Real Number Systems

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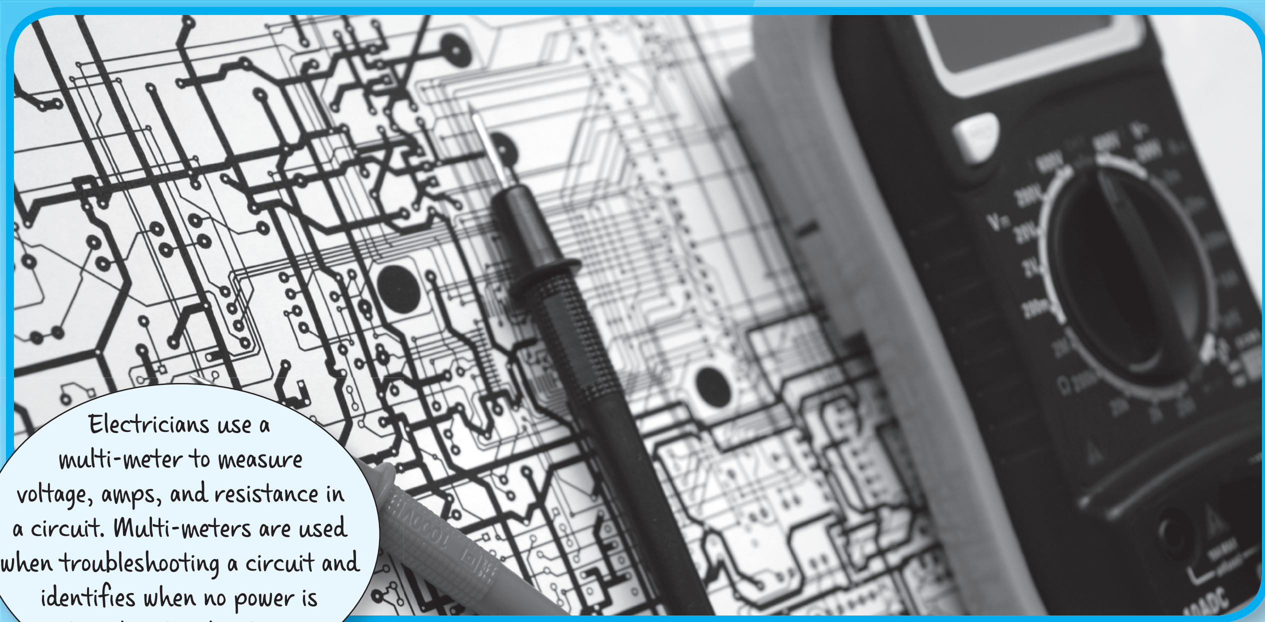

Chapter 15 Overview

This chapter begins by reviewing the real number system and then move to introducing the imaginary and ultimately the complex number system. Using the powers of exponents rules, students discover the necessity of the number *i*. This discovery leads to students exploring whether quadratic functions have one, two, or no real roots.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
15.1	The Numbers of the Real Number System	N.RN.3	1	This lesson reviews real number system. Questions ask students to compare natural and whole numbers, and integers, rational, and irrational numbers by completing a Venn diagram.	X	X	X		
15.2	Real Number Properties	N.RN.3	1	This lesson introduces the concept of closure under an operation. Questions then lead students to simplify expressions and provide rationale for each step performed.				X	
15.3	Imaginary and Complex Numbers	N.RN.1 N.RN.2 N.CN.1	1	This lesson introduces students to imaginary numbers. Questions ask students to use properties of exponent rules to discover the imaginary number *i*. Students then learn and interpret the complex number system.	X	X			
15.4	Complex Number Operations	N.CN.1 N.CN.2 N.CN.3(+) N.CN.8(+)	1	This lesson provides opportunities for students to understand how to operate with the imaginary number *i*. Questions ask students to explore various powers of *i* and practice rewriting expressions using *i*. Questions then focus students on adding, subtracting, multiplying, and dividing expressions containing *i*.		X	X		
15.5	Solving Quadratics with Complex Solutions	A.REI.4.b N.CN.1 N.CN.7	2	This lesson connects imaginary numbers to the concept of imaginary roots and imaginary zeros. Questions lead students to determine whether a quadratic function has one, two, or no real roots by examining the discriminant of the quadratic function.			X	X	

Skills Practice Correlation for Chapter 15

Lesson		Problem Set	Description
15.1	The Numbers of the Real Number System		Vocabulary
		1 – 6	Determine if numbers are included in a given number set
		7 – 12	Identify whether number sets are closed or not closed under addition, subtraction, multiplication, and division
		13 – 18	Identify equations that could be solved using given number sets
		19 – 28	Represent decimals as fractions
15.2	Real Number Properties	1 – 10	Identify properties demonstrated by given equations
		11 – 16	Identify properties, transformations, or simplifications used to simplify expressions
		17 – 26	Identify properties, transformations, or simplifications used to solve equations
15.3	Imaginary and Complex Numbers		Vocabulary
		1 – 10	Calculate powers of *i*
		11 – 18	Simplify numeric expressions using *i*
		19 – 24	Simplify algebraic expressions using *i*
		25 – 32	Determine the real and imaginary parts of complex numbers
		33 – 40	Identify sets that given numbers belong to
15.4	Complex Number Operations		Vocabulary
		1 – 6	Calculate powers of *i*
		7 – 14	Rewrite expressions using *i*
		15 – 22	Simplify expressions involving complex numbers
		23 – 28	Determine products of complex numbers
		29 – 34	Identify expressions as monomial, binomial, or trinomial
		35 – 40	Simplify expressions involving *i*
		41 – 48	Write conjugates of complex numbers
		49 – 54	Determine quotients of complex numbers
15.5	Solving Quadratics with Complex Solutions		Vocabulary
		1 – 6	Determine the number of roots of quadratic graphs then determine if the roots are real or imaginary
		7 – 14	Determine zeros of given functions

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The Real Numbers… For Realsies!

15.1

The Numbers of the Real Number System

LEARNING GOALS

In this lesson, you will:

- Define sets of natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers.
- Determine under which operations different sets of number are closed.
- Create a Venn diagram to show how different number sets are related.
- Determine which equations can be solved using different number sets.
- Write repeating decimals as fractions.

KEY TERMS

- natural numbers
- whole numbers
- closed (closure)
- counterexample
- integers
- rational numbers
- irrational numbers
- real numbers
- Venn diagram

ESSENTIAL IDEAS

- The set of natural numbers or counting numbers consists of the numbers that you use to count objects.
- The set of whole numbers consists of the set of natural numbers and the number 0.
- A number set has closure when you can perform any operation on any of the numbers in the set and the results in a number that is also in the same set.
- The set of integers consists of the set of whole numbers and their opposites.
- The set of rational numbers consists of all numbers that can be written as $\frac{a}{b}$, where a and b are integers.
- The set of irrational numbers consists of all numbers that cannot be written as $\frac{a}{b}$, where a and b are integers.
- The set of real numbers consists of the set of rational numbers and the set of irrational numbers.
- A Venn diagram uses circles to show how elements among sets of numbers of objects are related.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

N-RN The Real Number System

Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

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Overview

The sets of natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers are defined. The terms closure and Venn diagrams are introduced. Students determine which sets of numbers are closed under the operations of addition, subtraction, multiplication, and division. A Venn diagram is then used to show the relationship between the sets of numbers. An example of writing a repeating decimal as a fraction is provided and students use the example to represent other repeating decimals as fractions. Finally, students explore the relationship between 1 and the repeating decimal 0.999. . .

Warm Up

Represent each fraction as a decimal.

1. $\frac{1}{3}$

   $\frac{1}{3} = 3\overline{)1}$

   $3\overline{)1.000}$ = 0.333

2. $\frac{1}{9}$

   $\frac{1}{9} = 9\overline{)1}$

   $9\overline{)1.000}$ = 0.111

3. Describe the process of representing a fraction into a decimal.

   To represent a fraction into a decimal, divide the numerator by the denominator.

4. Describe the process of representing a decimal into a fraction.

   To represent a decimal into a fraction, count the digits to the right of the decimal and write those numbers in the numerator without a decimal. Then place a multiple of 10 in the denominator containing the same amount of zeros.

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15.1

The Real Numbers . . . For Realsies!

The Numbers of the Real Number System

LEARNING GOALS

In this lesson, you will:

- Define sets of natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers.
- Determine under which operations different sets of number are closed.
- Create a Venn diagram to show how different number sets are related.
- Determine which equations can be solved using different number sets.
- Write repeating decimals as fractions.

KEY TERMS

- natural numbers
- whole numbers
- closed (closure)
- counterexample
- integers
- rational numbers
- irrational numbers
- real numbers
- Venn diagram

Mathematicians have given some types of numbers special names. A repunit number is an integer with all 1's as digits. So, 11, 111, 1111, and so on, are all repunit numbers. A pronic number is a number that is the product of two consecutive counting numbers. The numbers 2, 6, and 12 are the first pronic numbers, because $1 \times 2 = 2$, $2 \times 3 = 6$, and $3 \times 4 = 12$.

Another set of numbers are referred to as the "lucky" numbers by some. To determine the lucky numbers, first start with all the counting numbers (1, 2, 3, 4, 5, and so on). Delete every second number. This will give you 1, 3, 5, 7, 9, and so on. The second number in that list is 3, so cross off every third number remaining. Now you have 1, 3, 7, 9, 13, and so on. The next number that is left is 7, so cross off every seventh number remaining.

Can you list all the "lucky" numbers less than 50?

Problem 1

Counting numbers or natural numbers are defined first. Then whole numbers and integers are introduced. Students determine which sets of numbers are closed under the operations of addition, subtraction, multiplication, and division. The terms rational numbers, irrational numbers, and real numbers are defined. Students design a Venn diagram to show the relationship between the various sets of numbers.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- Would you use the number 0 to count objects? Why or why not?
- What is the smallest number used to count?
- What is the difference between the set of natural numbers and the set of whole numbers?
- What is one example of needing the number 0 to represent something?

PROBLEM 1 Let's Take a Walk through Number History

Sometime in the history of humanity, it became necessary for people to count objects they saw or owned. This is how the idea of numbers came about. They probably began with the first set of numbers that you learned when you were very young, the counting numbers, or *natural numbers*.

The set of **natural numbers** consists of the numbers that you use to count objects.

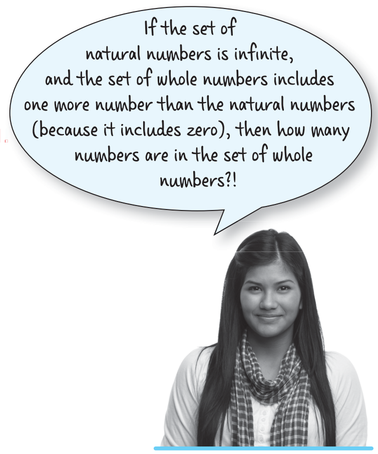
1. Is the set of natural numbers finite or infinite?
   The set of natural numbers is infinite.

2. How many natural numbers are between -1 and 1? List the natural number(s).
   There are no natural numbers between -1 and 1.

Once people recognized that they needed a number to represent the lack of any quantity, or zero, they used the set of *whole numbers*.

The set of **whole numbers** consists of the set of natural numbers and the number 0.

3. How many whole numbers are between -1 and 1? List the whole number(s).
   There is one whole number between -1 and 1. The whole number is 0.



It was probably not until people began performing operations on numbers that they discovered more numbers than those included in the set of whole numbers.

When an operation is performed on any of the numbers in a set and the result is a number that is also in the same set, the set is said to be **closed** (or to have **closure**) under that operation. For instance, the set of natural numbers is closed under addition because any natural numbers that are added together result in a sum that is a natural number.

- What is one example of needing the number -1 to represent something?
- Can you think of a situation when the use of negative numbers important?

Grouping

- Ask a student to read the information and definition that follows Question 3. Discuss as a class.
- Have students complete Questions 4 through 11 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 4 through 8

- When you multiply two natural numbers together, do you always get a product that is a natural number?
- What is an example of subtracting two natural numbers that gives you a difference that is not a natural number?
- What is an example of dividing two natural numbers that gives you a quotient that is not a natural number?
- What do natural numbers and whole numbers have in common?
- How is the set of natural numbers different than the set of whole numbers?
- What do integers and whole numbers have in common?
- How is the set of integers different than the set of whole numbers?
- What do integers and natural numbers have in common?
- How is the set of integers different than the set of natural numbers?

4. Under which other operations—subtraction, multiplication, division—is the set of natural numbers closed? If the set is closed under an operation, explain why. If the set is not closed, provide at least one example to show the set is not closed.

The set of natural numbers is closed under multiplication because multiplying any natural numbers results in a product that is a natural number.

The set of natural numbers is not closed under subtraction because $4 - 4 = 0$, and 0 is not a natural number.

The set of natural numbers is not closed under division because $2 \div 4 = 0.5$, and 0.5 is not a natural number.

To show that a set is *not* closed under an operation, you only need to determine *one* example that shows the result is not part of that set. This is called a **counterexample**.

5. Under which operations is the set of whole numbers closed? If the set is closed under an operation, explain why. If the set is not closed, provide at least one counterexample.

The set of whole numbers of whole numbers is closed under addition and multiplication because when I add or multiply any whole numbers, the result is always a whole number.

The set of whole numbers is not closed under subtraction because $4 - 5 = -1$, and -1 is not a whole number.

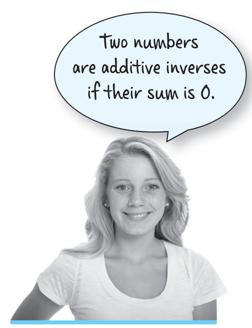
The set of whole numbers is not closed under division because $2 \div 4 = 0.5$, and 0.5 is not a whole number.

So, the need for people to perform certain operations on numbers is what most likely led to the set of numbers called *integers*.

The set of **integers** consists of the set of whole numbers and their opposites.

6. How many integers are between -2 and 2? List the integer(s).

There are three integers between -2 and 2. They are -1, 0, and 1.



7. Under which operations is the set of integers closed? If the set is closed under an operation(s), explain why. If the set is not closed under an operation(s), provide at least one counterexample for each operation.

The set of integers is closed under addition, subtraction, and multiplication because when I add, subtract, or multiply any integers, the result is always an integer.

The set is not closed under division because $2 \div 4 = 0.5$, and 0.5 is not an integer.

8. Compare the closure properties of whole numbers with the closure properties of integers. What do you notice?

They are the same except for subtraction. Integers provide closure under subtraction, while whole numbers do not.

- What is an example of dividing two integers that gives you a quotient that is also an integer?
- What is an example of dividing two integers that gives you a quotient that is not an integer?

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Guiding Questions for Share Phase, Questions 9 through 11

- Are all natural numbers considered rational numbers? Why or why not?
- Are all whole numbers considered rational numbers? Why or why not?
- Are all integers considered rational numbers? Why or why not?
- Can you think of a number that is not a rational number? What is it?
- Do you think pi is a rational number?

At some point, people were confronted with the problem of having to divide one thing among more than one person. From this dilemma came the set of *rational numbers*.

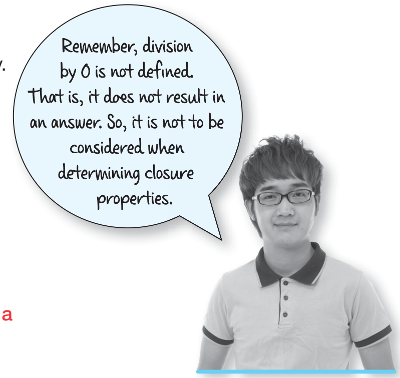
The set of **rational numbers** consists of all numbers that can be written as $\frac{a}{b}$, where a and b are integers, but b is not equal to 0.

9. How many rational numbers are between -1 and 1?

There are an infinite number of rational numbers between -1 and 1.

