Solving Quadratic Equations and Inequalities

14

Chapter 14 Overview

This chapter introduces the quadratic formula and emphasizes choosing an appropriate method to solve quadratic equations. Quadratic inequalities are solved using a coordinate plane, and then an algebraic strategy is introduced. Systems of equations involving one or more quadratic equations are solved.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
14.1	The Quadratic Formula	A.CED.1 A.CED.2 A.REI.4.a A.REI.4.b	2	This lesson introduces the Quadratic Formula as a strategy to solve any quadratic equation. Questions ask students to solve quadratic equations using the Quadratic Formula. They then analyze the discriminant to predict the number of real zeros of a quadratic function or the number of real roots of a quadratic equation.		X	X		X
14.2	Using a Calculator-Based Ranger to Model Quadratic Motion	A.REI.4.b F.IF.7.a	2	This lesson provides an opportunity for students to use a calculator-based ranger to model the trajectory of a ball. Questions ask students to gather data, sketch a graph, determine a quadratic regression, and then analyze the coefficient of determination.					X
14.3	Solving Quadratic Inequalities	A.CED.1 A.CED.2 A.REI.4.b	1	This lesson extends solving quadratic equations to include quadratic inequalities. Questions ask students to write a quadratic function to represent a problem situation. They then graph the quadratic function on a coordinate plane and consider different intervals that satisfy the quadratic inequality. An algebraic strategy for solving quadratic inequalities is then presented.		X	X		
14.4	Systems of Quadratic Equations	A.REI.7 A.CED.1 A.CED.2	1	This lesson extends the understanding of solving systems to include systems of quadratic equations. Questions ask students to solve systems algebraically and verify their solutions graphically.	X				

14

Skills Practice Correlation for Chapter 14

Lesson		Problem Set	Description
14.1	The Quadratic Formula		Vocabulary
		1 – 6	Determine the approximate zeros or roots of functions or equations
		7 – 12	Determine the exact zeros or roots of functions or equations
		13 – 18	Use the discriminant to determine the number of zeros or roots of functions or equations
14.2	Using a Calculator-Based Ranger to Model Quadratic Motion		Vocabulary
		1 – 6	Determine quadratic regression equations and coefficients of determination for given data sets using a graphing calculator
14.3	Solving Quadratic Inequalities	1 – 6	Determine roots of quadratic inequalities then use the interval method to determine solution sets
		7 – 12	Answer questions using a given vertical motion problem situation
14.4	Systems of Quadratic Equations	1 – 6	Solve linear and quadratic systems of equations algebraically and verify solutions graphically
		7 – 12	Solve quadratic systems of equations algebraically and verify solutions graphically

Ladies and Gentlemen: Please Welcome the Quadratic Formula!

14.1

The Quadratic Formula

LEARNING GOALS

In this lesson, you will:

- Use the Quadratic Formula to determine roots and zeros.
- Derive the Quadratic Formula from a quadratic equation written in standard form.
- Use the discriminant of a Quadratic Formula to determine the number of roots or zeros.
- Determine the most efficient method of calculating roots or zeros.

KEY TERMS

- Quadratic Formula
- discriminant

ESSENTIAL IDEAS

- For a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, the solutions can be calculated using the Quadratic Formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- The discriminant or $b^2 - 4ac$ is the radicand of the Quadratic Formula and the sign of the discriminant denotes the number of possible zeros of the function.
- If $b^2 - 4ac < 0$, the quadratic function has no real zeros, and the quadratic equation has no real roots.
- If $b^2 - 4ac = 0$, the quadratic function has one real zero, and the quadratic equation has two real roots.
- If $b^2 - 4ac > 0$, the quadratic function has two real zeros, and the quadratic equation has two real roots.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-CED Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

A-REI Reasoning with Equations and Inequalities

Solve equations and inequalities in one variable

4. Solve quadratic equations in one variable.

   a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

   b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Overview

A scenario described by a quadratic equation is given. The quadratic equation cannot be factored and the Quadratic Formula is introduced. An example of using the Quadratic Formula is provided and students will apply the formula to the equation described by the scenario. Next, students are guided through the process of deriving the Quadratic Formula from a quadratic equation written in standard form. Students then practice using the Quadratic Formula and identify errors in worked solutions. Students explore the relationship between the discriminant and the number of real roots and number of real zeros of quadratic functions and quadratic equations. At the completion of this lesson, students will know different methods for solving quadratic equations; factoring, completing the square, and the Quadratic Formula.

