

Similarity Through Transformations

4



The first U.S. mass-produced marbles were made in Akron, Ohio in the early 1890's. But marbles have been around a lot longer than that. Some of the earliest records of marbles are from ancient Rome and Egypt!



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Chapter 4 Overview

This chapter addresses similar triangles and establishes similar triangle theorems as well as theorems about proportionality. The chapter leads student exploration of the conditions for triangle similarity and opportunities for applications of similar triangles.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
4.1	Dilating Triangles to Create Similar Triangles	G.SRT.1.a G.SRT.1.b G.SRT.2 G.SRT.5 G.MG.1	1	<p>This lesson uses the dilation of triangles to introduce similar triangles.</p> <p>Questions ask students to describe the relationship between the image and the pre-image, and to determine the scale factor. Students will determine whether given information proves two triangles similar. They will also use transformations to map one triangle to another triangle.</p>	X				
4.2	Similar Triangle Theorems	G.SRT.3 G.SRT.5	2	<p>This lesson explores the Angle-Angle Similarity Theorem, Side-Angle-Side Similarity Theorem, and Side-Side-Side Similarity Theorem.</p> <p>Students will use construction tools to explore the similarity theorems. Questions then ask students to use these theorems to determine if two triangles are similar.</p>	X		X	X	
4.3	Theorems About Proportionality	G.GPE.7 G.SRT.4 G.SRT.5	2	<p>This lesson explores the Angle Bisector/Proportional Side Theorem, the Triangle Proportionality Theorem and its Converse, the Proportional Segments Theorem, and the Triangle Midsegment Theorem.</p> <p>Questions ask students to prove these theorems, and to apply the theorems to determine segment lengths.</p>	X		X		

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
4.4	More Similar Triangles	G.SRT.4 G.SRT.5 G.MG.1	2	<p>This lesson establishes the relationships created when an altitude is drawn to the hypotenuse of a right triangle.</p> <p>Questions ask students to prove the Right Triangle/Altitude Similarity Theorem and to use the geometric mean to solve for unknown lengths.</p>	X				
4.5	Proving the Pythagorean Theorem and the Converse of the Pythagorean Theorem	G.SRT.4	1	<p>The lesson explores the Pythagorean Theorem and its Converse.</p> <p>Questions ask students to prove the Pythagorean Theorem using similar triangles, and to prove its Converse with algebraic reasoning.</p>	x				
4.6	Application of Similar Triangles	G.SRT.5 G.MG.1	1	<p>The lesson uses similar triangles to determine indirect measurements.</p> <p>Students will set up proportions to solve problem situations.</p>	x	x			

Skills Practice Correlation for Chapter 4

Lesson		Problem Set	Objectives
4.1	Dilating Triangles to Create Similar Triangles		Vocabulary
		1 – 4	Determine the dimensions of rectangles and their dilations and express the corresponding measurements as ratios
		5 – 8	Determine the scale factor of triangle dilations on the coordinate plane
		9 – 12	Graph dilations of triangles given the pre-image, scale factor, and center of dilation
		13 – 20	Determine the coordinates of dilated triangles given the vertex coordinates, scale factor, and center of dilation
4.2	Similar Triangle Theorems		Vocabulary
		1 – 6	Explain why two given triangles are similar
		7 – 12	Determine the additional information needed to prove similarity given two triangles and a similarity theorem
		13 – 20	Determine whether two triangles are similar
4.3	Theorems About Proportionality		Vocabulary
		1 – 8	Calculate the length of indicated segments in triangles
		9 – 12	Use triangle proportionality to solve problems
		13 – 20	Write statements about given triangles that can be justified using the Proportional Segments Theorem, Triangle Proportionality Theorem, or the Converse of the Triangle Proportionality Theorem
		21 – 24	Use the Triangle Proportionality Theorem and the Proportional Segments Theorem to determine missing values
		25 – 28	Use given information to write statements about triangles that can be justified using the Triangle Midsegment Theorem
		29 – 30	Compare the measures of segments of inscribed figures

Lesson		Problem Set	Objectives
4.4	More Similar Triangles		Vocabulary
		1 – 4	Construct an altitude to the hypotenuse of right triangles
		5 – 8	Use similarity statements to write corresponding sides of triangles as proportions
		9 – 12	Use the Right Triangle/Altitude Similarity Theorem to write similarity statements given triangles
		13 – 18	Solve for x given triangle diagrams
		19 – 22	Solve for x , y , and z given triangle diagrams
		23 – 26	Use triangle similarity to solve problems
4.5	Proving the Pythagorean Theorem and the Converse of the Pythagorean Theorem	1 – 6	Write Given and Prove statements for given triangles and theorems
		7 – 12	Prove theorems given diagrams
4.6	Application of Similar Triangles		Vocabulary
		1 – 4	Explain why given triangles are similar
		5 – 12	Use indirect measurement to calculate missing distances

Big and Small

Dilating Triangles to Create Similar Triangles

LEARNING GOALS

In this lesson, you will:

- Prove that triangles are similar using geometric theorems.
- Prove that triangles are similar using transformations.

ESSENTIAL IDEAS

- Dilation is a transformation that enlarges or reduces a pre-image to create a similar image.
- The center of dilation is a fixed point at which the figure is either enlarged or reduced.
- The scale factor is the ratio of the distance from the center of dilation to a point on the image to the distance from the center of dilation to the corresponding point on the pre-image.
- When the scale factor is greater than one, the dilation is an enlargement. When the scale factor is between zero and one, the dilation is a reduction.
- In dilation, the coordinates of a point (x, y) transform into the coordinates (kx, ky) where k is the scale factor. If $0 < k < 1$, the dilation is a reduction. If $k > 1$, the dilation is an enlargement.

KEY TERM

- similar triangles

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G.SRT Similarity, Right Triangles, and Trigonometry

Understand similarity in terms of similarity transformations

1. Verify experimentally the properties of dilations given by a center and a scale factor:
 - a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
 - b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

Prove theorems involving similarity

5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

G.MG Modeling with Geometry

Apply geometric concepts in modeling situations

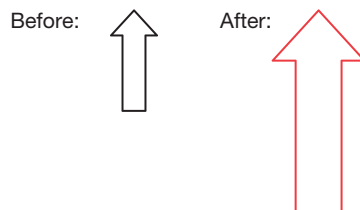
1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

Students perform dilations on triangles both on and off of the coordinate plane. Some problem situations require the use of construction tools and others require measurement tools. They explore the ratios formed as a result of dilation and discover the importance of scale factor. Similar triangles are defined and students explore the relationships between the corresponding sides and between the corresponding angles. Students then use similarity statements to draw similar triangles, and describe the transformations necessary to map one triangle onto another for an alternate approach to showing similarity.

Warm Up

1. Redraw the given figure such that it is twice its size.



2. Redraw the given figure such that it is half its size.



3. When you redrew the figures in Questions 1 and 2, did the shape of the figure change?
No, the shape of the figure did not change.
4. When you redrew the figures in Questions 1 and 2, did the size of the figure change?
Yes, the size of the figure changed.

Big and Small

Dilating Triangles to Create Similar Triangles

LEARNING GOALS

In this lesson, you will:

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- Prove that triangles are similar using transformations.

KEY TERM

- similar triangles

Making hand shadow puppets has a long history. This activity goes back to ancient China and India. Before the invention of television, or even radio, hand shadows were used to entertain people by telling stories.

Today, you can find tutorials online that will show you how to create really complicated and interesting shadow puppets. Groups of people can get together and create entire landscapes and scenes—all with the shadows made by their hands!

Problem 1

Students are given a scenario involving marbles and a dilation factor of 2. They conclude that this dilation takes a line that does not pass through the center of dilation to a line parallel to the original line.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

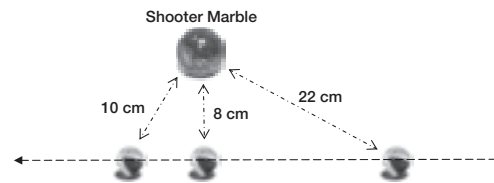
- How many marbles did you draw in the second row?
- Should the distance between the shooter marble and the marbles in the first row be greater than, less than, or the same as the distance from the marbles in the first row to the marbles in the second row? Why?
- If the distance between the shooter marble and the marbles in the first row was more than the distance from the marbles in the first row to the marbles in the second row, what would this tell you about the dilation factor?
- If the distance between the shooter marble and the marbles in the first row was less than the distance from the marbles in the first row to the marbles in the second row, what would this tell you about the dilation factor?

PROBLEM 1 A New Marbles Game?

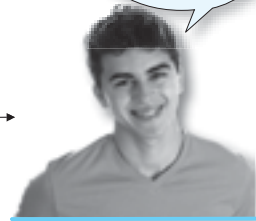


The game of marbles is played in a circle. The goal is to knock your opponents marbles outside of the circle by flicking a shooter marble at the other marbles in the circle. The shooter marble is often larger than the other marbles.

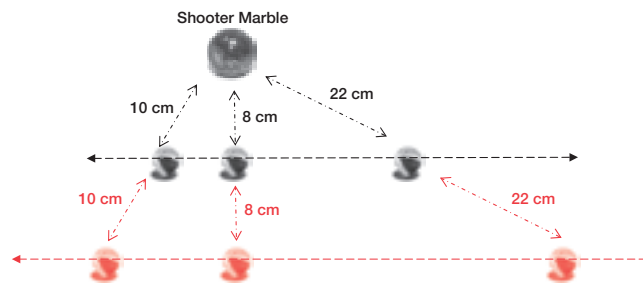
John placed a shooter marble near three smaller marbles as shown.



Can you describe the 3 smaller marbles as being collinear?



1. Draw another row of three marbles under the first row of marbles using a dilation factor of 2 with the shooter marble as the center of the dilation.



- If the distance between the shooter marble and the marbles in the first row was the same as the distance from the marbles in the first row to the marbles in the second row, what would this tell you about the dilation factor?
- Does the first row of marbles appear to be parallel to the second row of marbles?
- If the line that is dilated by a factor of 2 passes through the center of the dilation, how does this affect the distance the line is dilated?

2. Explain how you located the positions of each additional marble. Label the distances between the marbles in the first row and in the second row.

To locate each additional marble, I extended the line segment through the original marble and placed the new marble on this line so that the distance from the shooter marble to the new marble was twice the distance from the shooter marble to the original marble.

3. Describe the relationship between the first and second rows of marbles.

The marbles in each row appear to be collinear. The lines containing the marbles appear to be parallel.

4. Use a ruler to compare the length of the line segments connecting each original marble to the line segments connecting each additional marble.

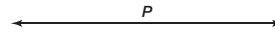
The line segments connecting each additional marble are twice the length of the line segments connecting each original marble.

5. What can you conclude about dilating a line that does not pass through the center of a dilation?

A dilation takes a line that does not pass through the center to a line parallel to the original line.



6. Consider line P . How could you show a dilation of this line by a factor of 2 using P as the center of dilation? Explain your reasoning.



The line would not change. Because the line passes through the center of dilation, its distance from the center is 0. Dilating a distance of 0 by a factor of 2 still gives you a distance of 0.

Problem 2

The scenario uses cutout figures taped to craft sticks and a flashlight to create large shadows puppets. A rabbit (pre-image) is drawn illuminated by a flashlight and the image of the rabbit is an enlargement of the pre-image. The initial questions focus on the size and shape of the rabbit as a result of the enlargement. Next, a triangle and its shadow are drawn and the image is larger than the pre-image. Students use measuring tools to compare the lengths of the corresponding sides and the measures of the corresponding angles. They conclude all pairs of corresponding sides have the same ratio and all pairs of the corresponding angles are congruent, therefore the triangles formed are similar. Students also conclude an enlargement is associated with a scale factor greater than one and a reduction is associated with a scale factor greater than zero but less than one.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- Is the size of the shadow puppet larger, smaller, or the same as the size of the poster board rabbit?

PROBLEM 2 Making a Shadow

You have volunteered to help at the children's booth at an art festival. The children that visit the booth will be able to create objects, like animals or people, out of poster board and craft sticks. Then, they will use a flashlight to create shadow puppets. Your job is to show the children how to use a flashlight and a wall to make their own puppet show.



1. How does the size of the shadow puppet compare to the size of the object made out of poster board and craft sticks?

The size of the shadow puppet is larger than the size of the object.

2. How does the shape of the shadow puppet compare to the shape of the object made out of poster board and craft sticks?

The shape of the shadow is the same as the shape of the puppet.



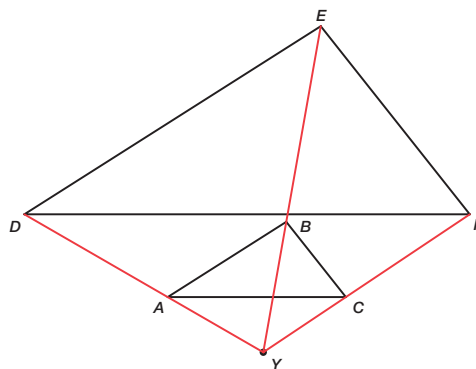
3. Do you think that the shadow is a transformation of the object? Why or why not?

Yes. The shadow is a transformation because the points that make up the outline of the object are mapped to form the outline of the shadow puppet.

- Is it possible for the shape of the shadow puppet to be different than the shape of the poster board rabbit? How?
- Could the shadow puppet be described as a dilation of the poster board puppet? How so?



Consider $\triangle ABC$, $\triangle DEF$, and point Y . Imagine that point Y is the flashlight and $\triangle DEF$ is the shadow of $\triangle ABC$.



Grouping

Have students complete Questions 4 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 7

- Is triangle ABC an enlargement of triangle DEF or is triangle DEF an enlargement of triangle ABC ?
- Which vertices are considered corresponding vertices in triangles ABC and DEF ?
- Which sides are considered corresponding sides in triangles ABC and DEF ?
- Why do you suppose the ratios are equal?
- If the ratios were not equal, what information would this provide?

4. Draw the line segments \overline{YD} , \overline{YE} , \overline{YF} on the figure shown. These line segments show the path of the light from the flashlight.

Describe what these line segments connect.

These line segments connect the corresponding vertices of the triangles.

5. Use a metric ruler to determine the actual lengths of \overline{YA} , \overline{YB} , \overline{YC} , \overline{YD} , \overline{YE} , and \overline{YF} to the nearest tenth of a centimeter.

The length of \overline{YA} is 2.0 centimeters.

The length of \overline{YB} is 2.4 centimeters.

The length of \overline{YC} is 1.8 centimeters.

The length of \overline{YD} is 5.0 centimeters.

The length of \overline{YE} is 6.0 centimeters.

The length of \overline{YF} is 4.5 centimeters.

6. Express the ratios $\frac{YD}{YA}$, $\frac{YE}{YB}$, and $\frac{YF}{YC}$ as decimals.

$$\frac{YD}{YA} = \frac{5.0}{2.0} = 2.5$$

$$\frac{YE}{YB} = \frac{6.0}{2.4} = 2.5$$

$$\frac{YF}{YC} = \frac{4.5}{1.8} = 2.5$$

Guiding Questions for Discuss Phase, Questions 8 through 11

- What is a dilation?
- How is the center of dilation determined?
- Is the resulting image of a dilation always an image that is larger or smaller than the pre-image?

7. What do you notice about the ratios?

The ratios are equal.

Remember that a ratio is a comparison of two numbers that uses division. The ratio of two numbers a and b (b cannot be zero), can be written in three ways.

a to b
 $a : b$
 $\frac{a}{b}$



8. Use a protractor to measure the corresponding angles in the triangles. What can you conclude?

The corresponding angles are congruent because the measures of the corresponding angles are the same.



9. What is the relationship between the image and pre-image in a dilation?

The image and pre-image are similar.

The shadow created by the flashlight is a dilation.

10. In any dilation:

a. how will the corresponding angles in the image and pre-image compare?

The corresponding angles will be congruent.

b. how will the ratios of the lengths of the corresponding sides compare?

The ratios will be equal.



11. What is the center of the dilation shown in Question 4?

The center of dilation is point Y .

Grouping

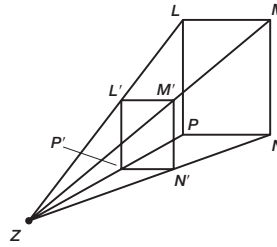
Have students complete Questions 12 and 13 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 12 and 13

- What are the pairs of corresponding sides of the rectangles?
- What are the pairs of corresponding angles of the rectangles?
- If the ratios are equal, what information does this provide?
- Why do you suppose the center of dilation is used to determine the ratio relationships?
- Is the ratio and the scale factor the same thing?
- If the ratios were not the same, does that mean different scale factors were used?
- If different scale factors were used, could it be considered a dilation?



12. Rectangle $L'M'N'P'$ is a dilation of rectangle $LMNP$. The center of dilation is point Z .



Recall that an image that is the result of a dilation is not always larger than the pre-image.



- a. Use a metric ruler to determine the actual lengths of \overline{ZL} , \overline{ZN} , \overline{ZM} , \overline{ZP} , $\overline{ZL'}$, $\overline{ZN'}$, $\overline{ZM'}$, and $\overline{ZP'}$ to the nearest tenth of a centimeter. Then express the ratios $\frac{ZL'}{ZL}$, $\frac{ZN'}{ZN}$, $\frac{ZM'}{ZM}$, and $\frac{ZP'}{ZP}$ as decimals.

$$\frac{ZL'}{ZL} = \frac{3}{5} = 0.6 \quad \frac{ZM'}{ZM} = \frac{3.6}{6} = 0.6 \quad \frac{ZN'}{ZN} = \frac{2.7}{4.5} = 0.6 \quad \frac{ZP'}{ZP} = \frac{1.8}{3} = 0.6$$

- b. What do you notice about the ratios?

The ratios are equal.

13. How does the image compare to the pre-image when:

- a. the scale factor is greater than 1?

When the scale factor is greater than 1, the image is larger than the pre-image.

- b. the scale factor is less than 1?

When the scale factor is less than 1, the image is smaller than the pre-image.



The ratios that you wrote in Questions 6 and 12 are the scale factors of each dilation.



Problem 3

A triangle is drawn in the first quadrant. Students use a compass and a straightedge to dilate the figure with a scale factor of 2. Students identify the coordinates of corresponding vertices and compare the coordinates of the image and pre-image. Next, the pre-image and image of a triangle are given. Students identify the coordinates of the vertices and conclude the scale factor used is one-half.

Grouping

- Ask a student to read aloud the information before Question 1.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

4

Guiding Questions for Share Phase, Questions 1 through 4

- What information does the scale factor provide about the diagram?
- Does the dilation result in an enlargement? How do you know?
- Where is the origin?
- How did you calculate the distance from the origin to point G' ?
- Could the same dilation have been performed without the use of the coordinate plane? Why or why not?