10. Under which operations is the set of rational numbers closed? If the set is closed under an operation(s), explain why. If the set is not closed under an operation(s), provide at least one counterexample for each operation.

The set of rational numbers is closed under addition, subtraction, multiplication, and division (division by zero is not defined) because if you complete any of these operations on rational numbers, the solution is always a rational number.



11. Compare the closure properties of integers with the closure properties of rational numbers. What do you notice?

They are the same except for division.

Rational numbers provide for closure under division, while integers do not.

Grouping

- Ask a student to read the information and definition that follows Question 11. Discuss as a class.
- Have students complete Questions 12 and 13 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 12 and 13

- Why do you suppose irrational numbers have no exact numerical representation?
- If a number is not rational, is it automatically considered irrational? Why or why not?
- Can you think of a number that is not a real number? If so, what does it look like?
- Have you ever seen a Venn diagram used to represent something? What was the situation?
- Is the set of natural numbers smaller than the set of whole numbers? Why or why not?
- If the set of numbers has fewer numbers in it, is it considered a smaller set of numbers? Why or why not?
- Do any of the sets of numbers described have a first number? Which ones?
- Do any of the sets of numbers described have a last number? Why not?

Eventually, people realized there are some numbers that are not rational numbers. Some examples are π, $\sqrt{2}$, and $\sqrt{3}$. These numbers cannot be written as fractions. They are called *irrational numbers*.

The set of **irrational numbers** consists of all numbers that cannot be written as $\frac{a}{b}$, where a and b are integers.

Irrational numbers can be represented by a symbol, such as π, or by using other notation, such as $\sqrt{2}$. Irrational numbers are often approximated by a decimal or fraction, such as $\pi \approx 3.14$ or $\pi \approx \frac{22}{7}$. However, they have no exact numerical representation.

The set of **real numbers** consists of the set of rational numbers and the set of irrational numbers.

12. Under which operations is the set of real numbers closed? If the set is closed under an operation, explain why. If the set is not closed, provide at least one counterexample.

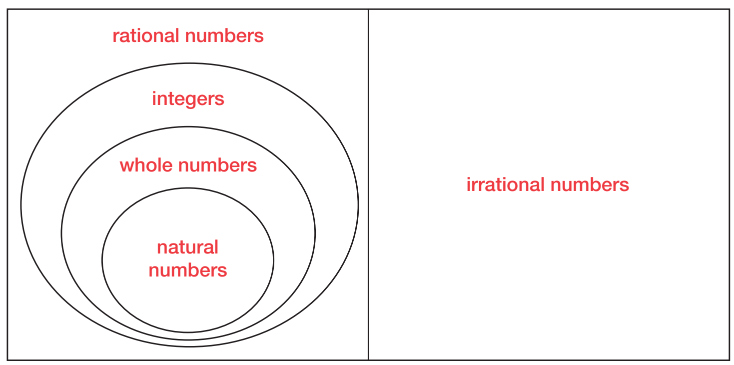
The set of real numbers is closed under addition, subtraction, multiplication, and division (division by zero is not defined).

A **Venn diagram** uses circles to show how elements among sets of numbers or objects are related.

13. Write each number set from the word box in its appropriate place in the Venn diagram.

natural numbers	whole numbers	integers
rational numbers	irrational numbers	

Real Numbers



Problem 2

Several equations are given and students choose which equations can be solved knowing only natural numbers, whole numbers, integers, rational numbers, or real numbers.

Grouping

Have students complete Questions 1 through 5 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- Which equations result in answers that are not included in the set of natural numbers?
- Which equations result in answers that are not included in the set of whole numbers?
- Which equations result in answers that are not integers?
- Which equations result in answers that are irrational numbers?

PROBLEM 2 Real Numbers and Equations

Consider the equations shown.

- Equation A: $3x = 9$
- Equation B: $x + 1 = 5$
- Equation C: $8x = 4$
- Equation D: $x^2 = 2$
- Equation E: $x + 5 = 1$
- Equation F: $x^2 = 4$
- Equation G: $x + 6 = 6$

1. Suppose that the only numbers you may use to evaluate the given equations are the natural numbers. Which equations could you solve?
   I could solve Equations A and B.

2. Suppose that the only numbers you may use to evaluate the given equations are the whole numbers. Which equations could you solve?
   I could solve Equations A, B, and G.

3. Suppose that the only numbers you may use to evaluate the given equations are the integers. Which equations could you solve?
   I could solve Equations A, B, E, F, and G.

4. Suppose that the only numbers you can use to evaluate the given equations are the rational numbers. Which equations could you solve?
   I could solve Equations A, B, C, E, F, and G.

5. Suppose that the only numbers you can use to evaluate the given equations are real numbers. Which equations could you solve?
   I could solve all the equations.

Problem 3

Rational numbers represented by repeating decimals are introduced and an example of changing a repeating decimal into a fraction is given. Students represent other repeating decimals as fractions. They then explore the relationship between the repeating decimal 0.999. . . and 1.

Grouping

- Ask a student to read the information and worked example. Discuss any student observations as a class.
- Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 1

- How do you know what multiple of 10 to multiply the equation by?
- If you multiplied by the wrong multiple of 10, how would it affect the answer? Would it still be correct?

PROBLEM 3 It's Repeating, It's Repeating, It's Repeating . . .

All irrational numbers have an infinite number of non-repeating decimal places. Any infinite decimal that repeats single digits or blocks of digits can be written as a fraction; therefore, it is a rational number.

You can use algebra to write a repeating decimal as a fraction. Consider the repeating decimal 0.313131 . . .

Step 1: Set the decimal equal to a variable.

$$v = 0.313131\ldots$$

Step 2: Multiply both sides of the equation by a multiple of 10 that has the same number of zeros as repeating digits in the decimal. In this case, the decimal has 2 repeating digits, so multiply both sides of the equation by 100.

$$100v = 31.313131\ldots$$

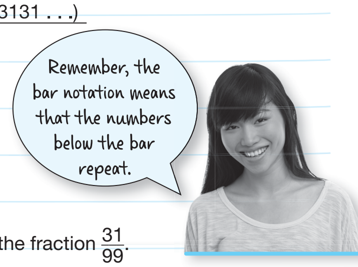
Step 3: Subtract the first equation from the second equation.

$$100v = 31.313131\ldots$$
$$\underline{-(v = 0.313131\ldots)}$$
$$99v = 31$$

Step 4: Solve for v.

$$\frac{99v}{99} = \frac{31}{99}$$
$$v = \frac{31}{99}$$

So, the repeating decimal $0.\overline{31}$ is equal to the fraction $\frac{31}{99}$.



1. Represent each decimal as a fraction. Show your work.

   a. 0.2222 . . .

   Let $x = 0.2222\ldots$

   $10x = 2.2222\ldots$

   $\underline{-(x = 0.2222\ldots)}$

   $9x = 2$

   $\frac{9x}{9} = \frac{2}{9}$

   $x = \frac{2}{9}$

   b. 0.512512 . . .

   Let $x = 0.512512\ldots$

   $1000x = 512.512512\ldots$

   $\underline{-(x = 0.512512\ldots)}$

   $999x = 512$

   $\frac{999x}{999} = \frac{512}{999}$

   $x = \frac{512}{999}$

15 Guiding Questions for Share Phase, Question 2

- If Donnie multiplied the equation by 100, would he still arrive at the same conclusion?
- Donnie's, Kris's, and Nathaniel's work supports the conclusion that $1 = 0.999\ldots$, can you think of an argument that does not support this conclusion?

2. Consider each student's work shown.

Donnie

Suppose that $x = 0.999\ldots$.

$10x = 9.999\ldots$

$-(x = 0.999\ldots)$

$9x = 9$

$\frac{9x}{9} = \frac{9}{9}$

$x = 1$

So, $0.999\ldots$ is equal to 1.

Kris

$\frac{1}{9} = 0.111\ldots$

$9 \cdot \frac{1}{9} = 9 \cdot 0.111\ldots$

$1 = 0.999\ldots$

The number 1 is equal to $0.999\ldots$.

Nathaniel

I know that $\frac{1}{3} = 0.333\ldots$ and $\frac{2}{3} = 0.666\ldots$.

$\frac{1}{3} + \frac{2}{3} = 0.333\ldots + 0.666\ldots$

$\frac{3}{3} = 0.999\ldots$

$1 = 0.999\ldots$

So, 1 equals $0.999\ldots$.

Are any of the students correct in their reasoning? Are any of the students incorrect in their reasoning? Do you think that $0.\bar{9}$ is equal to 1? Explain your reasoning.

Answers will vary.

There is no correct or incorrect answer for this question. The purpose for this question is to think about why this happens.
You may want to encourage students to do their own research about whether $0.\bar{9}$ is equal to 1.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Represent each decimal as a fraction.

1. 0.4444 . . .

$\frac{4}{9}$

2. 0.5555 . . .

$\frac{5}{9}$

3. 0.6666 . . .

$\frac{6}{9} = \frac{2}{3}$

4. 0.7777 . . .

$\frac{7}{9}$

5. 0.8888 . . .

$\frac{8}{9}$

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15.2

Getting Real, and Knowing How...

Real Number Properties

LEARNING GOALS

In this lesson, you will:

- Learn set notation.
- Make statements about real number properties using set notation.
- Identify the properties of the real number system including: commutative, associative, distributive, additive identity, multiplicative identity, additive inverse, and multiplicative inverse.

ESSENTIAL IDEAS

- The Commutative Property of Addition states that for all real numbers a and b, $a + b = b + a$. In other words, numbers can be added in any order and the sum is the same.
- The Commutative Property of Multiplication states that for all real numbers a and b, $a \cdot b = b \cdot a$. In other words, numbers can be multiplied in any order and the product is the same.
- The Associative Property of Addition states that for all real numbers a, b and c, $(a + b) + c = a + (b + c)$. In other words, numbers that are added can be grouped in any way and the sum is the same.
- The Associative Property of Multiplication states that for all real numbers a, b and c, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. In other words, numbers that are multiplied can be grouped in any way and the result is the same.
- The Distributive Property of Multiplication over Addition states that for all real numbers a, b and c, $a(b + c) = ab + ac$.
- The Distributive Property of Multiplication over Subtraction states that for all real numbers a, b, and c, $a(b - c) = ab - ac$.
- The Distributive Property of Division over Addition states that for all real numbers a, b, and c, $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.
- The Distributive Property of Division over Subtraction states that for all real numbers a, b, and c, $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$.
- The additive identity is 0 because $a + 0 = a$ for any real number a.
- The multiplicative identity is 1 because $a \cdot 1 = a$ for any real number a.
- The additive inverse of any real number a is $-a$.
- The multiplicative inverse of any real number a is $\frac{1}{a}$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

N-RN The Real Number System

Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

15

Overview

The symbols for "for all," "is an element of," and "the set of real numbers" are given. Students then use these symbols to describe various algebraic properties. Students explore the Commutative Property of Addition, the Commutative Property of Subtraction, the Associative Property of Addition, the Associative Property of Multiplication, the Distributive Property of Multiplication over Addition, the Distributive Property of Multiplication over Subtraction, the Distributive Property of Division over Addition, and the Distributive Property of Division over Subtraction. The terms additive identity, multiplicative identity, additive inverse, and multiplicative inverse are defined. Students solve simple equations and identify the appropriate algebraic property used in each step of the solution path. They finally complete a graphic organizer that lists each algebraic property by providing a description of the property using symbolic notation.

Warm Up

Determine if each mathematical statement is true or false. If the statement is false, explain why it is false.

1. $5 + 8 = 8 + 5$

   True

2. $5 - 8 = 8 - 5$

   False. The statement is false because $5 - 8 = -3$ and $8 - 5 = 3$.

3. $5 \cdot 8 = 8 \cdot 5$

   True

4. $5 \div 8 = 8 \div 5$

   False. The statement is false because $5 \div 8 = \frac{5}{8}$ and $8 \div 5 = \frac{8}{5}$.

15.2

Getting Real, and Knowing How . . .

Real Number Properties

LEARNING GOALS

In this lesson, you will:

- Learn set notation.
- Make statements about real number properties using set notation.
- Identify the properties of the real number system including: commutative, associative, distributive, additive identity, multiplicative identity, additive inverse, and multiplicative inverse.

Yikes! What did our teacher say?

At some point in school, you probably have had to take notes during class. More than likely, you didn't write word-for-word what your teacher said, but you might have written or typed keywords.

People who take notes often, such as secretaries or court reporters, may use shorthand. Shorthand is an abbreviated symbolic writing method that increases the speed of writing in comparison to writing full words. Therefore, a person writing in shorthand may be able to write more keywords or ideas during a presentation. Newspaper reporters also use shorthand to ensure that a quote they hear during a press conference can be captured and then added to a story.

If you use a cell phone or chat on your computer, you probably know something about shorthand. Where do you think most people use a type of shorthand? Do you think mathematics uses a type of shorthand? as well?

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Problem 1

The Commutative, Associative, and Distributive properties are reviewed. Students describe the algebraic properties using symbolic notation that is introduced in this problem. The terms additive identity, multiplicative identity, additive inverse, and multiplicative inverse are defined to conclude the problem.

Grouping

- Ask a student to read the information and complete Questions 1 and 2 as a class.
- Have students complete Questions 3 through 9 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 3 through 5

- How is $(a + b)$ read?
- How is $(a + b) + c$ read?
- How is $(a \cdot b)$ read?
- How is $(a \cdot b) \cdot c$ read?
- Is $8 - (6 - 4) = (8 - 6) - 4$?
- Is $(40 \div 2) \div 5 = 40 \div (2 \div 5)$?
- Is $3(10 - 2) = 3 \cdot 10 - 3 \cdot 2$?
- Is $\frac{2 + 10}{3} = \frac{2}{3} + \frac{10}{3}$?
- Is $\frac{10 - 1}{3} = \frac{10}{3} - \frac{1}{3}$?

PROBLEM 1 Operations on Real Numbers

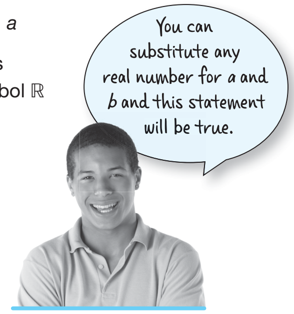
Addition and multiplication are commutative. Recall, commutative means that numbers can be added or multiplied in any order and the sum or product will be the same.

To show the Commutative Property of Addition, you can use the set notation shown:

$$\forall\, a, b \in \mathbb{R},\, a + b = b + a$$

The symbol ∀ is read as "for all." The symbol ∈ is read as "is an element of," or "are elements of." Finally, the symbol ℝ is read as "the set of real numbers."

So, the entire statement is read as:
For all numbers *a* and *b* that are elements of the set of real numbers, *a* plus *b* equals *b* + *a*.



1. Write the Commutative Property of Multiplication using similar set notation.
   $\forall\, a, b \in \mathbb{R},\, a \cdot b = b \cdot a$

2. Are subtraction and division commutative?
   If so, write the properties in similar set notation. If not, give counterexamples.
   Subtraction is not commutative because $7 - 4 \neq 4 - 7$. The resulting differences are 3 and -3.

   Division is not commutative because $9 \div 3 \neq 3 \div 9$. The resulting quotients are 3 and $\frac{1}{3}$.

Addition and multiplication are also associative. Recall, associative means that numbers that are added or multiplied can be grouped in any way and the sum or product is the same. The Associative Property of Addition can be written this way:

$$\forall\, a, b, c \in \mathbb{R},\, (a + b) + c = a + (b + c)$$

3. Write this statement in words.
   For all numbers *a*, *b*, and *c* that are elements of the set of real numbers, the sum of *a* and *b* plus *c* is equal to *a* plus the sum of *b* and *c*.

   a. Write the Associative Property of Multiplication using similar notation.
      $\forall\, a, b, c \in \mathbb{R},\, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

   b. Are division and subtraction associative? If so, write the properties in similar set notation. If not, give counterexamples.
      Subtraction is not associative: $9 - (7 - 4) \neq (9 - 7) - 4$, because $9 - 3$, or 6, is not equal to $2 - 4$, or -2.