Warm Up

Factor each expression.

1. $h^2 + 3h$

   $h(h + 3)$

2. $-0.2h^2 + 0.8h$

   $-0.2h(h - 0.4)$

3. $0.3h^2 - 0.6h$

   $0.3h(h - 0.2)$

4. $-0.1h^2 - 0.5h$

   $-0.1h(h + 0.5)$

Ladies and Gentlemen: Please Welcome the Quadratic Formula!

14.1

The Quadratic Formula

LEARNING GOALS

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- Derive the Quadratic Formula from a quadratic equation written in standard form.
- Use the discriminant of a Quadratic Formula to determine the number of roots or zeros.
- Determine the most efficient method of calculating roots or zeros.

KEY TERMS

- Quadratic Formula
- discriminant

It seems like some companies will do almost anything to get attention—and to get loyal patrons! Typically, companies will create all sorts of promotions to generate business. Sometimes these promotions can be as simple as a discount on a product price. Other times, *zany* is the only adjective that can describe the promotion.

Professional sports teams seem to have frequent promotions to grow stadium attendance during a season. In 2012, the National Football League's Cincinnati Bengals offered a 2-for-1 ticket deal to ensure a sell-out crowd. If a person bought x number of tickets, they would receive the same number of tickets for free!

Of course this isn't as zany as Major League Baseball's Los Angeles Dodgers offering tickets to canines. The Dodgers' "Bark in the Park" sold tickets to dogs and their loving owners as part of a Memorial Day promotion. And it worked. The game was sold out!

What interesting or zany promotions have you heard about? Have you or your friends ever participated in an event with a unique promotion?

Problem 1

A quadratic equation that cannot be factored is given to model a situation. The Quadratic Formula is introduced and an example provided. Students will use the example to solve the quadratic equation described in the scenario. A step by step procedure is outlined to derive the quadratic formula from a quadratic equation written in standard form. Students then follow the verbal descriptions attached to each step and perform the indicated mathematics.

Grouping

Ask a student to read the information and example. Complete Question 1 as a class.

PROBLEM 1 Get Your Free T-Shirts!

The Perris Pandas baseball team has a new promotional activity to encourage fans to attend games: launching free T-shirts! They can launch a T-shirt in the air with an initial velocity of 91 feet per second from $5\frac{1}{2}$ feet off the ground (the height of the team mascot).

A T-shirt's height can be modeled with the quadratic function $h(t) = -16t^2 + 91t + 5.5$, where t is the time in seconds and $h(t)$ is the height of the launched T-shirt in feet. They want to know how long it will take for a T-shirt to land back on the ground after being launched.

1. What methods can you use to determine how long it will take for a T-shirt to reach the ground?

   I might be able to graph the function on a coordinate plane, but I do not know if I can determine an exact value for the x-intercepts.

   It does not seem that I can factor the function to determine its roots.

When you encounter a quadratic equation or function that is difficult to factor, you can always use the *Quadratic Formula* to determine the roots or zeros.

For a quadratic equation of the form $ax^2 + bx + c = 0$, the solutions can be calculated using the

Quadratic Formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

14

Grouping

Have students complete Question 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 2

- Is the function written in standard form? How do you know?
- Is the value of *a* positive or negative?
- Quadratic functions have how many zeros?
- What are the signs of the zeros of this function?
- Which of the two zeros describes the amount of time elapsed?
- What unit of measurement is associated with time? How is time measured?
- How would you describe the graph of this situation?
- Does the parabola open upward or downward? How do you know?
- Do you think the parabola passes through the origin? Why or why not?
- What is the *y*-intercept of the parabola? How do you know?

