceles triangle theorems to complete two-column proofs
		17 – 22	Use isosceles triangle theorems to solve problems
6.4	Inverse, Contrapositive, Direct Proof, and Indirect Proof		Vocabulary
		1 – 8	Write the converse of conditional statements and determine if the converse is true
		9 – 16	Write the inverse of conditional statements and determine if the inverse is true
		17 – 24	Write the contrapositive of conditional statements and determine if the contrapositive is true
		25 – 28	Write indirect proofs to prove statements
		29 – 32	Use the Hinge Theorem to write conclusions about triangles using inequalities

6.1

Time to Get Right

Right Triangle Congruence Theorems

LEARNING GOALS

In this lesson, you will:

- Prove the Hypotenuse-Leg Congruence Theorem using a two-column proof and construction.
- Prove the Leg-Leg, Hypotenuse-Angle, and Leg-Angle Congruence Theorems by relating them to general triangle congruence theorems.
- Apply right triangle congruence theorems.

KEY TERMS

- Hypotenuse-Leg (HL) Congruence Theorem
- Leg-Leg (LL) Congruence Theorem
- Hypotenuse-Angle (HA) Congruence Theorem
- Leg-Angle (LA) Congruence Theorem

ESSENTIAL IDEAS

- The Hypotenuse-Leg Congruence Theorem states: "If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of a second right triangle, then the triangles are congruent." Using the Pythagorean Theorem, this situation is directly related to SSS.
- The Leg-Leg Congruence Theorem states: "If two legs of one right triangle are congruent to two legs of a second right triangle, then the triangles are congruent." This situation is directly related to SAS.
- The Hypotenuse-Angle Congruence Theorem states: "If the hypotenuse and acute angle of one right triangle are congruent to the hypotenuse and acute angle of a second right triangle, then the triangles are congruent." This situation is directly related to AAS.
- The Leg-Angle Congruence Theorem states: "If the leg and acute angle of one right triangle are congruent to the leg and acute angle of a second right triangle, then the triangles are congruent." This situation is directly related to AAS.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Understand congruence in terms of rigid motions

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Prove geometric theorems

10. Prove theorems about triangles.

Make geometric constructions

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

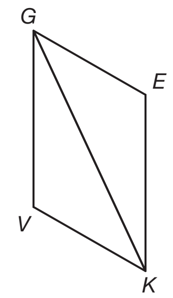
Four right triangle congruency theorems are introduced: Hypotenuse-Leg, Leg-Leg, Hypotenuse-Angle, and Leg-Angle.

Students are asked to prove each right triangle congruence theorem. The Pythagorean Theorem in conjunction with the AA Similarity Postulate is used to establish the third side of each triangle congruent in the Hypotenuse-Leg Congruence Theorem. Either SSS or SAS can be used to conclude the triangles are congruent. Students may need a few hints to guide them through this proof. This theorem is proved using a two-column format, a construction proof, and transformations. The remaining right triangle theorems, LL, HA, and LA are proven through the use of transformations, mapping a triangle onto itself. After proving the right triangle congruence theorems, students determine if there is enough information to prove two triangles congruent in different situations and identify the congruence theorem when appropriate.

Warm Up

1. Given: $\overline{GE} \parallel \overline{KV}$

Is $\triangle GVK \cong \triangle KEG$? Explain.

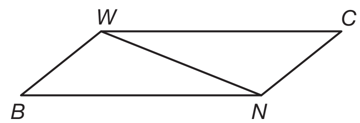


There is not enough information to determine if the triangles are congruent. It can be determined that $\overline{GK} \cong \overline{GK}$ using the Reflexive Property and $\angle EGK \cong \angle VKG$ because they are alternate interior angles, but that is not enough information to conclude triangle congruency.

2. Given: $\overline{WC} \parallel \overline{NB}$

$\overline{WC} \cong \overline{NB}$

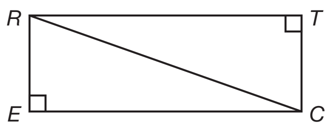
Is $\triangle CWN \cong \triangle BNW$? Explain.



Yes, $\triangle CWN \cong \triangle BNW$ by the SAS Congruence Theorem. It can be determined that $\overline{WN} \cong \overline{WN}$ using the Reflexive Property and $\angle TRC \cong \angle ECR$ because they are alternate interior angles.

3. Given: $\overline{RE} \cong \overline{CT}$

Is $\triangle REC \cong \triangle CTR$? Explain.



Yes, $\triangle REC \cong \triangle CTR$ by the SSS Congruence Theorem. It can be determined that $\overline{RC} \cong \overline{RC}$ using the Reflexive Property and $\overline{RT} \cong \overline{CE}$ using the Pythagorean Theorem.

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6.1

Time to Get Right

Right Triangle Congruence Theorems

LEARNING GOALS

In this lesson, you will:

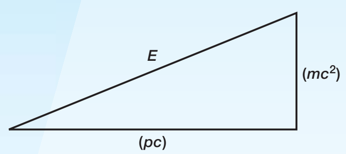
- Prove the Hypotenuse-Leg Congruence Theorem using a two-column proof and construction.
- Prove the Leg-Leg, Hypotenuse-Angle, and Leg-Angle Congruence Theorems by relating them to general triangle congruence theorems.
- Apply right triangle congruence theorems.

KEY TERMS

- Hypotenuse-Leg (HL) Congruence Theorem
- Leg-Leg (LL) Congruence Theorem
- Hypotenuse-Angle (HA) Congruence Theorem
- Leg-Angle (LA) Congruence Theorem

You know the famous equation $E = mc^2$. But this equation is actually incomplete. The full equation is $E^2 = (m^2)^2 + (pc)^2$, where E represents energy, m represents mass, p represents momentum, and c represents the speed of light.

You can represent this equation on a right triangle.



So, when an object's momentum is equal to 0, you get the equation $E = mc^2$.

But what about a particle of light, which has no mass? What equation would describe its energy?

Problem 1

Students review the triangle congruency theorems already proven and explain how fewer congruent corresponding parts are needed to prove right triangles congruent. The Hypotenuse-Leg Congruence Theorem (HL) is a lengthy proof when using a two-column format. The Pythagorean Theorem in conjunction with the AA Similarity Postulate is used to establish the third side of each triangle congruent. Either SSS or SAS can be used to conclude the triangles are congruent. Students may need a few hints to guide them through this proof. A second method for proving this theorem is using construction tools, and a third method of the proof employs the use of transformations, mapping one triangle onto itself.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- How many theorems have you proven related to triangle congruency?
- Do those theorems apply to right triangles?
- What do all right triangles have in common?
- What is the measure of a right angle?

PROBLEM 1 Hypotenuse-Leg (HL) Congruence Theorem

1. List all of the triangle congruence theorems you explored previously.

   The triangle congruence theorems are
   - Side-Side-Side
   - Side-Angle-Side
   - Angle-Side-Angle
   - Angle-Angle-Side

The congruence theorems apply to all triangles. There are also theorems that only apply to right triangles. Methods for proving that two right triangles are congruent are somewhat shorter. You can prove that two right triangles are congruent using only two measurements.

2. Explain why only two pairs of corresponding parts are needed to prove that two right triangles are congruent.

   Every right triangle contains a 90° angle. I already know that two corresponding angles are congruent, because all right angles are congruent.

3. Are all right angles congruent? Explain your reasoning.

   All right angles are congruent, because every right angle has a measure of 90° by the definition of a right angle, and all angles of equal measure are congruent.

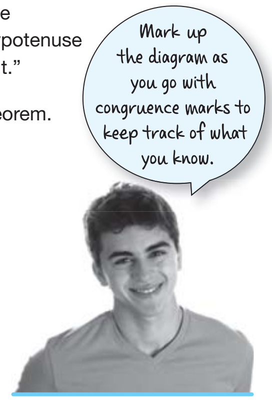
Grouping

Have students complete Question 4 with a partner. Then have students share their responses as a class.

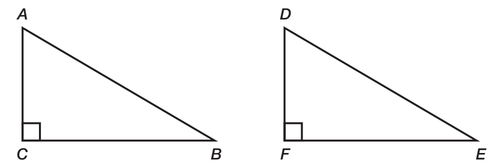
Guiding Questions for Share Phase, Question 4

- What do you know about all right angles?
- What is the definition of congruent segments?
- If two segments are congruent, does that mean they are equal in measure?
- How is the Pythagorean Theorem helpful in this situation?
- If $AB = DE$, does that mean $AB^2 = DE^2$? How do you know?
- If $CB^2 = FE^2$, does that mean $CB = FE$? How do you know?
- Which corresponding parts were stated congruent in the two triangles?
- Can you use HL as a method for proving any two triangles are congruent? Why or why not?

The **Hypotenuse-Leg (HL) Congruence Theorem** states: "If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent."



4. Complete the two-column proof of the HL Congruence Theorem.



Given: $\angle C$ and $\angle F$ are right angles

$\overline{AC} \cong \overline{DF}$

$\overline{AB} \cong \overline{DE}$

Prove: $\triangle ABC \cong \triangle DEF$

Statements	Reasons
1. $\angle C$ and $\angle F$ are right angles	1. Given
2. $\angle C \cong \angle F$	2. All right angles are congruent
3. $\overline{AC} \cong \overline{DF}$	3. Given
4. $\overline{AB} \cong \overline{DE}$	4. Given
5. $AC = DF$	5. Definition of congruent segments
6. $AB = DE$	6. Definition of congruent segments
7. $AC^2 + CB^2 = AB^2$	7. Pythagorean Theorem
8. $DF^2 + FE^2 = DE^2$	8. Pythagorean Theorem
9. $AC^2 + CB^2 = DF^2 + FE^2$	9. Substitution Property
10. $CB^2 = FE^2$	10. Subtraction Property of Equality
11. $CB = FE$	11. Square Root Property
12. $\overline{CB} \cong \overline{FE}$	12. Definition of congruent segments
13. $\triangle ABC \cong \triangle DEF$	13. SAS Congruence Theorem

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Grouping

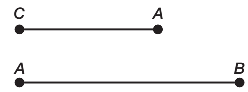
Have students complete Question 5 with a partner. Then have students share their responses as a class.

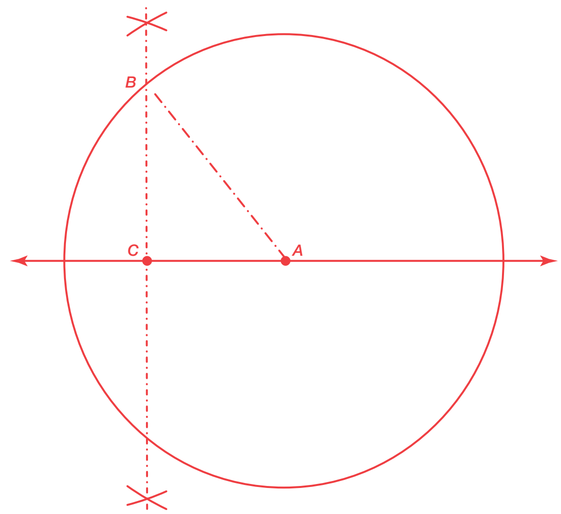
Guiding Questions for Share Phase, Question 5

- Which point did you locate first?
- What is the location of the first point?
- Which line segment did you duplicate first?
- Did you duplicate the line segment on the starter line? Why?
- Did you construct a line perpendicular to the starter line and through the first point you located?
- Did you construct a circle? Which line segment is the radius of the circle?
- How did you locate point *B*?
- How do you know you constructed a right triangle?
- If all of your classmates constructed the same triangle, does this prove that given the hypotenuse and leg of a right triangle, only one unique triangle can be constructed? Does this prove the HL Theorem?

You can also use construction to demonstrate the Hypotenuse-Leg Theorem.

5. Construct right triangle *ABC* with right angle *C*, given leg $\overline{CA}$ and hypotenuse $\overline{AB}$. Then, write the steps you performed to construct the triangle.





- Draw a starter line.
- Plot point *C* on the starter line.
- Duplicate leg $\overline{CA}$ on the starter line to locate point *A*.
- Construct a line perpendicular to the starter line through point *C*.
- Construct circle *A* with radius $\overline{AB}$.
- Label a point at which Circle *A* intersects the perpendicular line point *B*.
- Connect points *A*, *C*, and *B* to form right triangle *ABC*.

a. How does the length of side $\overline{CB}$ compare to the lengths of your classmates' sides $\overline{CB}$?

The length of side $\overline{CB}$ is the same in everyone's triangle.

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b. Use a protractor to measure $\angle A$ and $\angle B$ in triangle ABC. How do the measures of these angles compare to the measures of your classmates' angles A and B?

We all have congruent angles.

c. Is your triangle congruent to your classmates' triangles? Why or why not?

All of the triangles are congruent because all of the corresponding sides and all of the corresponding angles are congruent.

Grouping

Have students complete Question 6 with a partner. Then have students share their responses as a class.

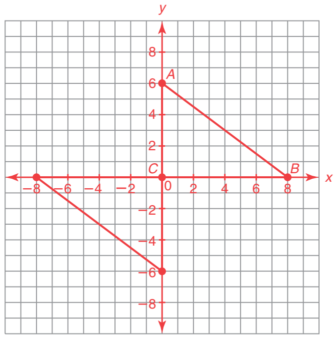
Guiding Questions for Share Phase, Question 6

- Was the Pythagorean Theorem or the Distance Formula needed to determine the lengths of the three sides of triangle ABC? Why not?
- Did rotating only the leg and hypotenuse of a right triangle result in forming a unique right triangle? How do you know?
- Is the new right triangle congruent to the original right triangle? How do you know?