      Division is not associative: $(12 \div 3) \div 2 \neq 12 \div (3 \div 2)$, because $4 \div 2$, or 2, is not equal to $12 \div \frac{3}{2}$, or 8.

Multiplication is distributive over addition. You can write the Distributive Property of Multiplication over Addition this way:

$$\forall\ a, b, c \in \mathbb{R},\ a(b + c) = ab + ac$$

4. Is multiplication distributive over subtraction? If so, write the property using similar set notation. If not, give a counterexample.

Yes. $\forall\ a, b, c \in \mathbb{R},\ a(b - c) = ab - ac$

5. Is division distributive over addition and subtraction? If so, write the property that is true in similar set notation. If not, provide a counterexample for the property that is not true.

Division is distributive over both addition and subtraction.

$\forall\ a, b, c \in \mathbb{R},\ \frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$

$\forall\ a, b, c \in \mathbb{R},\ \frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$

There is a number that when added to any real number *a*, the sum is equal to *a*. Recall, this number is called the additive identity.

6. What number is the additive identity? Explain your reasoning.

Zero is the additive identity because $a + 0 = a$ for any real number *a*.

Similarly, there is a number called the multiplicative identity that when multiplied by any real number *a*, the product is equal to *a*.

7. What number is the multiplicative identity? Explain your reasoning.

The multiplicative identity is 1 because $a \cdot 1 = a$ for any real number *a*.

When a number is added to its additive inverse, the sum is the additive identity.

8. For any real number *a*, what is its additive inverse? Explain your reasoning.

The additive inverse is the opposite, so for *a*, the additive inverse is $-a$. When the additive inverse is added to any real number *a*, the sum is 0.

When a number is multiplied by its multiplicative inverse, the product is the multiplicative identity.

9. For any real number *a*, what is its multiplicative inverse? Explain your reasoning.

The multiplicative inverse is the reciprocal, so for *a*, the multiplicative inverse is $\frac{1}{a}$.

When the multiplicative inverse is multiplied by any real number *a*, the product is 1.

Guiding Questions for Share Phase, Questions 6 through 9

- Does 0 added to any number result in the same number?
- Does 1 multiplied by any number result in the same number?
- What is the sign of the additive inverse of a negative number?
- What is the additive inverse of -12?
- What is the sign of the multiplicative inverse of a negative number?
- What is the multiplicative inverse of -12?
- What is the multiplicative inverse of $\frac{1}{7}$?
- What is the multiplicative inverse of $-\frac{2}{5}$?

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Problem 2

Students are given expressions that have been simplified step by step. They must identify the appropriate algebraic property associated with each step of the solution path. Students then complete a graphic organizer that lists each algebraic property by providing the meaning using symbolic notations.

Grouping

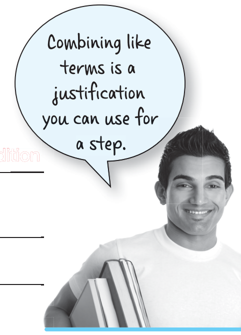
Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- How can you tell that the Distributive Property is used?
- How can you tell that the Commutative Property is used?
- How do you know if like terms are combined?
- How can you tell which Distributive Property is used?
- How can you tell which Associative Property is used?
- Is combining like terms always the last step when solving or simplifying? Why or why not?
- Will the Distributive Property always be used in the first step when solving or simplifying? Why or why not?

PROBLEM 2 Simplifying Expressions

1. Each expression has been simplified or solved one step at a time. Next to each step, identify the property, transformation, or simplification used in the step.



a. $4 + 5(x + 7)$

$4 + (5x + 35)$	Distributive Property of Multiplication over Addition
$5x + 4 + 35$	Commutative Property of Addition
$5x + (4 + 35)$	Associative Property of Addition
$5x + 39$	Combine like terms

b. $7x + 4 - 3(2x - 7)$

$7x + 4 - 6x + 21$	Distributive Property of Multiplication over Subtraction
$7x - 6x + 4 + 21$	Commutative Property of Addition
$(7x - 6x) + (4 + 21)$	Associative Property of Addition
$x + 25$	Combine like terms

c. $4(x + 3) - 5(x - 2)$

$4(x + 3) - 5x + 10$	Distributive Property of Multiplication over Subtraction
$4x + 12 - 5x + 10$	Distributive Property of Multiplication over Addition
$4x + -5x + 12 + 10$	Commutative Property of Addition
$-x + 22$	Combine like terms

2. Complete the graphic organizer. Write each property of the real number system using set notation from the lesson in the appropriate sections. Then provide examples of each property.

Real Number Properties

Commutative Properties

$\forall\ a, b \in \mathbb{R},\ a + b = b + a$

$\forall\ a, b \in \mathbb{R},\ a \cdot b = b \cdot a$

Associative Properties

$\forall\ a, b, c \in \mathbb{R},\ (a + b) + c = a + (b + c)$

$\forall\ a, b, c \in \mathbb{R},\ (a \cdot b) \cdot c = a \cdot (b \cdot c)$

Distributive Properties

$\forall\ a, b, c \in \mathbb{R},\ a(b + c) = ab + ac$

$\forall\ a, b, c \in \mathbb{R},\ a(b - c) = ab - ac$

$\forall\ a, b, c \in \mathbb{R},\ \frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$

$\forall\ a, b, c \in \mathbb{R},\ \frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$

Multiplicative Identity

The multiplicative identity is 1 because $a \cdot 1 = a$ for any real number a.

Multiplicative Inverse

The multiplicative inverse is the reciprocal, so for a, the multiplicative inverse is $\frac{1}{a}$.

Additive Identity

Zero is the additive identity because $a + 0 = a$ for any real number a.

Additive Inverse

The additive inverse is the opposite, so for a, the additive inverse is $-a$.

Be prepared to share your solutions and methods.

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Check for Students' Understanding

Match each example with the appropriate property.

Example	Algebraic Property
1. $3(5x + 4) = 3 \cdot 5x + 3 \cdot 4$ (H)	**A.** Combine Like Terms
2. $6x + x - 10 = 5x - 10$ (A)	**B.** Commutative Property of Multiplication
3. $8x + 2 - 3x - 6 = 8x - 3x + 2 - 6$ (J)	**C.** Distributive Property of Multiplication over Subtraction
4. $13 + 11 - 4 = 13 + (11 - 4)$ (K)	**D.** Additive Identity
5. $7 \cdot (5x \cdot 6) = (7 \cdot 5x) \cdot 6$ (G)	**E.** Additive Inverse
6. $3x \cdot 5 = 5 \cdot 3x$ (B)	**F.** Distributive Property of Division over Subtraction
7. $3(5x - 4) = 3 \cdot 5x - 3 \cdot 4$ (C)	**G.** Associative Property of Multiplication
8. $15 + 0 = 15$ (D)	**H.** Distributive Property of Multiplication over Addition
9. $19 + (-19) = 0$ (E)	**I.** Multiplicative Inverse
10. $14 \cdot \frac{1}{14} = 1$ (I)	**J.** Commutative Property of Addition
11. $\frac{15 - 3}{4} = \frac{15}{4} - \frac{3}{4}$ (F)	**K.** Associative Property of Addition
12. $-16 \cdot 1 = -16$ (L)	**L.** Multiplicative Identity

15.3

Imagine the Possibilities

Imaginary and Complex Numbers

LEARNING GOALS

In this lesson, you will:

- Determine powers of i.
- Simplify expressions involving imaginary numbers.
- Understand properties of the set of complex numbers.
- Determine the number sets to which numbers belong.

KEY TERMS

- exponentiation
- the number i
- imaginary numbers
- pure imaginary number
- complex numbers
- real part of a complex number
- imaginary part of a complex number

ESSENTIAL IDEAS

- Exponentiation means to raise a quantity to a power.
- Any real number raised to an integer exponent is a real number.
- The number i is a number such that $i^2 = -1$.
- The set of imaginary numbers is the set of all numbers written in the form $a + bi$, where a and b are real numbers and b is not equal to 0.
- A pure imaginary number is a number of the form bi, where b is not equal to 0.
- The set of complex numbers is the set of all numbers written in the form $a + bi$, where a and b are real numbers.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

N-RN The Real Number System

Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

N-CN The Complex Number System

Perform arithmetic operations with complex numbers.

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

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Overview

The imaginary number, i is defined such that $i^2 = -1$ and students compute the value of i to the first four powers. They then explore other possible powers of i and conclude that all powers of i result in one of four possible values, $\sqrt{-1}$, -1, $-\sqrt{-1}$, or 1. Students practice calculating the value of different powers of i, and simplify expressions using i. Next, the sets of imaginary numbers and complex numbers are introduced. A flow chart shows the connections between all possible sets of numbers and students complete statements and identify numbers using their cumulative knowledge of each set of numbers.

Warm Up

Write each number two different ways as a product of two numbers each raised to a power.

1. 3^5

   $3^5 = 3^4 \cdot 3^1$

   $3^5 = 3^3 \cdot 3^2$

2. 2^6

   $2^6 = 2^4 \cdot 2^2$

   $2^6 = 2^5 \cdot 2^1$

3. 5^{10}

   $5^{10} = 5^2 \cdot 5^8$

   $5^{10} = 5^4 \cdot 5^6$

4. Describe the power rule used to answer Questions 1 through 3.

   When multiplying two numbers with the same base and different exponents, the exponents are added and the base number remains the same.

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15.3

Imagine the Possibilities

Imaginary and Complex Numbers

LEARNING GOALS

In this lesson, you will:

- Determine powers of *i*.
- Simplify expressions involving imaginary numbers.
- Understand properties of the set of complex numbers.
- Determine the number sets to which numbers belong.

KEY TERMS

- exponentiation
- the number *i*
- imaginary numbers
- pure imaginary number
- complex numbers
- real part of a complex number
- imaginary part of a complex number

There are numbers in mathematics known as *imaginary numbers*. The concept of "imaginary" numbers may seem absurd, but imaginary numbers are used by people such as electrical engineers and airplane designers. For example, electrical engineers use imaginary numbers to analyze voltage and currents of various electronic devices and electronic outputs. Imaginary numbers are as real to these people as other numbers are to you.

Are imaginary numbers as real as other numbers? Or are all numbers equally imaginary?

15

Problem 1

The term exponentiation is introduced and students examine the value of numbers having integer exponents. They determine the set of real numbers is not closed for all real number exponents when a situation results in a negative radicand. The imaginary number, *i* is then defined such that $i^2 = -1$ and students compute the value of *i* to the first four powers. They explore other possible powers of *i* and conclude that all powers of *i* result in one of four possible values, $\sqrt{-1}$, -1, $-\sqrt{-1}$ or 1. Students then practice calculating the value of different powers of *i*, and simplify expressions using *i*.

Grouping

- Ask a student to read the information and definition. Complete Question 1 as a class.
- Have students complete Questions 2 and 3 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 2 and 3

- When the base number has a whole number exponent, what does that tell you about the value of the number?
- When the base number has a fractional exponent, what does that tell you about the value of the number?

PROBLEM 1 Imagine, if You Will . . .

You determined that the set of real numbers is closed under the operations of addition, subtraction, multiplication, and division. Let's explore whether the set of real numbers is closed under another operation called *exponentiation*. **Exponentiation** means to raise a quantity to a power.

1. Are the real numbers closed for all integer exponents? Explain why or why not. If not, provide a counterexample.

   The real numbers are closed for all integer exponents. Any real number raised to an integer exponent is a real number.

Now, let's consider real number exponents.

2. Simplify each expression, if possible.

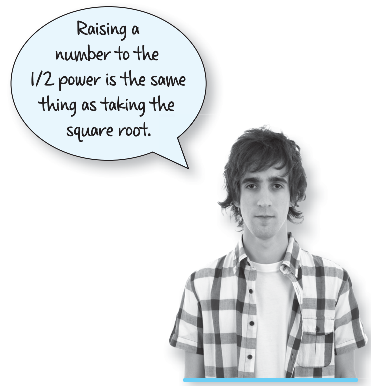
   a. $4^2 =$ 16

   b. $4^{\frac{1}{2}} =$ 2

   c. $(-4)^2 =$ 16

   d. $-(4)^2 =$ −16

   e. $(-4)^{\frac{1}{2}} =$?



3. Is the set of real numbers closed for all real number exponents? Explain why or why not.

   The real numbers are not closed for all real number exponents.

   Counterexample: $(-4)^{\frac{1}{2}} = \sqrt{-4}$ is not possible in the real number system.

In order for the real numbers to be closed for all real number exponents, there must be some way to calculate the square root of a negative number. That is, there must be a number such that when it is squared, it is equal to a negative number.

- When the base number is negative, when is the value of the number negative?
- When the base number is negative, when is the value of the number positive?
- What effect does a negative symbol have on the value of a positive base number within a parentheses?
- Why is the value of part (e) not a real number?

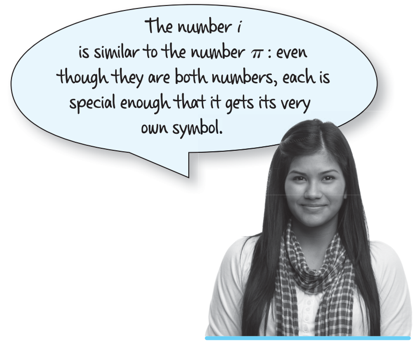
Grouping

- Ask a student to read the information and definition. Complete Questions 4 and 5 as a class.
- Have students complete Questions 6 and 7 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 6 and 7

- What does the word imaginary mean?
- Why do you suppose these numbers are called imaginary numbers?
- How are imaginary numbers immediately recognizable?
- What is the value of i^0?
- What is the difference between i and i^1?
- Why does $i^5 = i^4 \cdot i^1$?
- Why does $i^6 = i^4 \cdot i^2$?
- Why does $i^7 = i^4 \cdot i^3$?
- Why does $i^8 = i^4 \cdot i^4$?
- If you know the first four powers of i, how can you determine any power of i?
- How would you factor i^9? Predict the value of i^9.
- How would you factor i^{10}? Predict the value of i^{10}.
- How would you factor i^{11}? Predict the value of i^{11}.
- How would you factor i^{12}? Predict the value of i^{12}.

If a definition exists, then it must be possible to calculate any root of any real number. For this reason, mathematicians defined what is called *the number i*. **The number *i*** is a number such that $i^2 = -1$.



4. If $i^2 = -1$, then what is the value of i?

$i^2 = -1$

$\sqrt{i^2} = \sqrt{-1}$

$i = \sqrt{-1}$

If $i^2 = -1$, then $i = \sqrt{-1}$.

5. Write the values of the first four powers of i.

a. $i^1 = \sqrt{-1}$

b. $i^2 = -1$

c. $i^3 = (\sqrt{-1})^3 = \sqrt{-1} \cdot \sqrt{-1} \cdot \sqrt{-1} = -1 \cdot \sqrt{-1} = -\sqrt{-1}$

d. $i^4 = (\sqrt{-1})^4 = \sqrt{-1} \cdot \sqrt{-1} \cdot \sqrt{-1} \cdot \sqrt{-1} = -1 \cdot -1 = 1$

6. Use your answers in Question 5 to calculate each power of i. Part (a) has been done for you.

a. $i^5 = i^4 \cdot i^1 = 1 \cdot \sqrt{-1} = \sqrt{-1}$

b. $i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1$

c. $i^7 = i^4 \cdot i^3 = 1 \cdot (-\sqrt{-1}) = -\sqrt{-1}$

d. $i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1$

7. Compare your answers in Question 5 to your answers in Question 6. What do you notice?

I noticed that i^1 and i^5 are equal, i^2 and i^6 are equal, i^3 and i^7 are equal, and i^4 and i^8 are equal.