PROBLEM 3 Larger or Smaller?

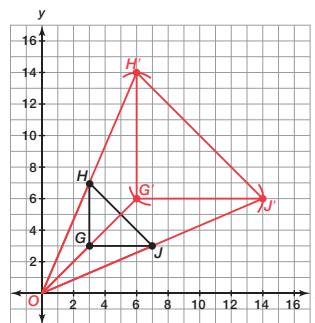


You can use your compass and a straightedge to perform a dilation. Consider $\triangle GHJ$ shown on the coordinate plane. You will dilate the triangle by using the origin as the center and by using a scale factor of 2.



1. How will the distance from the center of dilation to a point on the image of $\triangle G'H'J'$ compare to the distance from the center of dilation to a corresponding point on $\triangle GHJ$? Explain your reasoning.

The distance from the center of dilation to a point on the image of $\triangle G'H'J'$ will be two times the distance from the center of dilation to a corresponding point on $\triangle GHJ$ because the scale factor is 2:1.



2. For each vertex of $\triangle GHJ$, draw a ray that starts at the origin and passes through the vertex.

See diagram.

3. Use the duplicate segment construction to locate the vertices of $\triangle G'H'J'$.

See diagram.



4. List the coordinates of the vertices of $\triangle GHJ$ and $\triangle G'H'J'$. How do the coordinates of the image compare to the coordinates of the pre-image?

The coordinates of the pre-image are $G(3, 3)$, $H(3, 7)$, and $J(7, 3)$.

The coordinates of the image are $G'(6, 6)$, $H'(6, 14)$, and $J'(14, 6)$.

Each coordinate of the image is two times the corresponding coordinate of the pre-image.

Grouping

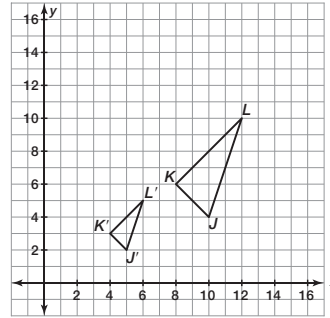
Have students complete Questions 5 and 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 and 6

- How are the coordinates of the image and pre-image helpful in determining the scale factor used in the dilation?
- How is the scale factor helpful when determining the coordinates of the image and pre-image?



5. Triangle $J'K'L'$ is a dilation of $\triangle JKL$. The center of dilation is the origin.



- a. List the coordinates of the vertices of $\triangle JKL$ and $\triangle J'K'L'$. How do the coordinates of the image compare to the coordinates of the pre-image?

The coordinates of $\triangle JKL$ are $J(10, 4)$, $K(8, 6)$, and $L(12, 10)$

The coordinates of $\triangle J'K'L'$ are $J'(5, 2)$, $K'(4, 3)$, and $L'(6, 5)$.

Each coordinate of the image is one-half of the corresponding coordinate of the pre-image.

- b. What is the scale factor of the dilation? Explain your reasoning.

The scale factor is $\frac{1}{2}$ because the distance from the center of dilation to a point on the image is one-half of the distance from the center of dilation to a corresponding point on the pre-image.

- c. How do you think you can use the scale factor to determine the coordinates of the vertices of an image?

The coordinates of the vertices of an image can be found by multiplying the coordinates of the vertices of the pre-image by the scale factor.



6. Use coordinate notation to describe the dilation of point (x, y) when the center of dilation is at the origin using a scale factor of k .

Using coordinate notation, the point (x, y) dilated can be described as (kx, ky) .

Problem 4

The definition of similar triangles is provided. Students use a similarity statement to draw the situation and list all of the pairs of congruent angles and proportional sides. Next, they are given a diagram and explain the conditions necessary to show the triangles are similar. Then students decide if given information is enough to determine a similarity relationship.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

4

Guiding Questions for Share Phase, Questions 1 and 2

- How is the similarity statement helpful when drawing the triangles?
- Is there more than one correct diagram for this similarity statement?
- Must all three pairs of angles be proven congruent to show the triangles are similar? Why or why not?
- If only two pairs of corresponding angles are congruent, are the triangles similar?
- If the ratios of corresponding sides are equal, are the triangles similar?

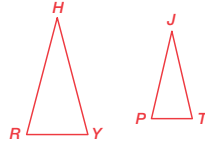
PROBLEM 4 Geometric Theorems and Similar Triangles

Similar triangles are triangles that have all pairs of corresponding angles congruent and all corresponding sides are proportional. Similar triangles have the same shape but not always the same size.



1. Triangle $HR Y \sim$ Triangle JPT

Draw a diagram that illustrates this similarity statement and list all of the pairs of congruent angles and all of the proportional sides.

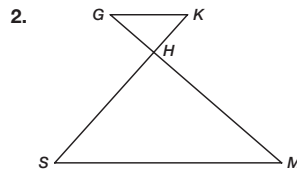


$$\angle H \cong \angle J$$

$$\angle R \cong \angle P$$

$$\angle Y \cong \angle T$$

$$\frac{HR}{JP} = \frac{RY}{PT} = \frac{HY}{JT}$$



- a. What conditions are necessary to show triangle $G H K$ is similar to triangle $M H S$?

$$\angle G \cong \angle M$$

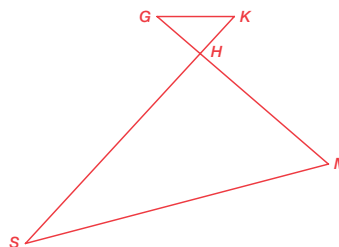
$$\angle G H K \cong \angle M H S$$

$$\angle K \cong \angle S$$

$$\frac{GH}{MH} = \frac{HK}{HS} = \frac{GK}{MS}$$

- What is the Vertical Angle Theorem?
- How is the Vertical Angle Theorem helpful when proving the triangles similar?

- b. Suppose $4GH = HM$. Determine whether this given information is enough to prove that the two triangles are similar. Explain why you think they are similar or provide a counter-example if you think the triangles are not similar.



This is not enough information to prove similarity. A counter-example could be made by extending SH , while increasing the measure of $\angle M$ as shown. Point S could be located such that $KH = 6SH$. Now triangles GKH and MHS satisfy the given conditions but are not similar.

- c. Suppose \overline{GK} is parallel to \overline{MS} . Determine whether this given information is enough to prove that the two triangles are similar. Explain why you think they are similar or provide a counter-example if you think the triangles are not similar.

This is enough information to prove similarity. Using the Alternate Interior Angle Theorem, it can be proven that $\angle G \cong \angle M$, and $\angle K \cong \angle S$. Using the Vertical Angle Theorem, it can be proven that $\angle GHK \cong \angle MHS$. If the three pair of corresponding angles are congruent, the triangles must be the same shape, so the triangles are similar.



- d. Suppose $\angle G \cong \angle S$. Determine whether this given information is enough to prove that the two triangles are similar. Explain why you think they are similar or provide a counter-example if you think the triangles are not similar.

Using the Vertical Angle Theorem, it can be proven that $\angle GHK \cong \angle MHS$. Using the Triangle Sum Theorem, it can be proven that $\angle K \cong \angle M$. If the three pairs of corresponding angles are congruent, the triangles must be the same shape, so the triangles are similar.

Problem 5

Students describe the sequences of transformations necessary to map one triangle onto another triangle. The problem situations were taken from Problem 4 and already established as having similar relationships.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

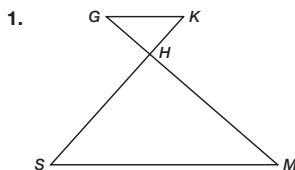
Guiding Questions for Share Phase, Question 1

- Which transformations are used in this situation?
- Did your classmates use the same transformations?
- Does it matter which transformation is done first?
- Does the order the transformations are performed change the result?

4

PROBLEM 5 Transformations and Similar Triangles

In each of the following situations you have concluded that given the information provided, the triangles could be proven similar using geometric theorems. The triangles could also be proven similar using a sequence of transformations. These transformations result in mapping one triangle to the other.



- a. Suppose \overline{KG} is parallel to \overline{MS} .

Describe a sequence of transformations that maps one triangle to the other triangle.

First, rotate triangle SHM so that angle SHM coincides with angle KHG . Then the image of side \overline{MS} under this rotation is parallel to the original side \overline{MS} , so the new side \overline{MS} is still parallel to side \overline{KG} . Now, apply a dilation about point H that moves the vertex M to point G . This dilation moves \overline{MS} to a line segment through G parallel to the previous line segment \overline{MS} . We already know that \overline{KG} is parallel to \overline{MS} , so the dilation must move \overline{MS} onto \overline{KG} . Since the dilation moves S to a point on \overline{HK} and on \overline{KG} , point S must move to point K . Therefore, the rotation and dilation map the triangle SHM to the triangle KHG .

- b. Suppose $\angle G \cong \angle S$.

Describe a sequence of transformations that maps one triangle to the other triangle.

First, draw the bisector of angle KHM , and reflect the triangle MHS across this angle bisector. This maps \overline{HM} onto \overline{HK} ; and since reflections preserve angles, it also maps \overline{HS} onto \overline{HG} . Since angle HMS is congruent to angle HKG , we also know that the image of side \overline{MS} is parallel to side \overline{KG} . Therefore, if we apply a dilation about point H that takes the new point M to K , then the new line segment \overline{MS} will be mapped onto \overline{KG} , by the same reasoning used in part (a). Therefore, the new point S is mapped to point G , and thus the triangle HMS is mapped to triangle HKG . So, triangle HMS is similar to triangle HKG .

- c. Suppose $(KH)(GH) = (SH)(MH)$

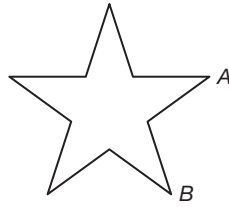
Describe a sequence of transformations that maps one triangle to the other triangle.

First, rewrite this proportion as $(KH)(SH) = (GH)(MH)$. Let the scale factor, $k = (KH)(SH)$. Suppose we rotate the triangle SHM 180 degrees about point H , so that the angle SHM coincides with angle KHG . Then, dilate the triangle SHM by a factor of k about the center H . This dilation moves point S to point K , since $k(SH) = KH$, and moves point M to point G , since $k(MH) = GH$. Then, since the dilation fixes point H , and dilations take line segments to line segments, we see that the triangle SHM is mapped to triangle KHG . So, the original triangle DXC is similar to triangle AXE .

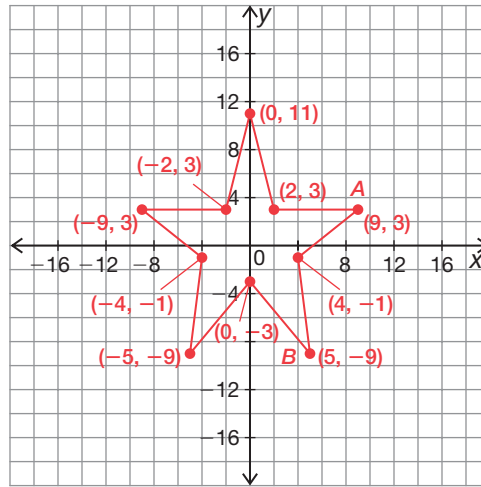


Be prepared to share your solutions and methods.

Check for Students' Understanding



- Graph the given coordinates to transfer the star onto the coordinate plane.
 $(0, 11)$, $(2, 3)$, $(9, 3)$, $(4, -1)$, $(5, -9)$, $(0, -3)$, $(-5, -9)$, $(-4, -1)$, $(-9, 3)$, $(-2, 3)$



- Connect the coordinates to form the star and also label vertices A and B on the star.
- Without graphing, if the star is dilated with its center of dilation at the origin and a scale factor of 3, what are the new coordinates of vertex A ?
 $(27, 9)$
- Without graphing, if the star is dilated with its center of dilation at the origin and a scale factor of 3, what are the new coordinates of the vertex B ?
 $(15, -27)$
- Without graphing, if the star is dilated with its center of dilation at the origin and a scale factor of 0.5, what are the new coordinates of the vertex A ?
 $(4.5, 1.5)$
- Without graphing, if the star is dilated with its center of dilation at the origin and a scale factor of 0.5, what are the new coordinates of the vertex B ?
 $(2.5, -4.5)$

Similar Triangles or Not?

Similar Triangle Theorems

LEARNING GOALS

In this lesson, you will:

- Use constructions to explore similar triangle theorems.
- Explore the Angle-Angle (AA) Similarity Theorem.
- Explore the Side-Side-Side (SSS) Similarity Theorem.
- Explore the Side-Angle-Side (SAS) Similarity Theorem.

ESSENTIAL IDEAS

- The Angle-Angle Similarity Theorem states: “If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.”
- The Side-Side-Side Similarity Theorem states: “If all three corresponding sides of two triangles are proportional, then the triangles are similar.”
- The Side-Angle-Side Similarity Theorem states: “If two of the corresponding sides of two triangles are proportional and the included angles are congruent, then the triangles are similar.”

KEY TERMS

- Angle-Angle Similarity Theorem
- Side-Side-Side Similarity Theorem
- included angle
- included side
- Side-Angle-Side Similarity Theorem

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G.SRT Similarity, Right Triangles, and Trigonometry

Understand similarity in terms of similarity transformations

3. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

Prove theorems involving similarity

5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Overview

Students explore shortcuts for proving triangles similar using construction tools and measuring tools. Then, the Angle-Angle Similarity Theorem, Side-Side-Side Similarity Theorem, and Side-Angle-Side Similarity Theorem are stated and students use these theorems to determine the similarity of triangles. The terms included angle and included side are defined in this lesson.

Warm Up

1. Draw an example of two polygons that have corresponding angles congruent, but do not have corresponding sides proportional.

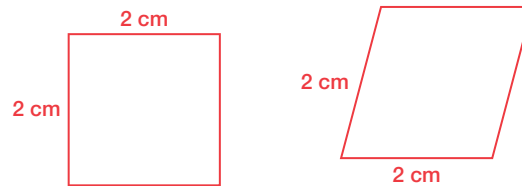


The two rectangles have congruent corresponding angles, but the sides are not proportional.

2. Are the two polygons similar? Why or why not?

The polygons are not similar, because the corresponding sides are not proportional.

3. Draw an example of two polygons that have corresponding sides proportional, but do not have corresponding angles congruent.



The length of each side in the square and the rhombus are equal, so the sides must be proportional, but the corresponding angles are not congruent.

4. Are the two polygons similar? Why or why not?

The polygons are not similar, because the corresponding angles are not congruent.

Similar Triangles or Not?

Similar Triangle Theorems

LEARNING GOALS

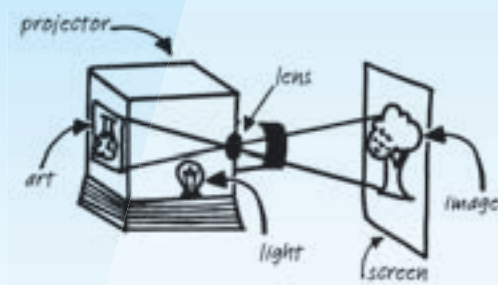
In this lesson, you will:

- Use constructions to explore similar triangle theorems.
- Explore the Angle-Angle (AA) Similarity Theorem.
- Explore the Side-Side-Side (SSS) Similarity Theorem.
- Explore the Side-Angle-Side (SAS) Similarity Theorem.

KEY TERMS

- Angle-Angle Similarity Theorem
- Side-Side-Side Similarity Theorem
- included angle
- included side
- Side-Angle-Side Similarity Theorem

An art projector is a piece of equipment that artists have used to create exact copies of artwork, to enlarge artwork, or to reduce artwork. A basic art projector uses a light bulb and a lens within a box. The light rays from the art being copied are collected onto a lens at a single point. The lens then projects the image of the art onto a screen as shown.



If the projector is set up properly, the triangles shown will be similar polygons. You can show that these triangles are similar without measuring all of the side lengths and all of the interior angles.

Problem 1

Students construct a triangle using two given angles to determine it is enough information to conclude the two triangles are similar. The Angle-Angle Similarity Theorem states: "If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar."

Grouping

- Ask students to read Question 2. Discuss as a class.
- Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

4

Guiding Questions for Discuss Phase

- Could you have identified all of the corresponding congruent angles and all of the corresponding proportional sides using only the similarity statement without the given diagrams?
- How does the length of side DE compare to the length of side $D'E'$?
- What ratios are formed by the corresponding side lengths?
- Are all of the ratios the same?
- Are the ratios formed by the pairs of corresponding sides equal?

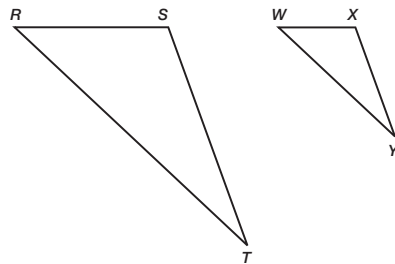
PROBLEM 1 Using Two Angles



In the previous lesson, you used transformations to prove that triangles are similar when their corresponding angles are congruent and their corresponding sides are proportional. In this problem, you will explore the similarity of two triangles using construction tools.

1. Identify all of the corresponding congruent angles and all of the corresponding proportional sides using the similar triangles shown.

$$\triangle RST \sim \triangle WXY$$



The corresponding congruent angles are $\angle R \cong \angle W$, $\angle S \cong \angle X$, $\angle T \cong \angle Y$.

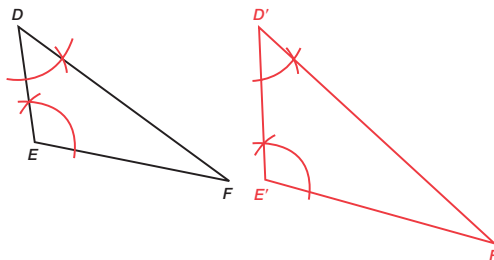
The corresponding proportional sides are $\frac{RS}{WX} = \frac{ST}{XY} = \frac{TR}{YW}$.



You can conclude that two triangles are similar if you are able to prove that all of their corresponding angles are congruent and all of their corresponding sides are proportional.

Let's use constructions to see if you can use fewer pairs of angles or fewer pairs of sides to show that triangles are similar.

2. Construct triangle $D'E'F'$ using only $\angle D$ and $\angle E$ in triangle DEF as shown. Make all the corresponding side lengths of triangle $D'E'F'$ different from the side lengths of triangle DEF .



Guiding Questions for Share Phase, Questions 3 and 4

- Is the third interior angle of each triangle congruent?
- If two interior angles of a triangle are congruent to two interior angles of a second triangle, does the third interior angle in each triangle have to be congruent as well? Why or why not?



3. Measure the angles and sides of triangle $D'E'F'$ and triangle DEF . Are the two triangles similar? Explain your reasoning.