Through your two-column proof and your construction proof, you have proven that Hypotenuse-Leg is a valid method of proof for any right triangle. Now let's prove the Hypotenuse-Leg Theorem on the coordinate plane using algebra.

6. Consider right triangle ABC with right angle C and points A (0, 6), B (8, 0), and C (0, 0).

a. Graph right triangle ABC.



b. Calculate the length of each line segment forming the sides of triangle ABC and record the measurements in the table.

Sides of Triangle *ABC*	Lengths of Sides of Triangle *ABC* (units)
$\overline{AB}$	10
$\overline{BC}$	8
$\overline{AC}$	6

$a^2 + b^2 = c^2$

$6^2 + 8^2 = c^2$

$c^2 = 36 + 64$

$c^2 = 100$

$c = \sqrt{100} = 10$

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c. Rotate side AB, side AC, and $\angle C$ 180° counterclockwise about the origin. Then, connect points B' and C' to form triangle $A'B'C'$. Use the table to record the coordinates of triangle $A'B'C'$.

Coordinates of Triangle *ABC*	Coordinates of Triangle *A′B′C′*
$A(0,6)$	$A'(0, -6)$
$B(8,0)$	$B'(-8, 0)$
$C(0,0)$	$C'(0, 0)$

d. Calculate the length of each line segment forming the sides of triangle $A'B'C'$, and record the measurements in the table.

Sides of Triangle *A′B′C′*	Lengths of Sides of Triangle *A′B′C′* (units)
$\overline{A'B'}$	10
$\overline{B'C'}$	8
$\overline{A'C'}$	6

$a^2 + b^2 = c^2$

$6^2 + 8^2 = c^2$

$c^2 = 36 + 64$

$c^2 = 100$

$c = \sqrt{100} = 10$

e. What do you notice about the side lengths of the image and pre-image?

The side lengths of triangle ABC are the same length as the corresponding sides of triangle $A'B'C'$.

f. Use a protractor to measure $\angle A$, $\angle A'$, $\angle B$, and $\angle B'$. What can you conclude about the corresponding angles of triangle ABC and triangle $A'B'C'$?

The corresponding angles of the two triangles are congruent.

You have shown that the corresponding sides and corresponding angles of the pre-image and image are congruent. Therefore, the triangles are congruent.

Problem 2

Rigid motion is used to prove the remaining right triangle congruency theorems. Students graph three coordinates to form a right triangle. They calculate the length of the three sides using the Pythagorean Theorem or Distance Formula. Next, students translate the two legs of the right triangle both vertically and horizontally and connect endpoints to form a second triangle. After calculating the lengths of the sides of the second triangle and using a protractor to measure angles, it can be concluded that two right triangles map onto each other, therefore they are congruent (LL). This activity is repeated for the LA and HA right triangle congruence theorems using reflections or translations.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Where is the right angle of triangle *ABC* located on the coordinate plane?
- Was the Pythagorean Theorem or the Distance Formula needed to determine the lengths of the three sides of triangle *ABC*? Why not?

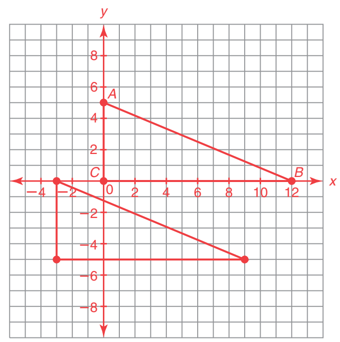
PROBLEM 2 Proving Three More Right Triangle Theorems

You used a two-column proof, a construction, and rigid motion to prove the Hypotenuse-Leg Congruent Theorem. There are three more right triangle congruence theorems that we are going to explore. You can prove each of them using the same methods but you'll focus on rigid motion in this lesson.

The **Leg-Leg (LL) Congruence Theorem** states: "If two legs of one right triangle are congruent to two legs of another right triangle, then the triangles are congruent."

1. Consider right triangle *ABC* with right angle *C* and points *A* (0, 5), *B* (12, 0), and *C* (0, 0).

a. Graph right triangle *ABC*.



b. Calculate the length of each line segment forming the sides of triangle *ABC,* and record the measurements in the table.

Sides of Triangle *ABC*	Lengths of Sides of Triangle *ABC* (units)
$\overline{AB}$	13
$\overline{BC}$	12
$\overline{AC}$	5

$a^2 + b^2 = c^2$

$5^2 + 12^2 = c^2$

$c^2 = 25 + 144$

$c^2 = \sqrt{169} = 13$

- Did translating only the two legs of a right triangle result in forming a unique right triangle? How do you know?
- Is the new right triangle congruent to the original right triangle? How do you know?
- Can you use LL as a method for proving any two triangles are congruent? Why or why not?

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c. Translate side AC, and side BC, to the left 3 units, and down 5 units. Then, connect points A', B' and C' to form triangle $A'B'C'$. Use the table to record the image coordinates.

Coordinates of Triangle *ABC*	Coordinates of Triangle *A′B′C′*
$A(0, 5)$	$A'(-3, -0)$
$B(12, 0)$	$B'(9, -5)$
$C(0, 0)$	$C'(-3, -5)$

d. Calculate the length of each line segment forming the sides of triangle $A'B'C'$, and record the measurements in the table.

Sides of Triangle *A′B′C′*	Lengths of Sides of Triangle *A′B′C′* (units)
$\overline{A'B'}$	13
$\overline{B'C'}$	12
$\overline{A'C'}$	5

$a^2 + b^2 = c^2$

$5^2 + 12^2 = c^2$

$c^2 = 25 + 144$

$c^2 = \sqrt{169} = 13$

e. What do you notice about the side lengths of the image and pre-image?

The side lengths of triangle *ABC* are the same length as the corresponding sides of triangle *A′B′C′*.

f. Use a protractor to measure $\angle A$, $\angle A'$, $\angle B$, and $\angle B'$. What can you conclude about the corresponding angles of triangle *ABC* and triangle *A′B′C′*?

The corresponding angles of the two triangles are congruent.

You have shown that the corresponding sides and corresponding angles of the pre-image and image are congruent. Therefore, the triangles are congruent.

Grouping

Have students complete Question 2 with a partner. Then have students share their responses as a class.

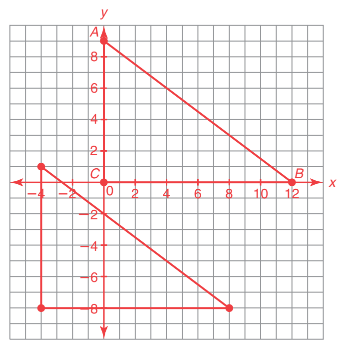
Guiding Questions for Share Phase, Question 2

- Where is the right angle of triangle *ABC* located on the coordinate plane?
- Was the Pythagorean Theorem or the Distance Formula needed to determine the lengths of the three sides of triangle *ABC*? Why not?
- Did translating only one acute angle and the hypotenuse of a right triangle result in forming a unique right triangle? How do you know?
- Is the new right triangle congruent to the original right triangle? How do you know?
- Can you use HA as a method for proving any two triangles are congruent? Why or why not?

The **Hypotenuse-Angle (HA) Congruence Theorem** states: "If the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and acute angle of another right triangle, then the triangles are congruent."

2. Consider right triangle *ABC* with right angle *C* and points *A* (0, 9), *B* (12, 0), and *C* (0, 0).

a. Graph right triangle *ABC* with right $\angle C$, by plotting the points *A* (0, 9), *B* (12, 0), and *C* (0, 0).



b. Calculate the length of each line segment forming the sides of triangle *ABC,* and record the measurements in the table.

Sides of Triangle *ABC*	Lengths of Sides of Triangle *ABC* (units)
$\overline{AB}$	15
$\overline{BC}$	12
$\overline{AC}$	9

$a^2 + b^2 = c^2$

$9^2 + 12^2 = c^2$

$c^2 = 81 + 144$

$c^2 = \sqrt{225} = 15$

c. Translate side *AB*, and $\angle A$, to the left 4 units, and down 8 units. Then, connect points A', B' and C' to form triangle $A'B'C'$. Use the table to record the image coordinates.

Coordinates of Triangle *ABC*	Coordinates of Triangle $A'B'C'$
$A(0, 9)$	$A'(-4, 1)$
$B(12, 0)$	$B'(8, -8)$
$C(0, 0)$	$C'(-4, -8)$

d. Calculate the length of each line segment forming the sides of triangle $A'B'C'$, and record the measurements in the table.

Sides of Triangle $A'B'C'$	Lengths of Sides of Triangle $A'B'C'$ (units)
$\overline{A'B'}$	15
$\overline{B'C'}$	12
$\overline{A'C'}$	9

$a^2 + b^2 = c^2$

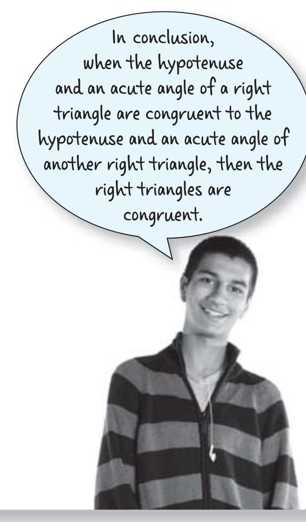
$9^2 + 12^2 = c^2$

$c^2 = 81 + 144$

$c^2 = \sqrt{225} = 15$

e. What do you notice about the side lengths of the image and pre-image?

The side lengths of triangle *ABC* are the same length as the corresponding sides of triangle $A'B'C'$.



f. Use a protractor to measure $\angle A$, $\angle A'$, $\angle B$, and $\angle B'$. What can you conclude about the corresponding angles of triangle *ABC* and triangle $A'B'C'$?

The corresponding angles of the two triangles are congruent.

You have shown that the corresponding sides and corresponding angles of the pre-image and image are congruent. Therefore, the triangles are congruent.

Grouping

Have students complete Question 3 with a partner. Then have students share their responses as a class.

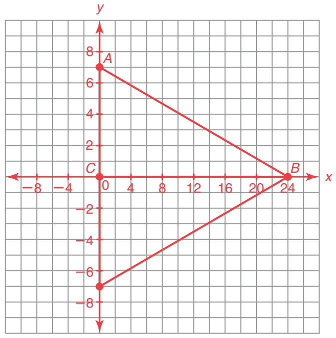
Guiding Questions for Share Phase, Question 3

- Where is the right angle of triangle ABC located on the coordinate plane?
- Was the Pythagorean Theorem or the Distance Formula needed to determine the lengths of the three sides of triangle ABC? Why not?
- Did translating only one leg and one acute angle of a right triangle result in forming a unique right triangle? How do you know?
- Is the new right triangle congruent to the original right triangle? How do you know?
- Can you use LA as a method for proving any two triangles are congruent? Why or why not?

The **Leg-Angle (LA) Congruence Theorem** states: "If a leg and an acute angle of one right triangle are congruent to a leg and an acute angle of another right triangle, then the triangles are congruent."

3. Consider right triangle ABC with right angle C and points A (0, 7), B (24, 0), and C (0, 0).

a. Graph right triangle ABC with right $\angle C$, by plotting the points A (0, 7), B (24, 0), and C (0, 0).



b. Calculate the length of each line segment forming the sides of triangle ABC, and record the measurements in the table.

Sides of Triangle *ABC*	Lengths of Sides of Triangle *ABC* (units)
$\overline{AB}$	25
$\overline{BC}$	24
$\overline{AC}$	7

$a^2 + b^2 = c^2$

$7^2 + 24^2 = c^2$

$c^2 = 49 + 576$

$c^2 = \sqrt{625} = 25$

c. Reflect side AC, and $\angle B$ over the x-axis. Then, connect points A', B' and C' to form triangle $A'B'C'$. Use the table to record the image coordinates.

Coordinates of Triangle ***ABC***	Coordinates of Triangle ***A'B'C'***
$A(0, 7)$	$A'(0, -7)$
$B(24, 0)$	$B'(24, 0)$
$C(0, 0)$	$C'(0, 0)$

d. Calculate the length of each line segment forming the sides of triangle $A'B'C'$, and record the measurements in the table.

Sides of Triangle ***A'B'C'***	Lengths of Sides of Triangle ***A'B'C'*** (units)
$\overline{A'B'}$	25
$\overline{B'C'}$	24
$\overline{A'C'}$	7

$a^2 + b^2 = c^2$

$7^2 + 24^2 = c^2$

$c^2 = 49 + 576$

$c^2 = \sqrt{625} = 25$

e. What do you notice about the side lengths of the image and pre-image?

The side lengths of triangle *ABC* are the same length as the corresponding sides of triangle *A'B'C'*.

f. Use a protractor to measure $\angle A$, $\angle A'$, $\angle B$, and $\angle B'$. What can you conclude about the corresponding angles of triangle ABC and triangle $A'B'C'$?

The corresponding angles of the two triangles are congruent.

You have shown that the corresponding sides and corresponding angles of the pre-image and image are congruent. Therefore, the triangles are congruent.

Problem 3

Students determine if there is enough information to prove two triangles congruent and identify the triangle congruence theorem when appropriate.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

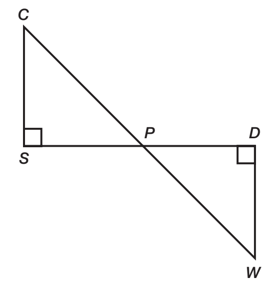
Guiding Questions for Share Phase, Questions 1 through 4

- If two lines are perpendicular to the same line, what can you conclude?
- If *P* is the midpoint of line segment *CW*, what can you conclude?
- Are triangles *CSP* and *WDP* right triangles? How do you know?
- Do the two triangles share a common side?
- Do the two triangles share a common angle?
- Are triangles *RFP* and *BPF* right triangles? How do you know?
- Is there more than one way to prove these triangles congruent?
- Do you know if the lengths of the treads are the same?
- Do you know if the heights of the risers are the same?
- Are triangles *JYM* and *AYM* right triangles? How do you know?