8. Calculate each power of i.

a. $i^{101} = i^{100} \cdot i^1 = (i^4)^{25} \cdot i^1 = (1)^{25} \cdot \sqrt{-1} = 1 \cdot \sqrt{-1} = \sqrt{-1}$

b. $i^{102} = i^{100} \cdot i^2 = (i^4)^{25} \cdot i^2 = (1)^{25} \cdot (-1) = 1 \cdot (-1) = -1$

c. $i^{103} = i^{100} \cdot i^3 = (i^4)^{25} \cdot i^3 = (1)^{25} \cdot (-\sqrt{-1}) = 1 \cdot (-\sqrt{-1}) = -\sqrt{-1}$

d. $i^{104} = (i^4)^{26} = (1)^{26} = 1$

Grouping

Have students complete Questions 8 and 9 with a partner. Then share the responses as a class.

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Guiding Questions for Share Phase, Questions 8 and 9

- How did you factor i^{101}? Which power of i did you need to know?
- How did you factor i^{102}? Which power of i did you need to know?
- How did you factor i^{103}? Which power of i did you need to know?
- How did you factor i^{104}? Which power of i did you need to know?

Grouping

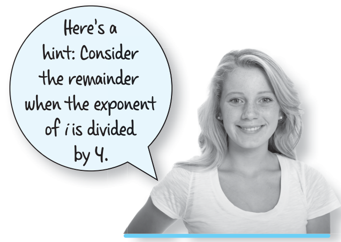
- Ask a student to read the worked example. Discuss as a class.
- Have students complete Question 10 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 10

- How is $\sqrt{-4}$ rewritten as a product of two radicals?
- How is $\sqrt{-12}$ rewritten as a product of two radicals?
- How is $\sqrt{-50}$ rewritten as a product of two radicals?
- How is $\sqrt{-8}$ rewritten as a product of two radicals?

9. Describe how you can calculate the value of any integer power of i.

To calculate the value of any integer power of i, divide the exponent of i by 4; the remainder is the equivalent power of i.



You can simplify expressions involving negative roots by using i.

The expression $\sqrt{-25}$ can be simplified.

Factor out -1.	$\sqrt{-25} = \sqrt{(-1)(25)}$
Rewrite the radical expression.	$= \sqrt{-1} \cdot \sqrt{25}$
Simplify $\sqrt{25}$.	$= 5\sqrt{-1}$
Rewrite $\sqrt{-1}$ as i.	$= 5i$

So, $\sqrt{-25}$ simplifies to $5i$.

10. Simplify each expression by using i.

a. $\sqrt{-4} = \sqrt{4} \cdot \sqrt{-1}$
$= 2i$

b. $\sqrt{-12} = \sqrt{12} \cdot \sqrt{-1}$
$= \sqrt{4} \cdot \sqrt{3} \cdot \sqrt{-1}$
$= 2\sqrt{3}\, i$

c. $5 + \sqrt{-50} = 5 + \sqrt{50} \cdot \sqrt{-1}$
$= 5 + \sqrt{25} \cdot \sqrt{2} \cdot \sqrt{-1}$
$= 5 + 5\sqrt{2}\, i$

d. $\frac{6 - \sqrt{-8}}{2} = \frac{6 - \sqrt{8} \cdot \sqrt{-1}}{2}$
$= \frac{6 - \sqrt{4} \cdot \sqrt{2} \cdot \sqrt{-1}}{2}$
$= \frac{6 - 2\sqrt{2}}{2}$
$= \frac{6}{2} - \frac{2\sqrt{2}\, i}{2}$
$= 3 - \sqrt{2}\, i$

Grouping

- Ask a student to read the example. Discuss as a class.
- Have students complete Question 11 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 11

- How many xi's are $1xi$ plus $1xi$?
- How many xi's are $1xi$ minus $1xi$?
- Are $3x$ and $-xi$ like terms? Can they be combined?
- Are x and $-xi$ like terms? Can they be combined?

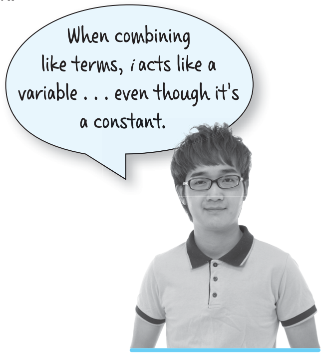
You can also simplify certain algebraic expressions involving i.

Consider the expression $(x - i)(x + i)$. To simplify the expression follow the steps shown.

Multiply binomials.	$(x - i)(x + i) = x^2 + xi - xi - i^2$
Group like terms.	$= x^2 + (xi - xi) - i^2$
Combine like terms.	$= x^2 - i^2$
Use the powers of i to rewrite i^2 as -1.	$= x^2 - (-1)$
Simplify.	$= x^2 + 1$

So, $(x - i)(x + i) = x^2 + 1$.

11. Simplify each algebraic expression. Show your work.



a. $xi + xi = 2xi$

b. $xi - xi = 0$

c. $3x + 5i - 2 - 3i - xi + x =$
$(3x + x) + (5i - 3i) - xi - 2 = 4x + 2i - xi - 2$

d. $(x + i)^2 = (x + i)(x + i)$
$= x^2 + xi + xi + i^2$
$= x^2 + 2xi + (-1)$
$= x^2 + 2xi - 1$

e. $(2x - i)(x - 3i) = 2x^2 - 6xi - xi + 3i^2$
$= 2x^2 - 7xi + 3i^2$
$= 2x^2 - 7xi + 3(-1)$
$= 2x^2 - 7xi - 3$

Problem 2

The set of imaginary numbers and the set of complex numbers are defined. The terms pure imaginary number, real part of a complex number, and imaginary part of a complex number are introduced and the notations associated with these terms are provided. Students complete several statements focusing on imaginary, real, and complex numbers.

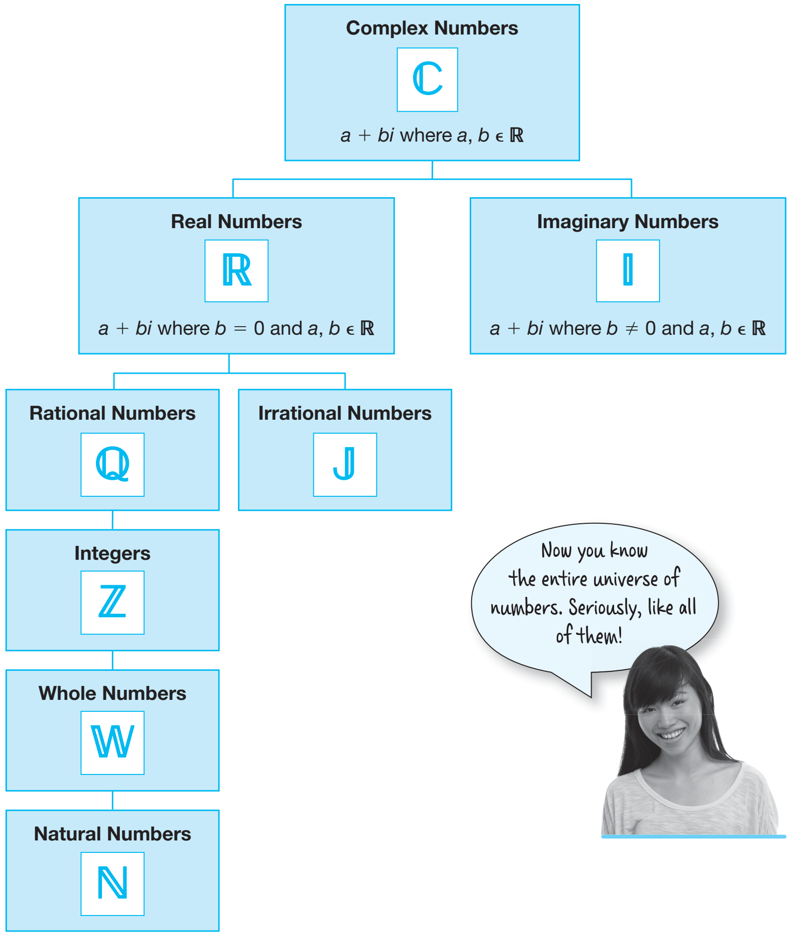
Grouping

Ask a student to read the information and definitions. Discuss any student observations of the given flowchart as a class.

PROBLEM 2 It's Getting a Bit Complex

The set of **imaginary numbers** is the set of all numbers written in the form $a + bi$, where a and b are real numbers and b is not equal to 0. The set of imaginary numbers is represented by the notation $\mathbb{I}$. A **pure imaginary number** is a number of the form bi, where b is not equal to 0.

The set of **complex numbers** is the set of all numbers written in the form $a + bi$, where a and b are real numbers. The term a is called the **real part of a complex number**, and the term bi is called the **imaginary part of a complex number**. The set of complex numbers is represented by the notation $\mathbb{C}$.



Grouping

Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- What is the difference between the set of imaginary numbers and the set of complex numbers?
- How would you describe the complex numbers that are not real numbers?
- How would you describe the complex numbers that are not imaginary numbers?
- What determines if a complex number is a real number or an imaginary number?

1. Complete each statement with *always*, *sometimes*, or *never*.

a. If a number is an imaginary number, then it is ___always___ a complex number.

b. If a number is a complex number, then it is ___sometimes___ an imaginary number.

c. If a number is a real number, then it is ___always___ a complex number.

d. If a number is a real number, then it is ___never___ an imaginary number.

e. If a number is a complex number, then it is ___sometimes___ a real number.

2. List *all* number sets from the word box that describe each given number.

natural number	imaginary number	irrational number	integer
rational number	whole number	complex number	real number

a. 3

natural number, whole number, integer, rational number, real number, complex number

b. $\sqrt{7}$

irrational number, real number, complex number

c. $3i$

imaginary number, complex number

d. $5.\overline{45}$

rational number, real number, complex number

e. $\frac{7}{8}$

rational number, real number, complex number

f. $6 - i$

imaginary number, complex number

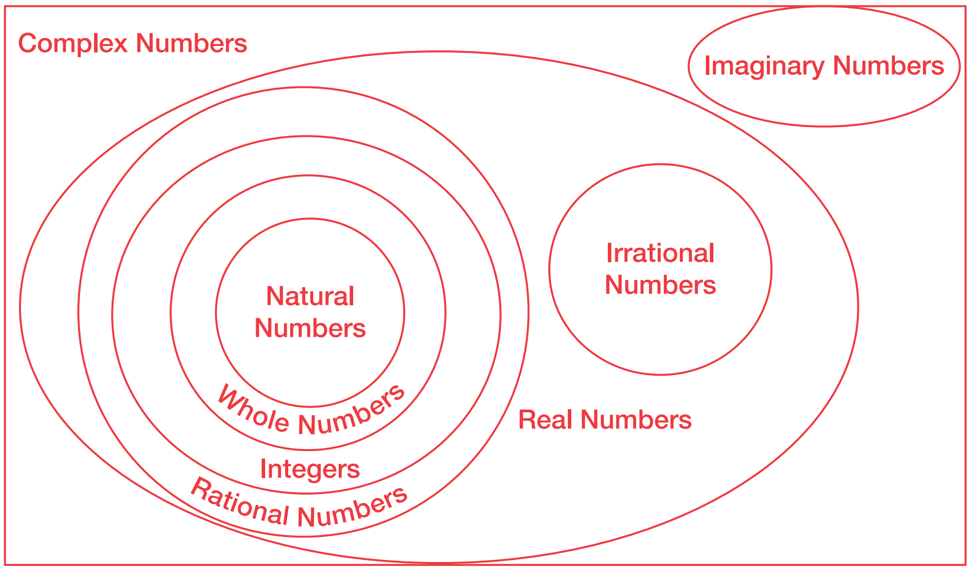
Be prepared to share your solutions and methods.

15

Check for Students' Understanding

Have students create a Venn diagram to show the relationships between the sets of numbers shown.

- Natural Numbers
- Whole Numbers
- Integers
- Rational Numbers
- Irrational Numbers
- Real Numbers
- Imaginary Numbers
- Complex Numbers



Complex Numbers
Imaginary Numbers
Real Numbers
Irrational Numbers
Rational Numbers
Integers
Whole Numbers
Natural Numbers

Now It's Getting Complex . . . But It's Really Not Difficult!

15.4

Complex Number Operations

LEARNING GOALS

In this lesson, you will:

- Calculate powers of *i*.
- Interpret the real numbers as part of the complex number system.
- Add, subtract, and multiply complex numbers.
- Add, subtract, and multiply complex polynomial expressions.
- Understand that the product of complex conjugates is a real number.
- Rewrite quotients of complex numbers.

KEY TERMS

- the imaginary number *i*
- principal square root of a negative number
- set of imaginary numbers
- pure imaginary number
- set of complex numbers
- real part of a complex number
- imaginary part of a complex number
- complex conjugates
- monomial
- binomial
- trinomial

ESSENTIAL IDEAS

- The imaginary number i is a number such that $i^2 = -1$.
- The set of imaginary numbers is the set of all numbers written in the form $a + bi$, where a and b are real numbers and b is not equal to 0.
- A pure imaginary number is a number of the form bi where b is not equal to 0.
- The set of complex numbers is the set of all numbers written in the form $a + bi$, where a and b are real numbers. The term a is called the real part of a complex number, and the term bi is called the imaginary part of a complex number.
- Complex conjugates are pairs of numbers of the form $a + bi$ and $a - bi$.
- The product of a pair of complex conjugates is always a real number.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

N–CN The Complex Number System

Perform arithmetic operations with complex numbers.

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Use complex numbers in polynomial identities and equations.

8. (+) Extend polynomial identities to the complex numbers.

Overview

The imaginary number *i* is defined as a number such that $i^2 = -1$ and students will compute the powers of *i* to the first four powers. They then explore other possible powers of *i* and conclude that all powers of *i* result in one of four possible values, *i*, −1, −*i*, or 1. Students then practice calculating the value of different powers of *i*, and rewriting expressions using *i*. Next, the sets of imaginary numbers and complex numbers are introduced. Students will create a flow chart showing the connections between all possible sets of numbers. They add, subtract, multiply, and divide algebraic expressions containing *i*. The terms complex conjugates, monomial, binomial, and trinomial are defined. Students then identify expressions as monomial, binomial, and trinomial. They simplify polynomials and analyze student work to determine the most efficient methods to simplify the polynomials.

Warm Up

Write each number two different ways as a product of two numbers each raised to a power.

1. 3^5

   $3^5 = 3^4 \cdot 3^1$

   $3^5 = 3^3 \cdot 3^2$

2. 2^6

   $2^6 = 2^4 \cdot 2^2$

   $2^6 = 2^5 \cdot 2^1$

3. 5^{10}

   $5^{10} = 5^2 \cdot 5^8$

   $5^{10} = 5^4 \cdot 5^6$

4. Describe the power rule used to answer Questions 1 through 3.

   When multiplying two numbers with the same base and different exponents, the exponents are added and the base number remains the same.

15

15.4

Now It's Getting Complex . . . But It's Really Not Difficult!

Complex Number Operations

LEARNING GOALS

In this lesson, you will:

- Calculate powers of *i*.
- Interpret the real numbers as part of the complex number system.
- Add, subtract, and multiply complex numbers.
- Add, subtract, and multiply complex polynomial expressions.
- Understand that the product of complex conjugates is a real number.
- Rewriting quotients of complex numbers.

KEY TERMS

- the imaginary number *i*
- principal square root of a negative number
- set of imaginary numbers
- pure imaginary number
- set of complex numbers
- real part of a complex number
- imaginary part of a complex number
- complex conjugates
- monomial
- binomial
- trinomial

"Let me hear the downbeat!" might be something you hear the lead singer tell the band to start a song during a performance. In fact for centuries, bands, ensembles, barber shop quartets, and orchestras relied on tempo and beats to sync up with other band members. Well, this is true for band members today, but there is also music that doesn't have any band members—but a single musician mixing it up on turntables or on a laptop! Of course, these solo musicians are called DJs who have been mixing it since the late 1960s.