Yes. The third pair of angles is congruent and the corresponding sides are proportional.

Did everyone construct the same triangle?

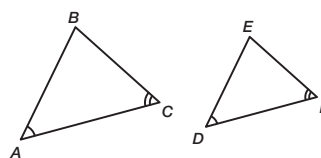


4. In triangles DEF and $D'E'F'$, two pairs of corresponding angles are congruent. Determine if this is sufficient information to conclude that the triangles are similar.

Yes. Knowing that two pairs of corresponding angles are congruent is sufficient information to conclude that the triangles are similar.



The **Angle-Angle Similarity Theorem** states: "If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar."



If $m\angle A = m\angle D$ and $m\angle C = m\angle F$, then $\triangle ABC \sim \triangle DEF$.

5. Explain why this similarity theorem is Angle-Angle instead of Angle-Angle-Angle.

If I know that two pairs of corresponding angles are congruent, I can use the triangle sum to show that the third pair of corresponding angles must also be congruent.

Grouping

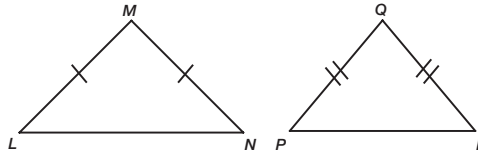
Have students complete Questions 6 and 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 6 and 7

- Is there enough information to determine angle L congruent to angle P ?
- Is there enough information to determine angle L congruent to angle N ?
- Is there enough information to determine side ML congruent to side QP ?
- Is there enough information to determine side ML congruent to side MN ?
- Is there enough information to determine angle S congruent to angle V ?
- Is there enough information to determine angle U congruent to angle X ?
- Is there enough information to determine angle S congruent to angle U ?
- Is there enough information to determine angle V congruent to angle X ?
- Is there enough information to determine if the sides of the two triangles are proportional?
- Do we need to know if the sides of the two triangles are proportional to determine if the triangles are similar?
- Is there an Angle-Angle Congruence Theorem?



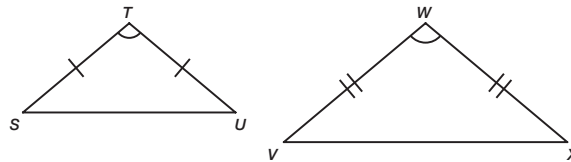
6. The triangles shown are isosceles triangles. Do you have enough information to show that the triangles are similar? Explain your reasoning.



No. Because I do not know anything about the relationship between the corresponding angles of the triangles, I cannot determine if the triangles are similar.



7. The triangles shown are isosceles triangles. Do you have enough information to show that the triangles are similar? Explain your reasoning.



Yes. In an isosceles triangle, the angles opposite congruent sides are congruent. Since the vertex angles of the isosceles triangles are congruent, the sum of the measures of the angles opposite the congruent sides in each triangle must be the same. So by the Angle-Angle Similarity Theorem, the triangles are similar.

Problem 2

Students construct a triangle using a scale factor of two given sides to determine it is not enough information to conclude the two triangles are similar. Then they construct a triangle using a scale factor of three given sides to determine it is enough information to conclude the two triangles are similar. The Side-Side-Side Similarity Theorem states: “If the corresponding sides of two triangles are proportional, then the triangles are similar.”

Grouping

- Discuss and complete Question 1 as a class.
- Have students complete Questions 2 through 6 with a partner. Then have students share their responses as a class.

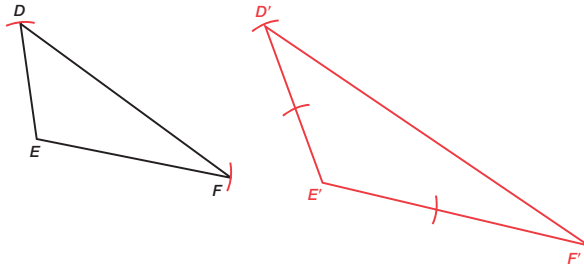
Guiding Questions for Share Phase, Questions 2 through 6

- How does the length of side DF compare to the length of side $D'F'$?
- Is the ratio $\frac{DF}{D'F'}$ equal to the ratios formed using the other two pairs of corresponding sides?
- If two pairs of corresponding sides have the same ratio, does that ensure the third pair of corresponding sides will have the same ratio?

PROBLEM 2 Using Two and Three Proportional Sides



1. Construct triangle $D'E'F'$ by doubling the lengths of sides \overline{DE} and \overline{EF} . Construct the new $D'E'$ and $E'F'$ separately and then construct the triangle. This will ensure a ratio of 2:1. Do not duplicate angles.



2. Measure the angles and sides of triangle $D'E'F'$ and triangle DEF . Are the two triangles similar? Explain your reasoning.

No. The third pair of corresponding sides does not have a 2:1 ratio and the corresponding angles are not congruent.

Two pairs of corresponding sides have the same ratio, but all three pairs of corresponding sides must have the same ratio for the triangles to be similar and the corresponding angles must be congruent.

Did everyone construct the same triangle?



3. Two pairs of corresponding sides are proportional. Determine if this is sufficient information to conclude that the triangles are similar.

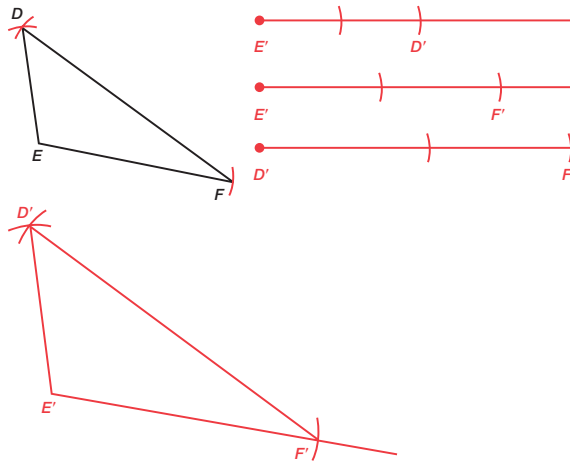
No. All three pairs of corresponding sides must have the same ratio for the triangles to be similar and the corresponding angles must be congruent.

Not having sufficient information doesn't mean that the triangle are NOT similar. It just means that you can't know for sure whether the triangles are or are not similar.



- Are all pairs of corresponding angles congruent?
- How does doubling the length of all three sides of the triangle ensure the ratios formed by the three pairs of corresponding sides will be the same?

4. Construct triangle $D'E'F'$ by doubling the lengths of sides \overline{DE} , \overline{EF} , and \overline{FD} . Construct the new side lengths separately, and then construct the triangle. Do not duplicate angles.



What about now? Did everyone construct the same triangle?



5. Measure the angles and sides of triangle $D'E'F'$ and triangle DEF . Are the two triangles similar? Explain your reasoning.

Yes. The corresponding angles are congruent and the three pairs of corresponding sides have the same ratio, or are proportional.

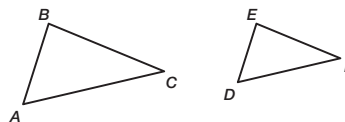


6. Three pairs of corresponding sides are proportional. Determine if this is sufficient information to conclude that the triangles are similar.

Yes. Knowing that three pairs of corresponding sides are proportional is sufficient information to conclude that the triangles are similar.



The **Side-Side-Side Similarity Theorem** states: "If all three corresponding sides of two triangles are proportional, then the triangles are similar."



If $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$, then $\triangle ABC \sim \triangle DEF$.

Grouping

Have students complete Questions 7 through 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 7 through 9

- What is the relationship between the side of a triangle and the angle opposite the side?
- If the three pairs of corresponding sides of two triangles have the same proportional relationship, why must the corresponding pairs of angles be congruent?
- Do you need to know anything about the angle measures of triangle VUW and triangle YXZ to determine if they are similar? Why or why not?
- Can one triangle be mapped onto the other triangle using reflections and dilations?
- What is the difference between the Side-Side-Side Similarity Theorem and the Side-Side-Side Congruence Theorem?

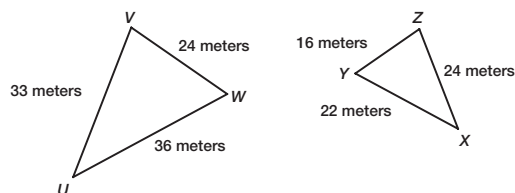


Stacy says that the Side-Side-Side Similarity Theorem tells us that two triangles can have proportional sides, but not congruent angles, and still be similar. Michael doesn't think that's right, but he can't explain why.

7. Is Stacy correct? If not, explain why not.

Stacy is not correct. If all three pairs of corresponding sides in two triangles are proportional, that means the corresponding angles must be congruent because the sides would determine the angles.

8. Determine whether $\triangle UVW$ is similar to $\triangle XYZ$. If so, use symbols to write a similarity statement.



$$\frac{UV}{XY} = \frac{33}{22} = \frac{3}{2}, \frac{VW}{YZ} = \frac{24}{16} = \frac{3}{2}, \frac{UW}{XZ} = \frac{36}{24} = \frac{3}{2}$$

The triangles are similar because the ratios of the corresponding sides are equal. So $\triangle UVW \sim \triangle XYZ$.



9. Describe how transformations could be used to determine whether two triangles are similar when all pairs of corresponding sides are proportional.

I could place the triangles on a coordinate plane and use a sequence of rotations, translations, reflections, and dilations with an appropriate scale factor to map one triangle onto the other.

Problem 3

Students construct a triangle using a scale factor of two given sides and a given included angle to determine it is enough information to conclude the two triangles are similar. The Side-Angle-Side Similarity Theorem states: "If two of the corresponding sides of two triangles are proportional, and the included angles are congruent, then the triangles are similar."

Grouping

- Discuss and complete Question 1 as a class.
- Have students complete Questions 2 through 4 with a partner. Then have students share their responses as a class.

4

Guiding Questions for Share Phase, Questions 2 through 4

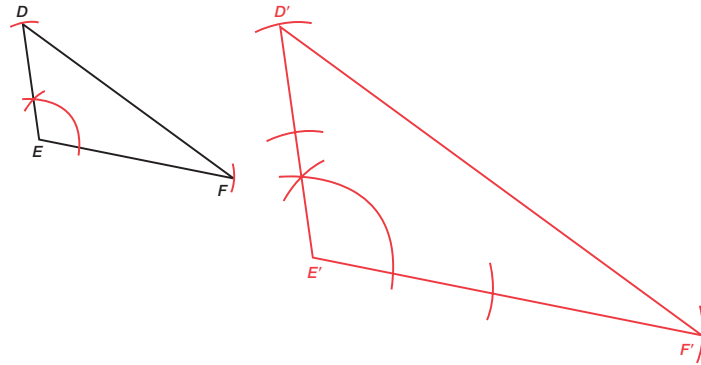
- In triangle ABC if the lengths of sides AB and BC were given, what angle would be considered the included angle?
- In triangle ABC if the measures of angles A and B were given, what side would be considered the included side?
- What two sides did you double in length?
- Which angle did you consider as the included angle?

PROBLEM 3 Using Two Proportional Sides and an Angle



An **included angle** is an angle formed by two consecutive sides of a figure. An **included side** is a line segment between two consecutive angles of a figure.

1. Construct triangle $D'E'F'$ by duplicating an angle and doubling the length of the two sides that make up that angle. Construct the new side lengths separately, and then construct the triangle.



2. Measure the angles and sides of triangle $D'E'F'$ and triangle DEF . Are the two triangles similar? Explain your reasoning.

Yes. The corresponding angles are congruent and the corresponding sides are proportional.

The two pairs of corresponding sides have the same ratio, or are proportional, and the corresponding angles between those sides are congruent.

- Did your classmates double in length the same two sides and use the same included angle?
- What is the difference between the Side-Angle-Side Similarity Theorem and the Side-Angle-Side Congruence Theorem?

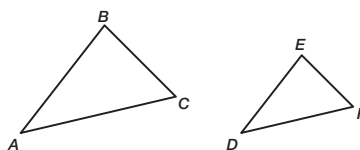
3. Two pairs of corresponding sides are proportional and the corresponding included angles are congruent. Determine if this is sufficient information to conclude that the triangles are similar.

Yes. This is sufficient information to conclude that the triangles are similar.

4. Describe how transformations could be used to determine whether two triangles are similar when two pairs of corresponding sides are proportional and the included angles are congruent.

We could place the triangles on a coordinate plane and use a sequence of rotations, translations, and dilations with an appropriate scale factor to map one triangle onto the other.

The **Side-Angle-Side Similarity Theorem** states: "If two of the corresponding sides of two triangles are proportional and the included angles are congruent, then the triangles are similar."



If $\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle A \cong \angle D$, then $\triangle ABC \sim \triangle DEF$.

Talk the Talk

Students complete a graphic organizer by providing examples of given sides and angles necessary to draw a duplicate triangle.

Grouping

Have students complete Talk the Talk graphic organizer with a partner. Then have students share their responses as a class.

Talk the Talk



1. Gaelin is thinking of a triangle and he wants everyone in his class to draw a similar triangle. Complete the graphic organizer to describe the sides and angles of triangles he could provide.



Be prepared to share your solutions and methods.

Angle-Angle (AA)

If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

This means that the third pair of corresponding angles will also be congruent.

Side-Angle-Side (SAS)

If two pairs of corresponding sides are proportional and the included angles are congruent, then the triangles are similar.

Similar Triangles

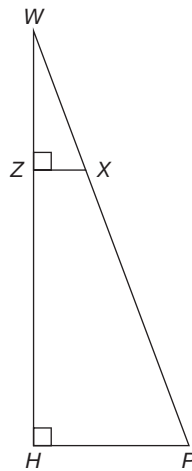
If all three corresponding sides of two triangles are proportional, then the triangles are similar.

Side-Side-Side (SSS)

- One pair of corresponding sides of two triangles are proportional (S).
- Two pairs of corresponding sides of two triangles are proportional (SS).
- One angle of one triangle is congruent to one angle of another triangle (A).
- Two pairs of corresponding sides are proportional and the non-included angles are congruent (SSA).

Combinations of Sides and Angles that Do Not Ensure Similarity

Check for Students' Understanding



Is $\triangle WZX \sim \triangle WHP$?

Explain your reasoning.

Yes. $\angle WZX \sim \angle WHP$ using the AA Similarity Theorem. $\angle W \cong \angle W$ and $\angle WZX \cong \angle WHP$ because they are both right angles and all right angles are congruent.

Keep It in Proportion

Theorems About Proportionality

LEARNING GOALS

- In this lesson, you will:
- Prove the Angle Bisector/Proportional Side Theorem.
- Prove the Triangle Proportionality Theorem.
- Prove the Converse of the Triangle Proportionality Theorem.
- Prove the Proportional Segments Theorem associated with parallel lines.
- Prove the Triangle Midsegment Theorem.

ESSENTIAL IDEAS

- The Angle Bisector/Proportional Side Theorem states: “A bisector of an angle in a triangle divides the opposite side into two segments whose lengths are in the same ratio as the lengths of the sides adjacent to the angle.”
- The Triangle Proportionality Theorem states: “If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally.”
- The Converse of the Triangle Proportionality Theorem states: “If a line divides the two sides proportionally, then it is parallel to the third side.”
- The Proportional Segments Theorem states: “If three parallel lines intersect two transversals, then they divide the transversals proportionally.”
- The Triangle Midsegment Theorem states: “The midsegment of a triangle is parallel to the third side of the triangle and half the measure of the third side of the triangle.”

KEY TERMS

- Angle Bisector/Proportional Side Theorem
- Triangle Proportionality Theorem
- Converse of the Triangle Proportionality Theorem
- Proportional Segments Theorem
- Triangle Midsegment Theorem

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-GPE Expressing Geometric Properties with Equations

Use coordinates to prove simple geometric theorems algebraically

7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

G.SRT Similarity, Right Triangles, and Trigonometry

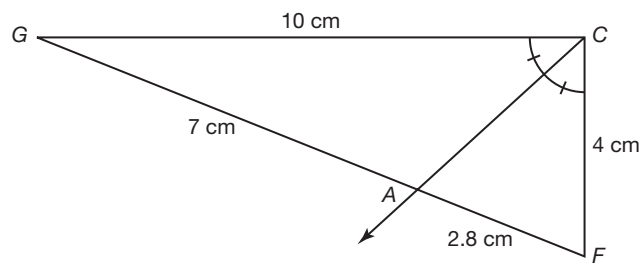
Prove theorems involving similarity

4. Prove theorems about triangles.
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Overview

Students prove the Angle Bisector/Proportional Side Theorem, the Triangle Proportionality Theorem, the Converse of the Triangle Proportionality Theorem, the Proportional Segments Theorem, and the Triangle Midsegment Theorem. Students use these theorems to solve problems.

Warm Up



1. Calculate the ratio of the two sides of triangle GCF adjacent to the angle bisector.

$$\frac{10}{4} \text{ or } \frac{4}{10}$$

2. Calculate the ratio of the two segments formed by the angle bisector.

$$\frac{7}{2.8} \text{ or } \frac{2.8}{7}$$

3. What do you notice about the two ratios?

The two ratios are equivalent.

Keep It in Proportion

Theorems About Proportionality

LEARNING GOALS

In this lesson, you will:

- Prove the Angle Bisector/Proportional Side Theorem.
- Prove the Triangle Proportionality Theorem.
- Prove the Converse of the Triangle Proportionality Theorem.
- Prove the Proportional Segments Theorem associated with parallel lines.
- Prove the Triangle Midsegment Theorem.

KEY TERMS

- Angle Bisector/Proportional Side Theorem
- Triangle Proportionality Theorem
- Converse of the Triangle Proportionality Theorem
- Proportional Segments Theorem
- Triangle Midsegment Theorem

Although geometry is a mathematical study, it has a history that is very much tied up with ancient and modern religions. Certain geometric ratios have been used to create religious buildings, and the application of these ratios in construction even extends back into ancient times.

Music, as well, involves work with ratios and proportions.

Problem 1

The Angle Bisector/Proportional Side Theorem states: “A bisector of an angle in a triangle divides the opposite side into two segments whose lengths are in the same ratio as the lengths of the sides adjacent to the angle.” Auxiliary lines are used to prove this theorem. Students provide some statements and reasons to complete the two-column proof of this theorem.

Grouping

- Ask a student to read aloud the information and Angle Bisector/Proportional Side Theorem and complete Question 1 as a class.
- Have students complete Question 2 with a partner. Then have students share their responses as a class.

4

PROBLEM 1 Proving the Angle Bisector/Proportional Side Theorem



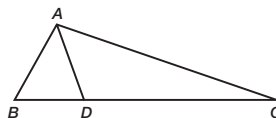
When an interior angle of a triangle is bisected, you can observe proportional relationships among the sides of the triangles formed. You will be able to prove that these relationships apply to all triangles.