PROBLEM 3 Applying Right Triangle Congruence Theorems

Determine if there is enough information to prove that the two triangles are congruent. If so, name the congruence theorem used.

1. If $\overline{CS} \perp \overline{SD}$, $\overline{WD} \perp \overline{SD}$, and P is the midpoint of $\overline{CW}$, is $\triangle CSP \cong \triangle WDP$?



Yes. There is enough information to conclude that $\triangle CSP \cong \triangle WDP$ by HA.

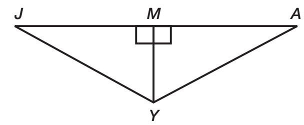
Point P is the midpoint so I know that $\overline{SP} \cong \overline{PD}$. Also, $\angle CPS \cong \angle DPW$ because they are vertical angles and vertical angles are congruent.

2. Pat always trips on the third step and she thinks that step may be a different size. The contractor told her that all the treads and risers are perpendicular to each other. Is that enough information to state that the steps are the same size? In other words, if $\overline{WN} \perp \overline{NZ}$ and $\overline{ZH} \perp \overline{HK}$, is $\triangle WNZ \cong \triangle ZHK$?

No. Triangle WNZ might not be congruent to $\triangle HKZ$. The lengths of the treads may be different or heights of the risers may be different.

- Is there more than one way to prove these triangles congruent?
- Are triangles *STR* and *ATR* right triangles? How do you know?
- Is there more than one way to prove these triangles congruent?

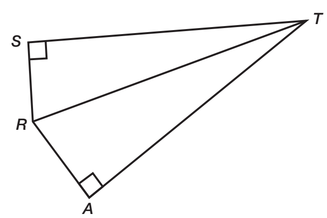
3. If $\overline{JA} \perp \overline{MY}$ and $\overline{JY} \cong \overline{AY}$, is $\triangle JYM \cong \triangle AYM$?



Yes. There is enough information to conclude that $\triangle JYM \cong \triangle AYM$ by HL.

I know that $\overline{MY}$ is congruent to itself by the Reflexive Property.

4. If $\overline{ST} \perp \overline{SR}$, $\overline{AT} \perp \overline{AR}$, and $\angle STR \cong \angle ATR$, is $\triangle STR \cong \triangle ATR$?



Yes. There is enough information to conclude that $\triangle STR \cong \triangle ATR$ by HA.

I know that $\overline{RT}$ is congruent to itself by the Reflexive Property.

Grouping

- Ask a student to read aloud the information before Question 5 and discuss as a class.
- Have students complete Questions 5 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 7

- Do the two triangles share a common side?
- Do the two triangles share a common angle?
- Are the two triangles right triangles?
- What is the definition of perpendicular bisector?
- What is the definition of perpendicular lines?
- What is the definition of bisect?
- Is there more than one way to prove these triangles congruent?
- Where is the right angle in the dock situation?
- Where is the right triangle in the dock situation?

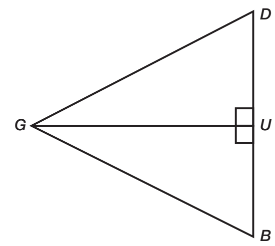
It is necessary to make a statement about the presence of right triangles when you use the Right Triangle Congruence Theorems. If you have previously identified the right angles, the reason is the definition of right triangles.

5. Create a proof of the following.

Given: $\overline{GU} \perp \overline{DB}$

$\overline{GB} \cong \overline{GD}$

Prove: $\triangle GUD \cong \triangle GUB$

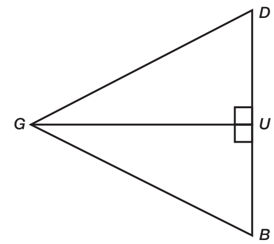


Statements	Reasons
1. $\overline{GU} \perp \overline{DB}$	**1.** Given
2. $\angle GUD$ and $\angle GUB$ are right angles	**2.** Definition of perpendicular lines
3. $\triangle GUD$ and $\triangle GUB$ are right triangles	**3.** Definition of right triangles
4. $\overline{GB} \cong \overline{GD}$	**4.** Given
5. $\overline{GU} \cong \overline{GU}$	**5.** Reflexive Property of $\cong$
6. $\triangle GUD \cong \triangle GUB$	**6.** HL Congruence Theorem

6. Create a proof of the following.

Given: $\overline{GU}$ is the $\perp$ bisector of $\overline{DB}$

Prove: $\triangle GUD \cong \triangle GUB$



Statements	Reasons
1. $\overline{GU}$ is the $\perp$ bisector of $\overline{DB}$	1. Given
2. $\overline{GU} \perp \overline{DB}$	2. Definition of perpendicular bisector
3. $\overline{GU}$ bisects $\overline{DB}$	3. Definition of perpendicular bisector
4. $\angle GUD$ and $\angle GUB$ are right angles	4. Definition of perpendicular lines
5. $\triangle GUD$ and $\triangle GUB$ are right triangles	5. Definition of right triangles
6. $\overline{DU} \cong \overline{BU}$	6. Definition of bisect
7. $\overline{GU} \cong \overline{GU}$	7. Reflexive Property of Equality
8. $\triangle GUD \cong \triangle GUB$	8. LL Congruence Theorem

7. A friend wants to place a post in a lake 20 feet to the right of the dock. What is the minimum information you need to make sure the angle formed by the edge of the dock and the post is a right angle?

I only need to be given one more piece of information. I could use the length of the dock (Leg-Leg), or I could use the measure of an acute angle (LA).

Suppose I am given the length of the dock. I could then calculate the distance from the post to the furthest end of the dock to know the length of the hypotenuse. If I measure that distance to see if it matches, I know it is a right triangle.

6

Talk the Talk

Students answer a series of questions focusing on triangle congruence theorems.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Talk the Talk

1. Which triangle congruence theorem is most closely related to the LL Congruence Theorem? Explain your reasoning.

   The SAS Congruence Theorem is most closely related to the LL Congruence Theorem because the right angles in the two right triangles are the included angle between the two given legs.

2. Which triangle congruence theorem is most closely related to the HA Congruence Theorem? Explain your reasoning.

   The AAS Congruence Theorem is most closely related to the HA Congruence Theorem because the right angles are in each of the right triangles as well as the given acute angles and the given hypotenuses.

3. Which triangle congruence theorem is most closely related to the LA Congruence Theorem? Explain your reasoning.

   The AAS Congruence Theorem is most closely related to the LA Congruence Theorem because the right angles are in each of the right triangles as well as the given acute angles and the given legs.

4. Which triangle congruence theorem is most closely related to the HL Congruence Theorem? Explain your reasoning.

   The SSS Congruence Theorem is most closely related to the HL Congruence Theorem because I can use the Pythagorean Theorem to show that the other legs are also congruent.

Be prepared to share your solutions and methods.

6

Check for Students' Understanding

1. List the four triangle congruence theorems associated with right triangles.

1. HL Congruence Theorem
2. LL Congruence Theorem
3. HA Congruence Theorem
4. LA Congruence Theorem

2. List the four triangle congruence theorems associated with all triangles.

1. SAS Congruence Theorem
2. ASA Congruence Theorem
3. SSS Congruence Theorem
4. AAS Congruence Theorem

3. Associate each theorem in Question 1 with its related theorem in Question 2.

HL can be associated to the SSS Congruence Theorem.

LL can be associated to the SAS Congruence Theorem.

HA can be associated to the AAS Congruence Theorem.

LA can be associated to the AAS Congruence Theorem.

6

6.2

CPCTC

Corresponding Parts of Congruent Triangles are Congruent

LEARNING GOALS

In this lesson, you will:

- Identify corresponding parts of congruent triangles.
- Use corresponding parts of congruent triangles are congruent to prove angles and segments are congruent.
- Use corresponding parts of congruent triangles are congruent to prove the Isosceles Triangle Base Angle Theorem.
- Use corresponding parts of congruent triangles are congruent to prove the Isosceles Triangle Base Angle Converse Theorem.
- Apply corresponding parts of congruent triangles.

KEY TERMS

- corresponding parts of congruent triangles are congruent (CPCTC)
- Isosceles Triangle Base Angle Theorem
- Isosceles Triangle Base Angle Converse Theorem

ESSENTIAL IDEAS

- Corresponding parts of congruent triangles are congruent. Abbreviated CPCTC.
- The Isosceles Triangle Base Angle Theorem states: “If two sides of a triangle are congruent, then the angles opposite these sides are congruent.”
- The Isosceles Triangle Base Angle Converse Theorem states: “If two angles of a triangle are congruent, then the sides opposite these angles are congruent.”

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Prove geometric theorems

10. Prove theorems about triangles.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

Students identify corresponding parts of congruent triangles and CPCTC (Corresponding Parts of Congruent Triangles are Congruent) is used as a reason in proof problems to conclude segments and angles are congruent. The Isosceles Base Angle Theorem and the Isosceles Base Angle Converse Theorem are stated. Students prove the theorems by constructing a line that bisects the vertex angle of an isosceles triangle. Students apply the theorems to several problem situations to solve for unknown measurements.

6

Warm Up

1. $\triangle COR \cong \triangle ESP$

   Name a part of $\triangle COR$ that corresponds to $\triangle SEP$.

   Answers will vary.

   $\angle SEP$ corresponds to $\angle OCR$

2. $\triangle OND \cong \triangle ING$

   Name a part of $\triangle OND$ that corresponds to $\overline{GI}$.

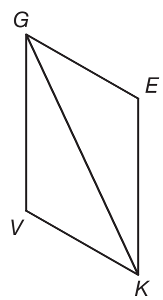
   Answers will vary.

   $\overline{GI}$ corresponds to $\overline{DO}$

3. Given: $\overline{GE} \parallel \overline{KV}$ and $\overline{VK} \cong \overline{EG}$

   Is $\angle V \cong \angle E$? Explain.

   Yes, $\angle V \cong \angle E$ because the triangles can be proven congruent by the SAS Congruence Theorem. If the triangles are congruent then all of their corresponding parts are congruent.



6

CPCTC
Corresponding Parts of Congruent Triangles are Congruent

6.2

LEARNING GOALS

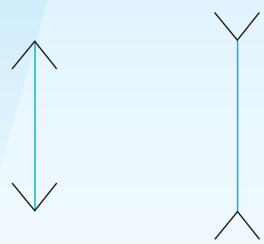
In this lesson, you will:

- Identify corresponding parts of congruent triangles.
- Use corresponding parts of congruent triangles are congruent to prove angles and segments are congruent.
- Use corresponding parts of congruent triangles are congruent to prove the Isosceles Triangle Base Angle Theorem.
- Use corresponding parts of congruent triangles are congruent to prove the Isosceles Triangle Base Angle Converse Theorem.
- Apply corresponding parts of congruent triangles.

KEY TERMS

- corresponding parts of congruent triangles are congruent (CPCTC)
- Isosceles Triangle Base Angle Theorem
- Isosceles Triangle Base Angle Converse Theorem

Which of the blue lines shown is longer? Most people will answer that the line on the right appears to be longer.



But in fact, both blue lines are the exact same length! This famous optical illusion is known as the Mueller-Lyer illusion. You can measure the lines to see for yourself. You can also draw some of your own to see how it almost always works!

Problem 1

The abbreviation for Corresponding Parts of Congruent Triangles are Congruent (CPCTC) is introduced. In the first proof, students use CPCTC to prove segments congruent. In the second proof, students use CPCTC to prove two angles congruent. The use of CPCTC in conjunction with other definitions, theorems, and postulates creates a multitude of proof possibilities.

Grouping

- Ask a student to read aloud the information. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- What is the definition of bisect?
- Do the triangles share a common angle?
- Do the triangles share a common side?
- Do the triangles contain vertical angles?
- What is the triangle congruency statement?
- Which congruence theorem was used to prove the triangles congruent?

PROBLEM 1 CPCTC

If two triangles are congruent, then each part of one triangle is congruent to the corresponding part of the other triangle. "**Corresponding parts of congruent triangles are congruent**," abbreviated as **CPCTC**, is often used as a reason in proofs. CPCTC states that corresponding angles or sides in two congruent triangles are congruent. This reason can only be used after you have proven that the triangles are congruent.

To use CPCTC in a proof, follow these steps:

Step 1: Identify two triangles in which segments or angles are corresponding parts.

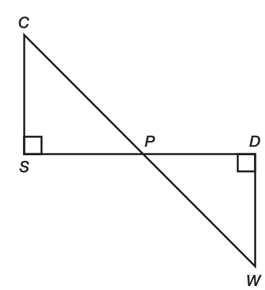
Step 2: Prove the triangles congruent.

Step 3: State the two parts are congruent using CPCTC as the reason.