The cornerstone of almost any DJ's music is the art of sampling. Sampling is taking a portion or a "sample" of one sound recording and repurposing it into another song. One of the most common samples is taking the drum beats. Many DJs will take four measures of drum beats (with each measure having 4 beats per measure), and reuse it to become the spine of their new piece. Sometimes those four drum-beat measures are repeated for an entire piece—and sometimes these pieces can last 20 to 30 minutes in duration with the DJ infusing other samples of vinyl noise, ambient sound effects, record scratches, and lyrics.

Even more recently, artists have been using technology to create mashups. Mashups generally use two or more pre-recorded songs (not just samples, but entirely mixed songs) and arranging them together to create a new musical piece. Do you think that mashups use this same concept of 4 measures of music to create new musical pieces? Why do you think "4" is so special in creating music?

15

Problem 1

The imaginary number i is defined as a number such that $i^2 = -1$. Students will compute the powers of i, record the values in a table and discover patterns. They conclude the values repeat after every four powers of i. Three methods for computing large powers of i are shown and students use any of these methods to compute powers of i. Students then use i to rewrite expressions that have negative square roots.

Grouping

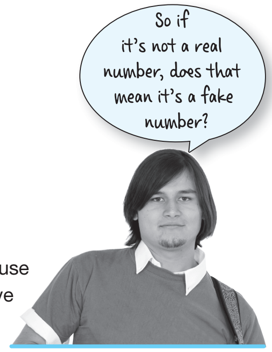
- Ask a student to read the information and definition. Complete Questions 1 and 2 as a class.
- Have students complete Questions 3 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- Which powers of i are equivalent to 1?
- Which powers of i are equivalent to -1?
- Which powers of i are equivalent to $\sqrt{-1}$?
- Which powers of i are equivalent to $\sqrt{-1}$?
- How many values are associated with powers of i?
- The values repeat after how many powers of i?
- Who used the division algorithm to solve for i^{15}?

PROBLEM 1 The Powers of *i*

So far within this course, you have worked within the set of real numbers and determined real number solutions. Remember, the set of real numbers includes the sets of rational and irrational numbers.



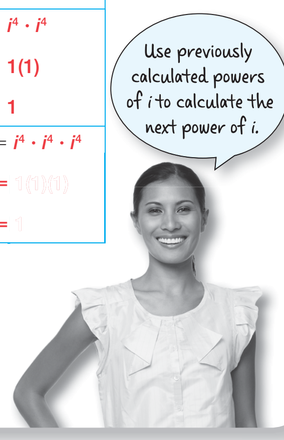
1. Consider the equation $x^2 = -1$.
   Is there a real number solution to this equation?
   Explain why or why not.
   No. I cannot take the square root of a negative number, so there is no real number solution to this equation.

The imaginary number *i* is a number such that $i^2 = -1$. Because no real number exists such that its square is equal to a negative number, the number i is not a part of the real number system.

2. If $i^2 = -1$, what is the value of i?
   $i^2 = -1$
   $\sqrt{i^2} = \sqrt{-1}$
   $i = \sqrt{-1}$

3. Use the values of i and i^2 and the properties of exponents to calculate each power of i. Enter your results in the table and show your work.

Powers of *i*			
$i = \sqrt{-1}$	$i^2 = -1$	$i^3 = i^1 \cdot i^2$ $= \sqrt{-1}(-1)$ $= -\sqrt{-1}$	$i^4 = i^2 \cdot i^2$ $= -1(-1)$ $= 1$
$i^5 = i^4 \cdot i^1$ $= 1(\sqrt{-1})$ $= \sqrt{-1}$	$i^6 = i^4 \cdot i^2$ $= 1(-1)$ $= -1$	$i^7 = i^4 \cdot i^3$ $= 1(-\sqrt{-1})$ $= -\sqrt{-1}$	$i^8 = i^4 \cdot i^4$ $= 1(1)$ $= 1$
$i^9 = i^4 \cdot i^4 \cdot i^1$ $= 1(1)(\sqrt{-1})$ $= \sqrt{-1}$	$i^{10} = i^4 \cdot i^4 \cdot i^2$ $= 1(1)(-1)$ $= -1$	$i^{11} = i^4 \cdot i^4 \cdot i^3$ $= 1(1)(-\sqrt{-1})$ $= -\sqrt{-1}$	$i^{12} = i^4 \cdot i^4 \cdot i^4$ $= 1(1)(1)$ $= 1$



4. Describe any patterns you see in the table.
   The values repeat after every four powers of i.

- Who used the definition of exponent and $i^2 = 1$ to solve for i^{15}?
- Who used the Power to a Power Rule and the Product Rule for Exponents to solve for i^{15}?

Guiding Questions for Share Phase, Question 5

- Whose method is most cumbersome?
- Why did Kira divide 15 by 4?

5. Tristan, Kira, and Libby calculated the power i^{15} using different methods as shown.

   a. Explain why each student's method is correct.

Tristan

$i^{15} = (i^4)^3 \cdot i^3$

$= (1)^3(-\sqrt{-i})$

$= 1(-\sqrt{-i})$

$= -\sqrt{-i}$

Kira

The exponent of i^{15} is 15. When I divide 15 by 4, I have a remainder of 3. I know $i^3 = -\sqrt{-i}$. So, $i^{15} = i^3 = -\sqrt{-i}$.

Tristan used the Power to a Power Rule and the Product Rule for Exponents to rewrite the expression with i^4, since $i^4 = 1$.

Kira recognized a pattern with every fourth power of i and used a division algorithm.

Libby

$i^{15} = i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i$

$= \underbrace{\sqrt{-i} \cdot \sqrt{-i} \cdot \sqrt{-i}}_{-i} \cdot \underbrace{\sqrt{-i} \cdot \sqrt{-i} \cdot \sqrt{-i}}_{-i} \cdot \underbrace{\sqrt{-i} \cdot \sqrt{-i} \cdot \sqrt{-i}}_{-i} \cdot \underbrace{\sqrt{-i} \cdot \sqrt{-i} \cdot \sqrt{-i}}_{-i} \cdot \underbrace{\sqrt{-i} \cdot \sqrt{-i} \cdot \sqrt{-i}}_{-i}$

$= -i \cdot \sqrt{-i}$

$= -i\sqrt{-i}$

Libby used the definition of exponent and the fact that $i^2 = -1$.

   b. If you had to calculate i^{99}, whose method would you use and why?

   Answers will vary. Students should choose an efficient method, like Tristan's or Kira's.

6. Explain how to calculate any integer power of i.

   Answers will vary.

   Students may describe a method from Questions 5, such as Tristan's or Libby's method, or they may come up with a method of their own.

15

Grouping

Have students complete Questions 7 through 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 7 and 8

- Which method did you use to calculate the power of i?
- Did your classmates use the same method?
- Will all methods result in the same answer?
- How did you work with the negative exponent?
- How did Georgette factor out -1? Why can't this be done?
- Does the i appear inside or outside of the radical sign?

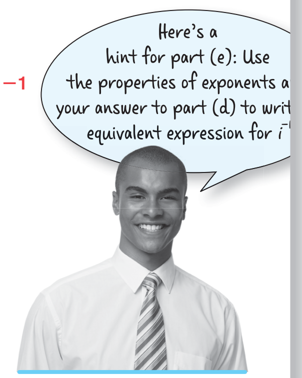
7. Calculate each power of i.

a. $i^{93} = (i^4)^{23} \cdot i^1$
$= 1^{23} i = i$

b. $i^{206} = (i^4)^{51} \cdot i^2$
$= 1^{23}(-1) = -1$

c. $i^{400} = (i^4)^{100}$
$= 1^{400} = 1$

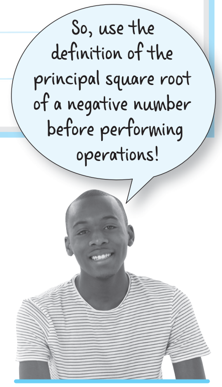
d. $i^{-2} = \frac{1}{i^2}$
$= \frac{1}{-1} = -1$



Now that you know about i, you can rewrite expressions involving negative roots. For any positive real number n, the **principal square root of a negative number,** $-n$, is defined by $\sqrt{-n} = i\sqrt{n}$.

Determine the value of $\sqrt{-75}$.

$\sqrt{-75} = \sqrt{(-1)(75)}$
$= \sqrt{-1} \cdot \sqrt{75}$
$= i\sqrt{25 \cdot 3}$
$= 5\sqrt{3}\,i$



8. Analyze Georgette's work.

Georgette

$\sqrt{-32} = -\sqrt{32}$
$= -\sqrt{16(2)}$
$= -4\sqrt{2}$

Explain why she is incorrect.

Georgette incorrectly factored out -1 from the square root.

Guiding Question for Share Phase, Question 10

How did you use the definition of the principal square root in each expression?

9. Jen and Tami each rewrote the expression $\sqrt{-4} \cdot \sqrt{-4}$.

Jen

$\sqrt{-4} \cdot \sqrt{-4}$

$= \sqrt{(-4)(-4)}$

$= \sqrt{16}$

$= 4$

Tami

$\sqrt{-4} \cdot \sqrt{-4}$

$= 2i \cdot 2i$

$= 4i^2$

$= -4$

Who's correct? Explain the error in the other student's reasoning.

Tami is correct. Jen did not use the definition of the principal square root of a negative number before performing multiplication.

10. Rewrite each expression using i.

a. $\sqrt{64} - \sqrt{-63} = 8 - \sqrt{9(7)(-1)} = 8 - 3\sqrt{7}\,i$

b. $\sqrt{-13} + 10 = \sqrt{13(-1)} + 10$

$= \sqrt{13}i + 10$

c. $\dfrac{1 - \sqrt{-44}}{2} = \dfrac{1}{2} - \dfrac{\sqrt{4(11)(-1)}}{2}$

$= \dfrac{1}{2} - \dfrac{2\sqrt{11}i}{2} = \dfrac{1}{2} - \sqrt{11}i$

15

Problem 2

The terms for the set of imaginary numbers, pure imaginary number, the set of complex numbers, the real part of a complex number, and the imaginary part of a complex number are defined. Students will identify complex numbers and explain the difference between complex numbers and imaginary numbers. They then create a diagram to show the relationship between various sets of numbers.

Grouping

- Ask a student to read the information and definitions. Complete Questions 1 through 3 as a class.
- Have students complete Questions 4 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- How can i be written in the form $a + bi$?
- How can 3 be written in the form $a + bi$?
- How can -5.5216 be written in the form $a + bi$?
- Is π a real number?

PROBLEM 2 I Must Have Only Imagined this Was Complex

The **set of imaginary numbers** is the set of all numbers written in the form $a + bi$, where a and b are real numbers and b is not equal to 0. A **pure imaginary number** is a number of the form bi, where b is not equal to 0.

1. Write the imaginary number i in the form $a + bi$. What are the values of a and b?

   $i = 0 + 1i$, so $a = 0$ and $b = 1$

2. Give an example of a pure imaginary number.

   Answers will vary.

   The number should be of the form bi, where b is a real number that is not equal to 0.

3. Can a number be both real and imaginary? Explain why or why not.

   A number $a + bi$ cannot be both real and imaginary because b must be equal to 0 if it is a real number and b must not be equal to 0 if it is an imaginary number.

The **set of complex numbers** is the set of all numbers written in the form $a + bi$, where a and b are real numbers. The term a is called the **real part of a complex number**, and the term bi is called the **imaginary part of a complex number**.

4. Identify whether each number is a complex number. Explain your reasoning.

   a. i

   This is a complex number because it can be written as $0 + 1i$, and 0 and 1 are real numbers.

   b. 3

   This is a complex number because it can be written as $3 + 0i$, and 3 and 0 are real numbers.

   c. -5.5216

   This is a complex number because it can be written as $-5.5216 + 0i$, and -5.5216 and 0 are real numbers.

   d. $\pi + 3.2i$

   This is a complex number because it can be written as $\pi + 3.2i$, and π and 3.2 are real numbers.

Guiding Questions for Share Phase, Questions 5 through 7

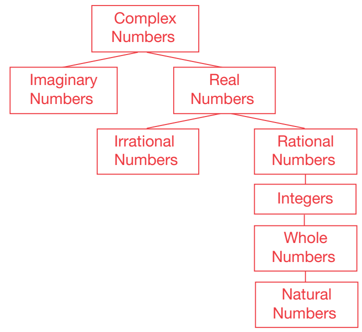
- If a number is imaginary, is it also complex? Why?
- Does the set of complex numbers include both the real numbers and the imaginary numbers?
- Are all numbers considered complex? Why?
- Are natural numbers, whole numbers, integers, and rational numbers all real numbers?
- Are imaginary numbers considered rational or irrational?

5. What is the difference between a complex number and an imaginary number?

If a number is imaginary, then it is also a complex number. If a number is complex, then it is either an imaginary number or a real number. The set of complex numbers includes both the real numbers and the imaginary numbers. Only some numbers are imaginary, while all numbers are complex.

6. Create a diagram to show the relationship between each set of numbers shown.

- complex numbers
- imaginary numbers
- integers
- irrational numbers
- natural numbers
- rational numbers
- real numbers
- whole numbers



7. Use the word box to complete each statement. Explain your reasoning.

always	sometimes	never

a. If a number is an imaginary number, then it is __always__ a complex number.

b. If a number is a complex number, then it is __sometimes__ an imaginary number.

c. If a number is a real number, then it is __always__ a complex number.

d. If a number is a real number, then it is __never__ an imaginary number.

e. If a number is a complex number, then it is __sometimes__ a real number.

15

Problem 3

Students will simplify expressions containing *i* by combining like terms and performing basic operations such as addition, subtraction, and multiplication. The terms complex conjugates, monomial, binomial, and trinomial is defined. Students then identify expressions as monomial, binomial, and trinomial. They simplify polynomials and analyze student work to determine the most efficient methods of solution.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- What algebraic properties were used to simplify the expression?
- Which terms are considered like terms?
- Which terms can be combined?
- What operations were used to simplify the expression?
- Is there more than one correct answer?
- What method did you use to determine the product?
- Are the products always equal to a whole number? How did that happen? What happened to the *i* terms?

PROBLEM 3 Call the Doctor, Stat! It's Time to Operate!

You know how to perform the basic operations of addition, subtraction, multiplication, and division on the set of real numbers. You can also perform these operations on the set of complex numbers.

When operating with complex numbers involving *i*, combine like terms by treating *i* as a variable (even though it is a constant).

1. Simplify each expression. Show your work.

   a. $(3 + 2i) - (1 - 6i) = 3 + 2i - 1 + 6i$
   $= (3 - 1) + (2i + 6i) = 2 + 8i$

   b. $4i + 3 - 6 + i - 1 = (4i + i) + (3 - 6 - 1)$
   $= 5i - 4$

   c. $5i(3 - 2i) = 15i - 10i^2$
   $= 15i - 10(-1)$
   $= 15i + 10$

   d. $(5 + 3i)(2 - 3i) = 10 - 15i + 6i - 9i^2$
   $= 10 - 15i + 6i - 9(-1)$
   $= (10 + 9) + (-15i + 6i) = 19 - 9i$

2. Determine each product.

   a. $(2 + i)(2 - i) = 4 - 2i + 2i - i^2$
   $= 4 - (-1)$
   $= 5$

   b. $\left(\frac{1}{2} + i\right)\left(\frac{1}{2} - i\right) = \frac{1}{4} - \frac{1}{2}i + \frac{1}{2}i - i^2$
   $= \frac{1}{4} - (-1)$
   $= \frac{5}{4}$

   c. $(3 + 2i)(3 - 2i) = 9 - 6i + 6i - 4i^2$
   $= 9 - 4(-1)$
   $= 13$

   d. $(1 - 3i)(1 + 3i) = 1 - 3i + 3i - 9i^2$
   $= 1 - 9(-1)$
   $= 10$

   e. What do you notice about each product?
   Each factor is the same, except for the sign between the real and imaginary parts. Also, the product is equal to a whole number.