The **Angle Bisector/Proportional Side Theorem** states: “A bisector of an angle in a triangle divides the opposite side into two segments whose lengths are in the same ratio as the lengths of the sides adjacent to the angle.”

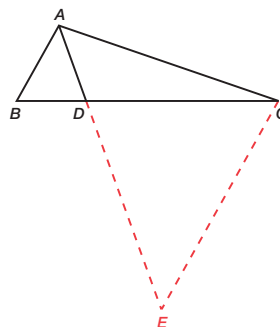
To prove the Angle Bisector/Proportional Side Theorem, consider the statements and figure shown.

Given: \overline{AD} bisects $\angle BAC$

Prove: $\frac{AB}{AC} = \frac{BD}{CD}$



1. Draw a line parallel to \overline{AB} through point C. Extend \overline{AD} until it intersects the line. Label the point of intersection, point E.



Guiding Questions for Share Phase, Question 2

- How many different lines can be drawn through a given point parallel to a given line segment?
- How many different lines can be drawn through point C parallel to line segment AB?
- Angles BAE and CEA are what type of angles with respect to parallel lines?
- Which two sides are opposite the two congruent angles in the triangle?
- What is the definition of congruent line segments?
- What is the Alternate Interior Angle Theorem?
- Which theorem is used to prove the two triangles similar?
- Are corresponding sides are proportional? Which proportion can be used when proving this theorem?
- How is the substitution property used to prove this theorem?



2. Complete the proof of the Angle Bisector/Proportional Side Theorem.

Statements	Reasons
1. \overline{AD} bisects $\angle BAC$	1. Given
2. $\overline{CE} \parallel \overline{AB}$	2. Construction
3. $\angle BAE \cong \angle EAC$	3. Definition of angle bisector
4. $\angle BAE \cong \angle CEA$	4. Alternate Interior Angle Theorem
5. $\angle EAC \cong \angle CEA$	5. Transitive Property of \cong
6. $\overline{AC} \cong \overline{EC}$	6. If two angles of a triangle are congruent, then the sides opposite the angles are congruent.
7. $AC = EC$	7. Definition of congruent segments
8. $\angle BCE \cong \angle ABC$	8. Alternate Interior Angle Theorem
9. $\triangle DAB \sim \triangle DEC$	9. AA Similarity Postulate
10. $\frac{AB}{EC} = \frac{BD}{CD}$	10. Corresponding sides of similar triangles are proportional.
11. $\frac{AB}{BD} = \frac{EC}{CD}$	11. Rewrite as an equivalent proportion
12. $\frac{AB}{BD} = \frac{AC}{CD}$	12. Substitution Property



Problem 2

Students use the Angle Bisector/Proportional Side Theorem to solve for unknown lengths.

Grouping

Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- What proportion best represents the problem situation?
- What operations are used to solve the proportion?
- Is the solution an exact answer or an approximate answer?

4

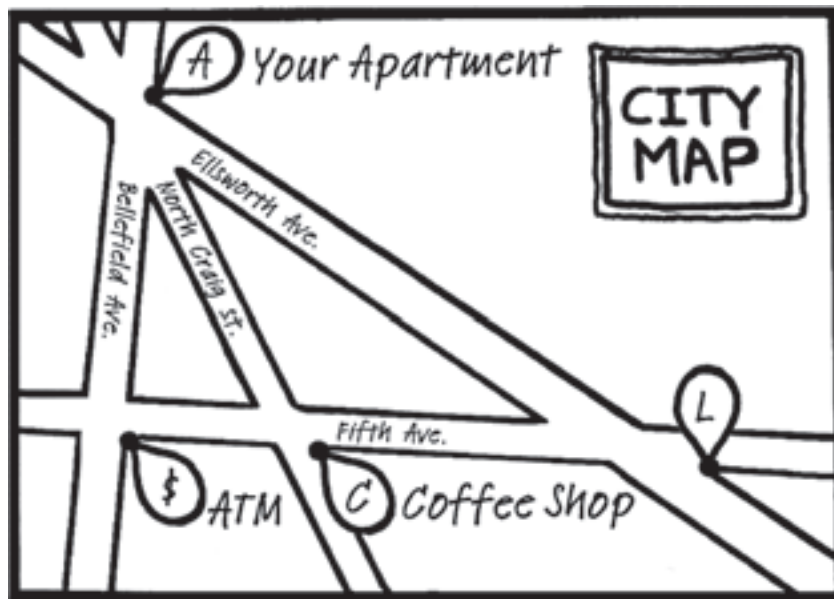
PROBLEM 2 Applying the Angle Bisector/Proportional Side Theorem



1. On the map shown, North Craig Street bisects the angle formed between Bellefield Avenue and Ellsworth Avenue.

- The distance from the ATM to the Coffee Shop is 300 feet.
- The distance from the Coffee Shop to the Library is 500 feet.
- The distance from your apartment to the Library is 1200 feet.

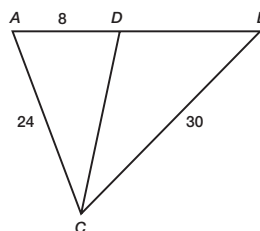
Determine the distance from your apartment to the ATM.



$$\frac{x}{300} = \frac{1200}{500}$$
$$x = 720 \text{ ft}$$

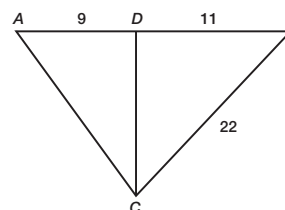
The distance from your apartment to the ATM is 720 feet.

2. \overline{CD} bisects $\angle C$. Solve for DB .



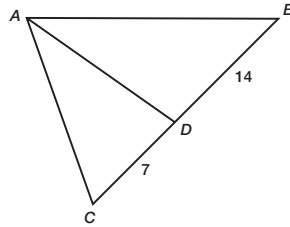
$$\frac{24}{8} = \frac{30}{DB}$$
$$DB = 10$$

3. \overline{CD} bisects $\angle C$. Solve for AC .



$$\frac{9}{AC} = \frac{11}{22}$$
$$AC = 18$$

4. \overline{AD} bisects $\angle A$. $AC + AB = 36$. Solve for AC and AB .



$$\frac{AC}{7} = \frac{AB}{14}$$

$$2AC = AB$$

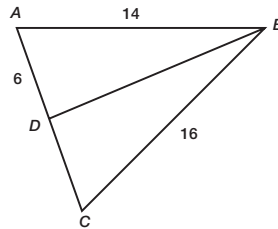
$$AC + 2AC = 36$$

$$AC = 12$$

$$AB = 24$$



5. \overline{BD} bisects $\angle B$. Solve for AC .



$$\frac{14}{6} = \frac{16}{DC}$$

$$DC \approx 6.9$$

$$AC \approx 6 + 6.9 \approx 12.9$$

Problem 3

The Triangle Proportionality Theorem states: “If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally.” Students cut out several statements and reasons written on slips of paper and arrange them in an appropriate order by numbering them to construct a two-column proof for this theorem.

Grouping

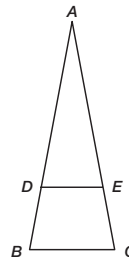
Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Which statement and reason appears in the first step of this proof?
- Which statement appears in the last step of this proof?
- What is the triangle similarity statement?
- Why are the triangles similar?
- What proportional statement will help to prove this theorem?
- What is the Segment Addition Postulate?
- How is the Segment Addition Postulate used to prove this theorem?
- How is substitution used to prove this theorem?

PROBLEM 3 Triangle Proportionality Theorem

The **Triangle Proportionality Theorem** states: “If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally.”

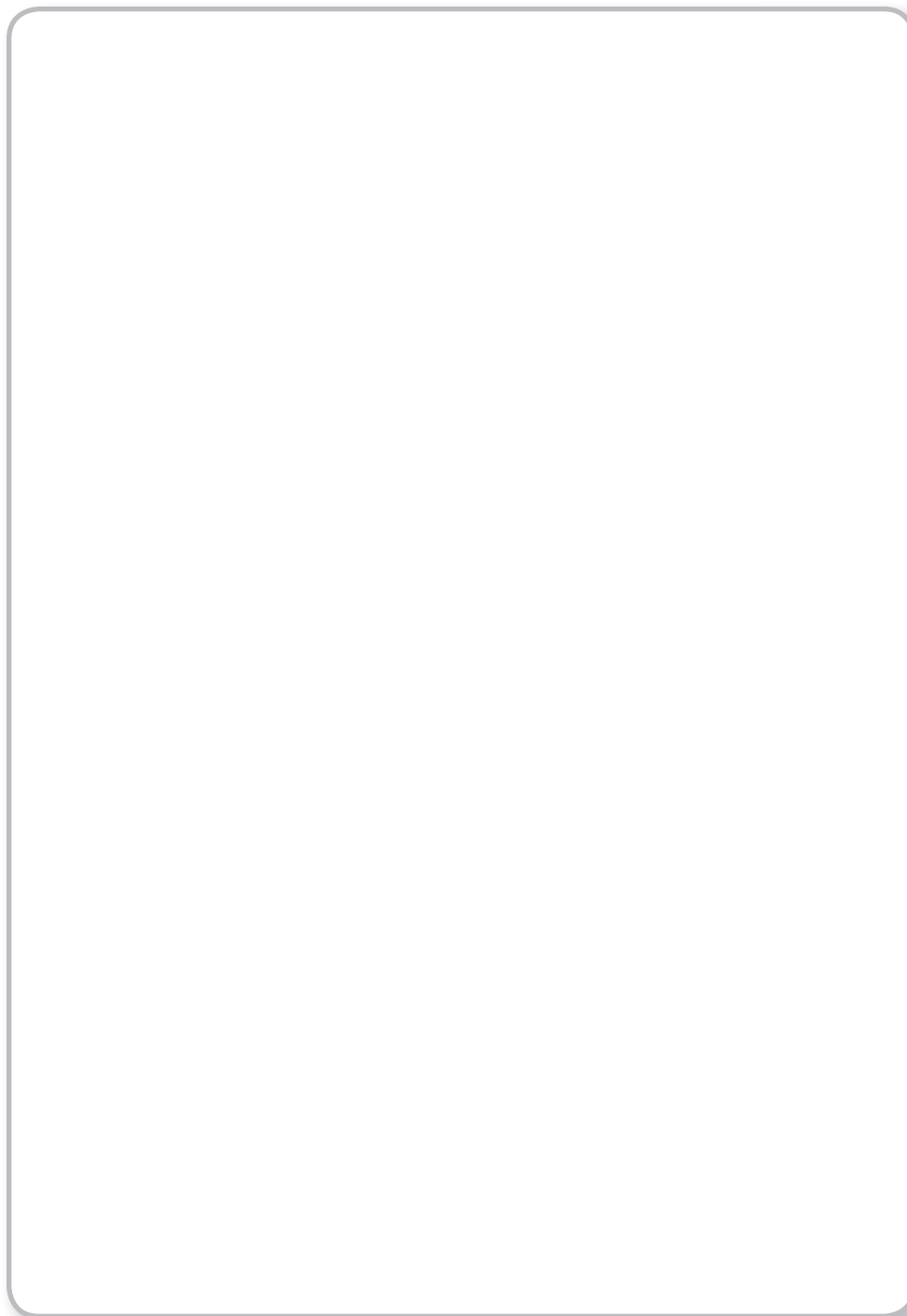


Given: $\overline{BC} \parallel \overline{DE}$

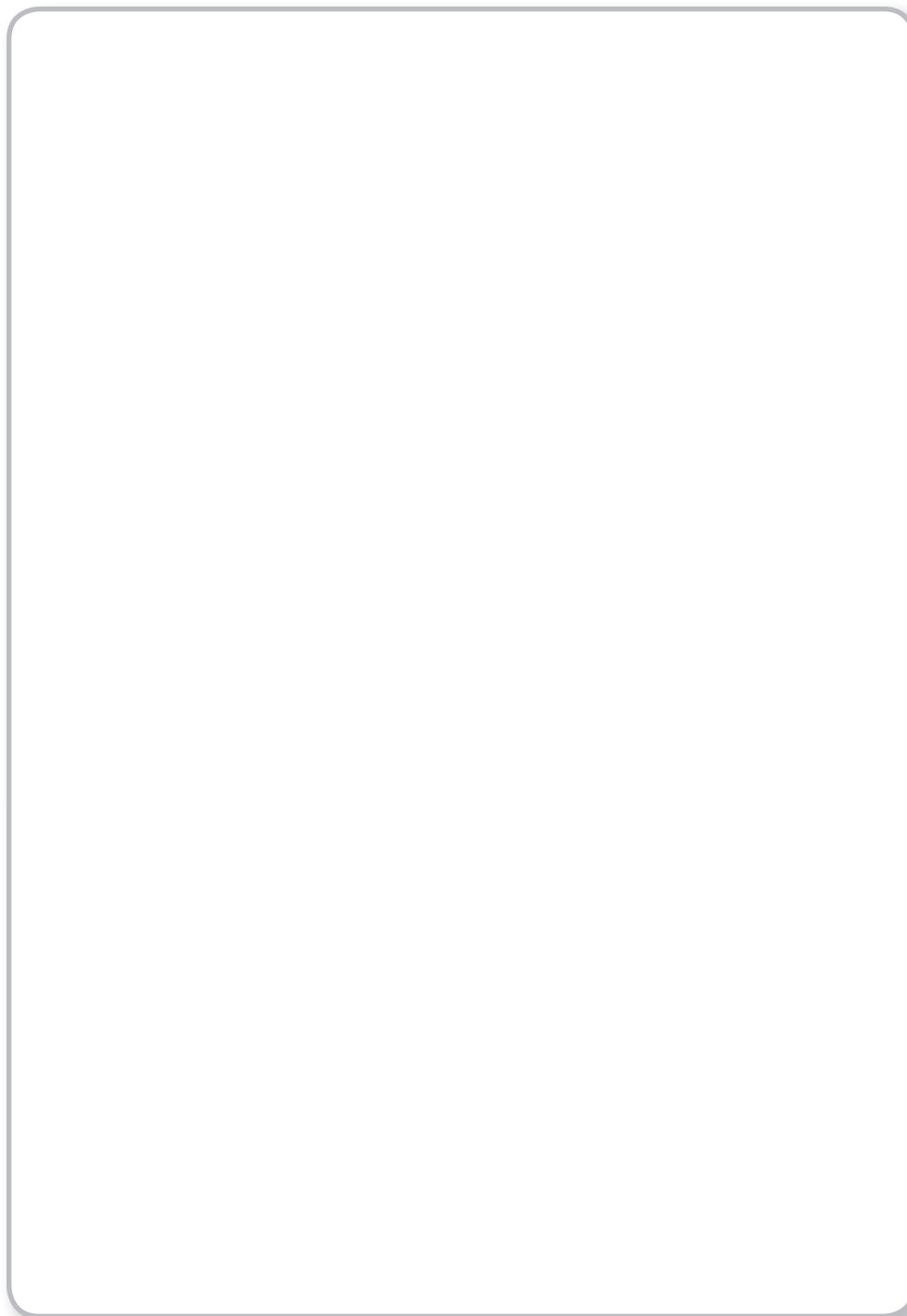
Prove: $\frac{BD}{DA} = \frac{CE}{EA}$



1. Write a paragraph proof to prove triangle ADE is similar to triangle ABC .
 $\angle ADE \cong \angle B$ and $\angle AED \cong \angle C$ because they are pairs of corresponding angles formed by parallel lines. Using the AA Similarity Theorem, triangle ADE is similar to triangle ABC .
2. Cut out each statement and reason on the next page. Match them together, and then rearrange them in an appropriate order by numbering them to create a proof for the Triangle Proportionality Theorem.



Triangle ADE is similar to triangle ABC	Corresponding sides of similar triangles are proportional
$\frac{BD}{DA} = \frac{CE}{EA}$	Corresponding Angle Postulate
$\angle AED \cong \angle C$	Given
$\overline{BC} \parallel \overline{DE}$	Corresponding Angle Postulate
$\frac{BA}{DA} = \frac{CA}{EA}$	AA Similarity Theorem
$BA = BD + DA$ and $CA = CE + EA$	Substitution
$\frac{BD + DA}{DA} = \frac{CE + EA}{EA}$	Segment Addition Postulate
$\angle ADE \cong \angle B$	Simplify



Statements	Reasons
1. $\overline{BC} \parallel \overline{DE}$	1. Given
2. $\angle ADE \cong \angle B$	2. Corresponding Angle Postulate
3. $\angle AED \cong \angle C$	3. Corresponding Angle Postulate
4. Triangle ADE is similar to triangle ABC .	4. AA Similarity Theorem
5. $\frac{BA}{DA} = \frac{CA}{EA}$	5. Corresponding sides of similar triangles are proportional.
6. $BA = BD + DA$ and $CA = CE + EA$	6. Segment Addition Postulate
7. $\frac{BD + DA}{DA} = \frac{CE + EA}{EA}$	7. Substitution
8. $\frac{BD}{DA} = \frac{CE}{EA}$	8. Simplify



Problem 4

The Converse of the Triangle Proportionality Theorem states: “If a line divides the two sides proportionally, then it is parallel to the third side.” Students write a paragraph proof to prove this theorem.

Grouping

Have students complete the problem with a partner. Then have students share their responses as a class.

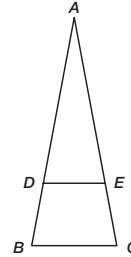
Guiding Questions for Share Phase

- If 1 is added to both sides of the proportion, is the resulting proportion also equal?
- What forms of 1 could be added to both sides of the proportion, so that the proportion would simplify to $\frac{AB}{DA} = \frac{AC}{CE}$?
- How is the transitive property used to help prove this theorem?
- Which angle is shared by both triangles?
- What is the triangle similarity statement?
- Why are the triangles similar?
- Which pair of congruent angles support the Prove statement?

PROBLEM 4 Converse of the Triangle Proportionality Theorem



The **Converse of the Triangle Proportionality Theorem** states: “If a line divides two sides of a triangle proportionally, then it is parallel to the third side.”



Given: $\frac{BD}{DA} = \frac{CE}{EA}$

Prove: $\overline{BC} \parallel \overline{DE}$

Prove the Converse of the Triangle Proportionality Theorem.

First, state $\frac{AD + BD}{BD} = \frac{EA + CE}{CE}$, then simplify it to $\frac{AB}{BD} = \frac{AC}{CE}$. Use proportions to solve for $\frac{BD}{CE}$ and use the transitive property of equality. Show triangle ADE similar to triangle ABC using the SAS Similarity Theorem. Note that $\angle A$ is shared by both triangles. Then, show $\angle ADE \cong \angle B$ by definition of similar triangles. Finally, use the Corresponding Angles Converse Theorem to show $\overline{BC} \parallel \overline{DE}$.