1. Create a proof of the following.

Given: $\overline{CW}$ and $\overline{SD}$ bisect each other

Prove: $\overline{CS} \cong \overline{WD}$

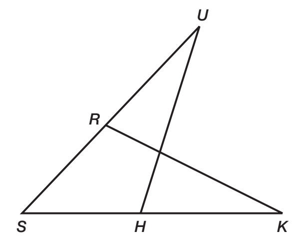


Statements	Reasons
1. $\overline{CW}$ and $\overline{SD}$ bisect each other	1. Given
2. $\overline{CP} \cong \overline{WP}$	2. Definition of bisect
3. $\overline{SP} \cong \overline{DP}$	3. Definition of bisect
4. $\angle CPS \cong \angle WPD$	4. Vertical angles are congruent
5. $\triangle CPS \cong \triangle WPD$	5. SAS Congruence Theorem
6. $\overline{CS} \cong \overline{WD}$	6. CPCTC

2. Create a proof of the following.

Given: $\overline{SU} \cong \overline{SK}$, $\overline{SR} \cong \overline{SH}$

Prove: $\angle U \cong \angle K$



Statements	Reasons
1. $\overline{SU} \cong \overline{SK}$	**1.** Given
2. $\overline{SR} \cong \overline{SH}$	**2.** Given
3. $\angle S \cong \angle S$	**3.** Reflexive Property of $\cong$
4. $\triangle USH \cong \triangle KSR$	**4.** SAS Congruence Theorem
5. $\angle U \cong \angle K$	**5.** CPCTC

Problem 2

Using CPCTC, students prove the Isosceles Triangle Base Angle Theorem: If two sides of a triangle are congruent, then the angles opposite these sides are congruent. Students also prove the Isosceles Triangle Base Angle Converse Theorem: If two angles of a triangle are congruent, then the sides opposite these angles are congruent. Both proofs are done with the use of a line segment connecting the vertex angle of the isosceles triangle to the base.

Grouping

- Discuss the information about the Isosceles Triangle Base Angle Theorem as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Where should a line be drawn that divides the isosceles triangle into two triangles?
- Can you draw a line segment that bisects angle G?
- Do the triangles share a common angle?
- Do the triangles share a common side?
- Do the triangles contain vertical angles?

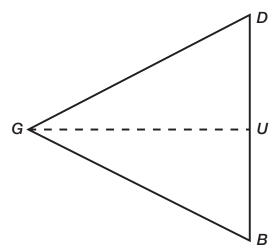
PROBLEM 2 Isosceles Triangle Base Angle Theorem and Its Converse

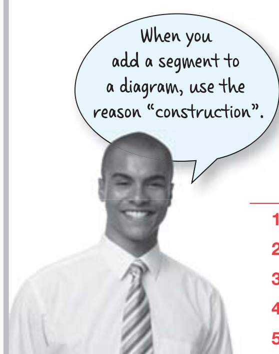
CPCTC makes it possible to prove other theorems.

The **Isosceles Triangle Base Angle Theorem** states: "If two sides of a triangle are congruent, then the angles opposite these sides are congruent."

To prove the Isosceles Triangle Base Angle Theorem, you need to add a line to an isosceles triangle that bisects the vertex angle as shown.

1. Create a proof of the following.
   Given: $\overline{GB} \cong \overline{GD}$
   Prove: $\angle B \cong \angle D$





Statements	Reasons
1. $\overline{GB} \cong \overline{GD}$	1. Given
2. $\overline{GU}$ bisects $\angle BGD$	2. Construction
3. $\angle BGU \cong \angle DGU$	3. Definition of bisect
4. $\overline{GU} \cong \overline{GU}$	4. Reflexive Property of Congruence
5. $\triangle GUD \cong \triangle GUB$	5. SAS Congruence Theorem
6. $\angle B \cong \angle D$	6. CPCTC

- What is the triangle congruency statement?
- Which congruence theorem was used to prove the triangles congruent?

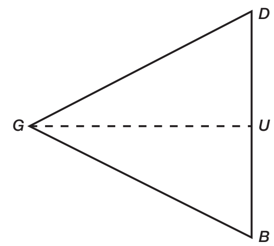
The **Isosceles Triangle Base Angle Converse Theorem** states: "If two angles of a triangle are congruent, then the sides opposite these angles are congruent."

To prove the Isosceles Triangle Base Angle Converse Theorem, you need to again add a line to an isosceles triangle that bisects the vertex angle as shown.

2. Create a proof of the following.

Given: $\angle B \cong \angle D$

Prove: $\overline{GB} \cong \overline{GD}$



Statements	Reasons
1. $\angle B \cong \angle D$	1. Given
2. $\overline{GU}$ bisects $\angle BGD$	2. Construction
3. $\angle BGU \cong \angle DGU$	3. Definition of bisect
4. $\overline{GU} \cong \overline{GU}$	4. Reflexive Property of Congruence
5. $\triangle GUD \cong \triangle GUB$	5. AAS Congruence Theorem
6. $\overline{GB} \cong \overline{GD}$	6. CPCTC

Problem 3

Students apply the theorems in this lesson to solve for unknown measurements.

Grouping

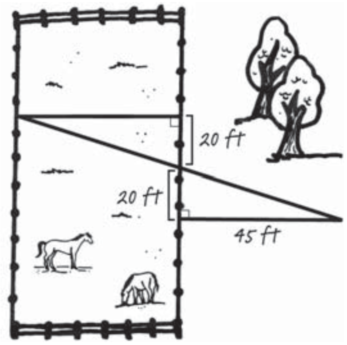
Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

- Are the two triangles congruent?
- What theorem can be used to prove the two triangles congruent?
- Was CPCTC used to determine the solution? How?
- What do you know about sides *AY* and *AP*? How do you know this?
- Which theorem is helpful when determining the relationship between sides *AY* and *AP*?
- On the Ferris wheel, what is the length of lighting on one beam?
- What do you know about the length of all four beams?
- What theorem is helpful in determining the total length of lighting?
- How did you determine the measure of angle *WMT*?
- What do you know about the relationship between angles *W* and *WMT*?

PROBLEM 3 Applications of CPCTC

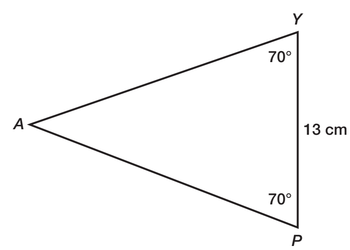
1. How wide is the horse's pasture?



The horse's pasture is 45 feet wide.

I know that one pair of legs are congruent. I also know that the vertical angles are congruent. So, the triangles are congruent by LA Congruence Theorem. Corresponding parts of the triangles are congruent, so the width of the pasture is 45 feet.

2. Calculate *AP* if the perimeter of $\triangle AYP$ is 43 cm.



$AP = 15$ cm

By the Isosceles Triangle Base Angles Converse Theorem, I know that triangle *AYP* is an isosceles triangle with $AY = AP$. Let x represent the length of $\overline{AP}$.

$x + x + 13 = 43$

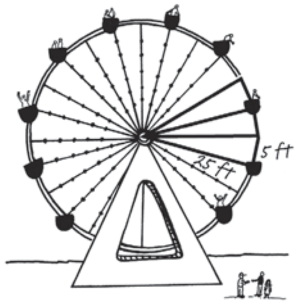
$2x + 13 = 43$

$2x = 30$

$x = 15$

- What do you know about the sum of the measures of the three interior angles of a triangle?
- What theorem is helpful in determining the total length of lighting?
- How did you determine the width of the river?
- What theorem was helpful in determining the width of the river?
- Where would an auxiliary line be helpful in this figure?

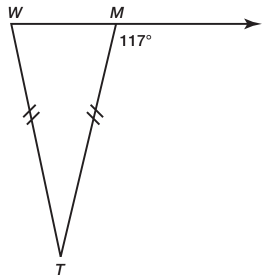
3. Lighting booms on a Ferris wheel consist of four steel beams that have cabling with light bulbs attached. These beams, along with three shorter beams, form the edges of three congruent isosceles triangles, as shown. Maintenance crews are installing new lighting along the four beams. Calculate the total length of lighting needed.



100 feet of lights are needed to illuminate the three triangles.

The three isosceles triangles are congruent so the congruent sides of each are equal to 25 feet. I will need 4 × 25, or 100 feet of lights for the four beams.

4. Calculate $m\angle T$.



$m\angle T = 54°$

The angle measuring 117° and angle *TMW* form a linear pair so they are supplementary. So, the measure of angle *TMW* is 180° − 117°, or 63°.

By the Isosceles Triangle Base Angle Theorem, I know that the base angles are congruent. So, the measure of angle *TWM* is also 63°.

Let x represent the measure of angle T.

$$63 + 63 + x = 180$$
$$126 + x = 180$$
$$x = 54$$

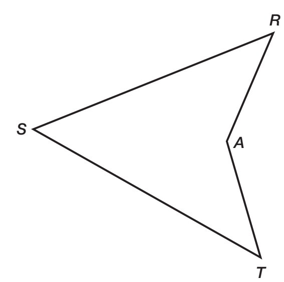
5. What is the width of the river?

The river is 65 feet wide.

I know that two pairs of corresponding sides are congruent. I also know that the included angles are vertical angles and congruent. So, the triangles are congruent by SAS Congruence Theorem. Corresponding parts of the triangles are congruent, so the width of the river is 65 feet.

6. Given: $\overline{ST} \cong \overline{SR}$, $\overline{TA} \cong \overline{RA}$

Explain why $\angle T \cong \angle R$.

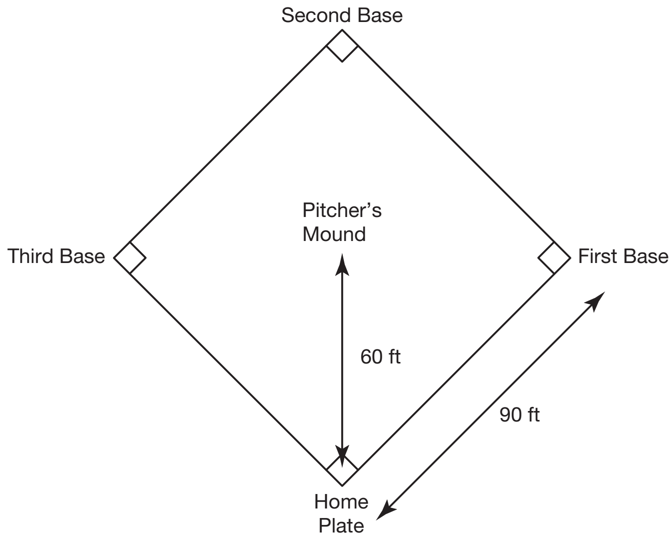


Construct a line from point S to point A to form $\triangle SAR$ and $\triangle SAT$. Two pairs of corresponding sides are congruent. I also know that segment SA is congruent to itself by the Reflexive Property. So, $\triangle SAR \cong \triangle SAT$ by SSS. Therefore, $\angle T \cong \angle R$ by CPCTC.

Be prepared to share your solutions and methods.

Check for Students' Understanding

On a baseball diamond, the bases are at the vertices of a square. Each side of the square is 90 feet. The Pitcher's mound is on the diagonal between home plate and 2nd base, 60 feet from home plate. Is the pitcher's mound 60 feet from 2nd base? Can CPCTC be used to solve for this distance? Explain your reasoning.



CPCTC could be used if the triangle formed by the home plate, the pitcher's mound and first base was congruent to the triangle formed by the second base, the pitcher's mound, and the first base. If those two triangles are congruent isosceles right triangles, the pitcher's mound would be 60 feet from 2nd base.

If the distance between the pitcher's mound and 2nd base was 60 feet, the pitcher's mound would be the middle point of the diagonal. If it was the middle point of one diagonal, it would be the middle point of the second diagonal connecting 3rd base to 1st base because the diagonals of a square perpendicular bisectors of each other and are congruent. The diagonals of the square would form four congruent isosceles right triangles with legs equal to 60 feet and hypotenuses equal to 90 feet. This is impossible:

$$\begin{aligned} c^2 &= 60^2 + 60^2 \\ &= 3600 + 3600 \\ &= 7200 \\ c &= \sqrt{7200} \approx 84.85 \text{ feet} \end{aligned}$$

$$\begin{aligned} 90^2 &= x^2 + x^2 \\ 8100 &= 2x^2 \\ 4050 &= x^2 \\ x &= \sqrt{4050} \approx 63.64 \text{ feet} \end{aligned}$$

If the legs are 60 feet, the hypotenuse must be approximately 84.85 feet, not 90 feet. Or if the hypotenuse is 90 feet, the legs must be approximately 63.64 feet, not 60 feet.

Therefore the pitcher's mound cannot be 60 feet from 2nd base. The triangle formed by the pitcher's mound, home plate and the 1st base is not an isosceles right triangle.

The distance from the pitcher's mound to 2nd base must be approximately

$2(63.64) - 60 = 67.28$ feet.

CPCTC could not be used to solve for the distance from the pitcher's mound to 2nd base.

6

6.3

Congruence Theorems in Action

Isosceles Triangle Theorems

LEARNING GOALS

In this lesson, you will:

- Prove the Isosceles Triangle Base Theorem.
- Prove the Isosceles Triangle Vertex Angle Theorem.
- Prove the Isosceles Triangle Perpendicular Bisector Theorem.
- Prove the Isosceles Triangle Altitude to Congruent Sides Theorem.
- Prove the Isosceles Triangle Angle Bisector to Congruent Sides Theorem.

KEY TERMS

- vertex angle of an isosceles triangle
- Isosceles Triangle Base Theorem
- Isosceles Triangle Vertex Angle Theorem
- Isosceles Triangle Perpendicular Bisector Theorem
- Isosceles Triangle Altitude to Congruent Sides Theorem
- Isosceles Triangle Angle Bisector to Congruent Sides Theorem

ESSENTIAL IDEAS

- The Isosceles Triangle Base Theorem states: "The altitude to the base of an isosceles triangle bisects the base."
- The Isosceles Triangle Vertex Angle Theorem states: "The altitude to the base of an isosceles triangle bisects the vertex angle."
- The Isosceles Triangle Vertex Bisector Theorem states: "The altitude from the vertex angle of an isosceles triangle is the perpendicular bisector of the base."
- The Isosceles Triangle Altitude to Congruent Sides Theorem states: "In an isosceles triangle, the altitudes to the congruent sides are congruent."
- The Isosceles Triangle Angle Bisector to Congruent Sides Theorems states: "In an isosceles triangle, the angle bisectors to the congruent sides are congruent."