Grouping

- Ask a student to read the definitions and information. Complete Question 3 as a class.
- Have students complete Questions 4 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- Is $3 + 5i$ considered one or two terms?
- Is $3 + 5i$ a constant term?
- How can the expression $-4xi + 5x - 5i + 1$ be rewritten to show one x-term and one constant term?
- How can this expression in part (c) be rewritten to show one x^2 term?

You may have noticed that the products in Question 2 did not contain an imaginary number, even though the original expression contained imaginary numbers. Each pair of expressions in Question 2 are called *complex conjugates*.

Complex conjugates are pairs of numbers of the form $a + bi$ and $a - bi$. The product of a pair of complex conjugates is always a real number and equal to $a^2 + b^2$.

Remember that a polynomial is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. The definition of a polynomial can now be extended to include imaginary numbers.

A polynomial in one variable is an expression of the form $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where the coefficients ($a_0, a_1, a_2, \ldots$) are complex numbers (real or imaginary) and the exponents are nonnegative integers.

A polynomial can have a special name, according to the number of terms it contains. A polynomial with one term is called **a monomial**. A polynomial with two terms is called a **binomial**. A polynomial with three terms is called a **trinomial**.

3. Maria says that the expression $3x + xi - 5$ is a trinomial because it has three terms. Dante says that the expression is a binomial because it can be rewritten as the equivalent expression $(3 + i)x - 5$, which has two terms. Jermaine says that it is not a polynomial. Who is correct? Explain your reasoning.

Dante is correct. The expression is a polynomial in one variable made up of x-terms and a constant term. Since i is a number, the x-terms can be combined as $(3 + i)x$, which is one x-term with a coefficient of $3 + i$. The constant term is -5.

4. Identify each expression as a monomial, binomial, trinomial, or other. Explain your reasoning.

a. $3 + 5i$

The expression is a monomial because $3 + 5i$ is the only term, and it is a constant term.

b. $-4xi + 2x - 5i + 1$

The expression is a binomial because it can be rewritten as $(-4i + 2)x + (-5i + 1)$, which shows one x-term and one constant term.

c. $\frac{3}{2}x^2 - \frac{1}{4}x^2i$

The expression is a monomial because it can be rewritten as $\left(\frac{3}{2} - \frac{1}{4}i\right)x^2$, which shows one x^2-term.

d. $1.5x + 3i + 0.5x^3i$

The expression is a trinomial because it has one x^3-term, one x-term, and one constant term.

You can simplify some polynomial expressions involving *i* using methods similar to those you used to operate with numerical expressions involving *i*.

5. Simplify each polynomial expression, if possible. Show your work.

a. $xi + xi = 2xi$

b. $xi + xy =$ This expression cannot be simplified.

c. $-2.5x + 3i - xi + 1.8i + 4x + 9 =$

$(-2.5x + 4x) + (3i + 1.8i) - xi + 9$

$= 1.5x + 4.8i - xi + 9$

d. $(x + 3i)^2 = x^2 + 3xi + 3xi + 9i^2$

$= x^2 + 6xi + 9(-1)$

$= x^2 + 6xi - 9$

e. $(2i - 4x)(i + x) = 2i^2 + 2xi - 4xi - 4x^2$

$= 2(-1) - 2xi - 4x^2$

$= -2 - 2xi - 4x^2$

Guiding Questions for Share Phase, Question 6

- What method did you use to multiply two binomials?
- Which student factored out the common factor of $(x + 3)$, and then multiplied from left to right?
- Which student multiplied first from left to right, and then, added like terms from left to right?
- Which student multiplied the factors from left to right?
- Which student rearranged the factors so they could multiply the complex conjugates first?

6. Analyze each method. Explain each student's reasoning. Then, identify which of the two methods seems more efficient and explain why.

a.

Shania

$$\begin{aligned}(2 - i)(1 + 2i)(2 + i) &= (2 - i)(2 + i)(1 + 2i)\\ &= (4 - i^2)(1 + 2i)\\ &= (4 - (-1))(1 + 2i)\\ &= 5(1 + 2i)\\ &= 5 + 10i\end{aligned}$$

Shania rearranged the factors so that she could multiply the complex conjugates first to calculate a real number.

Lindsay

$$\begin{aligned}(2 - i)(1 + 2i)(2 + i) &= (2 + 3i - 2i^2)(2 + i)\\ &= (2 + 3i - 2(-1))(2 + i)\\ &= (4 + 3i)(2 + i)\\ &= 8 + 10i + 3i^2\\ &= 8 + 10i + 3(-1)\\ &= 5 + 10i\end{aligned}$$

Lindsay multiplied the factors from left to right.

Shania used a more efficient method because she multiplied the complex conjugates first to get a real number, which made the multiplication simpler.

b.

Elijah

$(x + i) + (x + 3) + (x + 3i)(x + 3)$

$= (x^2 + 3x + xi + 3i) + (x^2 + 3x + 3xi + 9i)$

$= (x^2 + x^2) + (3x + 3x) + (xi + 3xi) + (3i + 9i)$

$= 2x^2 + 6x + 4xi + 12i$

Elijah multiplied first from left to right, and then, added like terms from left to right.

Aiden

$(x + i) + (x + 3) + (x + 3i)(x + 3) = (x + 3)(x + i + x + 3i)$

$= (x + 3)(2x + 4i)$

$= 2x^2 + 4xi + 6x + 12i$

Aiden factored out the common factor of $(x + 3)$, and then he multiplied from left to right.

Aiden used a more efficient method because he first factored out the common factor of $x + 3$, therefore, he did not have to perform as many operations.

Grouping

Have students complete Questions 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 7

- Does it make a difference which two of the three binomials are multiplied together first?
- Which two binomials are easiest to multiply together first? Why?

7. Simplify each expression.

a. $(2 - i)(1 + 2i)(2 + i)$

$= (2 - i)(2 + i)(1 + 2i)$

$= (4 - i^2)(1 + 2i)$

$= (4 - (-1))(1 + 2i)$

$= 5(1 + 2i)$

$= 5 + 10i$

b. $(x + i)(x + 3) + (x + 3i)(x + 3)$

$= (x + 3)(x + i + x + 3i)$

$= (x + 3)(2x + 4i)$

$= 2x^2 + 4xi + 6x + 12i$

Problem 4

Students will write the conjugate for a complex number, and then calculate the product. A worked example is provided to demonstrate how to rewrite the division of a complex number without a complex number in the denominator. Finally, students will rewrite a quotient without a complex number in the denominator.

Grouping

- Have students complete Question 1 with a partner. Then have students share their responses as a class.
- Ask a student to read the worked example and discuss as a class.

Guiding Questions for Share Phase, Question 1

- How do you identify the conjugate of a complex number?
- Describe the type of number that results from the product of complex conjugates.

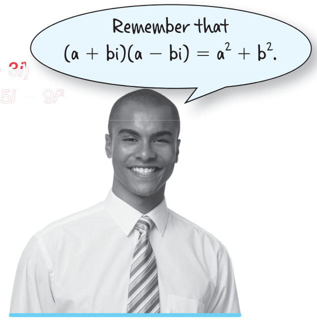
PROBLEM 4 Rewriting Quotients of Complex Numbers

Division of complex numbers requires the use of complex conjugates, thus changing the divisor into a real number. Recall, the complex conjugate of $a + bi$ is $a + bi$.

1. For each complex number, write its conjugate. Then calculate each product.

   a. $7 + i$
   $7 - i$
   $(7 + i)(7 - i)$
   $= 49 + 7i - 7i - i^2$
   $= 50$

   b. $-5 - 3i$
   $-5 + 3i$
   $(-5 - 3i)(-5 + 3i)$
   $= 25 - 15i + 15i - 9i^2$
   $= 34$



   c. $12 + 11i$
   $12 - 11i$
   $(12 + 11i)(12 - 11i)$
   $= 144 - 132i + 132i - 121i^2$
   $= 265$

   d. $-4i$
   $4i$
   $(-4i)(4i) = -16i^2$
   $= 16$

You can rewrite the division of a complex number by multiplying both the divisor and the dividend by the conjugate of the divisor, thus changing the divisor into a real number.

You can rewrite $\frac{3 - 2i}{4 + 3i}$ without a complex number in the denominator.

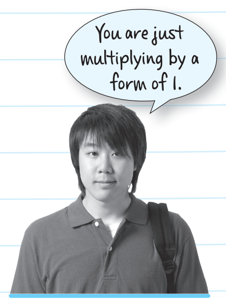
$$\frac{3 - 2i}{4 + 3i} = \frac{3 - 2i}{4 + 3i} \cdot \frac{4 - 3i}{4 - 3i}$$

$$= \frac{12 - 9i - 8i + 6i^2}{4^2 + 3^2}$$

$$= \frac{12 - 17i - 6}{16 + 9}$$

$$= \frac{6 - 17i}{25}$$

$$\frac{3 - 2i}{4 + 3i} = \frac{6}{25} - \frac{17}{25}i$$



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Grouping

Have students complete Question 2 with a partner. Then have students share their responses as a class.

Guiding Question for Share Phase, Question 2

What is the form of 1 you multiplied each expression by to remove the complex number from the denominator?

2. Rewrite each quotient without a complex number in the denominator.

a. $\frac{2-i}{3+2i} = \frac{2-i}{3+2i} \cdot \frac{3-2i}{3-2i}$

$= \frac{6-4i-3i+2i^2}{9-6i+6i-4i^2}$

$= \frac{4-7i}{13}$

$= \frac{4}{13} - \frac{7}{13}i$

b. $\frac{3-4i}{2-3i} = \frac{3-4i}{2-3i} \cdot \frac{2+3i}{2+3i}$

$= \frac{6+9i-8i-12i^2}{4+6i-6i-9i^2}$

$= \frac{18+i}{13}$

$= \frac{18}{13} + \frac{1}{13}i$

c. $\frac{5+2i}{1+i} = \frac{5+2i}{1+i} \cdot \frac{1-i}{1-i}$

$= \frac{5-5i+2i-2i^2}{1-i+i-i^2}$

$= \frac{7-3i}{2}$

$= \frac{7}{2} - \frac{3}{2}i$

d. $\frac{20-5i}{2-4i} = \frac{20-5i}{2-4i} \cdot \frac{2+4i}{2+4i}$

$= \frac{40+80i-10i-20i^2}{4+8i-8i-16i^2}$

$= \frac{60+70i}{20}$

$= 3 + \frac{7}{2}i$

Be prepared to share your solutions and methods.

Check for Students' Understanding

Consider the following sets of numbers:

- Natural numbers
- Rational numbers
- Imaginary numbers
- Whole numbers
- Irrational numbers
- Complex numbers
- Integers
- Real numbers

List all of the number sets form the above list that describes each given number.

1. 6

Natural number, Whole number, Integer, Rational number, Real number, Complex number

2. $\sqrt{6}$

Irrational number, Real number, Complex number

3. $6i$

Imaginary number, Complex number

4. $\frac{1}{6}$

Irrational number, Real number, Complex number

5. $6 - i$

Imaginary number, Complex number

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It's Not Complex—Just Its Solutions Are Complex!

15.5

Solving Quadratics with Complex Solutions

LEARNING GOALS

In this lesson, you will:

- Calculate complex roots of quadratic equations and complex zeros of quadratic functions.
- Interpret complex roots of quadratic equations and complex zeros of quadratic functions.
- Determine whether a function has complex solutions from a graph and from an equation in radical form.
- Determine the number of roots of a quadratic equation from a graph and from an equation in radical form.

KEY TERMS

- imaginary roots
- imaginary zeros

ESSENTIAL IDEAS

- Functions and equations that have imaginary solutions have imaginary roots or imaginary zeros.
- A quadratic function with a negative discriminant has imaginary zeros.
- A quadratic function written in standard form is $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers.
- A quadratic function written in factored form is $f(x) = a(x - b)(x - c)$, where a, b, and c are real numbers.
- A quadratic function written in vertex form is $f(x) = a(x - h)^2 + k$, with vertex (h, k).
- When the discriminant of a quadratic equation is a positive number, the equation has two real roots.
- When the discriminant of a quadratic equation is a negative number, the equation has two imaginary roots.
- When the discriminant of a quadratic equation is equal to zero, the equation has a double root (real).

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-REI Reasoning with Equations and Inequalities

Solve equations and inequalities in one variable

4. Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

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N-CN The Complex Number System

Perform arithmetic operations with complex numbers.

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

Use complex numbers in polynomial identities and equations.

7. Solve quadratic equations with real coefficients that have complex solutions.

Overview

The lesson begins with quadratic functions having one, two, or no x-intercepts are graphed on a coordinate plane. Students list the key characteristics of each graph. In the second activity, students identify the imaginary roots or zeros of a quadratic function using the Quadratic Formula. They will conclude that when the discriminant is a negative number the zeros of the function are imaginary. Students then explore the connections between a quadratic function with imaginary zeros written in standard form, factored form, and vertex form. In the last activity, students conclude that when the discriminant is equal to zero, the quadratic equation has a double real root, when the discriminant is equal to a positive number the equation has two real roots, and when the discriminant is equal to a negative number the equation has two imaginary roots.

Warm Up

Without graphing the quadratic functions, identify any obvious key characteristics you notice.

1. $f(x) = 3x^2 + 6$

   The parabola opens upward and the y-intercept is (0, 6).

2. $f(x) = -5x^2 - 2$

   The parabola opens downward and the y-intercept is (0, −2).

3. $f(x) = 7(x - 4)(x + 5)$

   The parabola opens upward and the x-intercept are (4, 0) and (−5, 0).

4. $f(x) = -2(x - 4)^2 + 8$

   The parabola opens downward, the axis of symmetry is $x = 4$, and the vertex is (4, 8).

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15.5

It's Not Complex—Just Its Solutions Are Complex!

Solving Quadratics with Complex Solutions

LEARNING GOALS

In this lesson, you will:

- Calculate complex roots of quadratic equations and complex zeros of quadratic functions.
- Interpret complex roots of quadratic equations and complex zeros of quadratic functions.
- Determine whether a function has complex solutions from a graph and from an equation in radical form.
- Determine the number of roots of a quadratic equation from a graph and from an equation in radical form.

KEY TERMS

- imaginary roots
- imaginary zeros

So, all this talk about real and imaginary numbers can appear to be quite *complex*—no pun intended—but really, it's a matter of determining where the x-intercepts occur—or if they even occur. You've successfully gone through this course determining roots, intercepts, and zeros of quadratics; and with the exception of one question, all the solutions you have encountered so far have been real. So, don't be worried: imaginary numbers are nothing to fret about—they really aren't *that* complex!

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Problem 1

Pairs of quadratic functions having one, two, or no x-intercepts are graphed on a coordinate plane. Students list the number of roots, coordinates of the x-intercepts, coordinates of the y-intercepts, the axis of symmetry, and the vertex of each graph.

Grouping

Have students complete Questions 1 through 3 with a partner. Then share the responses as a class.

Note

The Guiding Questions provided can be asked and answered for Questions 1 through 3. Students will be able to determine differences between the graphs in each question.

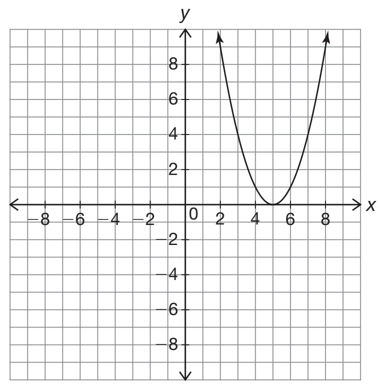
Guiding Questions for Share Phase, Questions 1 through 3

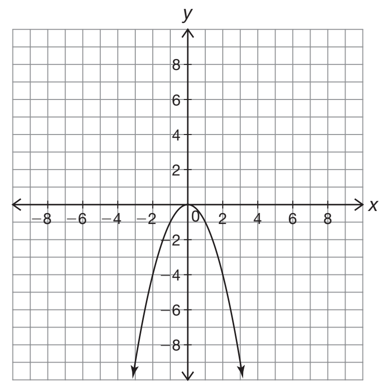
- Which graph describes a positive a-value in the quadratic function? Is the parabola drawn upward or downward?
- Which graph describes a negative a-value in the quadratic function? Is the parabola drawn upward or downward?
- What does the c-value in the quadratic function tell you about the graphical behavior of the parabola?