Problem 5

The Proportional Segments Theorem states: “If three parallel lines intersect two transversals, then they divide the transversals proportionally.” Students use the Triangle Proportionality Theorem to prove this theorem.

Grouping

- Ask students to read the Proportional Segments Theorem. Discuss as a class.
- Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- How is the Triangle Proportionality Theorem helpful in proving this theorem?
- What ratio is equal to $\frac{AB}{BC}$?
- What ratio is equal to $\frac{DH}{HC}$?
- How is the transitive property helpful in proving this theorem?

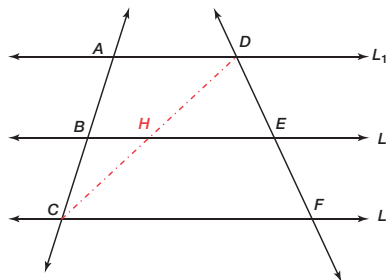
PROBLEM 5 Proportional Segments Theorem



The **Proportional Segments Theorem** states: “If three parallel lines intersect two transversals, then they divide the transversals proportionally.”

Given: $L_1 \parallel L_2 \parallel L_3$

Prove: $\frac{AB}{BC} = \frac{DE}{EF}$



1. Through any two points there is exactly one line. Draw line segment CD to form triangle ACD and triangle FDC .

2. Let H be the point at which L_2 intersects line segment CD . Label point H .

3. Using the Triangle Proportionality Theorem and triangle ACD , what can you conclude?
 $\frac{AB}{BC} = \frac{DH}{HC}$

4. Using the Triangle Proportionality Theorem and triangle FDC , what can you conclude?
 $\frac{DH}{HC} = \frac{DE}{EF}$



5. What property of equality will justify the prove statement?
Using the transitive property, we can conclude $\frac{AB}{BC} = \frac{DE}{EF}$.

Problem 6

The Triangle Midsegment Theorem states: “The midsegment of a triangle is parallel to the third side of the triangle and half the measure of the third side of the triangle.” Students identify the Given and Prove statements and use a two-column proof to prove the theorem.

Grouping

- Ask students to read the Triangle Midsegment Theorem. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

4

Guiding Questions for Share Phase, Questions 1 and 2

- How many given statements are in this proof?
- How many prove statements are in this proof?
- What is the definition of midpoint?
- What is the value of the ratio $\frac{MJ}{DJ}$?
- What is the value of the ratio $\frac{MG}{SG}$?
- Which angle is shared by both triangles?
- What triangle similarity statement is used in the proof of this theorem?

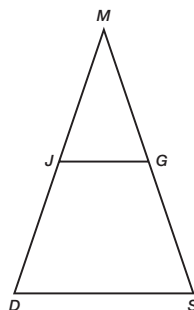
PROBLEM 6 Triangle Midsegment Theorem



The **Triangle Midsegment Theorem** states: “The midsegment of a triangle is parallel to the third side of the triangle and is half the measure of the third side of the triangle.”



1. Use the diagram to write the “Given” and “Prove” statements for the Triangle Midsegment Theorem.



Given: J is the midpoint of \overline{MD} , G is the midpoint of \overline{MS}

Prove: $\overline{JG} \parallel \overline{DS}$

$$JG = \frac{1}{2} DS$$

2. Prove the Triangle Midsegment Theorem.

Statements	Reasons
1. J is the midpoint of \overline{MD} , G is the midpoint of \overline{MS} .	1. Given
2. $MJ = DJ$ and $MG = SG$	2. Definition of midpoint
3. $\frac{MJ}{DJ} = 1$, $\frac{MG}{SG} = 1$	3. Equality Property of Division
4. $\frac{MJ}{DJ} = \frac{MG}{SG}$	4. Substitution Property
5. $\angle M \cong \angle M$	5. Reflexive Property
6. $\triangle MJG \sim \triangle MDS$	6. SAS Similarity Postulate
7. $\angle MJG \cong \angle MDS$	7. Corresponding angles of similar triangles are congruent.
8. $\overline{JG} \parallel \overline{DS}$	8. Corresponding Angle Postulate
9. $MD = MJ + JD$	9. Segment Addition
10. $MD = 2MJ$	10. Substitution Property
11. $MJ = \frac{1}{2} MD$	11. Division
12. $JG = \frac{1}{2} DS$	12. Corresponding sides of similar triangles are proportional.



- Why is $MD = MJ + JD$?
- Which pair of angles are used to prove the lines parallel?

Grouping

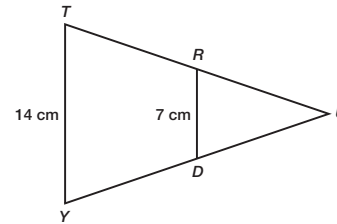
Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- Is there a way to draw segment RD so it is 7 cm but does not look parallel to segment TY ? How?
- Keeping segment RD 7 cm, can it be drawn inside the triangle in such a way that it doesn't appear to be parallel to side TY ? How?



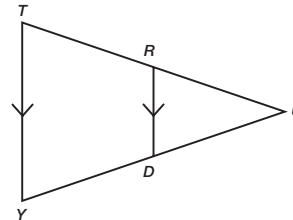
3. Ms. Zoid asked her students to determine whether \overline{RD} is a midsegment of $\triangle TUY$, given $TY = 14$ cm and $RD = 7$ cm.



Carson told Alicia that using the Triangle Midsegment Theorem, he could conclude that \overline{RD} is a midsegment. Is Carson correct? How should Alicia respond if Carson is incorrect?

Carson is not correct. Alicia should draw several 7 cm line segments connecting side \overline{TU} to side \overline{YU} . The diagram does not show $\overline{RD} \parallel \overline{TY}$.

4. Ms. Zoid drew a second diagram on the board and asked her students to determine if \overline{RD} is a midsegment of triangle TUY , given $\overline{RD} \parallel \overline{TY}$.



Alicia told Carson that using the Triangle Midsegment Theorem, she could conclude that \overline{RD} is a midsegment. Is Alicia correct? How should Carson respond if Alicia is incorrect?

Alicia is not correct. Carson should draw several line segments parallel to side \overline{TY} connecting side \overline{TU} to side \overline{YU} . The diagram does not show that R is a midpoint of \overline{TU} or that D is a midpoint of \overline{YU} .

Grouping

Have students complete Questions 5 and 6 with a partner. Then have students share their responses as a class.

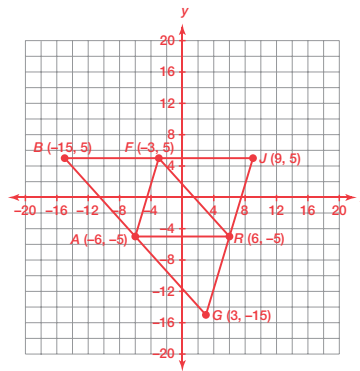
Guiding Questions for Share Phase, Questions 5 and 6

- How does determining the slope of each midsegment help to locate the sides of the triangle?
- Which of the midsegments has a slope equal to 0?
- What formula is used to determine the perimeter of the triangles?
- Is the answer an exact answer or an approximate answer?



In $\triangle BJG$, the midpoint of \overline{BJ} is $F(-3, 5)$. The midpoint of \overline{BG} is $A(-6, -5)$. The midpoint of \overline{GJ} is $R(6, -5)$.

5. Use the Triangle Midsegment Theorem to determine the coordinates of the vertices of $\triangle BJG$. Show all of your work.



The slope of \overline{FA} is $\frac{10}{3}$, so I drew a line through point R with a slope of $\frac{10}{3}$.

The slope of \overline{FR} is $\frac{10}{-9}$, so I drew a line through point A with a slope of $\frac{10}{-9}$.

The slope of \overline{BJ} is 0, so I drew a line through point F with a slope of 0.

The three lines intersected to form $\triangle BJG$. The coordinates of the vertices of $\triangle BJG$ are $B(-15, 5)$, $J(9, 5)$, and $G(3, -15)$.

6. Determine the perimeter of $\triangle BJG$ and the perimeter of $\triangle FAR$. Round each radical to the nearest tenth. Show all of your work.

Length of midsegment \overline{FR} :

$$\begin{aligned}d &= \sqrt{(-3 - 6)^2 + (5 - (-5))^2} \\ &= \sqrt{81 + 100} \\ &= \sqrt{181} \approx 13.5\end{aligned}$$

Length of midsegment \overline{RA} :

$$\begin{aligned}d &= \sqrt{(-6 - 6)^2 + (-5 - (-5))^2} \\ &= \sqrt{144 + 0} \\ &= \sqrt{144} = 12\end{aligned}$$

Length of midsegment \overline{AF} :

$$\begin{aligned}d &= \sqrt{(-3 - (-6))^2 + (5 - (-5))^2} \\ &= \sqrt{9 + 100} \\ &= \sqrt{109} \approx 10.4\end{aligned}$$

The perimeter of $\triangle FAR$ is approximately 35.9 units.

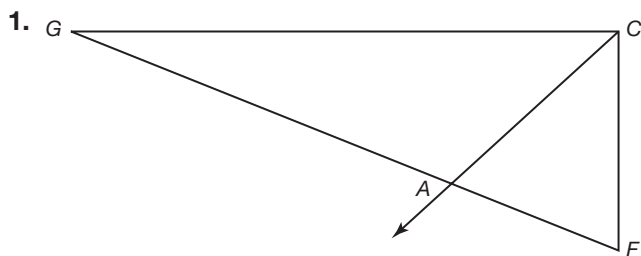
The perimeter of $\triangle BJG$ is approximately 2(35.9) or 71.8 units.



Be prepared to share your solutions and methods.

Check for Students' Understanding

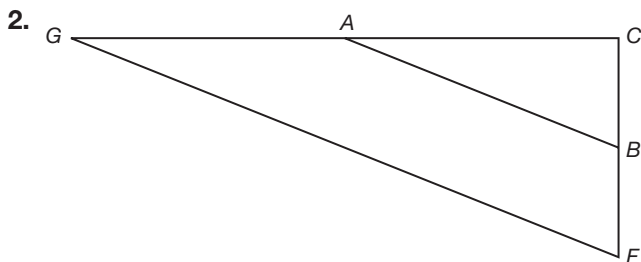
Match each problem situation with the appropriate theorem.



Given: \overrightarrow{CA} bisects $\angle GCF$

Prove: $\frac{CG}{CF} = \frac{GA}{FA}$

Angle Bisector/Proportional Side Theorem

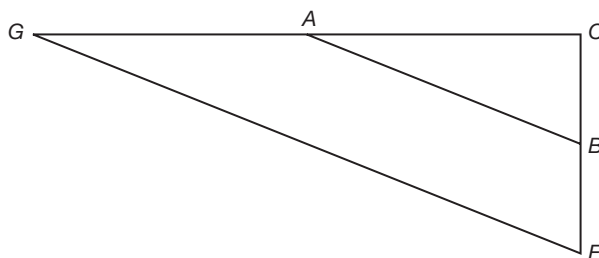


Given: $\overline{AB} \parallel \overline{GF}$

Prove: $\frac{GA}{AC} = \frac{FB}{BC}$

Triangle Proportionality Theorem

3. Converse of the Triangle Proportionality Theorem

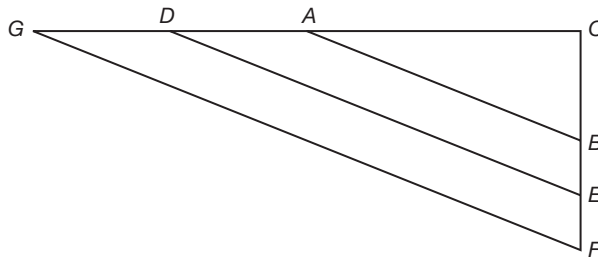


Given: $\frac{GA}{AC} = \frac{FB}{BC}$

Prove: $\overline{AB} \parallel \overline{GF}$

Converse of the Triangle Proportionality Theorem

4. Proportional Segments Theorem

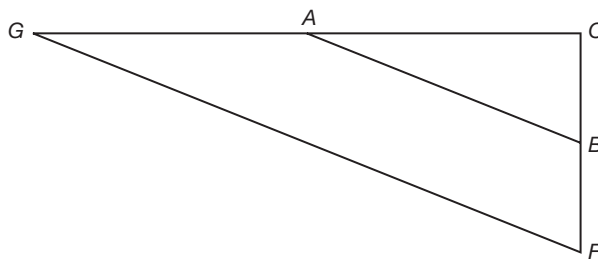


Given: $\overline{AB} \parallel \overline{DE} \parallel \overline{GF}$

Prove: $\frac{AD}{DG} = \frac{BE}{EF}$

Proportional Segments Theorem

5. Triangle Midsegment Theorem



Given: A is the midpoint of \overline{GC} , B is the midpoint of \overline{CF}

Prove: $\overline{AB} \parallel \overline{GF}$, $AB = \frac{1}{2}GF$

Triangle Midsegment Theorem

Geometric Mean

More Similar Triangles

LEARNING GOALS

In this lesson, you will:

- Explore the relationships created when an altitude is drawn to the hypotenuse of a right triangle.
- Prove the Right Triangle/Altitude Similarity Theorem.
- Use the geometric mean to solve for unknown lengths.

ESSENTIAL IDEAS

- The Right Triangle/Altitude Similarity Theorem states: “If an altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other.”
- The geometric mean of two numbers a and b is the number x such that $\frac{a}{x} = \frac{x}{b}$.
- The Right Triangle Altitude/Hypotenuse Theorem states: “The measure of the altitude drawn from the vertex of the right angle of a right triangle to its hypotenuse is the geometric mean between the measures of the two segments of the hypotenuse.”
- The Right Triangle Altitude/Leg Theorem states: “If the altitude is drawn to the hypotenuse of a right triangle, each leg of the right triangle is the geometric mean between the hypotenuse and the segment of the hypotenuse adjacent to the leg.”

KEY TERMS

- Right Triangle/Altitude Similarity Theorem
- geometric mean
- Right Triangle Altitude/Hypotenuse Theorem
- Right Triangle Altitude/Leg Theorem

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G.SRT Similarity, Right Triangles, and Trigonometry

Prove theorems involving similarity

4. Prove theorems about triangles.
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

The term geometric mean is defined and is used in triangle theorems to solve for unknown measurements. Students practice using the Right Triangle/Altitude Similarity Theorem, the Right Triangle/Altitude Theorem, and the Right Triangle/Leg Theorem to solve problems.

Warm Up

Solve for x .

$$\begin{aligned} 1. \quad \frac{10}{x} &= \frac{7}{x-3} \\ 7x &= 10(x-3) \\ 7x &= 10x - 30 \\ 3x &= 30 \\ x &= 10 \end{aligned}$$

$$\begin{aligned} 2. \quad \frac{x}{2} &= \frac{8}{x} \\ x^2 &= 16 \\ x &= 4 \end{aligned}$$

$$\begin{aligned} 3. \quad \frac{x}{7} &= \frac{6}{x} \\ x^2 &= 42 \\ x &= \sqrt{42} \approx 6.5 \end{aligned}$$

Geometric Mean

More Similar Triangles

LEARNING GOALS

In this lesson, you will:

- Explore the relationships created when an altitude is drawn to the hypotenuse of a right triangle.
- Prove the Right Triangle/Altitude Similarity Theorem.
- Use the geometric mean to solve for unknown lengths.

KEY TERMS

- Right Triangle/Altitude Similarity Theorem
- geometric mean
- Right Triangle Altitude/Hypotenuse Theorem
- Right Triangle Altitude/Leg Theorem

People have been building bridges for centuries so that they could cross rivers, valleys, or other obstacles. The earliest bridges probably consisted of a log that connected one side to the other—not exactly the safest bridge!

The longest bridge in the world is the Danyang-Kunshan Grand Bridge in China. Spanning 540,700 feet, it connects Shanghai to Nanjing. Construction was completed in 2010 and employed 10,000 people, took 4 years to build, and cost approximately \$8.5 billion.

Lake Pontchartrain Causeway is the longest bridge in the United States. Measuring *only* 126,122 feet, that's less than a quarter of the Danyang-Kunshan Grand Bridge. However, it currently holds the record for the longest bridge over continuous water. Not too shabby!

Problem 1

A scenario is introduced as an application of the geometric mean. Students will not be able to solve the problem until they understand the relationships between the three triangles formed by an altitude drawn to the hypotenuse of a right triangle. This problem steps through the proof of the Right Triangle/Altitude Similarity Theorem: "If an altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other." This theorem serves as an introduction to the geometric mean.

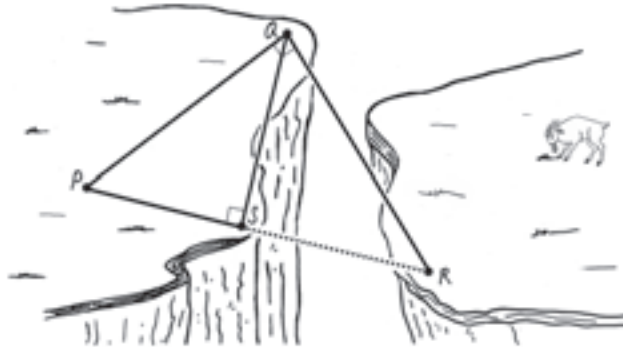
Grouping

- Ask a student to read aloud the information and complete Question 1 as a class.
- Have students complete Questions 2 through 5 with a partner. Then have students share their responses as a class.

PROBLEM 1 Right Triangles



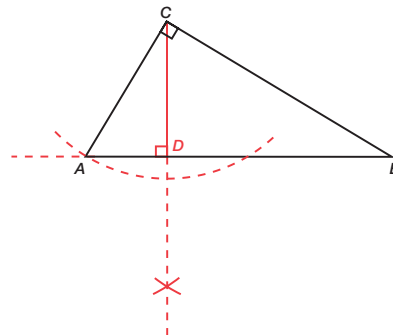
A bridge is needed to cross over a canyon. The dotted line segment connecting points S and R represents the bridge. The distance from point P to point S is 45 yards. The distance from point Q to point S is 130 feet. How long is the bridge?



To determine the length of the bridge, you must first explore what happens when an altitude is drawn to the hypotenuse of a right triangle.

When an altitude is drawn to the hypotenuse of a right triangle, it forms two smaller triangles. All three triangles have a special relationship.

1. Construct an altitude to the hypotenuse in the right triangle ABC . Label the altitude CD .



We're going to explore these relationships in several triangles first. Then we can answer the question at the beginning of this problem.