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Prove geometric theorems

10. Prove theorems about triangles.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

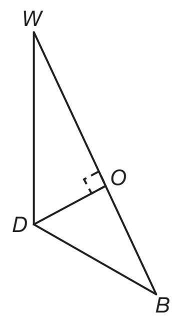
Overview

Students prove several theorems associated with an isosceles triangle involving the altitude to the base, the altitude from the vertex angle, the altitudes to the congruent sides, and the angle bisectors to the congruent sides. The formats of the proofs include flow-chart, two-column, and paragraph.

6

Warm Up

$\overline{DO}$ is an altitude in $\triangle WDE$.



1. Is $\angle DOW \cong \angle DOB$? Explain your reasoning.

   Yes, they are congruent. If $\overline{DO}$ is an altitude, $\overline{DO}$ is perpendicular to $\overline{WB}$. Perpendicular line segments determine right angles. $\angle DOW$ and $\angle DOB$ are right angles. All right angles are congruent.

2. Is $\overline{OW} \cong \overline{OB}$? Explain your reasoning.

   There is not enough information to determine $\overline{OW} \cong \overline{OB}$.

3. Is $\angle WDO \cong \angle BDO$? Explain your reasoning.

   There is not enough information to determine $\angle WDO \cong \angle BDO$.

6

6

6.3

Congruence Theorems in Action

Isosceles Triangle Theorems

LEARNING GOALS

In this lesson, you will:

- Prove the Isosceles Triangle Base Theorem.
- Prove the Isosceles Triangle Vertex Angle Theorem.
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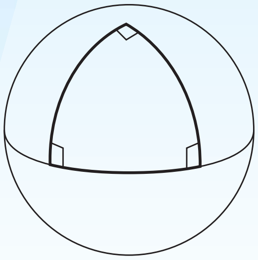
KEY TERMS

- vertex angle of an isosceles triangle
- Isosceles Triangle Base Theorem
- Isosceles Triangle Vertex Angle Theorem
- Isosceles Triangle Perpendicular Bisector Theorem
- Isosceles Triangle Altitude to Congruent Sides Theorem
- Isosceles Triangle Angle Bisector to Congruent Sides Theorem

You know that the measures of the three angles in a triangle equal 180°, and that no triangle can have more than one right angle or obtuse angle.

Unless, however, you're talking about a spherical triangle. A spherical triangle is a triangle formed on the surface of a sphere. The sum of the measures of the angles of this kind of triangle is always greater than 180°. Spherical triangles can have two or even three obtuse angles or right angles.



The properties of spherical triangles are important to a certain branch of science. Can you guess which one?

Problem 1

The Isosceles Triangle Base Theorem states: "The altitude to the base of an isosceles triangle bisects the base." Students prove this theorem using a two-column proof. Next, they extend their reasoning to describe the proof of the Isosceles Triangle Vertex Angle Theorem which states: "The altitude to the base of an isosceles triangle bisects the vertex angle" and the Isosceles Triangle Perpendicular Bisector Theorem which states: "The altitude from the vertex angle of an isosceles triangle is the perpendicular bisector of the base."

Grouping

- Discuss the definition of *vertex angle of an isosceles triangle* and the Isosceles Triangle Base Theorem as a class.
- Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

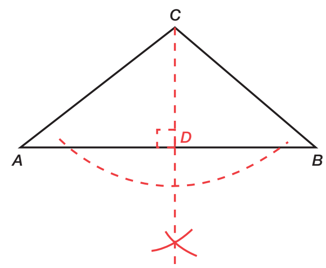
- What is the definition of altitude?
- How many altitudes are in a triangle?
- Are perpendicular line relationships always associated with altitudes?

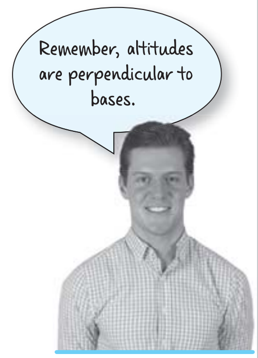
PROBLEM 1 Isosceles Triangle Theorems

You will prove theorems related to isosceles triangles. These proofs involve altitudes, perpendicular bisectors, angle bisectors, and *vertex angles*. A **vertex angle of an isosceles triangle** is the angle formed by the two congruent legs in an isosceles triangle.

The **Isosceles Triangle Base Theorem** states: "The altitude to the base of an isosceles triangle bisects the base."

1. Given: Isosceles $\triangle ABC$ with $\overline{CA} \cong \overline{CB}$.
   a. Construct altitude $\overline{CD}$ from the vertex angle to the base.





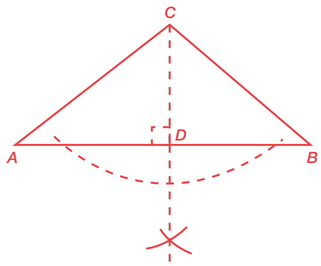
2. Prove the Isosceles Triangle Base Theorem.

Statements	Reasons
1. $\overline{CA} \cong \overline{CB}$	1. Given
2. $\overline{CD}$ is an altitude	2. Construction
3. $\overline{CD} \perp \overline{AB}$	3. Definition of altitude
4. $\angle CDA$ and $\angle CDB$ are right angles	4. Definition of perpendicular
5. $\triangle CDA$ and $\triangle CDB$ are right triangles	5. Definition of right triangle
6. $\angle A \cong \angle B$	6. Isosceles Triangle Base Angle Theorem
7. $\triangle CDA \cong \triangle CDB$	7. HA
8. $\overline{AD} \cong \overline{BD}$	8. CPCTC

- To state that a triangle is a right triangle, what must first be stated?
- Are the triangles formed by the altitude drawn to the base of the isosceles triangle congruent?
- Is CPCTC helpful when proving this theorem?
- What is the difference between the Isosceles Triangle Base Theorem and the Isosceles Triangle Vertex Angle Theorem?

The **Isosceles Triangle Vertex Angle Theorem** states: "The altitude to the base of an isosceles triangle bisects the vertex angle."

3. Draw and label a diagram you can use to help you prove the Isosceles Triangle Vertex Angle Theorem. State the "Given" and "Prove" statements.



Given: Isosceles $\triangle ABC$ with $\overline{CA} \cong \overline{CB}$

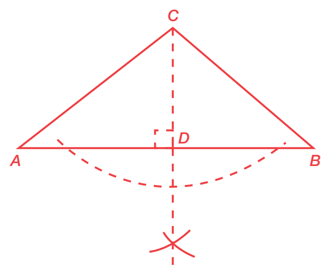
Prove: $\overline{CD}$ bisects $\angle ACB$

4. Prove the Isosceles Triangle Vertex Angle Theorem.

Statements	Reasons
1. $\overline{CA} \cong \overline{CB}$	1. Given
2. $\overline{CD}$ is an altitude	2. Construction
3. $\overline{CD} \perp \overline{AB}$	3. Definition of altitude
4. $\angle CDA$ and $\angle CDB$ are right angles	4. Definition of perpendicular
5. $\triangle CDA$ and $\triangle CDB$ are right triangles	5. Definition of right triangle
6. $\angle A \cong \angle B$	6. Isosceles Triangle Base Angle Theorem
7. $\triangle CDA \cong \triangle CDB$	7. HA Congruence Theorem
8. $\angle ACD \cong \angle BCD$	8. CPCTC
9. $\overline{CD}$ bisects $\angle ACB$	9. Definition of bisect

The **Isosceles Triangle Perpendicular Bisector Theorem** states: "The altitude from the vertex angle of an isosceles triangle is the perpendicular bisector of the base."

5. Draw and label a diagram you can use to help you prove the Isosceles Triangle Perpendicular Bisector Theorem. State the "Given" and "Prove" statements.



Given: Isosceles $\triangle ABC$ with $\overline{CA} \cong \overline{CB}$

Prove: $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$

6. Prove the Isosceles Triangle Perpendicular Bisector Theorem.

Statements	Reasons
1. $\overline{CA} \cong \overline{CB}$	1. Given
2. $\overline{CD}$ is an altitude	2. Construction
3. $\overline{CD} \perp \overline{AB}$	3. Definition of altitude
4. $\overline{AD} \cong \overline{BD}$	4. Isosceles Triangle Base Theorem
5. $\overline{CD}$ bisects $\overline{AB}$	5. Definition of bisect
6. $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$	6. Definition of perpendicular bisector

Problem 2

Students use a two-column proof to prove the Isosceles Triangle Altitude to Congruent Sides Theorem which states: "In an isosceles triangle, the altitudes to the congruent sides are congruent" and the Isosceles Triangle Angle Bisector to Congruent Sides Theorem which states: "In an isosceles triangle, the angle bisectors to the congruent sides are congruent."

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

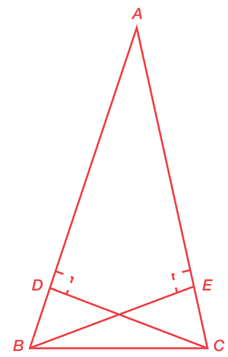
Guiding Questions for Share Phase, Questions 1 through 4

- How would you reword the theorems as conditional statements?
- What is the hypothesis or given information?
- What is the conclusion or prove statement?
- Do the two triangles share a common side or angle?
- Which congruence theorems are helpful when proving this theorem?
- How can CPCTC be helpful when proving this theorem?
- Is it helpful creating a paragraph proof before creating a two-column proof? Why or why not?

PROBLEM 2 More Isosceles Triangle Theorems

The **Isosceles Triangle Altitude to Congruent Sides Theorem** states: "In an isosceles triangle, the altitudes to the congruent sides are congruent."

1. Draw and label a diagram you can use to help you prove this theorem. State the "Given" and "Prove" statements.



Given: $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{AC}$, $\overline{CD}$ and $\overline{BE}$ are altitudes

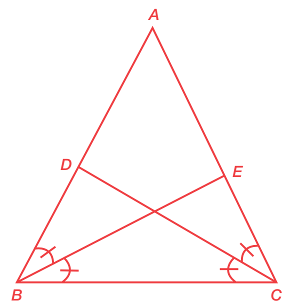
Prove: $\overline{CD} \cong \overline{BE}$

2. Prove the Isosceles Triangle Altitude to Congruent Sides Theorem.

Statements	Reasons
1. $\overline{AB} \cong \overline{AC}$	1. Given
2. $\overline{CD}$ and $\overline{BE}$ are altitudes	2. Given
3. $\overline{CD} \perp \overline{AB}$ and $\overline{BE} \perp \overline{AC}$	3. Definition of altitude
4. $\angle CDB$ and $\angle BEC$ are right angles	4. Definition of perpendicular
5. $\triangle CDB$ and $\triangle BEC$ are right triangles	5. Definition of right triangle
6. $\angle ABC \cong \angle ACB$	6. Isosceles Triangle Base Angle Theorem
7. $\overline{BC} \cong \overline{BC}$	7. Reflexive Property of Congruence
8. $\triangle CDB \cong \triangle BEC$	8. HA
9. $\overline{CD} \cong \overline{BE}$	9. CPCTC

The **Isosceles Triangle Angle Bisector to Congruent Sides Theorem** states:
"In an isosceles triangle, the angle bisectors to the congruent sides are congruent."

3. Draw and label a diagram you can use to help you prove this theorem. State the "Given" and "Prove" statements.



Given: $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{AC}$,
$\overline{BE}$ bisects $\angle ABC$, $\overline{CD}$ bisects $\angle ACB$

Prove: $\overline{CD} \cong \overline{BE}$

4. Prove the Isosceles Triangle Angle Bisector to Congruent Sides Theorem.

Statements	Reasons
1. $\overline{AB} \cong \overline{AC}$	1. Given
2. $\overline{BE}$ bisects $\angle ABC$	2. Given
3. $\angle ABE \cong \angle EBC$	3. Definition of angle bisector
4. $\overline{CD}$ bisects $\angle ACB$	4. Given
5. $\angle ACD \cong \angle DCB$	5. Definition of angle bisector
6. $\angle ABC \cong \angle ACB$	6. Isosceles Triangle Base Angle Theorem
7. $\overline{BC} \cong \overline{BC}$	7. Reflexive Property of Congruence
8. $\triangle CDB \cong \triangle BEC$	8. ASA
9. $\overline{CD} \cong \overline{BE}$	9. CPCTC

Talk the Talk

Students use the theorems in this lesson to answer questions.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Talk the Talk

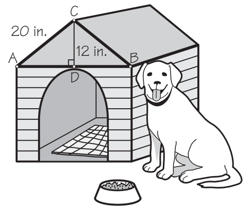
1. Solve for the width of the dog house.

   $\overline{CD} \perp \overline{AB}$

   $\overline{AC} \cong \overline{BC}$

   $CD = 12''$

   $AC = 20''$



   $AD^2 + 12^2 = 20^2$

   $AD^2 + 144 = 400$

   $AD^2 = 256$

   $AD = \sqrt{256} = 16''$

   The width of the dog house is 32 inches.

Use the theorems you have just proven to answer each question about isosceles triangles.

2. What can you conclude about an altitude drawn from the vertex angle to the base?

   I can conclude that the altitude bisects the vertex angle, and is the perpendicular bisector of the base.

3. What can you conclude about the altitudes to the congruent sides?

   I can conclude that the altitudes drawn to the congruent sides are congruent.