PROBLEM 1 Does It Intersect the x?

1. Consider the two quadratic functions and their graphs shown.

$f(x) = x^2 - 10x + 25$ $g(x) = -x^2$





a. List all the key characteristics you know about $f(x)$. Be sure to include the number of zeros, the x-intercept(s), the y-intercept, the axis of symmetry, and the vertex.

There is one zero and one x-intercept, (5, 0).
The y-intercept is (0, 25).
The axis of symmetry is $x = 5$.
The vertex is (5, 0).

b. List all the key characteristics you know about $g(x)$. Be sure to include the number of zeros, the x-intercept(s), the y-intercept, the axis of symmetry, and the vertex.

There is one zero and one x-intercept, (0, 0).
The y-intercept is (0, 0).
The axis of symmetry is $x = 0$.
The vertex is (0, 0).

c. Compare $f(x)$ and $g(x)$. What do they have in common? What is different about the two functions?

The functions $f(x)$ and $g(x)$ both have one zero and one x-intercept. All the other characteristics of the functions are different.

- Looking at the graph of each function, where is the root(s) of the quadratic equation?
- How many roots does each quadratic equation have?
- What is the difference between the root of the quadratic equation and the x-intercept of the graph of each quadratic function?
- Looking at the graph of each function, is the y-intercept of the quadratic equation viewable?

- Looking at the graph of each function, how would you know the location of the *y*-intercept of the quadratic equation?
- Looking at the equation of each function, how would you know the location of the *y*-intercept of the quadratic function?
- How do you determine the axis of symmetry of the quadratic function? Where is it located with respect to the graph of the function?
- How do you determine the coordinates of the vertex of the quadratic function? Where is it located with respect to the graph of the function?

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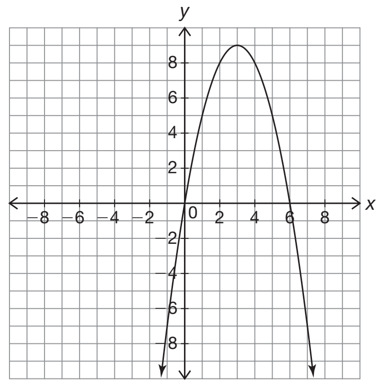
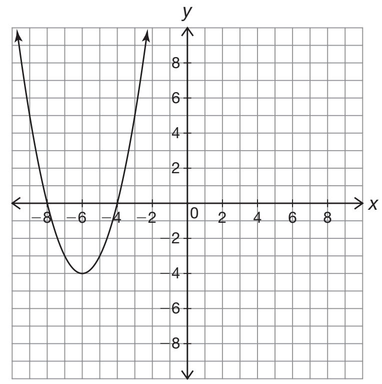
2. Consider the two quadratic functions and their graphs shown.

$c(x) = -x^2 + 6x$ $d(x) = x^2 + 12x + 32$





a. List all the key characteristics you know about $c(x)$. Be sure to include the number of zeros, the *x*-intercept(s), the *y*-intercept, the axis of symmetry, and the vertex.

There are two zeros and two *x*-intercepts, (0, 0) and (6, 0).
The *y*-intercept is (0, 0).
The axis of symmetry is $x = 3$.
The vertex is (3, 9).

b. List all the key characteristics you know about $d(x)$. Be sure to include the number of zeros, the *x*-intercept(s), the *y*-intercept, the axis of symmetry, and the vertex.

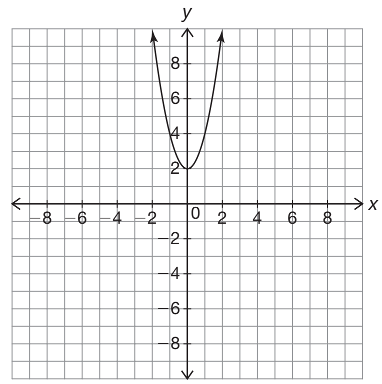
There are two zeros and two *x*-intercepts, (−8, 0) and (−4, 0).
The *y*-intercept is (0, 32).
The axis of symmetry is $x = -6$.
The vertex is (−6, −4).

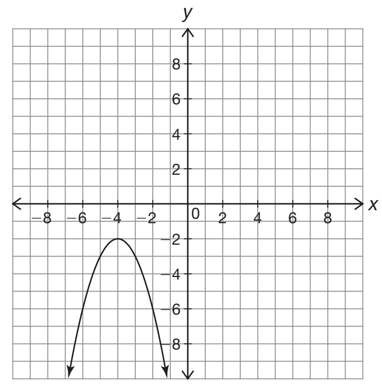
c. Compare $c(x)$ and $d(x)$. What do they have in common? What is different about the two functions?

The functions $c(x)$ and $d(x)$ both have two zeros and two *x*-intercepts. All the other characteristics of the functions are different.

3. Consider the two quadratic functions and their graphs shown.

$p(x) = 2x^2 + 2$ $\qquad$ $q(x) = -x^2 - 8x - 18$





a. List all the key characteristics you know about $p(x)$. Be sure to include the number of zeros, the x-intercept(s), the y-intercept, the axis of symmetry, and the vertex.

There are no real zeros and no x-intercepts.
The y-intercept is (0, 2).
The axis of symmetry is $x = 0$.
The vertex is (0, 2).

b. List all the key characteristics you know about $q(x)$. Be sure to include the number of zeros, the x-intercept(s), the y-intercept, the axis of symmetry, and the vertex.

There are no real zeros and no x-intercepts.
The y-intercept is (0, −18).
The axis of symmetry is $x = -4$.
The vertex is (−4, −2).

c. Compare $p(x)$ and $q(x)$. What do they have in common? What is different about the two functions?

The functions $p(x)$ and $q(x)$ both have no real zeros and no x-intercepts. All the other characteristics of the functions are different.

Problem 2

The terms imaginary roots or imaginary zeros are introduced. Students use the Quadratic Formula to determine the imaginary roots or zeros of a quadratic function. Students conclude that a quadratic function with a negative discriminant has imaginary zeros.

Grouping

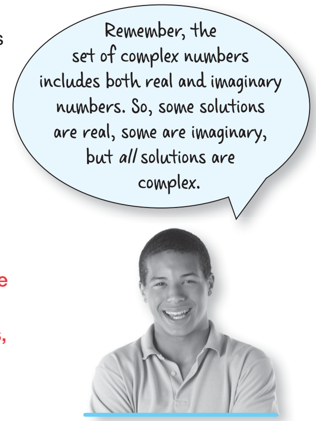
- Ask a student to read the information and definitions. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- If the parabola has two x-intercepts, what does this imply about the roots of the equation?
- If the parabola has one x-intercept, what does this imply about the roots of the equation?
- If the parabola has no x-intercepts, what does this imply about the roots of the equation?
- In Question 3, are the roots of this quadratic equation real or imaginary? How do you know?
- What is the graphic behavior of the parabola if the value of a in the quadratic equation is positive?
- What is the graphic behavior of the parabola if the value of a in the quadratic equation is negative?
- What is the graphic behavior of the parabola if the value of c in the quadratic equation is positive?
- What is the graphic behavior of the parabola if the value of c in the quadratic equation is negative?

PROBLEM 2 I See! No x-intercept Means Imaginary!

Before learning about the set of complex numbers, you probably would have said that the quadratic equations in Problem 1 Question 3 had no real solutions. Now you know that they have imaginary solutions. Functions and equations that have imaginary solutions have **imaginary roots** or **imaginary zeros**, which are the solutions.



1. How can you tell from the graph of a quadratic equation whether or not it has real solutions or imaginary solutions?

   If the graph of the quadratic equation intersects the x-axis, the equation has a real solution or real solutions. If the graph does not intersect the x-axis, the equation has imaginary solutions.

2. Do you think you can determine the imaginary solutions by examining the graph? Explain your reasoning.

   No. I cannot determine imaginary solutions by just examining a graph. I might be able to determine the approximate imaginary solutions; however, to determine the exact solutions will require algebra.

3. Recall the function $p(x) = 2x^2 + 2$.

   a. Use any method to solve $2x^2 + 2 = 0$.

   $a = 2, b = 0, c = 2$

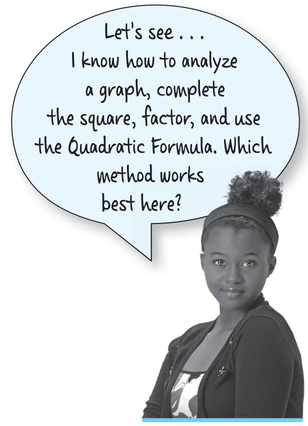
   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   $$x = \frac{0 \pm \sqrt{0^2 - 4(2)(2)}}{2(2)}$$

   $$x = \frac{0 \pm \sqrt{-16}}{4}$$

   $$x = \frac{0 \pm 4i}{4}$$

   $$x = \pm i$$



b. Consider a function written in the form $ax^2 + c = 0$. Complete the table to show when the solutions of a function are real or imaginary.

	c is positive	*c* is negative
a is positive	imaginary	real
a is negative	real	imaginary

4. Recall the function $q(x) = -x^2 - 8x - 18$.

a. Use any method to solve $-x^2 - 8x - 18 = 0$.

$a = -1, b = -8, c = -18$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

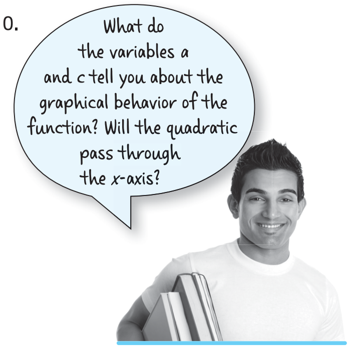
$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(-1)(-18)}}{2(-1)}$$

$$x = \frac{8 \pm \sqrt{64 - 72}}{-2}$$

$$x = \frac{8 \pm \sqrt{-8}}{-2}$$

$$x = \frac{8 \pm 2\sqrt{2}i}{-2}$$

$$x = -4 \pm \sqrt{2}i$$



b. Suppose you use the Quadratic Formula to solve the equation in part (a). How can you tell whether the solutions are real or imaginary?

If the quadratic function has a negative discriminant, it will have imaginary zeros.

Problem 3

Students first solve a quadratic function written in standard form using the Quadratic Formula to identify the imaginary roots. They then rewrite the function in factored form using the imaginary roots and simplify the factored form to get the original function. Students then identify the axis of symmetry to determine the coordinates of the vertex and rewrite the function again in vertex form.

Grouping

Have students complete Questions 1 through 7 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- What is the standard form of a quadratic function?
- What key characteristics are obvious when the quadratic function is written in standard form?
- What is the sign of the discriminant in the quadratic function? What does this imply with respect to the zeros of the function?

PROBLEM 3 Imaginary ≠ Impossible

Consider the function $f(x) = x^2 - 2x + 2$.

1. In what form is the quadratic function given?

   The function is given in standard form.

2. Determine the y-intercept of the function.

   The y-intercept is at (0, 2).

3. Use any method to determine the zeros of the function.

   $a = 1, b = -2, c = 2$

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   $$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

   $$x = \frac{2 \pm \sqrt{4 - 8}}{2}$$

   $$x = \frac{2 \pm \sqrt{-4}}{2}$$

   $$x = \frac{2 \pm 2i}{2}$$

   $$x = 1 \pm i$$

   The zeros of the function are $x = 1 + i$ and $x = 1 - i$.

4. Are the zeros of the function real or imaginary? Explain how you know.

   The zeros are imaginary. I know because the discriminant is negative and the solutions involve the number i.

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Guiding Questions for Share Phase, Questions 5 through 7

- What is the factored form of a quadratic function?
- What key characteristics are obvious when the quadratic function is written in factored form?

Recall that a quadratic function in factored form is written in the form $f(x) = a(x - r_1)(x - r_2)$.

5. What do r_1 and r_2 represent for a function written in this form?

The variables r_1 and r_2 represent the zeros of the function.

6. Use your answer to Question 3 to write the function $f(x) = x^2 - 2x + 2$ in factored form.

$f(x) = [x - (1 + i)][x - (1 - i)]$

7. Is the function you wrote in factored form the same as the original function in standard form? Simplify the function you wrote in Question 6 to verify your answer.

$f(x) = [x - (1 + i)][x - (1 - i)]$

$f(x) = x^2 - x(1 - i) - x(1 + i) + (1 + i)(1 - i)$

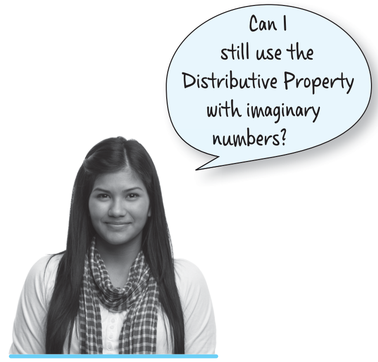
$f(x) = x^2 - x + xi - x - xi + 1 - i + i - i^2$

$f(x) = x^2 - 2x - i^2 + 1$

$f(x) = x^2 - 2x + 1 + 1$

$f(x) = x^2 - 2x + 2$

When I multiply, I get the original function in standard form.



Grouping

Have students complete Questions 8 through 12 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 8 through 12

- Why is the axis of symmetry of a parabola the average of the x-coordinates of the zeros of the function?
- How is the axis of symmetry of a parabola used to determine the coordinates of the vertex?
- What is the vertex form of a quadratic function?
- What key characteristics are obvious when the quadratic function is written in vertex form?
- If a quadratic function is written in standard form, is it ever obvious that the zeros will be imaginary zeros or real zeros?
- If the quadratic function is written in factored form, is it ever obvious that the zeros will be imaginary zeros or real zeros?
- If the quadratic function is written in vertex form, is it ever obvious that the zeros will be imaginary zeros or real zeros?

Recall that the axis of symmetry is the vertical line that passes through the vertex of a parabola and divides it in half.

8. Explain how to determine the axis of symmetry using the zeros of the function.

The axis of symmetry is the average of the x-coordinates of the zeros of the function, so I can add the x-coordinates of the zeros and divide by 2.

9. Determine the axis of symmetry for $f(x)$. Show your work.

$\frac{(1 + i) + (1 - i)}{2}$

$\frac{1 + i + 1 - i}{2}$

$\frac{2}{2} = 1$

The axis of symmetry is $x = 1$.

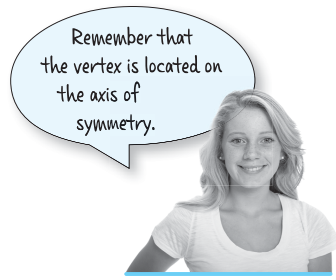
10. Use the axis of symmetry to determine the vertex of $f(x)$. Show your work.

$f(1) = (1)^2 - 2(1) + 2$

$= 1 - 2 + 2$

$= 1$

The vertex of the graph is at (1, 1).



Recall that a quadratic function in vertex form is written in the form $f(x) = a(x - h)^2 + k$ with vertex (h, k).

11. Rewrite the function $f(x) = x^2 - 2x + 2$ in vertex form.

$f(x) = (x - 1)^2 + 1$

12. Is the function you wrote in vertex form the same as the original function in standard form? Simplify the function you wrote in Question 11 to check.

$f(x) = (x-1)^2 + 1$

$f(x) = (x - 1)(x - 1) + 1$

$f(x) = x^2 - x - x + 1 + 1$

$f(x) = x^2 - 2x + 2$

Yes. The function I wrote in Question 11 is the same as the given function.