Before using the Quadratic Formula, be sure to write the quadratic function in standard form: $ax^2 + bx + c$.

Consider the function $f(x) = -4x^2 - 40x - 99$.

First, determine the values of a, b, and c.

$$a = -4,\ b = -40,\ c = -99$$

Next, substitute the values into the Quadratic Formula.

$$x = \frac{-(-40) \pm \sqrt{(-40)^2 - 4(-4)(-99)}}{2(-4)}$$

Then, simplify to determine the zeros of the function.

$$x = \frac{40 \pm \sqrt{1600 - 1584}}{-8}$$

$$x = \frac{40 \pm \sqrt{16}}{-8}$$

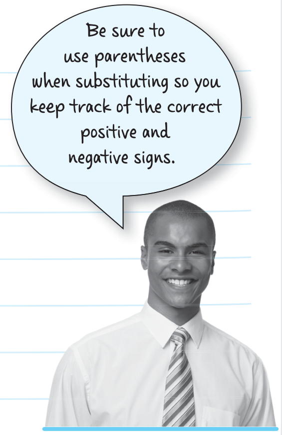
$$x = \frac{40 \pm 4}{-8}$$

$$x = \frac{40 + 4}{-8} \quad \text{or} \quad x = \frac{40 - 4}{-8}$$

$$x = \frac{44}{-8} \quad \text{or} \quad x = \frac{36}{-8}$$

$$x = -5.5 \quad \text{or} \quad x = -4.5$$

The zeros of the function $f(x) = -4x^2 - 40x - 99$ are -5.5 and -4.5.



Let's go back to the *Get Your Free T-Shirts*! problem situation.

2. Use the Quadratic Formula to determine how long it will take for a T-shirt to land back on the ground after being launched.

   a. Identify the values of a, b, and c for the function $h(t) = -16t^2 + 91t + 5.5$.

   $a = -16$, $b = 91$, $c = 5.5$

   b. Determine the zero(s) for the given function.

   $$x = \frac{-(91) \pm \sqrt{91^2 - 4(-16)(5.5)}}{2(-16)}$$

   $$x = \frac{-91 \pm \sqrt{8281 + 352}}{-32}$$

   $$x = \frac{-91 \pm \sqrt{8633}}{-32}$$

   $$x \approx -0.0598,\ 5.747$$

Guiding Questions for Discuss Phase, Quadratic Formula

- When c is moved to the right side of the equation is it a positive value or a negative value?
- How many terms did you divide by a?
- Is the left side of the equation a perfect square? How do you know?
- When the term $\left(\frac{b}{2a}\right)$ is squared, do you need to square both the numerator and denominator?
- Is $(2a)^2$ equal to $2a^2$ or $4a^2$?
- How do you determine the common denominator?
- How would you describe standard form in the numerator?
- When extracting square roots, what is done to both sides of the equation?
- Is the denominator of the radicand a perfect square?
- When the term $\left(\frac{b}{2a}\right)$ is moved to the right side of the equation, what is the sign of the term?

c. What do your solutions represent in the context of this problem?

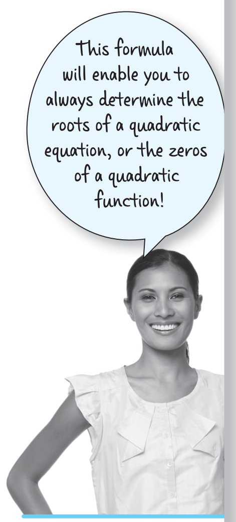
A T-shirt will land on the ground in approximately 5.7 seconds.

The solution -0.0598 does not make sense in terms of the problem.

So, where does the Quadratic Formula come from? You can derive the Quadratic Formula from a function or equation written in standard form.

These steps show how to derive the Quadratic Formula.