Guiding Questions for Share Phase, Questions 2 through 5

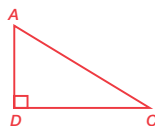
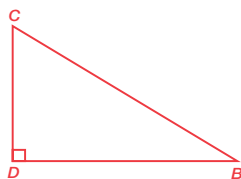
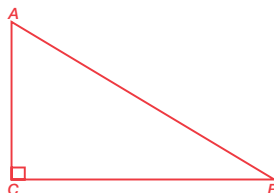
- Did you place the vertex labels on the front and back of each triangle?
- How would you describe the orientation of the three triangles?
- Triangle ABC and triangle ACD have which angle in common?
- Triangle ABC and triangle CBD have which angle in common?
- Triangle ACD and triangle CBD have which angle in common?
- Do all three triangles contain a right angle?
- What shortcut can be used to prove two of the triangles similar?
- If two triangles are similar to the same triangle, what can you conclude?



2. Name all right triangles in the figure.

The right triangles are triangles ABC , ACD , and CBD .

3. Trace each of the triangles on separate pieces of paper and label all the vertices on each triangle. Cut out each triangle. Label the vertex of each triangle. Arrange the triangles so that all of the triangles have the same orientation. The hypotenuse, the shortest leg, and the longest leg should all be in corresponding positions. You may have to flip triangles over to do this.



4. Name each pair of triangles that are similar. Explain how you know that each pair of triangles are similar.

$$\triangle ABC \sim \triangle ACD$$

Both triangles have one right angle and they share angle A so they are similar by the AA Similarity Theorem.

$$\triangle ABC \sim \triangle CBD$$

Both triangles have one right angle and they share angle B so they are similar by the AA Similarity Theorem.

$$\triangle ACD \sim \triangle CBD$$

Both triangles are similar to $\triangle ABC$ so they are also similar to each other.

5. Write the corresponding sides of each pair of triangles as proportions.

$\triangle ABC$ and $\triangle ACD$:

$$\frac{AC}{AD} = \frac{CB}{DC} = \frac{AB}{AC}$$

$\triangle ABC$ and $\triangle CBD$:

$$\frac{AC}{CD} = \frac{CB}{DB} = \frac{AB}{CB}$$

$\triangle ACD$ and $\triangle CBD$:

$$\frac{AD}{CB} = \frac{CD}{DB} = \frac{AC}{DB}$$



The **Right Triangle/Altitude Similarity Theorem** states: "If an altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other."

Problem 2

Geometric mean is defined. Two theorems associated with the altitude drawn to the hypotenuse of a right triangle are stated. The proofs of these theorems are homework assignments. Students apply these theorems to solve for unknown lengths.

Grouping

- Ask a student to read aloud the information, definition, and theorems. Complete Question 1 as a class.
- Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

PROBLEM 2 Geometric Mean



When an altitude of a right triangle is constructed from the right angle to the hypotenuse, three similar right triangles are created. This altitude is a *geometric mean*.

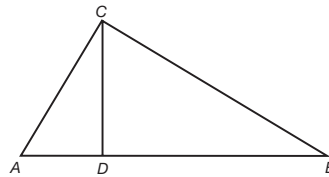
The **geometric mean** of two positive numbers a and b is the positive number x such that $\frac{a}{x} = \frac{x}{b}$.

Two theorems are associated with the altitude to the hypotenuse of a right triangle.

The **Right Triangle Altitude/Hypotenuse Theorem** states: “The measure of the altitude drawn from the vertex of the right angle of a right triangle to its hypotenuse is the geometric mean between the measures of the two segments of the hypotenuse.”

The **Right Triangle Altitude/Leg Theorem** states: “If the altitude is drawn to the hypotenuse of a right triangle, each leg of the right triangle is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to the leg.”

1. Use the diagram from Problem 1 to answer each question.



- a. Write a proportion to demonstrate the Right Triangle Altitude/Hypotenuse Theorem?

The altitude drawn from the vertex of the right angle is \overline{CD} .

The two segments of the hypotenuse are \overline{AD} and \overline{DB} .

The proportion is $\frac{AD}{CD} = \frac{CD}{DB}$.

- b. Write a proportion to demonstrate the Right Triangle Altitude/Leg Theorem?

The altitude drawn to the hypotenuse is \overline{CD} .

The two segments of the hypotenuse are \overline{AD} and \overline{DB} .

The proportion for leg \overline{AC} is $\frac{AB}{AC} = \frac{AC}{AD}$.

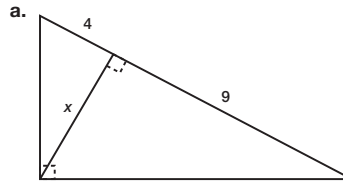
The proportion for leg \overline{BC} is $\frac{AB}{BC} = \frac{BC}{BD}$.

Guiding Questions for Share Phase, Questions 2 and 3

- Which theorem was used to solve each problem?
- What proportion was set up to solve each problem?
- Is the answer an exact answer or an approximate answer?
- Can there be more than one correct answer?
- Which variable did you solve for first?



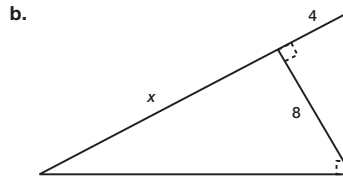
2. In each triangle, solve for x .



$$\frac{x}{4} = \frac{9}{x}$$

$$x^2 = 36$$

$$x = \sqrt{36} = 6$$

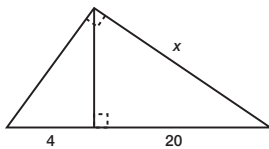


$$\frac{8}{4} = \frac{x}{8}$$

$$4x = 64$$

$$x = 16$$

c.

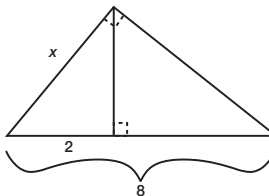


$$\frac{x}{20} = \frac{24}{x}$$

$$x^2 = 480$$

$$x = \sqrt{480} \approx 21.9$$

d.



$$\frac{x}{2} = \frac{8}{x}$$

$$x^2 = 16$$

$$x = \sqrt{16} = 4$$

Problem 3

The Bridge Over the Canyon question in Problem 1 is revisited. Students are now able to solve this problem.

Grouping

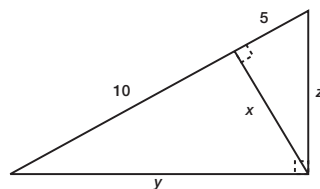
Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Which theorem was used to solve this problem?
- What proportion was set up to solve this problem?
- Is the answer an exact answer or an approximate answer?
- Can there be more than one correct answer?



3. In the triangle shown, solve for x , y , and z .



$$\frac{10}{x} = \frac{x}{5}$$

$$x^2 = 50$$

$$x = \sqrt{50} = 5\sqrt{2}$$

$$(\sqrt{50})^2 + (10)^2 = y^2$$

$$50 + 100 = y^2$$

$$y = \sqrt{150} = 5\sqrt{6}$$

$$(\sqrt{50})^2 + (5)^2 = z^2$$

$$50 + 25 = z^2$$

$$z = \sqrt{75} = 5\sqrt{3}$$

PROBLEM 3 Bridge Over the Canyon



1. Solve for the length of the bridge in Problem 1 using the geometric mean.

45 yards is equal to 3(45) or 135 feet.

$$\frac{135}{130} = \frac{130}{x}$$

$$135x = 16,900$$

$$x \approx 125.19 \text{ ft}$$

The bridge must be approximately 125.19 feet.

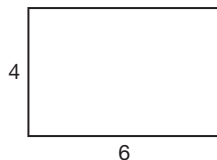
Now I know everything that I need to answer that "bridge" question. Bring it on!



Be prepared to share your solutions and methods.

Check for Students' Understanding

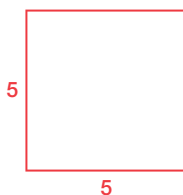
Use the given rectangle to answer each question.



1. Calculate the perimeter of the rectangle.

$$4 + 4 + 6 + 6 = 20$$

2. Draw a square with the same perimeter as the rectangle. What is the length of each side?



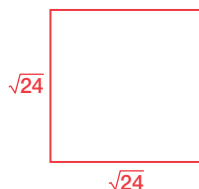
3. Calculate the arithmetic mean of 4, 4, 6, 6.

$$\frac{4 + 4 + 6 + 6}{4} = 5$$

4. Calculate the area of the rectangle.

$$4(6) = 24$$

5. Draw a square with the same area as the rectangle. What is the length of each side?



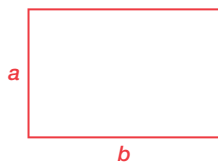
6. Write a proportion and solve for the geometric mean between 4 and 6.

$$\frac{4}{x} = \frac{x}{6}$$

$$x^2 = 24$$

$$x = \sqrt{24}$$

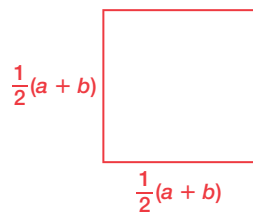
7. Use side lengths a and b instead of 4 and 6 to answer questions 1 through 6 again.



The perimeter of the rectangle is:

$$a + a + b + b = 2a + 2b$$

The square with the same perimeter as the rectangle is:



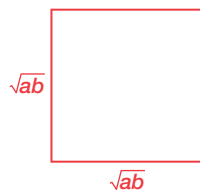
The arithmetic mean of a, a, b, b is:

$$\frac{a + a + b + b}{4} = \frac{2a + 2b}{4} = \frac{2(a + b)}{4} = \frac{a + b}{2}$$

The area of the rectangle is:

$$a(b) = ab$$

The square with the same area as the rectangle is:



The geometric mean between a and b is:

$$\frac{a}{x} = \frac{x}{b}$$

$$x^2 = ab$$

$$x = \sqrt{ab}$$

Proving the Pythagorean Theorem

4.5

Proving the Pythagorean Theorem and the Converse of the Pythagorean Theorem

LEARNING GOALS

In this lesson, you will:

- Prove the Pythagorean Theorem using similar triangles.
- Prove the Converse of the Pythagorean Theorem using algebraic reasoning.

ESSENTIAL IDEAS

- The Pythagorean Theorem states: “If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.”
- The Converse of the Pythagorean Theorem states: “If $a^2 + b^2 = c^2$, then triangle ABC is a right triangle where a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse.”

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G.SRT Similarity, Right Triangles, and Trigonometry

Prove theorems involving similarity

4. Prove theorems about triangles.

Overview

The Pythagorean Theorem is proven geometrically and algebraically. Students are guided through the steps necessary to prove the Pythagorean Theorem using similar triangles. Next, an area model is also used to prove the Pythagorean Theorem algebraically. In the last activity, students are guided through the steps necessary to prove the Converse of the Pythagorean Theorem using an area model once again.

Warm Up

Simplify.

1. $\frac{10\sqrt{2}}{2} = 5\sqrt{2}$

2. $\frac{6}{\sqrt{5}} = \frac{6\sqrt{5}}{5}$

Multiply.

3. $\frac{8}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$

4. $\frac{6}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{6\sqrt{5}}{5}$

5. $\frac{6}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$

Complete each of the following.

$1^2 = 1$ $\sqrt{1} = 1$

$2^2 = 4$ $\sqrt{4} = 2$

$3^2 = 9$ $\sqrt{9} = 3$

$4^2 = 16$ $\sqrt{16} = 4$

$5^2 = 25$ $\sqrt{25} = 5$

$6^2 = 36$ $\sqrt{36} = 6$

$7^2 = 49$ $\sqrt{49} = 7$

$8^2 = 64$ $\sqrt{64} = 8$

$9^2 = 81$ $\sqrt{81} = 9$

$10^2 = 100$ $\sqrt{100} = 10$

Use the information above to estimate each of the following.

6. $\sqrt{40} \approx 6.3$

7. $\sqrt{63} \approx 7.9$

8. $\sqrt{12} \approx 3.5$

9. $\sqrt{101} \approx 10.1$

10. $\sqrt{20} \approx 4.5$

Proving the Pythagorean Theorem

4.5

Proving the Pythagorean Theorem and the Converse of the Pythagorean Theorem

LEARNING GOALS

In this lesson, you will:

- Prove the Pythagorean Theorem using similar triangles.
- Prove the Converse of the Pythagorean Theorem using algebraic reasoning.

The Pythagorean Theorem is one of the most famous theorems in mathematics. And the proofs of the theorem are just as famous. It may be the theorem with the most different proofs. The book *Pythagorean Proposition* alone contains 370 proofs.

The scarecrow in the film *The Wizard of Oz* even tries to recite the Pythagorean Theorem upon receiving his brain. He proudly states, “The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side. Oh, joy! Oh, rapture! I’ve got a brain!”

Sadly the scarecrow’s version of the theorem is wrong—so much for that brain the wizard gave him!

4

Problem 1

Students use the Right Triangle/Altitude Similarity Theorem to prove the Pythagorean Theorem. They begin by constructing the altitude to the hypotenuse of a right triangle forming three similar triangles and create equivalent proportions using the sides of the similar triangles. They rewrite proportional statements as products, factor, and substitute to arrive at the Pythagorean Theorem.

Grouping

Have students complete Questions 1 through 10 with a partner. Then have students share their responses as a class.

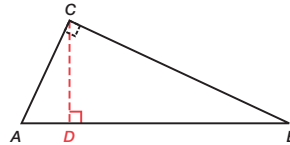
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Guiding Questions for Share Phase, Questions 1 through 10

- What is the Right Triangle/Altitude Similarity Theorem?
- Which side is the longest leg of triangle ABC ?
- Which side is the longest leg of triangle CBD ?
- Which side is the hypotenuse of triangle ABC ?
- Which side is the hypotenuse of triangle CBD ?
- How do you rewrite a proportional statement as a product?
- Which side is the shortest leg of triangle ABC ?
- Which side is the shortest leg of triangle CAD ?

PROBLEM 1 Proving the Pythagorean Theorem with Similar Triangles

Use the Right Triangle/Altitude Similarity Theorem to prove the Pythagorean Theorem.



Given: Triangle ABC with right angle C

Prove: $AC^2 + CB^2 = AB^2$



1. Construct altitude CD to hypotenuse AB .
2. Applying the Right Triangle/Altitude Similarity Theorem, what can you conclude?

$\text{Triangle } ABC \sim \text{Triangle } CBD \sim \text{Triangle } ACD$

3. Write a proportional statement describing the relationship between the longest leg and hypotenuse of triangle ABC and triangle CBD .

$$\frac{AB}{CB} = \frac{CB}{DB}$$

4. Rewrite the proportional statement you wrote in Question 3 as a product.

$$CB^2 = AB \times DB$$

5. Write a proportional statement describing the relationship between the shortest leg and hypotenuse of triangle ABC and triangle ACD .

$$\frac{AB}{AC} = \frac{AC}{AD}$$

- Which side is the hypotenuse of triangle CAD ?
- What factor do both terms have in common?
- What is the Segment Addition Postulate?

6. Rewrite the proportional statement you wrote in Question 5 as a product.

$$AC^2 = AB \times AD$$

7. Add the statement in Question 4 to the statement in Question 6.

$$CB^2 + AC^2 = AB \times DB + AB \times AD$$

8. Factor the statement in Question 7.

$$CB^2 + AC^2 = AB(DB + AD)$$

9. What is equivalent to $DB + AD$?

$$AB$$



10. Substitute the answer to Question 9 into the answer to Question 8 to prove the Pythagorean Theorem.

$$CB^2 + AC^2 = AB^2$$

Problem 2

Students are guided through the steps necessary to prove the Pythagorean Theorem algebraically using an area model.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

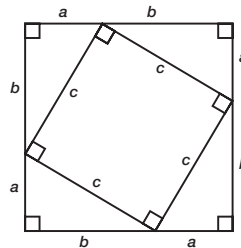
Guiding Questions for Share Phase, Questions 1 through 4

- What is the length and width of the largest square?
- What is the area formula for a square?
- What is the area formula for a triangle?
- What is the area of one right triangle?
- Does the area of the small square plus the area of the four right triangles equal the area of the largest square? Why?

4

PROBLEM 2 Proving the Pythagorean Theorem with Algebraic Reasoning

Use the diagram shown and the following questions to prove the Pythagorean Theorem.



1. What is the area of the larger square?

The area of the larger square is $a^2 + 2ab + b^2$.
 $(a + b)^2 = a^2 + 2ab + b^2$

2. What is the total area of the four right triangles?

The total area of the four right triangles is $2ab$.
 $4\left(\frac{1}{2}ab\right) = 2ab$

3. What is the area of the smaller square?

The area of the smaller square is c^2 .



4. What is the relationship between the area of the four right triangles, the area of the smaller square, and the area of the larger square?

The area of the four right triangles plus the area of the smaller square must equal the area of the larger square.

$$a^2 + 2ab + b^2 = 2ab + c^2$$
$$a^2 + b^2 = c^2$$

Problem 3

The Converse of the Pythagorean Theorem is stated. An area model is used to prove this theorem. Students use the Triangle Sum Theorem, the area formulas of a triangle and quadrilateral, and write expressions and equations which lead to proving the Converse of the Pythagorean Theorem.

Grouping

Have students complete Questions 1 through 9 with a partner. Then have students share their responses as a class.

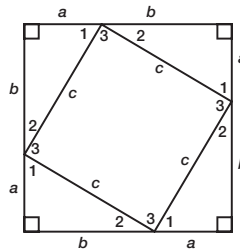
Guiding Questions for Share Phase, Questions 1 through 9

- What is the Triangle Sum Theorem?
- Why is $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$?
- Does the quadrilateral inside the large square have four congruent angles and four congruent sides?
- How is the Pythagorean Theorem different than the Converse of the Pythagorean Theorem?
- How is the proof of the Pythagorean Theorem different than the proof of the Converse of the Pythagorean Theorem?

PROBLEM 3 Proving the Converse of the Pythagorean Theorem

Use the diagram shown and the following questions to prove the Converse of the Pythagorean Theorem.

A large square is composed with four identical right triangles in its corners.



Recall, the Converse of the Pythagorean Theorem states: "If $a^2 + b^2 = c^2$, then a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse."



1. What can you conclude about $m\angle 1 + m\angle 2 + m\angle 3$?

I can conclude that $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$.

2. Use the Triangle Sum Theorem to determine $m\angle 1 + m\angle 2$.

Using the Triangle Sum Theorem, I determined that $m\angle 1 + m\angle 2 = 90^\circ$.

3. Knowing $m\angle 1 + m\angle 2$, what can you conclude about $m\angle 3$?

I concluded that $m\angle 3 = 90^\circ$.

4. What does $m\angle 3$ tell you about the quadrilateral inside of the large square?

The quadrilateral inside the large square is also a square because it has four congruent sides and four right angles.

5. What is the area of one of the right triangles?

The area of one of the right triangles is $A = \frac{1}{2}ab$.

6. What is the area of the quadrilateral inside the large square?

The area of the quadrilateral inside the larger square is $A = c^2$.