4. What can you conclude about the angle bisectors to the congruent sides?

   I can conclude that the angle bisectors drawn to the congruent sides are congruent.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Design a problem about this barn using one or more theorems from this lesson. Give the problem to your partner and ask them to solve it.

Answers will vary.

Given: $\triangle BAR$ is isosceles with $\overline{AB} \cong \overline{AR}$

$\overline{AN}$ is an altitude

$AB = 16$ feet

$AN = 10$ feet

Determine the width of the barn.

$16^2 = 10^2 + BN^2$

$BN^2 = 256 - 100$

$BN^2 = 156$

$BN = \sqrt{156} \approx 12.5$ feet

The width of the barn is approximately $2(12.6) = 26.2$ feet.

6

Making Some Assumptions

6.4

Inverse, Contrapositive, Direct Proof, and Indirect Proof

LEARNING GOALS

In this lesson, you will:

- Write the inverse and contrapositive of a conditional statement.
- Differentiate between direct and indirect proof.
- Use indirect proof.

KEY TERMS

- inverse
- contrapositive
- direct proof
- indirect proof or proof by contradiction
- Hinge Theorem
- Hinge Converse Theorem

ESSENTIAL IDEAS

- When using a conditional statement in the form "If *p*, then *q*", to state the converse, you reverse the hypothesis, *p*, and the conclusion, *q*. To state the inverse, you negate both parts. To state the contrapositive, you reverse and negate each part.
- An indirect proof, or proof by contradiction, uses the contrapositive by assuming the conclusion is false and then showing that one of the statements must be false.
- The Hinge Theorem states: "If two sides of one triangle are congruent to two sides of another triangle and the included angle of the first is larger than the included angle of the second, then the third side of the first is longer than the third side of the second."
- The Hinge Converse Theorem states: "If two sides of one triangle are congruent to two sides of another triangle and the third side of the first is longer than the third side of the second, then the included angle of the first is larger than the included angle of the second."

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-CO Congruence

Prove geometric theorems

10. Prove theorems about triangles.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

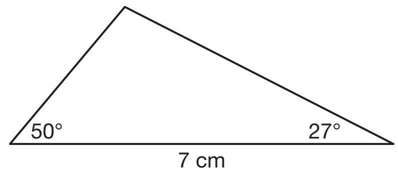
1. Use geometric shapes, their measures, and their properties to describe objects.

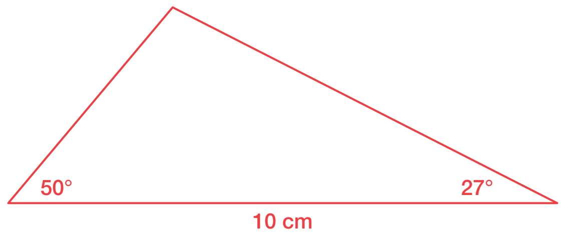
Overview

Students analyze conditional statements and write their inverses and contrapositives. Proof by contradiction, or indirect proof, is introduced and contrasted with direct proof. Students prove the Hinge Theorem and its converse using indirect proof.

Warm Up

1. Use a ruler and protractor to redraw this triangle increasing the 7 cm side to 10 cm.



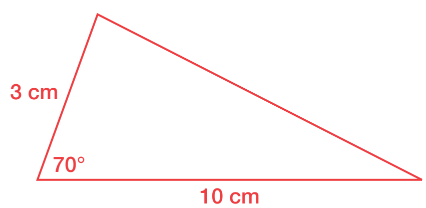


2. How did increasing the length of the side affect the measure of the third angle?
   Lengthening the side had no effect on the measure of the opposite angle.

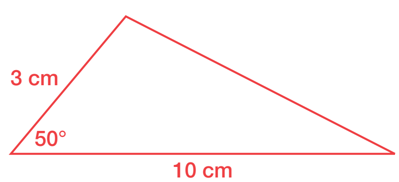
3. How did increasing the length of the side affect the length of the other two sides?
   Lengthening the side caused the other two sides to increase in length.

4. Is the triangle you drew similar to the original triangle? Explain.
   The two triangles are similar by the AA Similarity Postulate.

5. Use a ruler and protractor to draw a triangle that has two sides with the measure of 3 cm and 10 cm and an included angle with the measure of 70°.



6. Use a ruler and protractor to draw a triangle that has two sides with the measure of 3 cm and 10 cm and an included angle with the measure of 50°.



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7. Without measuring, can you predict which triangle in Questions 5 and 6 have the longest third side? Explain your reasoning.

The third side in the triangle in Question 5 has the longest side. The angle determining the length of the third side in Question 5 is greater than the angle determining the length of the third side in Question 6. Therefore, the side opposite the greater angle is longer.

8. Use a ruler to measure the third sides. Is your prediction correct?

Yes, my prediction is correct.

6

6.4

Making Some Assumptions

Inverse, Contrapositive, Direct Proof, and Indirect Proof

LEARNING GOALS

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- Write the inverse and contrapositive of a conditional statement.
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KEY TERMS

- inverse
- contrapositive
- direct proof
- indirect proof or proof by contradiction
- Hinge Theorem
- Hinge Converse Theorem

The Greek philosopher Aristotle greatly influenced our understanding of physics, linguistics, politics, and science. He also had a great influence on our understanding of logic. In fact, he is often credited with the earliest study of formal logic, and he wrote six works on logic which were compiled into a collection known as the *Organon*. These works were used for many years after his death. There were a number of philosophers who believed that these works of Aristotle were so complete that there was nothing else to discuss regarding logic. These beliefs lasted until the 19th century when philosophers and mathematicians began thinking of logic in more mathematical terms.

Aristotle also wrote another book, *Metaphysics*, in which he makes the following statement: "To say of what is that it is not, or of what is not that it is, is falsehood, while to say of what is that it is, and of what is not that it is not, is truth."

What is Aristotle trying to say here, and do you agree? Can you prove or disprove this statement?

Problem 1

The converse, inverse, and the contrapositive of a conditional statement are defined. Students use several conditional statements to write the converse, inverse, and contrapositive. They conclude that when the conditional statement is true, the inverse may be true or false and the contrapositive is always true. When the conditional statement is false, the inverse may be true or false and the contrapositive is always false.

Grouping

- Ask a student to read aloud the information. Discuss as a class.
- Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

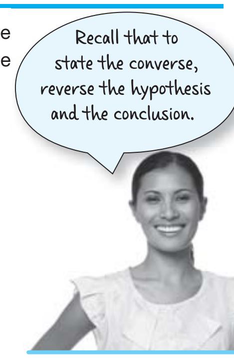
Guiding Questions for Share Phase, Questions 1 through 6

- What is a square?
- What is a rectangle?
- Are all squares considered rectangles? Why or why not?
- Are all rectangles considered squares? Why or why not?
- What is an integer?
- Is zero an integer?
- Is zero considered even or odd?
- What are even integers?
- Are all even integers divisible by two? Why or why not?

6

PROBLEM 1 The Inverse and Contrapositive

Every conditional statement written in the form "If p, then q" has three additional conditional statements associated with it: the converse, the *contrapositive*, and the *inverse*. To state the **inverse**, negate the hypothesis and the conclusion. To state the **contrapositive**, negate the hypothesis and conclusion, and reverse them.



Conditional Statement	If p, then q.
Converse	If q, then p.
Inverse	If not p, then not q.
Contrapositive	If not q, then not p.

1. If a quadrilateral is a square, then the quadrilateral is a rectangle.

   a. Hypothesis p:
   A quadrilateral is a square.

   b. Conclusion q:
   The quadrilateral is a rectangle.

   c. Is the conditional statement true? Explain your reasoning.
   The conditional statement is true. All squares are also rectangles.

   d. Not p:
   A quadrilateral is not a square.

   e. Not q:
   The quadrilateral is not a rectangle.

   f. Inverse:
   If a quadrilateral is not a square, then the quadrilateral is not a rectangle.

   g. Is the inverse true? Explain your reasoning.
   The inverse is false. A quadrilateral that is not a square could still be a rectangle.

   h. Contrapositive:
   If a quadrilateral is not a rectangle, then the quadrilateral is not a square.

   i. Is the contrapositive true? Explain your reasoning.
   The contrapositive is true. All squares are rectangles, so if the quadrilateral is not a rectangle, then it cannot be a square.

- What is a five-sided polygon called?
- What is a six-sided polygon called?
- When the conditional statement was true, was the inverse always true?
- When the conditional statement was false, was the inverse always false?
- When the conditional statement was true, was the contrapositive always true?
- When the conditional statement was false, was the contrapositive always false?

2. If an integer is even, then the integer is divisible by two.

a. Hypothesis p:

An integer is even.

b. Conclusion q:

The integer is divisible by two.

c. Is the conditional statement true? Explain your reasoning.

The conditional statement is true. All even integers are divisible by two.

d. Not p:

An integer is not even.

e. Not q:

The integer is not divisible by two.

f. Inverse:

If an integer is not even, then the integer is not divisible by two.

g. Is the inverse true? Explain your reasoning.

The inverse is true. If an integer is not even, then it is odd and odd integers are not divisible by two.

h. Contrapositive:

If an integer is not divisible by two, then the integer is not even.

i. Is the contrapositive true? Explain your reasoning.

The contrapositive is true. If an integer is not divisible by two, then it is odd, and an odd integer is not even.

3. If a polygon has six sides, then the polygon is a pentagon.

a. Hypothesis *p*:

A polygon has six sides.

b. Conclusion *q*:

The polygon is a pentagon.

c. Is the conditional statement true? Explain your reasoning.

The conditional statement is false. A polygon with six sides is a hexagon.

d. Not *p*:

A polygon does not have six sides.

e. Not *q*:

The polygon is not a hexagon.

f. Inverse:

If a polygon does not have six sides, then the polygon is not a hexagon.

g. Is the inverse true? Explain your reasoning.

The inverse is false. A polygon that does not have six sides could have five sides, which would make it a pentagon.

h. Contrapositive:

If a polygon is not a hexagon, then the polygon does not have six sides.

i. Is the contrapositive true? Explain your reasoning.

The contrapositive is false. A polygon that is not a pentagon could be a hexagon, which would have six sides.

6

4. If two lines intersect, then the lines are perpendicular.

   a. Hypothesis p:
      Two lines intersect.

   b. Conclusion q:
      The lines are perpendicular.

   c. Is the conditional statement true? Explain your reasoning.
      The conditional statement is false. Not all intersecting lines are perpendicular.

   d. Not p:
      Two lines do not intersect.

   e. Not q:
      The lines are not perpendicular.

   f. Inverse:
      If two lines do not intersect, then the lines are not perpendicular.

   g. Is the inverse true? Explain your reasoning.
      The inverse is true. Two lines that do not intersect cannot be perpendicular.

   h. Contrapositive:
      If two lines are not perpendicular, then the lines do not intersect.

   i. Is the contrapositive true? Explain your reasoning.
      The contrapositive is false. Two lines that are not perpendicular could intersect.

5. What do you notice about the truth value of a conditional statement and the truth value of its inverse?
   If a conditional statement is true, then its inverse may be either true or false. If a conditional statement is false, then its inverse may be either true or false.

6. What do you notice about the truth value of a conditional statement and the truth value of its contrapositive?
   If a conditional statement is true, then its contrapositive is true. If a conditional statement is false, then its contrapositive is also false.

Problem 2

Students are introduced to an indirect proof, or proof by contradiction, which uses the contrapositive by assuming the conclusion is false. Begin all indirect proofs by assuming the conclusion is false. Use the givens to arrive at a contradiction, thus completing the proof. An example of an indirect proof is provided. Students create two indirect proofs.

Grouping

- Discuss the information and worked example as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Note

The example of a two-column indirect proof can be used for a focused classroom discussion. Asking students to describe the differences and similarities between direct and indirect proof is a good place to start the discussion. Share with students that when a "not" statement appears in the problem set up, that is often viewed as a hint to use an indirect proof model.

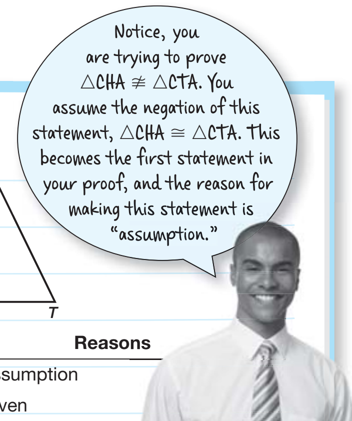
6

PROBLEM 2 Proof by Contradiction

All of the proofs up to this point were *direct proofs*. A **direct proof** begins with the given information and works to the desired conclusion directly through the use of givens, definitions, properties, postulates, and theorems.

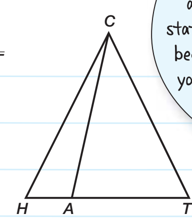
An **indirect proof**, or **proof by contradiction**, uses the contrapositive. If you prove the contrapositive true, then the original conditional statement is true. Begin by assuming the conclusion is false, and use this assumption to show one of the given statements is false, thereby creating a contradiction.

Let's look at an example of an indirect proof.



Given: In $\triangle CHT$, $\overline{CH} \cong \overline{CT}$,
$\overline{CA}$ does not bisect $\overline{HT}$

Prove: $\triangle CHA \ncong \triangle CTA$



Statements	Reasons
1. $\triangle CHA \cong \triangle CTA$	1. Assumption
2. $\overline{CA}$ does not bisect $\overline{HT}$	2. Given
3. $\overline{HA} \cong \overline{TA}$	3. CPCTC
4. $\overline{CA}$ bisects $\overline{HT}$	4. Definition of bisect
5. $\triangle CHA \cong \triangle CTA$ is false	5. This is a contradiction. Step 4 contradicts step 2; the assumption is false
6. $\triangle CHA \ncong \triangle CTA$ is true	6. Proof by contradiction

In step 5, the "assumption" is stated as "false." The reason for making this statement is "contradiction."