$ax^2 + bx + c = 0$	Write the original equation.
$ax^2 + bx = -c$	Subtract c from both sides.
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	Divide each side by a.
$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$	Add $\left(\frac{b}{2a}\right)^2$ to each side.
$x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$	Factor the left-hand side.
$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$	Power to a power rule.
$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$	Write the expression with common denominators.
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	Add and write the numerator in standard form.
$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{4a^2}$	Extract square roots.
$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$	Extract perfect squares from the denominator.
$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$	Subtract $-\frac{b}{2a}$ from each side.
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Write the right side as a single fraction.



Problem 2

Students will use the Quadratic Formula to solve quadratic equations. They also determine errors in worked solutions. Questions focus students on the number of possible solutions to a quadratic equation.

Grouping

Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- Is the quadratic function written in standard form? How do you know?
- What are the values of *a*, *b*, and *c* in the quadratic function?
- Is a calculator needed to determine the zeros? Why or why not?
- Is there any relationship between the signs of the terms in the quadratic function and the signs of the zeros of the function?
- Is Javier's quadratic equation written in standard form? Why or why not?
- What does it mean to write your solutions in exact form? How is this different than the way you have previously written the zeros?

PROBLEM 2 Using the Quadratic Formula

1. Javier was determining the exact zeros for $f(x) = x^2 - 14x + 19$. His work is shown.

Javier

$f(x) = x^2 - 14x + 19$

$a = 1, b = -14, c = 19$

$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(19)}}{2(1)}$

$x = \frac{14 \pm \sqrt{196 - 76}}{2}$

$x = \frac{14 \pm \sqrt{120}}{2}$

$x = 14 \pm \frac{\sqrt{30 \cdot 4}}{2}$

$x = \frac{14 \pm 2\sqrt{30}}{2}$

$x = 7 \pm 2\sqrt{30}$

a. Identify the error Javier made when determining the zeros.

Javier incorrectly simplified the expression. He should have also simplified the 2 in $\pm 2\sqrt{30}$.

b. Determine the correct zeros of the function.

The correct zeros are $7 \pm \sqrt{30}$.

2. Use the Quadratic Formula to determine the zeros for each function given. This time, leave your solutions in exact form.

a. $f(x) = -2x - 3x + 7$

$a = -2, b = -3, c = 7$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-2)(7)}}{2(-2)}$

$x = \frac{3 \pm \sqrt{9 + 56}}{-4}$

$x = \frac{3 \pm \sqrt{65}}{-4}$

b. $r(x) = -3x^2 + 19x - 7$

$a = -3, b = 19, c = -7$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-19 \pm \sqrt{19^2 - 4(-3)(-7)}}{2(-3)}$

$x = \frac{-19 \pm \sqrt{361 - 84}}{-6}$

$x = \frac{-19 \pm \sqrt{277}}{-6}$

14

Grouping

Have students complete Question 3 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 3

- Is Lauren's quadratic equation written in standard form? Why or why not?
- What do the intersection points of any two graphs represent?

3. Lauren was solving the quadratic equation $x^2 - 7x - 8 = -3$. Her work is shown.

Lauren

$x^2 - 7x - 8 = -3$

$a = 1,\ b = -7,\ c = -8$

$$x = \frac{-(-7)\sqrt{(-7)^2 - 4(1)(-8)}}{2(1)}$$

$$x = \frac{7 \pm \sqrt{49 + 32}}{2}$$

$$x = \frac{7 \pm \sqrt{81}}{2}$$

$$x = \frac{7 \pm 9}{2}$$

$$x = \frac{7 + 9}{2} \quad \text{or} \quad x = \frac{7 - 9}{2}$$

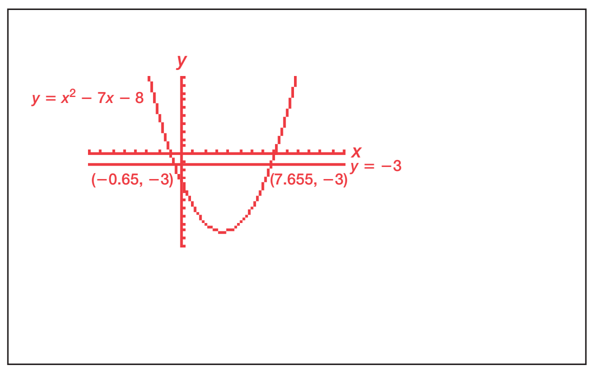
$$x = \frac{16}{2} = 8 \quad \text{or} \quad x = \frac{-2}{2} = -1$$

The roots are 8 and -1.

a. Identify Lauren's error.