7. Write an expression that represents the combined areas of the four right triangles and the quadrilateral inside the large square. Use your answers from Question 16, parts (e) and (f).

$$4\left(\frac{1}{2}ab\right) + c^2$$

8. Write an expression to represent the area of the large square, given that one side is expressed as $(a + b)$. Simplify your answer.

The area of the large square is $(a + b)^2 = a^2 + 2ab + b^2$.

9. Write an equation using the two different expressions representing the area of the large square from Questions 7 and 8. Then, solve the equation to prove the Converse of the Pythagorean Theorem.

$$4\left(\frac{1}{2}ab\right) + c^2 = a^2 + 2ab + b^2$$

$$2ab + c^2 = a^2 + 2ab + b^2$$

$$c^2 = a^2 + b^2$$

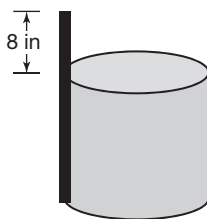


Be prepared to share your solutions and methods.

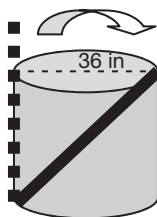
Check for Students' Understanding

There is a well in the ground. Use the three clues to calculate the depth of the water in the well.

Clue 1: When you place a stick vertically into the well, resting upon the inner well wall and perpendicular to ground as drawn below, the stick touches the bottom of the well and the stick rises 8 inches above the surface of the water.



Clue 2: Without moving the bottom of the stick, you take the very top of the stick and tilt it against the opposite wall of the well noticing that the stick is no longer above the surface of the water, it is now even with the surface of the water.



Clue 3: The diameter of the well, or the distance the stick moved across the surface of the water to reach the opposite side of the well is 36 inches.

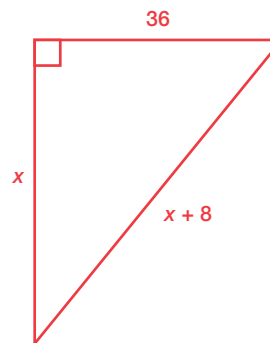
The well is 77 inches deep.

$$x^2 + 36^2 = (x + 8)^2$$

$$x^2 + 1296 = x^2 + 16x + 64$$

$$1232 = 16x$$

$$77 = x$$



Indirect Measurement

Application of Similar Triangles

LEARNING GOALS

In this lesson, you will:

- Identify similar triangles to calculate indirect measurements.
- Use proportions to solve for unknown measurements.

ESSENTIAL IDEAS

- Indirect measurement is the process of using proportions related to similar triangles to determine a measurement when direct measurement is undesirable or not possible.

KEY TERMS

- indirect measurement

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G.SRT Similarity, Right Triangles, and Trigonometry

Prove theorems involving similarity

5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

G.MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

Indirect measurement is an activity that takes students out of their classroom and school building. Students measure the height of objects such as flagpoles, tops of trees, telephone poles, or buildings using similar triangles. Each pair of students will need access to a tape measure, a marker, and a flat pocket mirror. In addition to the outside activity, students are given several situations in which they create proportions related to similar triangles to solve for unknown measurements.

Warm Up

1. How would you measure the height of a flagpole?

Answers will vary.

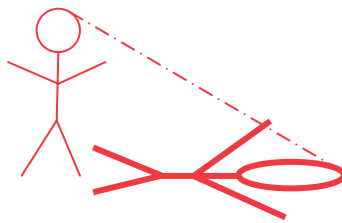
Measure the length of a rope that stretches from the top to the bottom of the flagpole.

2. How would you measure the height of a very tall building?

Answers will vary.

Use a measuring tool to determine the height of one story and multiply the height by the number of stories.

3. James is standing outside on a beautiful day. The sky is clear and the sun is blazing. James is looking at his shadow and thinking about geometry. Draw a picture of James and his shadow.



4. What is the measure of the angle formed by James and his shadow? Explain your reasoning.

A right angle is formed by James and his shadow. James is standing perpendicular to the ground. The shadow is on the ground.

5. Draw a line segment connecting the top of James' head with the top of his shadow's head. What geometric figure is formed?

A right triangle is formed.

Indirect Measurement

Application of Similar Triangles

LEARNING GOALS

In this lesson, you will:

- Identify similar triangles to calculate indirect measurements.
- Use proportions to solve for unknown measurements.

KEY TERM

- indirect measurement

You would think that determining the tallest building in the world would be pretty straightforward. Well, you would be wrong.

There is actually an organization called the Council on Tall Buildings and Urban Habitat that officially certifies buildings as the world's tallest. It was founded at Lehigh University in 1969 with a mission to study and report "on all aspects of the planning, design, and construction of tall buildings."

So, what does it take to qualify for world's tallest? The Council only recognizes a building if at least 50% of its height is made up of floor plates containing habitable floor area. Any structure that does not meet this criteria is considered a tower. These buildings might have to settle for being the world's tallest tower instead!

Problem 1

This is an outside activity. Step-by-step instructions for measuring the height of the school flagpole are given. In addition to a tape measure, a marker, and a pocket mirror, students should take paper and pencil to record their work. It is suggested that each pair of students switch roles so the measurements are done twice. This activity requires advance preparation to gather materials, check the weather forecast, and make the necessary arrangements to take the class outside. It is well worth it! The most common error students make when measuring is looking down at the mirror to see the reflection as their partner is measuring the height from the ground to their eyes. This can introduce error in their calculations. Check and make sure the students' eye sight is parallel to the ground as their partners measure their eye level height.

Grouping

Discuss the worked example as a class. Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

4

PROBLEM 1 How Tall Is That Flagpole?

At times, measuring something directly is impossible, or physically undesirable. When these situations arise, **indirect measurement**, the technique that uses proportions to calculate measurement, can be implemented. Your knowledge of similar triangles can be very helpful in these situations.



Use the following steps to measure the height of the school flagpole or any other tall object outside. You will need a partner, a tape measure, a marker, and a flat mirror.

Step 1: Use a marker to create a dot near the center of the mirror.

Step 2: Face the object you would like to measure and place the mirror between yourself and the object. You, the object, and the mirror should be collinear.

Step 3: Focus your eyes on the dot on the mirror and walk backward until you can see the top of the object on the dot, as shown.



Step 4: Ask your partner to sketch a picture of you, the mirror, and the object.

Step 5: Review the sketch with your partner. Decide where to place right angles, and where to locate the sides of the two triangles.



Step 6: Determine which segments in your sketch can easily be measured using the tape measure. Describe their locations and record the measurements on your sketch.

Guiding Questions for Share Phase, Questions 1 through 5

- What is the relationship between the person and the ground?
- What is the relationship between the object and the ground?
- Why are the angle of incidence and the angle of reflection in the mirror the same measure?
- What proportion is used to calculate the height of the object?
- When your partner measured the distance from you to the dot on the mirror, were you looking straight ahead or were you looking down?

1. How can similar triangles be used to calculate the height of the object?

The person and the flagpole are both perpendicular to the ground, so the angles are both right angles. The angle of incidence and the angle of reflection in the mirror are the same. Thus, the triangles are similar.

2. Use your sketch to write a proportion to calculate the height of the object and solve the proportion.

Answers will vary.

3. Compare your answer with others measuring the same object. How do the answers compare?

All answers should be relatively close.

4. What are some possible sources of error that could result when using this method?

Measurement can always include degrees of error. If you are looking down at the mirror while your partner is measuring the distance from you to the dot on the mirror, and then measure the height to your eyes when you are looking up, that alone will introduce a significant measurement error.



5. Switch places with your partner and identify a second object to measure. Repeat this method of indirect measurement to solve for the height of the new object.

Problem 2

Students are given information in three different situations, then create and solve proportions related to similar triangles to determine unknown measurements.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- How would you describe the location of the right angles in each triangle?
- What proportion was used to solve this problem?
- Is this answer exact or approximate? Why?
- How is using shadows to calculate the height of the tree different than the previous method you used?

4

PROBLEM 2 How Tall Is That Oak Tree?



1. You go to the park and use the mirror method to gather enough information to calculate the height of one of the trees. The figure shows your measurements. Calculate the height of the tree.

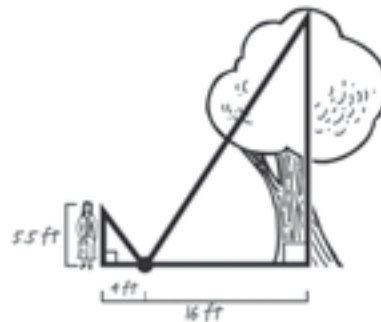
Let x be the height of the tree.

$$\frac{x}{5.5} = \frac{16}{4}$$

$$x = 5.5(4)$$

$$x = 22$$

The tree is 22 feet tall.



2. Stacey wants to try the mirror method to measure the height of one of her trees. She calculates that the distance between her and the mirror is 3 feet and the distance between the mirror and the tree is 18 feet. Stacey's eye height is 60 inches. Draw a diagram of this situation. Then, calculate the height of this tree.

Let x be the height of the tree.

$$\frac{x}{5} = \frac{18}{3}$$

$$\frac{x}{5} = 6$$

$$x = 5(6)$$

$$x = 30$$

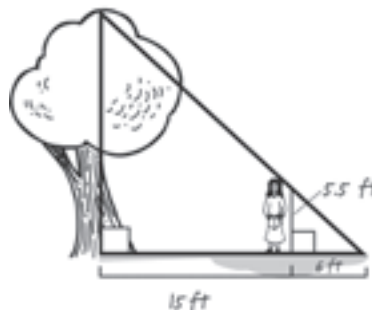
The tree is 30 feet tall.

Remember, whenever you are solving a problem that involves measurements like length (or weight), you may have to rewrite units so they are the same.





3. Stacey notices that another tree casts a shadow and suggests that you could also use shadows to calculate the height of the tree. She lines herself up with the tree's shadow so that the tip of her shadow and the tip of the tree's shadow meet. She then asks you to measure the distance from the tip of the shadows to her, and then measure the distance from her to the tree. Finally, you draw a diagram of this situation as shown below. Calculate the height of the tree. Explain your reasoning.



Let x be the height of the tree.

$$\frac{x}{5.5} = \frac{15 + 6}{6}$$

$$\frac{x}{5.5} = \frac{21}{6}$$

$$\frac{x}{5.5} = 3.5$$

$$x = 5.5(3.5)$$

$$x = 19.25$$

The tree is 19.25 feet tall.

The triangle formed by the tip of the shadow and the top and bottom of the tree and the triangle formed by the tip of the shadow and the top and bottom of my friend are similar by the Angle-Angle Similarity Theorem. So, I was able to set up and solve a proportion of the ratios of the corresponding side lengths of the triangles.

Problem 3

Using the given information, students determine the width of a creek, the width of a ravine, and the distance across the widest part of a pond using proportions related to similar triangles.

Grouping

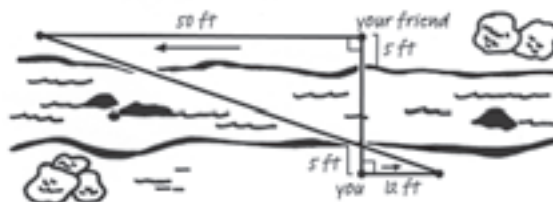
Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Are there two triangles in this diagram?
- Are the two triangles also right triangles? How do you know?
- Where is the location of the right angle in each triangle?
- How would you describe the location of each right triangle?
- How are vertical angles formed?
- Are there any vertical angles in the diagram? Where are they located?
- What do you know about vertical angles?
- What proportion was used to calculate the distance from your friend's starting point to your side of the creek?
- What operation is used to determine the width of the creek?

PROBLEM 3 How Wide Is That Creek?

1. You stand on one side of the creek and your friend stands directly across the creek from you on the other side as shown in the figure.



It is not reasonable for you to directly measure the width of a creek, but you can use indirect measurement to measure the width.



Your friend is standing 5 feet from the creek and you are standing 5 feet from the creek. You and your friend walk away from each other in opposite parallel directions. Your friend walks 50 feet and you walk 12 feet.



- a. Label any angle measures and any angle relationships that you know on the diagram. Explain how you know these angle measures.

The angles at the vertices of the triangles where my friend and I were originally standing are right angles because we started out directly across from each other and then we walked away from each other in opposite directions. The angles where the vertices of the triangles intersect are congruent because they are vertical angles.

- b. How do you know that the triangles formed by the lines are similar?

Because two pairs of corresponding angles are congruent, the triangles are similar by the Angle-Angle Similarity Theorem.

- c. Calculate the distance from your friend's starting point to your side of the creek. Round your answer to the nearest tenth, if necessary.

Let x be the distance from my friend's starting point to my side of the creek.

$$\frac{x}{5} = \frac{50}{12}$$

$$x = 5\left(\frac{50}{12}\right)$$

$$x \approx 20.8$$

The distance is approximately 20.8 feet.



- d. What is the width of the creek? Explain your reasoning.

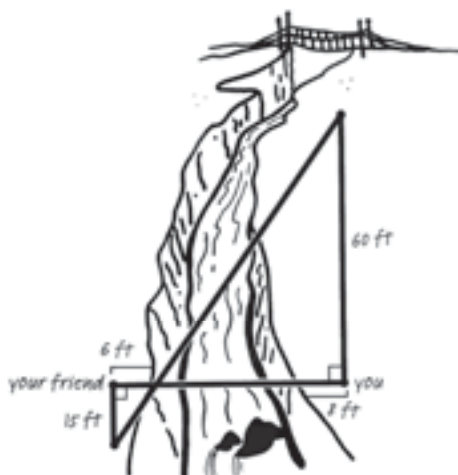
Width of creek: $20.8 - 5 = 15.8$

The width of the creek is about 15.8 feet.

The width of the creek is found by subtracting the distance from my friend's starting point to her side of the creek from the distance from my friend's starting point to my side of the creek.



2. There is also a ravine (a deep hollow in the earth) on another edge of the park. You and your friend take measurements like those in Problem 3 to indirectly calculate the width of the ravine. The figure shows your measurements. Calculate the width of the ravine.



The triangles are similar by the Angle-Angle Similarity Theorem. Let x be the distance from myself to the edge of the ravine on the other side.

$$\frac{x}{6} = \frac{60}{15}$$

$$x = 6(4)$$

$$x = 24$$

Width of ravine: $24 - 8 = 16$

The width of the ravine is 16 feet.

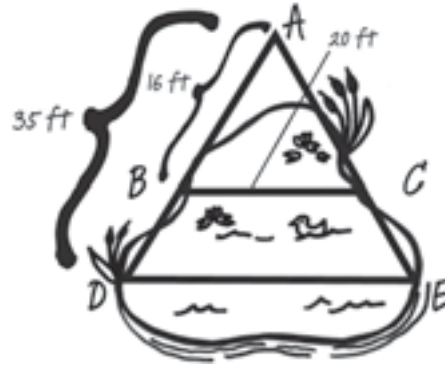
Grouping

Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 2 and 3

- What proportion was used to calculate the width of the ravine?
- How is this ravine situation similar to the creek situation?
- How does the pond situation compare to the creek or ravine situation?
- Are vertical angles formed in the pond situation?
- What other situation in this lesson was most like the pond situation?
- Are the two triangles in the pond situation similar? How do you know?

3. There is a large pond in the park. A diagram of the pond is shown below. You want to calculate the distance across the widest part of the pond, labeled as \overline{DE} . To indirectly calculate this distance, you first place a stake at point A . You chose point A so that you can see the edge of the pond on both sides at points D and E , where you also place stakes. Then, you tie a string from point A to point D and from point A to point E . At a narrow portion of the pond, you place stakes at points B and C along the string so that \overline{BC} is parallel to \overline{DE} . The measurements you make are shown on the diagram. Calculate the distance across the widest part of the pond.



Angle $\angle ABC$ and $\angle ADE$ are congruent because \overline{BC} is parallel to \overline{DE} (Corresponding Angles Postulate). Angle A is congruent to itself. So, by the Angle-Angle Similarity Theorem, $\triangle ABC$ is similar to $\triangle ADE$.

$$\frac{DE}{20} = \frac{35}{16}$$

$$DE = 20 \left(\frac{35}{16} \right)$$

$$DE = 43.75$$

The distance across the widest part of the pond is 43.75 feet.



Be prepared to share your solutions and methods.

Check for Students' Understanding

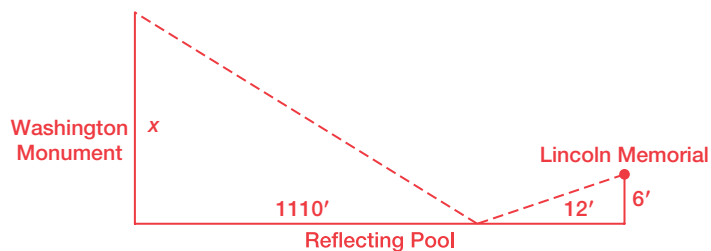
The Washington Monument is a tall obelisk built between 1848 and 1884 in honor of the first president of the United States, George Washington. It is the tallest free standing masonry structure in the world.

It was not until 1888 that the public was first allowed to enter the monument because work was still being done on the interior. During this time, the stairwell, consisting of 897 steps, was completed. The final cost of the project was \$1,817,710.

It is possible to determine the height of the Washington Monument using only a simple tape measure and a few known facts:

- Your eyes are 6 feet above ground level.
- The reflecting pool is located between the Washington Monument and the Lincoln Memorial.
- You are standing between the Washington Monument and the Lincoln Memorial facing the Monument with your back to the Lincoln Memorial while gazing into the reflecting pool at the reflection of the Washington Monument.
- You can see the top of the monument in the reflecting pool that is situated between the Lincoln Memorial and the Washington Monument.
- You measured the distance from the spot where you are standing to the spot where you see the top of the monument in the reflecting pool to be 12 feet.
- You measured the distance from the location in the reflecting pool where you see the top of the monument to the base of the monument to be 1110 feet.

With this information, calculate the height of the Washington Monument. Begin by drawing a diagram of the problem situation.



$$\frac{6}{12} = \frac{x}{1110}$$

$$12x = 6660$$

$$x = 555$$

The height of the Washington Monument is 555'.