In an indirect proof:

- State the assumption; use the negation of the conclusion, or "Prove" statement.
- Write the givens.
- Write the negation of the conclusion.
- Use the assumption, in conjunction with definitions, properties, postulates, and theorems, to prove a given statement is false, thus creating a contradiction.

Hence, your assumption leads to a contradiction; therefore, the assumption must be false. This proves the contrapositive.

Guiding Questions for Share Phase, Questions 1 and 2

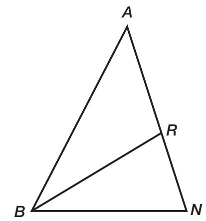
- What is the negation of the conclusion or the assumption?
- Do the triangles share a common side or common angle?
- What theorem is helpful in proving the triangles congruent?
- Which given statement will help arrive at a contradiction?
- Which corresponding part will contradict a given statement?
- Which corresponding parts will help arrive at a contradiction?

1. Create an indirect proof of the following.

Given: $\overline{BR}$ bisects $\angle ABN$,

$\angle BRA \ncong \angle BRN$

Prove: $\overline{AB} \ncong \overline{NB}$



Statements	Reasons
1. $\overline{AB} \cong \overline{NB}$	1. Assumption
2. $\overline{BR}$ bisects $\angle ABN$	2. Given
3. $\angle ABR \cong \angle NBR$	3. Definition of angle bisector
4. $\overline{BR} \cong \overline{BR}$	4. Reflexive Property of Congruence
5. $\triangle ABR \cong \triangle NBR$	5. SAS
6. $\angle BRA \cong \angle BRN$	6. CPCTC
7. $\angle BRA \ncong \angle BRN$	7. Given
8. $\overline{AB} \cong \overline{NB}$ is false	8. This is a contradiction. Step 7 contradicts step 6; the assumption is false
9. $\overline{AB} \ncong \overline{NB}$ is true	9. Proof by contradiction

2. Create an indirect proof to show that a triangle cannot have more than one right angle.

Given $\triangle ABC$. Begin by assuming $\triangle ABC$ has two right angles ($m\angle A = 90°$ and $m\angle B = 90°$). By the Triangle Sum Theorem, $m\angle A + m\angle B + m\angle C = 180°$. By substitution, $90° + 90° + m\angle C = 180°$, so by subtraction, $m\angle C = 0°$, which contradicts the given, $\triangle ABC$ is a triangle.

Problem 3

Students prove the Hinge Theorem and the Hinge Converse Theorem using two-column indirect proofs. To prove one angle measure is less than a second angle measure, two cases must be proven. It must be proven that the first angle is not congruent to the second angle and the first angle measure is not greater than the second angle measure. Each proof requires two cases.

Grouping

- Ask a student to read aloud the information. Discuss as a class.
- Have students complete Question 1 with a partner. Then have students share their responses as a class.

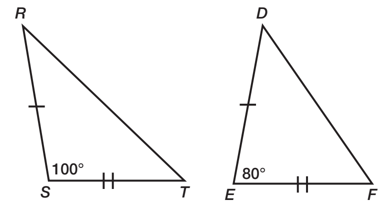
Guiding Questions for Share Phase, Question 1

- What is the negation of the conclusion or the assumption?
- Do the triangles share a common side or common angle?
- What theorem is helpful in proving the triangles congruent?
- Which given statement will help arrive at a contradiction?
- Which corresponding part will contradict a given statement?

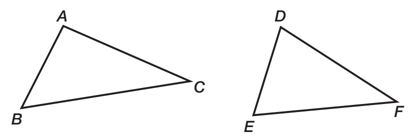
PROBLEM 3 Hinge Theorem and Its Converse

The **Hinge Theorem** states: "If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first pair is larger than the included angle of the second pair, then the third side of the first triangle is longer than the third side of the second triangle."

In the two triangles shown, notice that $RS = DE$, $ST = EF$, and $\angle S > \angle E$. The Hinge Theorem says that $RT > DF$.



1. Use an indirect proof to prove the Hinge Theorem.



Given: $AB = DE$
$AC = DF$
$m\angle A > m\angle D$

Prove: $BC > EF$

Negating the conclusion, $BC > EF$, means that either BC is equal to EF, or BC is less than EF. Therefore, this theorem must be proven for both cases.

Case 1: $BC = EF$

Case 2: $BC < EF$

- Which corresponding parts will help arrive at a contradiction?
- In the second case, what can be concluded using the Triangle Inequality Theorem?
- How is the Triangle Inequality Theorem used to arrive at a contradiction?

a. Write the indirect proof for Case 1, $BC = EF$.

Statements	Reasons
1. $BC \ngtr EF$, so $BC = EF$	1. Assumption
2. $AB = DE$	2. Given
3. $AC = DF$	3. Given
4. $\triangle ABC \cong \triangle DEF$	4. SSS Congruency Theorem
5. $\angle A \cong \angle D$	5. CPCTC
6. $m\angle A > m\angle D$	6. Given
7. $BC = EF$ is false	7. This is a contradiction. Step 6 contradicts step 5; the assumption is false
8. $BC > EF$ is true	8. Proof by contradiction

b. Write the indirect proof for Case 2, $BC < EF$.

Statements	Reasons
1. $BC \ngtr EF$, so $BC < EF$	1. Assumption
2. $AB = DE$	2. Given
3. $AC = DF$	3. Given
4. $\angle A < \angle D$	4. Triangle Inequality Theorem
5. $\angle A > \angle D$	5. Given
6. $BC < EF$ is false	6. This is a contradiction. Step 5 contradicts step 4; the assumption is false
7. $BC > EF$ is true	7. Proof by contradiction

6

Grouping

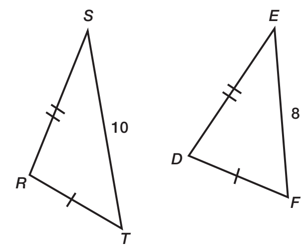
- Ask a student to read aloud the information. Discuss as a class.
- Have students complete Question 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

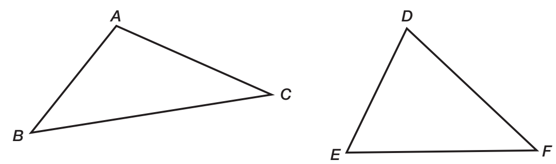
- What is the negation of the conclusion or the assumption?
- Do the triangles share a common side or common angle?
- What theorem is helpful in proving the triangles congruent?
- Which given statement will help arrive at a contradiction?
- Which corresponding part will contradict a given statement?
- Which corresponding parts will help arrive at a contradiction?
- In the second case, what can be concluded using the Hinge Theorem?
- How is the Triangle Inequality Theorem used to arrive at a contradiction?

The **Hinge Converse Theorem** states: "If two sides of one triangle are congruent to two sides of another triangle and the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first pair of sides is larger than the included angle of the second pair of sides."

In the two triangles shown, notice that $RT = DF$, $RS = DE$, and $ST > EF$. The Hinge Converse Theorem guarantees that $m\angle R > m\angle D$.



2. Create an indirect proof to prove the Hinge Converse Theorem.



Given: $AB = DE$
$AC = DF$
$BC > EF$

Prove: $m\angle A > m\angle D$

This theorem must be proven for two cases.

Case 1: $m\angle A = m\angle D$

Case 2: $m\angle A < m\angle D$

a. Create an indirect proof for Case 1, $m\angle A = m\angle D$.

Statements	Reasons
1. $\angle A \ngtr \angle D$ so $\angle A = \angle D$	1. Assumption
2. $AB = DE$	2. Given
3. $AC = DF$	3. Given
4. $\triangle ABC \cong \triangle DEF$	4. SAS Congruency Theorem
5. $BC \cong EF$	5. CPCTC
6. $BC > EF$	6. Given
7. $m\angle A = m\angle D$ is false	7. This is a contradiction. Step 6 contradicts step 5; the assumption is false
8. $m\angle A > m\angle D$ is true	8. Proof by contradiction

b. Create an indirect proof for Case 2, $m\angle A < m\angle D$.

Statements	Reasons
1. $m\angle A \ngtr m\angle D$ so $m\angle A < m\angle D$	1. Assumption
2. $AB = DE$	2. Given
3. $AC = DF$	3. Given
4. $BC < EF$	4. Hinge Theorem
5. $BC > EF$	5. Given
6. $m\angle A < m\angle D$ is false	6. This is a contradiction. Step 5 contradicts step 4; the assumption is false
7. $m\angle A > m\angle D$ is true	7. Proof by contradiction

Problem 4

Student apply the Hinge Theorem to solve problems.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

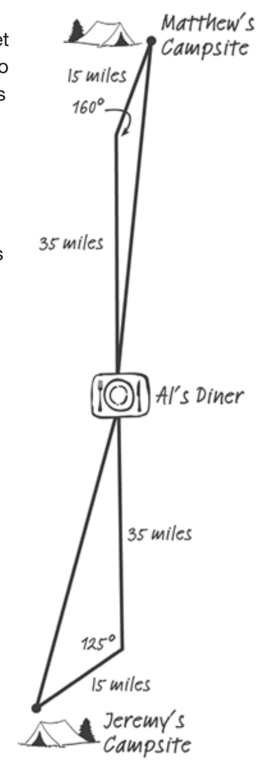
Guiding Questions for Share Phase, Questions 1 through 3

- How does the Hinge Theorem apply to this situation?
- How does the length *AH* compare to the length *WP*?
- Was the Hinge Theorem or the Hinge Converse Theorem helpful in determining the relationship between the length of *AH* and the length of *WP*?
- How does the measure of angle *E* compare to the measure of angle *R*?
- Was the Hinge Theorem or the Hinge Converse Theorem helpful in determining the relationship between the measure of angle *E* and the measure of angle *R*?

PROBLEM 4 It All Hinges on Reason

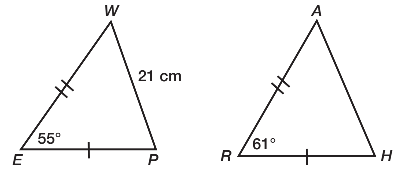
1. Matthew and Jeremy's families are going camping for the weekend. Before heading out of town, they decide to meet at Al's Diner for breakfast. During breakfast, the boys try to decide which family will be further away from the diner "as the crow flies." "As the crow flies" is an expression based on the fact that crows, generally fly straight to the nearest food supply.

   Matthew's family is driving 35 miles due north and taking an exit to travel an additional 15 miles northeast. Jeremy's family is driving 35 miles due south and taking an exit to travel an additional 15 miles southwest. Use the diagram shown to determine which family is further from the diner. Explain your reasoning.

   Because the included angle in the triangle representing Matthew's distance from Al's Diner is larger than the included angle in the triangle representing Jeremy's distance from Al's Diner, the direct distance from Al's Diner to Matthew's campsite is longer by the Hinge Theorem.

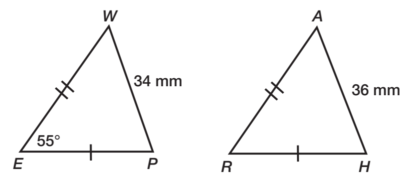
2. Which of the following is a possible length for AH: 20 cm, 21 cm, or 24 cm? Explain your choice.



Because the included angle in $\triangle ARH$ is larger than the included angle in $\triangle WEP$, the third side $\overline{AH}$ must be longer than $\overline{WP}$ by the Hinge Theorem.

So, of the three choices, the only possible length for $\overline{AH}$ is 24 cm.

3. Which of the following is a possible angle measure for $\angle ARH$: 54°, 55° or 56°? Explain your choice.

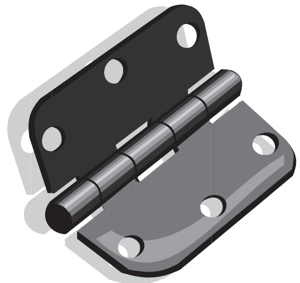


Because the third side in $\triangle ARH$ is longer than the third side in $\triangle WEP$, the included angle R must be larger than the included angle E by the Hinge Converse Theorem.

So, of the three choices, the only possible measure for angle R is 56°.

Be prepared to share your solutions and methods.

Check for Students' Understanding



Explain how this picture of a hinge is related to the Hinge Theorem.

Consider a door held onto the door frame with a hinge similar to the one pictured. One plate on the hinge is screwed into the door and the second plate is screwed into the door frame. As the door opens, the plates move closer together. As the plates move closer together, the measure of the angle formed by the plates decreases. As the door closes, the plates move further apart, the distance between the plates increase and the measure of the angle formed by the two plates increases. The plates represent sides of a triangle and the angle formed by the plates is the included angle. As the measure of the angle increases, the door closes and the length of the side opposite the angle increases. As the measure of the angle decreases, the door opens and the length of the side opposite the angle decreases.