Lauren determined the roots to the equation $y = x^2 - 7x - 8$, not to the equation $y = x^2 - 7x - 5$. To determine the roots of a quadratic equation using the Quadratic Formula, the equation must be set equal to 0.

b. Consider the original equation, $x^2 - 7x - 8 = 3$. Graph each side of the quadratic equation using a graphing calculator. Let $Y_1 = x^2 - 7x - 8$ and $Y_2 = 3$. Sketch your graph. Interpret the meaning of the intersection points.

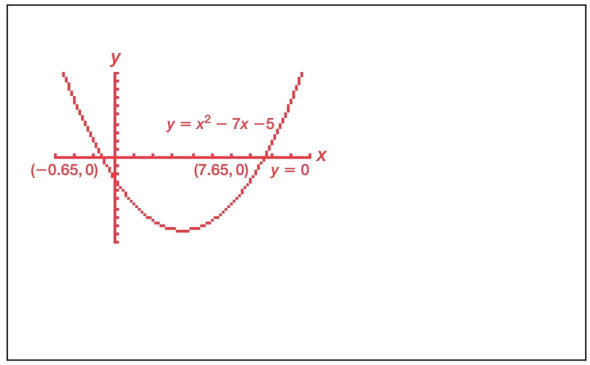


The x-values of the two points where the graphs intersect are the solutions to the quadratic equation.

Guiding Questions for Share Phase, Question 3

- How did you rewrite the original quadratic equation?
- What are the two equations you graphed in part (c)?
- Why are the solutions to parts (b) and (c) the same?
- What does the form of the quadratic equation have to look like in order to use the Quadratic Formula?
- Which graph represents the quadratic equation you used in part (d)?
- Why are all the solutions to parts (b), (c), and (d) the same?

c. Next, rewrite the given quadratic so that one side of the equation is equal to zero. Graph each side of the quadratic equation using a graphing calculator. Sketch your graph. Interpret the meaning of the intersection points.



The x-values of the intersection points are the roots of the quadratic equation.

d. Use the Quadratic Formula correctly to determine the solution to Lauren's quadratic equation.

$a = 1, b = -7, c = -5$

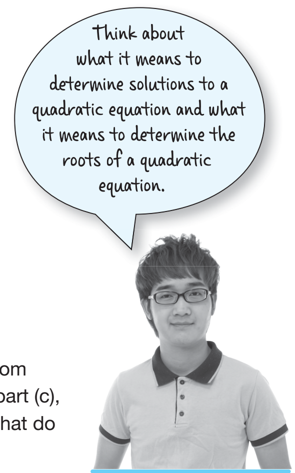
$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-5)}}{2(1)}$$

$$x = \frac{7 \pm \sqrt{49 + 20}}{2} = \frac{7 \pm \sqrt{69}}{2}$$

$$x = \frac{7 \pm 8.31}{2}$$

$$x = \frac{15.31}{2} \approx 7.655 \quad \text{or} \quad x = \frac{-1.31}{2} \approx -0.655$$

The roots are approximately 7.655 and -0.655.



e. Compare the x-values of the intersection points from part (b), the x-values of the intersection points in part (c), and the solutions using the Quadratic Formula. What do you notice?

The values are all the same.

Grouping

Have students complete Question 4 with a partner. Then share the responses as a class.