Chapter 4 Summary

KEY TERMS

- similar triangles (4.1)
- included angle (4.2)
- included side (4.2)
- geometric mean (4.4)
- indirect measurement (4.6)

POSTULATES AND THEOREMS

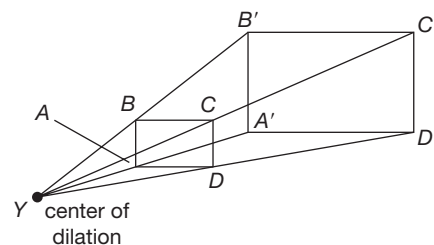
- Angle-Angle Similarity Theorem (4.2)
- Side-Side-Side Similarity Theorem (4.2)
- Triangle Proportionality Theorem (4.3)
- Converse of the Triangle Proportionality Theorem (4.3)
- Proportional Segments Theorem (4.3)
- Triangle Midsegment Theorem (4.3)
- Side-Angle-Side Similarity Theorem (4.2)
- Angle Bisector/Proportional Side Theorem (4.3)
- Right Triangle/Altitude Similarity Theorem (4.4)
- Right Triangle Altitude/Hypotenuse Theorem (4.4)
- Right Triangle Altitude/Leg Theorem (4.4)

4.1 Comparing the Pre-image and Image of a Dilation

A dilation increases or decreases the size of a figure. The original figure is the pre-image, and the dilated figure is the image. A pre-image and an image are similar figures, which means they have the same shape but different sizes.

A dilation can be described by drawing line segments from the center of dilation through each vertex on the pre-image and the corresponding vertex on the image. The ratio of the length of the segment to a vertex on the pre-image and the corresponding vertex on the image is the scale factor of the dilation. A scale factor greater than 1 produces an image that is larger than the pre-image. A scale factor less than 1 produces an image that is smaller than the pre-image.

Example



$$YD = 3.5 \quad YD' = 7.7$$

$$YB = 2.5 \quad YB' = ?$$

$$YC = ? \quad YC' = 3.3$$

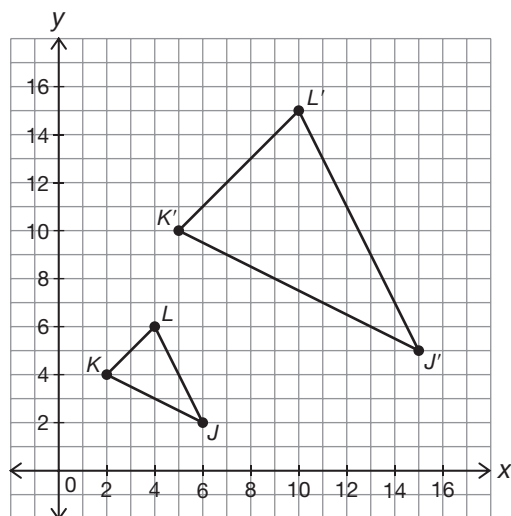
$$YA = 1.0 \quad YA' = ?$$

$$\begin{aligned} \text{scale factor} &= \frac{YD'}{YD} & \frac{YB'}{YB} &= 2.2 & \frac{YC'}{YC} &= 2.2 & \frac{YA'}{YA} &= 2.2 \\ &= \frac{7.7}{3.5} & YB' &= 2.2\overline{YB} & YC' &= 2.2YC & YA' &= 2.2YA \\ &= 2.2 & YB' &= 2.2(2.5) & \frac{YC'}{2.2} &= YC & YA' &= 2.2(1.0) \\ & & YB' &= 5.5 & \frac{3.3}{2.2} &= YC & YA' &= 2.2 \\ & & & & 1.5 &= \overline{YC} & & \end{aligned}$$

4.1 Dilating a Triangle on a Coordinate Grid

The length of each side of an image is the length of the corresponding side of the pre-image multiplied by the scale factor. On a coordinate plane, the coordinates of the vertices of an image can be found by multiplying the coordinates of the vertices of the pre-image by the scale factor. If the center of dilation is at the origin, a point (x, y) is dilated to (kx, ky) by a scale factor of k .

Example



The center of dilation is the origin.

The scale factor is 2.5.

$$J(6, 2) \rightarrow J'(15, 5)$$

$$K(2, 4) \rightarrow K'(5, 10)$$

$$L(4, 6) \rightarrow L'(10, 15)$$

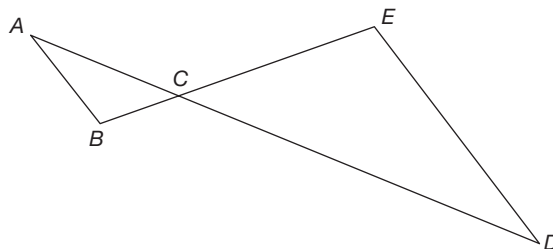
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4.1 Using Geometric Theorems to Prove that Triangles are Similar

All pairs of corresponding angles and all corresponding sides of similar triangles are congruent. Geometric theorems can be used to prove that triangles are similar. The Alternate Interior Angle Theorem, the Vertical Angle Theorem, and the Triangle Sum Theorem are examples of theorems that might be used to prove similarity.

Example

By the Alternate Interior Angle Theorem, $\angle A \cong \angle D$ and $\angle B \cong \angle E$. By the Vertical Angle Theorem, $\angle ACB \cong \angle ECD$. Since the triangles have three pairs of corresponding angles that are congruent, the triangles have the same shape and $\triangle ABC \cong \triangle DEC$.



4.1 Using Transformations to Prove that Triangles are Similar

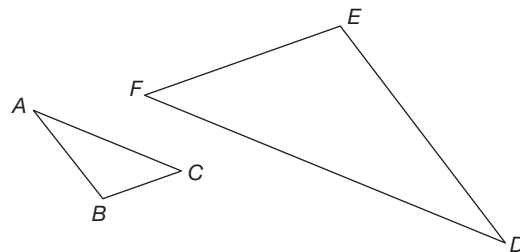
Triangles can also be proven similar using a sequence of transformations. The transformations might include rotating, dilating, and reflecting.

Example

Given: $\overline{AC} \parallel \overline{FD}$

Translate $\triangle ABC$ so that \overline{AC} aligns with \overline{FD} .

Rotate $\triangle ABC$ 180° about the point C so that \overline{AC} again aligns with \overline{FD} . Translate $\triangle ABC$ until point C is at point F . If we dilate $\triangle ABC$ about point C to take point B to point E , then \overline{AB} will be mapped onto \overline{ED} , and \overline{BC} will be mapped onto \overline{EF} . Therefore, $\triangle ABC$ is similar to $\triangle DEF$.

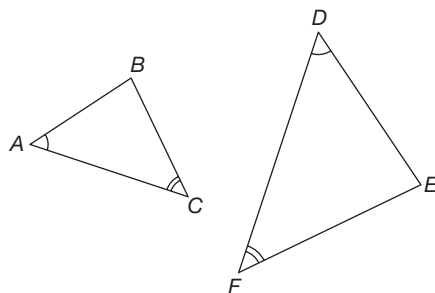


4.2 Using Triangle Similarity Theorems

Two triangles are similar if they have two congruent angles, if all of their corresponding sides are proportional, or if two of their corresponding sides are proportional and the included angles are congruent. An included angle is an angle formed by two consecutive sides of a figure. The following theorems can be used to prove that triangles are similar:

- The Angle-Angle (AA) Similarity Theorem—If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.
- The Side-Side-Side (SSS) Similarity Theorem—If the corresponding sides of two triangles are proportional, then the triangles are similar.
- The Side-Angle-Side (SAS) Similarity Theorem—If two of the corresponding sides of two triangles are proportional and the included angles are congruent, then the triangles are similar.

Example



Given: $\angle A \cong \angle D$

$\angle C \cong \angle F$

Therefore, $\triangle ABC \sim \triangle DEF$ by the AA Similarity Theorem.

4.3 Applying the Angle Bisector/Proportional Side Theorem

When an interior angle of a triangle is bisected, you can observe proportional relationships among the sides of the triangles formed. You can apply the Angle Bisector/Proportional Side Theorem to calculate side lengths of bisected triangles.

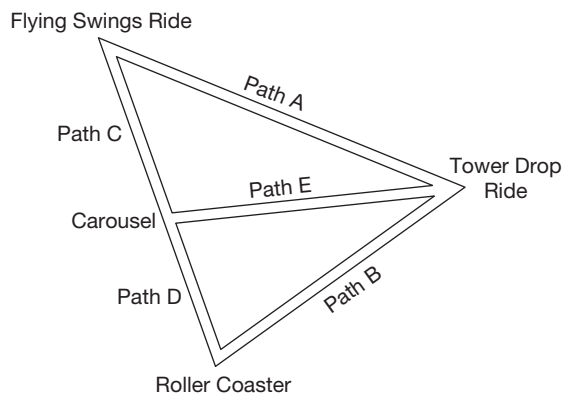
- Angle Bisector/Proportional Side Theorem—A bisector of an angle in a triangle divides the opposite side into two segments whose lengths are in the same ratio as the lengths of the sides adjacent to the angle.

Example

The map of an amusement park shows locations of the various rides.

Given:

- Path E bisects the angle formed by Path A and Path B.
- Path A is 143 feet long.
- Path C is 65 feet long.
- Path D is 55 feet long.



Let x equal the length of Path B.

$$\frac{x}{55} = \frac{143}{63} \quad \text{Path B is 121 feet long.}$$

$$x = 121$$

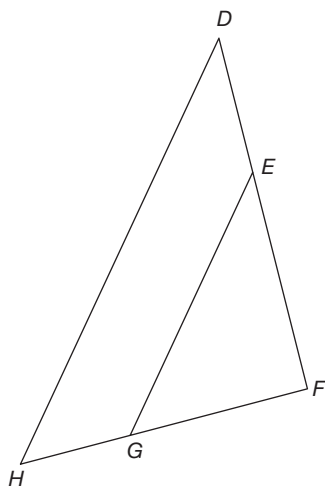
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4.3 Applying the Triangle Proportionality Theorem

The Triangle Proportionality Theorem is another theorem you can apply to calculate side lengths of triangles.

- Triangle Proportionality Theorem—If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally.

Example



Given: $DH \parallel EG$

$$DE = 30$$

$$EF = 45$$

$$GH = 25$$

$$FG = ?$$

$$\frac{DE}{EF} = \frac{GH}{FG}$$

$$FG = \frac{GH \cdot EF}{DE}$$

$$= \frac{(25)(45)}{30}$$

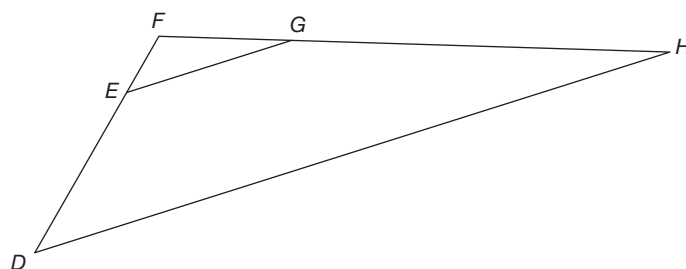
$$= 37.5$$

4.3 Applying the Converse of the Triangle Proportionality Theorem

The Converse of the Triangle Proportionality Theorem allows you to test whether two line segments are parallel.

- Converse of the Triangle Proportionality Theorem—If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

Example



$$\begin{aligned} \text{Given: } DE &= 33 & \frac{DE}{EF} &= \frac{GH}{FG} \\ EF &= 11 & \frac{33}{11} &= \frac{66}{22} \\ GH &= 22 & 3 &= 3 \\ FG &= 66 \\ \text{Is } \overline{DH} &\parallel \overline{EG}? \end{aligned}$$

Applying the Converse of the Triangle Proportionality, we can conclude that $\overline{DH} \parallel \overline{EG}$.

4.3 Applying the Proportional Segments Theorem

The Proportional Segments Theorem provides a way to calculate distances along three parallel lines, even though they may not be related to triangles.

- Proportional Segments Theorem—If three parallel lines intersect two transversals, then they divide the transversals proportionally.

Example

$$\text{Given: } L_1 \parallel L_2 \parallel L_3$$

$$AB = 52$$

$$BC = 26$$

$$DE = 40$$

$$EF = ?$$

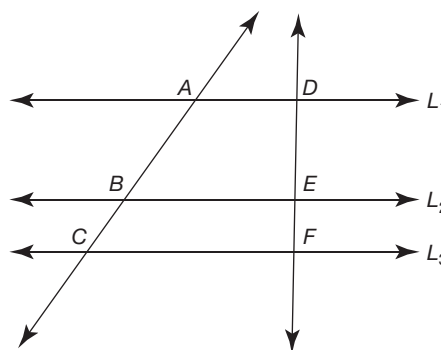
$$\frac{AB}{BC} = \frac{DE}{EF}$$

$$AB \cdot EF = DE \cdot BC$$

$$EF = \frac{DE \cdot BC}{AB}$$

$$= \frac{(40)(26)}{52}$$

$$= 20$$



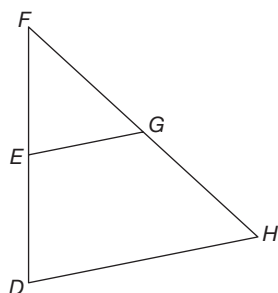
4.3

Applying the Triangle Midsegment Theorem

The Triangle Midsegment Theorem relates the lengths of the sides of a triangle when a segment is drawn parallel to one side.

- Triangle Midsegment Theorem—The midsegment of a triangle is parallel to the third side of the triangle and half the measure of the third side of the triangle.

Example



$$\begin{aligned} \text{Given: } DE &= 9 & EF &= 9 \\ FG &= 11 & GH &= 11 \\ DH &= 17 \end{aligned}$$

Since $DE = EF$ and $FG = GH$, point E is the midpoint of \overline{DF} , and G is the midpoint of \overline{FH} . \overline{EG} is the midsegment of $\triangle DFH$.

$$EG = \frac{1}{2}DH = \frac{1}{2}(17) = 8.5$$

4.4

Using the Geometric Mean and Right Triangle/Altitude Theorems

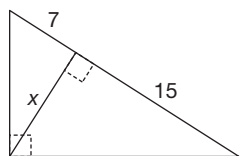
Similar triangles can be formed by drawing an altitude to the hypotenuse of a right triangle.

- Right Triangle/Altitude Similarity Theorem—If an altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other.

The altitude is the geometric mean of the triangle's bases. The geometric mean of two positive numbers a and b is the positive number x such as $\frac{a}{x} = \frac{x}{b}$. Two theorems are associated with the altitude to the hypotenuse as a geometric mean.

- The Right Triangle Altitude/Hypotenuse Theorem—The measure of the altitude drawn from the vertex of the right angle of a right triangle to its hypotenuse is the geometric mean between the measures of the two segments of the hypotenuse.
- The Right Triangle Altitude/Leg Theorem—If the altitude is drawn to the hypotenuse of a right triangle, each leg of the right triangle is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to the leg.

Example



$$\begin{aligned} \frac{7}{x} &= \frac{x}{8} \\ x^2 &= 56 \\ x &= \sqrt{56} \approx 7.5 \end{aligned}$$

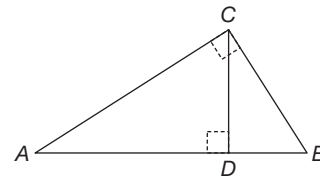
4.5 Proving the Pythagorean Theorem Using Similar Triangles

The Pythagorean Theorem relates the squares of the sides of a right triangle: $a^2 + b^2 = c^2$, where a and b are the bases of the triangle and c is the hypotenuse. The Right Triangle/Altitude Similarity Theorem can be used to prove the Pythagorean Theorem.

Example

Given: Triangle ABC with right angle C .

- Construct altitude CD to hypotenuse AB , as shown.
- According to the Right Triangle/Altitude Similarity Theorem, $\triangle ABC \sim \triangle CAD$.
- Since the triangles are similar, $\frac{AB}{CB} = \frac{CB}{DB}$ and $\frac{AB}{AC} = \frac{AC}{AD}$.
- Solve for the squares: $CB^2 = AB \times DB$ and $AC^2 = AB \times AD$.
- Add the squares: $CB^2 + AC^2 = AB \times DB + AB \times AD$
- Factor: $CB^2 + AC^2 = AB(DB + AD)$
- Substitute: $CB^2 + AC^2 = AB(AB) = AB^2$



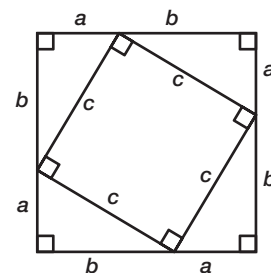
This proves the Pythagorean Theorem: $CB^2 + AC^2 = AB^2$

4.5 Proving the Pythagorean Theorem Using Algebraic Reasoning

Algebraic reasoning can also be used to prove the Pythagorean Theorem.

Example

- Write and expand the area of the larger square:
 $(a + b)^2 = a^2 + 2ab + b^2$
- Write the total area of the four right triangles:
 $4 \left(\frac{1}{2} ab \right) = 2ab$
- Write the area of the smaller square:
 c^2
- Write and simplify an equation relating the area of the larger square to the sum of the areas of the four right triangles and the area of the smaller square:
 $a^2 + 2ab + b^2 = 2ab + c^2$
 $a^2 + b^2 = c^2$



4.5

Proving the Converse of the Pythagorean Theorem

Algebraic reasoning can also be used to prove the Converse of the Pythagorean Theorem: “If $a^2 + b^2 = c^2$, then a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse.”

Example

Given: Triangle ABC with right angle C .

- Relate angles 1, 2, 3: $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$
- Use the Triangle Sum Theorem to determine $m\angle 1 + m\angle 2$.

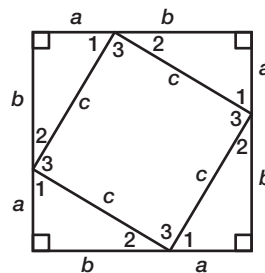
$$m\angle 1 + m\angle 2 = 90^\circ$$

- Determine $m\angle 3$ from the small right angles:

$$\text{Since } m\angle 1 + m\angle 2 = 90^\circ, m\angle 3 \text{ must also equal } 90^\circ.$$

- Identify the shape of the quadrilateral inside the large square: Since the quadrilateral has four congruent sides and four right angles, it must be a square.
- Determine the area of each right triangle: $A = \frac{1}{2}ab$
- Determine the area of the center square: c^2
- Write the sum of the areas of the four right triangles and the center square: $4\left(\frac{1}{2}ab\right) + c^2$
- Write and expand an expression for the area of the larger square: $(a + b)^2 = a^2 + 2ab + b^2$
- Write and simplify an equation relating the area of the larger square to the sum of the areas of the four right triangles and the area of the smaller square:

$$\begin{aligned} a^2 + 2ab + b^2 &= 2ab + c^2 \\ a^2 + b^2 &= c^2 \end{aligned}$$



4

4.6

Use Similar Triangles to Calculate Indirect Measurements

Indirect measurement is a method of using proportions to calculate measurements that are difficult or impossible to make directly. A knowledge of similar triangles can be useful in these types of problems.

Example

Let x be the height of the tall tree.

$$\begin{aligned} \frac{x}{20} &= \frac{32}{18} \\ x &= \frac{(32)(20)}{18} \\ x &\approx 35.6 \end{aligned}$$

The tall tree is about 35.6 feet tall.