6

Chapter 6 Summary

KEY TERMS

- corresponding parts of congruent triangles are congruent (CPCTC) (6.2)
- vertex angle of an isosceles triangle (6.3)
- inverse (6.4)
- contrapositive (6.4)
- direct proof (6.4)
- indirect proof or proof by contradiction (6.4)

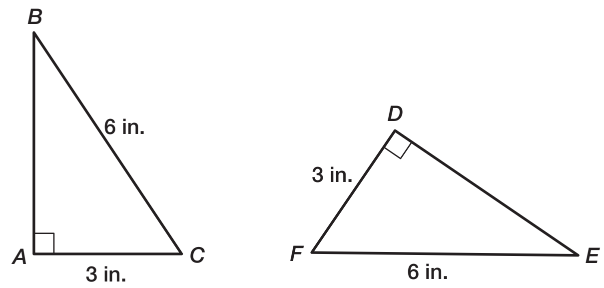
THEOREMS

- Hypotenuse-Leg (HL) Congruence Theorem (6.1)
- Leg-Leg (LL) Congruence Theorem (6.1)
- Hypotenuse-Angle (HA) Congruence Theorem (6.1)
- Leg-Angle (LA) Congruence Theorem (6.1)
- Isosceles Triangle Base Angle Theorem (6.2)
- Isosceles Triangle Base Angle Converse Theorem (6.2)
- Isosceles Triangle Base Theorem (6.3)
- Isosceles Triangle Vertex Angle Theorem (6.3)
- Isosceles Triangle Perpendicular Bisector Theorem (6.3)
- Isosceles Triangle Altitude to Congruent Sides Theorem (6.3)
- Isosceles Triangle Angle Bisector to Congruent Sides Theorem (6.3)
- Hinge Theorem (6.4)
- Hinge Converse Theorem (6.4)

6.1 Using the Hypotenuse-Leg (HL) Congruence Theorem

The Hypotenuse-Leg (HL) Congruence Theorem states: "If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent."

Example



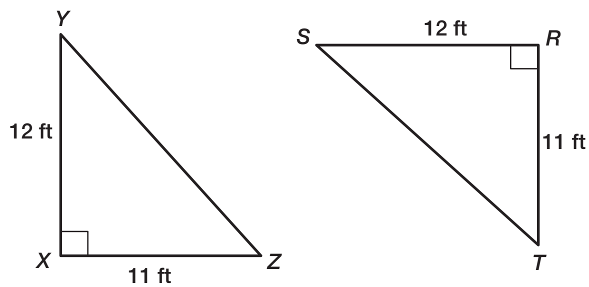
$\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$, and angles A and D are right angles, so $\triangle ABC \cong \triangle DEF$.

6

6.1 Using the Leg-Leg (LL) Congruence Theorem

The Leg-Leg (LL) Congruence Theorem states: "If two legs of one right triangle are congruent to two legs of another right triangle, then the triangles are congruent."

Example

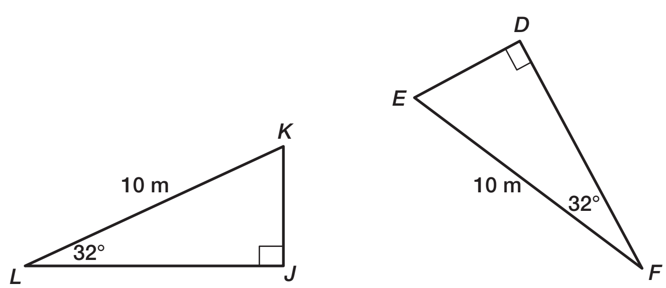


$\overline{XY} \cong \overline{RS}$, $\overline{XZ} \cong \overline{RT}$, and angles X and R are right angles, so $\triangle XYZ \cong \triangle RST$.

6.1 Using the Hypotenuse-Angle (HA) Congruence Theorem

The Hypotenuse-Angle (HA) Congruence Theorem states: "If the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and acute angle of another right triangle, then the triangles are congruent."

Example



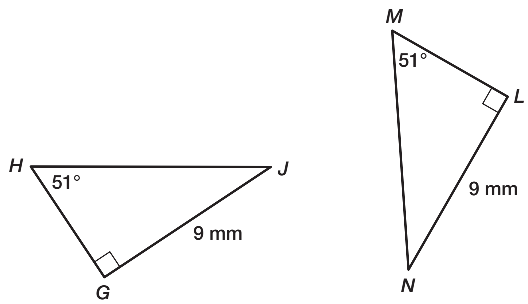
$\overline{KL} \cong \overline{EF}$, $\angle L \cong \angle F$, and angles J and D are right angles, so $\triangle JKL \cong \triangle DEF$.

6

6.1 Using the Leg-Angle (LA) Congruence Theorem

The Leg-Angle (LA) Congruence Theorem states: "If a leg and an acute angle of one right triangle are congruent to the leg and an acute angle of another right triangle, then the triangles are congruent."

Example



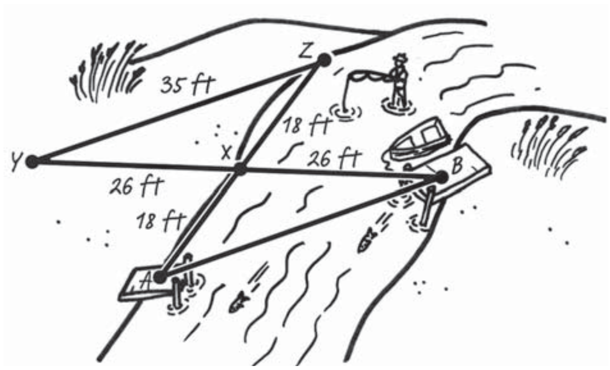
$\overline{GN} \cong \overline{LN}$, $\angle H \cong \angle M$, and angles G and L are right angles, so $\triangle GHJ \cong \triangle LMN$.

6.2 Using CPCTC to Solve a Problem

If two triangles are congruent, then each part of one triangle is congruent to the corresponding part of the other triangle. In other words, "corresponding parts of congruent triangles are congruent," which is abbreviated CPCTC. To use CPCTC, first prove that two triangles are congruent.

Example

You want to determine the distance between two docks along a river. The docks are represented as points *A* and *B* in the diagram below. You place a marker at point *X*, because you know that the distance between points *X* and *B* is 26 feet. Then, you walk horizontally from point *X* and place a marker at point *Y*, which is 26 feet from point *X*. You measure the distance between points *X* and *A* to be 18 feet, and so you walk along the river bank 18 feet and place a marker at point *Z*. Finally, you measure the distance between *Y* and *Z* to be 35 feet.

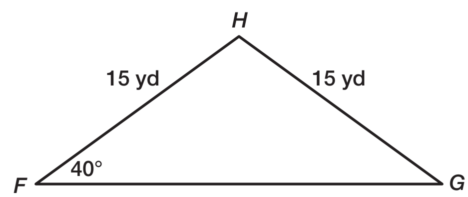


From the diagram, segments *XY* and *XB* are congruent and segments *XA* and *XZ* are congruent. Also, angles *YXZ* and *BXA* are congruent by the Vertical Angles Congruence Theorem. So, by the Side-Angle-Side (SAS) Congruence Postulate, $\triangle YXZ \cong \triangle BXA$. Because corresponding parts of congruent triangles are congruent (CPCTC), segment *YZ* must be congruent to segment *BA*. The length of segment *YZ* is 35 feet. So, the length of segment *BA*, or the distance between the docks, is 35 feet.

6.2 Using the Isosceles Triangle Base Angle Theorem

The Isosceles Triangle Base Angle Theorem states: "If two sides of a triangle are congruent, then the angles opposite these sides are congruent."

Example

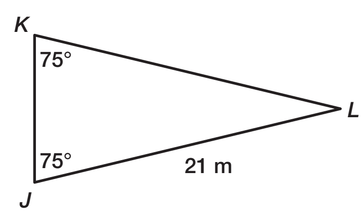


$\overline{FH} \cong \overline{GH}$, so $\angle F \cong \angle G$, and the measure of angle G is 40°.

6.2 Using the Isosceles Triangle Base Angle Converse Theorem

The Isosceles Triangle Base Angle Converse Theorem states: "If two angles of a triangle are congruent, then the sides opposite these angles are congruent."

Example

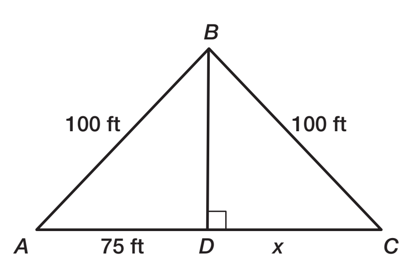


$\angle J \cong \angle K$, $\overline{JL} \cong \overline{KL}$, and the length of side KL is 21 meters.

6.3 Using the Isosceles Triangle Base Theorem

The Isosceles Triangle Base Theorem states: "The altitude to the base of an isosceles triangle bisects the base."

Example

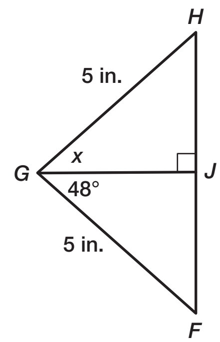


$CD = AD$, so $x = 75$ feet.

6.3 Using the Isosceles Triangle Vertex Angle Theorem

The Isosceles Triangle Base Theorem states: "The altitude to the base of an isosceles triangle bisects the vertex angle."

Example

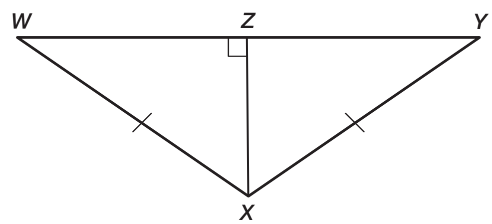


$m\angle FGJ = m\angle HGJ$, so $x = 48°$.

6.3 Using the Isosceles Triangle Perpendicular Bisector Theorem

The Isosceles Triangle Perpendicular Bisector Theorem states: "The altitude from the vertex angle of an isosceles triangle is the perpendicular bisector of the base."

Example

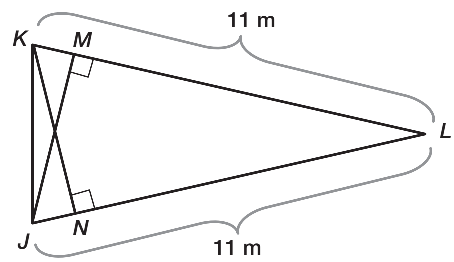


$\overline{WY} \perp \overline{XZ}$ and $WZ = YZ$

6.3 Using the Isosceles Triangle Altitude to Congruent Sides Theorem

The Isosceles Triangle Perpendicular Bisector Theorem states: "In an isosceles triangle, the altitudes to the congruent sides are congruent."

Example

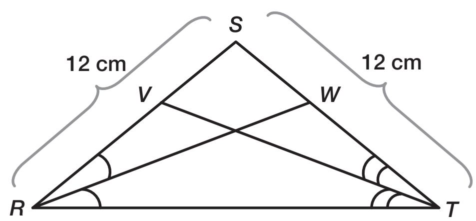


$\overline{KN} \cong \overline{JM}$

6.3 Using the Isosceles Triangle Bisector to Congruent Sides Theorem

The Isosceles Triangle Perpendicular Bisector Theorem states: "In an isosceles triangle, the angle bisectors to the congruent sides are congruent."

Example



$\overline{RW} \cong \overline{TV}$

6.4 Stating the Inverse and Contrapositive of Conditional Statements

To state the inverse of a conditional statement, negate both the hypothesis and the conclusion. To state the contrapositive of a conditional statement, negate both the hypothesis and the conclusion and then reverse them.

Conditional Statement: If p, then q.

Inverse: If not p, then not q.

Contrapositive: If not q, then not p.

Example

Conditional Statement: If a triangle is equilateral, then it is isosceles.

Inverse: If a triangle is not equilateral, then it is not isosceles.

Contrapositive: If a triangle is not isosceles, then it is not equilateral.

6.4 Writing an Indirect Proof

In an indirect proof, or proof by contradiction, first write the givens. Then, write the negation of the conclusion. Then, use that assumption to prove a given statement is false, thus creating a contradiction. Hence, the assumption leads to a contradiction, therefore showing that the assumption is false. This proves the contrapositive.

Example

Given: Triangle *DEF*

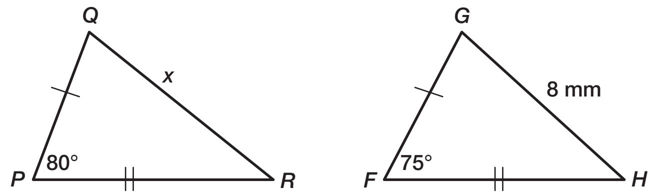
Prove: A triangle cannot have more than one obtuse angle.

Given $\triangle DEF$, assume that $\triangle DEF$ has two obtuse angles. So, assume $m\angle D = 91°$ and $m\angle E = 91°$. By the Triangle Sum Theorem, $m\angle D + m\angle E + m\angle F = 180°$. By substitution, $91° + 91° + m\angle F = 180°$, and by subtraction, $m\angle F = -2°$. But it is not possible for a triangle to have a negative angle, so this is a contradiction. This proves that a triangle cannot have more than one obtuse angle.

6.4 Using the Hinge Theorem

The Hinge Theorem states: "If two sides of one triangle are congruent to two sides of another triangle and the included angle of the first pair is larger than the included angle of the second pair, then the third side of the first triangle is longer than the third side of the second triangle."

Example

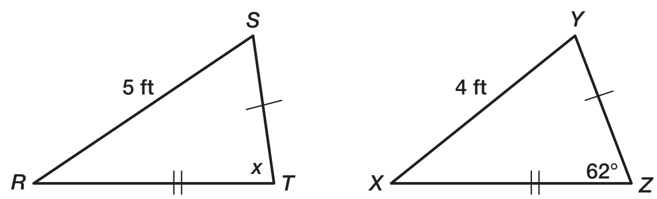


$QR > GH$, so $x > 8$ millimeters.

6.4 Using the Hinge Converse Theorem

The Hinge Converse Theorem states: "If two sides of one triangle are congruent to two sides of another triangle and the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first pair of sides is larger than the included angle of the second pair of sides."

Example



$m\angle T > m\angle Z$, so $x > 62°$.

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