4. Use the Quadratic Formula to determine the zeros for each function. Round your solutions to the nearest hundredth.

a. $f(x) = x^2 - 7x + 11$

$a = 1, b = -7, c = 11$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(11)}}{2(1)}$$

$$x = \frac{7 \pm \sqrt{49 - 44}}{2} = \frac{7 \pm \sqrt{5}}{2}$$

$$x = \frac{7 \pm 2.236}{2}$$

$$x = \frac{9.236}{2} \quad \text{or} \quad x \approx \frac{4.764}{2}$$

$$x \approx 4.62 \quad \text{or} \quad x \approx 2.38$$

b. $h(x) = 3x^2 - 11x - 2$

$a = 3, b = -11, c = -2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(3)(-2)}}{2(3)}$$

$$x = \frac{11 \pm \sqrt{121 + 24}}{6} = \frac{11 \pm \sqrt{145}}{6}$$

$$x = \frac{11 \pm 12.04}{6}$$

$$x = \frac{23.04}{6} \quad \text{or} \quad x = \frac{-1.04}{6}$$

$$x \approx 3.84 \quad \text{or} \quad x \approx -0.17$$

5. Let's think about the zeros of quadratic functions.

a. How many zeros did each quadratic function in this lesson have?

Each function I calculated had two zeros.

b. Do all quadratic functions have two zeros? Explain why or why not.

No. Not all quadratic functions have two zeros. For example, if a function has a vertex that intercepts the x-axis at one point, then there is only one zero.

c. Could a quadratic function have no zeros? Explain why or why not.

Yes. If a quadratic function has a vertex that is above or below the x-axis, then the function has no zeros.

d. Could a quadratic function can have more than two zeros? Explain why or why not.

No. Because a quadratic function has a graph in the shape of a parabola, the most number of zeros the function can have is two.

Problem 3

The discriminant is introduced and students will explore the relationship between the sign of the discriminant and the number of real zeros/real roots of the quadratic function/equation. They also sketch the graph of quadratic functions identifying the axis of symmetry and the vertex.

Grouping

Have students complete Questions 1 through 9 with a partner. Then share the responses as a class.

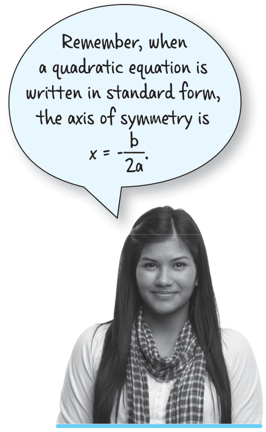
Guiding Questions for Share Phase, Questions 1 and 2

- How would the average of the zeros of a quadratic function help to identify the vertex and axis of symmetry?
- If both zeros of the function are equal to 0, what is the average of the zeros?
- How many x-intercepts does this function have?
- How can you determine the number of zeros in any quadratic function?

PROBLEM 3 The Discriminant

A quadratic function can have more than one zero, or at times, no zeros.

Use the Quadratic Formula to calculate the zero(s) of each quadratic function. Then, determine each function's vertex and axis of symmetry.



1. $y = 2x^2$

a. Zero(s):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-0 \pm \sqrt{0^2 - 4(2)(0)}}{2(2)} = 0$$

b. Axis of symmetry:

$x = 0$

c. Vertex:

$y = 2(0)^2$

(0, 0)

d. Determine the number of zeros in this function.

The function has one zero.

2. $y = 2x^2 - 2$

a. Zero(s):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-0 \pm \sqrt{0^2 - 4(2)(-2)}}{2(2)}$$

$$= \frac{\pm 4}{4} = \pm 1$$

(1, 0), (−1, 0)

b. Axis of symmetry:

$x = 0$

c. Vertex:

$y = 2(0)^2 - 2$

(0, −2)

d. Determine the number of zeros in this function.

The function has two zeros.

Guiding Questions for Share Phase, Questions 3 through 6

- What does a negative radicand indicate about the function?
- What do you notice about the graphs of the three functions, the y-intercepts, and the equations of the three functions?