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Talk the Talk

A "Who's Correct?" problem focuses on the number of possible real and/or imaginary zeros of a quadratic function. Students conclude that when the discriminant is equal to zero, the quadratic equation has a double root. Students also conclude that when the discriminant is equal to a positive number the equation has two real roots. They then determine that when the discriminant is equal to a negative number the equation has two imaginary roots.

Grouping

Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Do all quadratic functions have exactly two zeros? Why or why not?
- Is it possible for a quadratic function to have only one imaginary zero? Why or why not?
- Do imaginary roots always come in pairs? Why or why not?
- What is the difference between two unique real number solutions and two equal real number solutions?

Talk the Talk

1. Casey says that any quadratic equation has only one of these 3 types of solutions:
   - 2 unique real number solutions
   - 2 equal real number solutions (a double root)
   - 1 real and 1 imaginary solution

   Brandon says that any quadratic equation has only one of these 3 types of solutions:
   - 2 unique real number solutions
   - 2 equal real number solutions (a double root)
   - 2 imaginary solutions

   Karl says that any quadratic equation has only one of these 4 types of solutions:
   - 2 unique real number solutions
   - 2 equal real number solutions (a double root)
   - 2 imaginary solutions
   - 1 real and 1 imaginary solution

   Who's correct? Explain your reasoning.

   Brandon is correct. Casey and Karl are incorrect.

   For a quadratic equation, the discriminant can be one of the following: zero, positive, or negative. If the discriminant is zero the equation has 2 equal real number solutions (a double root), if the discriminant is positive the equation has 2 unique real number solutions, and if the discriminant is negative the equation has 2 imaginary solutions. So, a quadratic equation will have only one of the following three types of solutions: two unique real number solutions, 2 equal real number solutions (a double root), or 2 imaginary solutions.

2. Explain why it is not possible for a quadratic equation to have 2 equal imaginary solutions (double imaginary root).

   When a quadratic equation has a double root, the discriminant is 0, which means that the expression inside the radical is 0. The only way to get an imaginary solution is if the expression inside the radical is a negative number. So, when this expression is 0, it is not possible to have imaginary solutions.

Be prepared to share your solutions and methods.

Check for Students' Understanding

1. Without graphing, create a quadratic function that has two real zeros.

   $f(x) = (x + 5)(x - 8)$

   $f(x) = x^2 - 3x - 40$

2. Explain how you know the quadratic function you wrote has two real zeros.

   I know the quadratic function has two real zeros because I wrote the function first in factor form and wrote the two real zeros as factors.

3. Without graphing, create a quadratic function that has a double real zero.

   $f(x) = (x + 5)(x + 5)$

   $f(x) = x^2 + 10x + 25$

4. Explain how you know the quadratic function you wrote has two real zeros.

   I know the quadratic function has a double real zero because I wrote the function first in factor form and wrote the two real zeros as factors making sure each factor was the same.

5. Without graphing, create a quadratic function that has two imaginary zeros.

   $f(x) = x^2 + 40$

6. Explain how you know the quadratic function you wrote has two real zeros.

   I know the quadratic function has two imaginary zeros because the *a* value and the *c* value are both positive.

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Chapter 15 Summary

KEY TERMS

- natural numbers (15.1)
- whole numbers (15.1)
- closed (closure) (15.1)
- counterexample (15.1)
- integers (15.1)
- rational numbers (15.1)
- irrational numbers (15.1)
- real numbers (15.1)
- Venn diagram (15.1)
- exponentiation (15.3)
- the number *i* (15.3)
- imaginary numbers (15.3)
- pure imaginary number (15.3)
- complex numbers (15.3)
- real part of a complex number (15.3)
- imaginary part of a complex number (15.3)
- the imaginary number *i* (15.4)
- principal square root of a negative number
- set of imaginary numbers (15.4)
- pure imaginary number (15.4)
- set of complex numbers (15.4)
- real part of a complex number (15.4)
- imaginary part of a complex number (15.4)
- complex conjugates (15.4)
- monomial (15.4)
- binomial (15.4)
- trinomial (15.4)
- imaginary roots (15.5)
- imaginary zeros (15.5)

15.1 Defining Sets of Numbers in the Real Number System

Numbers can be classified into sets based on their characteristics.

Set Name	Description	Examples
Natural numbers	Counting numbers	1, 2, 3, . . .
Whole numbers	Natural numbers and the number zero	0, 1, 2, 3, . . .
Integers	Whole numbers and their additive inverses	−3, −2, −1, 0, 1, 2, 3, . . .
Rational numbers	All numbers that can be written as $\frac{a}{b}$, where *a* and *b* are integers	$\frac{1}{2}$, $-\frac{4}{7}$, 0.75, 0.01, 3, $6.\overline{51}$
Irrational numbers	All numbers that cannot be written as $\frac{a}{b}$, where *a* and *b* are integers	$\sqrt{2}$, π, 0.342359 . . .
Real numbers	All rational and irrational numbers	$-\frac{3}{2}$, 0, 4, $\sqrt{5}$, 3.222, −10

Examples

14 is a natural number, whole number, integer, rational number, and real number.

$2\frac{1}{2}$ is a rational number and a real number.

−1000 is an integer, rational number, and real number.

$\sqrt{3}$ is an irrational number and a real number.

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15.1 Determining Closure for Sets of Numbers in the Real Number System

When an operation is performed on any of the numbers in a set and the result is a number in that same set, the set is said to be closed, or have closure, under that operation. If a set is not closed under an operation, a counterexample can be used to show a result that is not part of the set.

Examples

Add any two integers and the sum is an integer. The set of integers is closed under addition.

Subtract any two natural numbers and the difference may not be a natural number. For example, $2 - 5 = -3$, and -3 is not a natural number. The set of natural numbers does not have closure under subtraction.

Multiply any two irrational numbers and the product may not be an irrational number. For example, $\sqrt{3} \cdot \sqrt{3} = 3$, and 3 is not an irrational number. The set of irrational numbers do not have closure under multiplication.

Divide any two whole numbers and the quotient may not be a whole number. For example, $3 \div 4 = 0.75$, and 0.75 is not a whole number. The set of whole numbers are not closed under division.

15.1 Writing Repeating Decimals as Fractions

All irrational numbers have an infinite number of non-repeating decimal places. Any infinite decimal that repeats single digits or blocks of digits can be written as a fraction and is therefore, a rational number. Algebra can be used to write a repeating decimal as a fraction by setting the decimal equal to a variable and multiplying both sides of the equation by a multiple of ten. Subtracting the first equation from the second equation and solving for the variable will result in the equivalent fraction.

Example

Write the decimal 0.414141 . . . as a fraction.

Let $x = 0.414141\ldots$

$100x = 41.4141\ldots$

$\underline{-(x = 0.414141\ldots)}$

$99x = 41$

$x = \frac{41}{99}$

The repeating decimal $0.\overline{41}$ is equal to the fraction $\frac{41}{99}$.

15.2 Understanding the Properties of Real Numbers

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The properties of the real numbers include:

- Commutative Property of Addition: $\forall\, a, b \in \mathbb{R}, a + b = b + a$
- Commutative Property of Multiplication: $\forall\, a, b \in \mathbb{R}, a \cdot b = b \cdot a$
- Associative Property of Addition: $\forall\, a, b, c \in \mathbb{R}, (a + b) + c = a + (b + a)$
- Associative Property of Multiplication: $\forall\, a, b, c \in \mathbb{R}, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- Distributive Property of Multiplication over Addition: $\forall\, a, b, c \in \mathbb{R}, a(b + c) = ab + ac$
- Distributive Property of Multiplication over Subtraction: $\forall\, a, b, c \in \mathbb{R}, a(b - c) = ab - ac$
- Distributive Property of Division over Addition: $\forall\, a, b, c \in \mathbb{R}, \frac{(a + b)}{c} = \frac{a}{c} + \frac{b}{c}$
- Distributive Property of Division over Subtraction: $\forall\, a, b, c \in \mathbb{R}, \frac{(a - b)}{c} = \frac{a}{c} - \frac{b}{c}$
- Additive Identity: a number that when added to any real number *a*, the sum is equal to *a*
- Multiplicative Identity: a number that when multiplied by any real number *a*, the product is equal to *a*
- Additive Inverse: a number that when added to any real number *a*, the sum is the additive identity
- Multiplicative Inverse: a number that when multiplied by any real number *a*, the product is the multiplicative identity

Examples

An example of the Distributive Property of Multiplication over Subtraction is $4(x - 3) = 4x - 12$.

An example of the multiplicative identity is $-35 \cdot 1 = -35$.

An example of the Associative Property of Addition is $14 + (11 + 5) = (14 + 11) + 5$.

An example of the additive inverse is $-8.5 + 8.5 = 0$.

An example of the Commutative Property of Multiplication is $4 \cdot (-15) = -15 \cdot 4$.

15.2 Simplifying Expressions Using the Properties of Real Numbers

The properties of real numbers can be used to simplify algebraic expressions.

Example

Simplify $5(3x - 7) + 8(x + 5)$.

$15x - 35 + 8x + 40$	Distributive Property of Multiplication over Subtraction and Addition
$15x + 8x - 35 + 40$	Associative Property of Addition
$23x + 5$	Combine like terms

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15.3 Simplifying Expressions Involving Imaginary Numbers

The set of real numbers is not closed under exponentiation, which is the operation used to raise a quantity to a power, unless the square root of a negative number can be calculated. The number i is defined such that $i^2 = -1$. Expressions involving negative roots can be simplified using i.

Examples

Simplify $-6 + \sqrt{-45}$.

$-6 + \sqrt{-45}$

$-6 + \sqrt{9} \cdot \sqrt{5} \cdot \sqrt{-1}$

$-6 + 3\sqrt{5}i$

Simplify $(5x - 8i)(x + 6i)$.

$(5x - 8i)(x + 6i)$

$5x^2 + 30xi - 8xi - 48i^2$

$5x^2 + 22xi - 48i^2$

$5x^2 + 22xi - 48(-1)$

$5x^2 + 22xi + 48$

15.3 Understanding the Properties of the Set of Complex Numbers

The set of imaginary numbers ($\mathbb{I}$) is the set of all numbers written in the form $a + bi$, where a and b are real numbers and b is not equal to zero. The set of complex numbers ($\mathbb{C}$) is comprised of the set of real numbers and the set of imaginary numbers.

Examples

25 is a natural number, whole number, integer, rational number, real number, and complex number.

$\sqrt{2}$ is an irrational number, real number, and complex number.

$4 + 7i$ is an imaginary number and a complex number.

$4.1\overline{66}$ is a rational number, real number, and complex number.

15.4 Calculating Powers of *i*

The imaginary number i is a number such that $i^2 = -1$. Because no real number exists such that its square is equal to a negative number, the number i is not a part of the real number system. The values of i^n repeat after every four powers of i, where $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -\sqrt{-1}$, and $i^4 = 1$.

Example

$i^{25} = (i^4)^6(i^1)$

$= (1)^6(\sqrt{-1})$

$= \sqrt{-1}$

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15.4 Rewriting Expressions with Negative Roots Using *i*

Expressions with negative roots can be rewritten. For any positive real number n, the principal square root of a negative number, $-n$, is defined by $\sqrt{-n} = i\sqrt{n}$.

Example

$$\sqrt{-63} + \sqrt{-24} = i\sqrt{63} + i\sqrt{24}$$
$$= i\sqrt{(9)(7)} + i\sqrt{(6)(4)}$$
$$= 3\sqrt{7}i + 2\sqrt{6}i$$

15.4 Adding, Subtracting, and Multiplying on the Set of Complex Numbers

The set of complex numbers is the set of all numbers written in the form $a + bi$, where a and b are real numbers. The term a is called the real part of a complex number, and the term bi is called the imaginary part of a complex number. When operating with complex numbers involving i, combine like terms by treating i as a variable (even though it's a constant). Complex conjugates are pairs of numbers of the form $a + bi$ and $a - bi$. The product of a pair of complex conjugates is always a real number in the form $a^2 + b^2$.

Example

$4x + (x + 3i)(x - 3i) - 5x + 7$

$$4x + (x + 3i)(x - 3i) - 5x + 7 = 4x + x^2 - 9i^2 - 5x + 7$$
$$= x^2 + (4x - 5x) + (7 - 9(-1))$$
$$= x^2 - x + 16$$

15.4 Identifying Complex Polynomials

A polynomial is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. The definition of a polynomial can be extended to include imaginary numbers. Some polynomials have special names, according to the number of terms they have. A polynomial with one term is called a monomial. A polynomial with two terms is called a binomial. A polynomial with three terms is called a trinomial. Combine like terms to name the polynomial.

Example

$5x - 3xi + x^2 - 3i + 9$

The expression is a trinomial because it can be rewritten as $x^2 + (5 - 3i)x + (9 - 3i)$, which shows one x^2 term, one x term, and one constant term.

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15.4 Adding, Subtracting, and Multiplying Complex Polynomials

Some polynomial expressions involving i can be simplified using methods similar to those used to operate with numerical expressions involving i. Whenever possible, multiply the complex conjugates first to get a real number.

Example

$(3x + 7i)(2x + 5i)$

$$\begin{aligned}(3x + 7i)(2x + 5i) &= 6x^2 + 15xi + 14xi + 35i^2\\ &= 6x^2 + 29xi + 35(-1)\\ &= 6x^2 + 29xi - 35\end{aligned}$$

15.4 Rewriting the Quotient of Complex Numbers

When rewriting the quotient of complex numbers, multiply both the divisor and the dividend by the complex conjugate of the divisor, thus changing the divisor into a real number. The product of a pair of complex conjugates is always a real number in the form $a^2 + b^2$.

Example

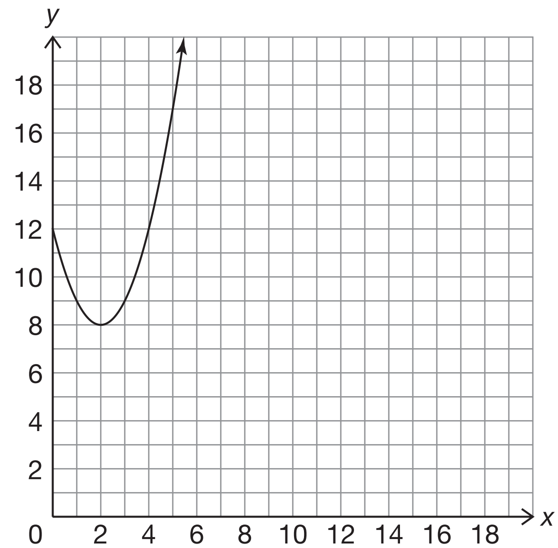
$$\begin{aligned}\frac{3 - i}{5 + 2i} &= \frac{3 - i}{5 + 2i} \cdot \frac{5 - 2i}{5 - 2i}\\ &= \frac{15 - 6i - 5i + 2i^2}{5^2 + 2^2}\\ &= \frac{15 - 11i - 2}{25 + 4}\\ &= \frac{13 - 11i}{29}\\ &= \frac{13}{29} - \frac{11}{29}i\end{aligned}$$

15.5 Determining Complex Roots of Quadratic Equations and Complex Zeros of Quadratic Functions

All quadratic equations have only one of 3 types of solutions: two unique real number solutions, two equal real number solutions (a double root), or two imaginary solutions. If the discriminant of the Quadratic Formula is positive, the equation has two unique real number solutions. If the discriminant is negative, the equation has two imaginary solutions.

Example

Consider the function $f(x) = x^2 - 4x + 12$.



The graph does not intersect the x-axis so the equation has two complex solutions. The quadratic equation can be used to determine the imaginary zeros.

$a = 1, b = -4, c = 12$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(12)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{16 - 48}}{2}$$

$$x = \frac{4 \pm \sqrt{-32}}{2}$$

$$x = \frac{4 \pm 4\sqrt{2}i}{2}$$

$$x = 2 \pm 2\sqrt{2}i$$

The zeros of the function are $x = 2 + 2\sqrt{2}i$ and $x = 2 - 2\sqrt{2}i$.