3. $y = 2x^2 + 4$

a. Zero(s):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-0 \pm \sqrt{0^2 - 4(2)(4)}}{2(2)} = \frac{\pm\sqrt{-32}}{4}$$

The function has no real zeros.

b. Axis of symmetry:

$x = 0$

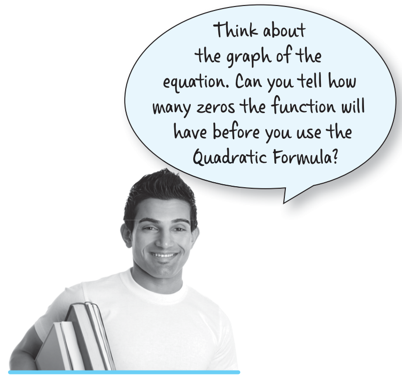
c. Vertex:

$y = 2(0)^2 + 4$

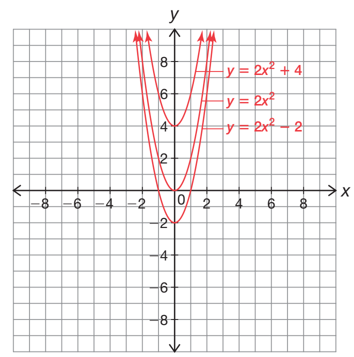
$(0, 4)$

d. Determine the number of zeros in this function.

The function has no zeros.



4. Sketch the functions from Question 1 through Question 3 on the coordinate plane shown.



5. What are the similarities among the graphs? The axes of symmetry? The vertices?

They all have the same shape. The axes of symmetry are all the same. The vertices are all on the y-axis.

6. Examine the work you did with the Quadratic Formula for each function.

a. What portion of the formula indicates the number of real zeros? How do you know?

The radicand indicates the number of zeros. If $b^2 - 4ac$ is zero, there is 1, if it is positive, there are 2, and if it is negative there are none.

Grouping

Ask students to read the table and then discuss as a class.

Guiding Questions for Discuss Phase

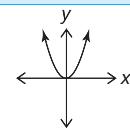
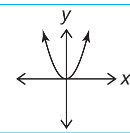
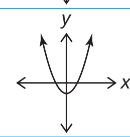
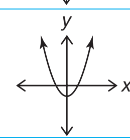
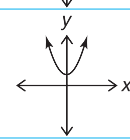
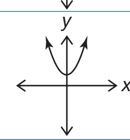
- Does a quadratic equation have roots or zeros?
- Does a quadratic function have roots or zeros?
- What is the difference between a root and a zero?
- What is the difference between a quadratic function and a quadratic equation?
- When the radicand or discriminant is positive, does it always have two real zeros?
- When the radicand or discriminant is negative, does it always have no real zeros?
- When the radicand or discriminant is equal to 0, does it always have one real zero?

Because this portion of the formula "discriminates" the number of zeros, or roots, it is called the **discriminant**.

b. Using the discriminant, write three inequalities to describe when a quadratic function has:

- no real roots/zeros.
  $b^2 - 4ac < 0$
- one real root/zero.
  $b^2 - 4ac = 0$
- two real roots/zeros.
  $b^2 - 4ac > 0$

The table shown summarizes the types of solutions for any quadratic equation or function.

		Interpretation of the Solutions			
Equation/ Function	**Solutions**	**Number of Real Roots**	**Number of Real Zeros**	**Number of x-Intercepts**	**Sketch**
$0 = 2x^2$	$x = \frac{-0 \pm \sqrt{0^2 - 4(2)(0)}}{2(2)}$ $= 0$	2	n/a	1	
$f(x) = 2x^2$	$x = \frac{-0 \pm \sqrt{0^2 - 4(2)(0)}}{2(2)}$ $= 0$	n/a	1	1	
$0 = 2x^2 - 2$	$x = \frac{-0 \pm \sqrt{0^2 - 4(2)(-2)}}{2(2)}$ $= \frac{\pm 4}{4} = \pm 1$	2	n/a	2	
$f(x) = 2x^2 - 2$	$x = \frac{-0 \pm \sqrt{0^2 - 4(2)(-2)}}{2(2)}$ $= \frac{\pm 4}{4} = \pm 1$	n/a	2	2	
$0 = 2x^2 + 4$	$x = \frac{-0 \pm \sqrt{0^2 - (4)(2)(4)}}{2(2)}$ $= \frac{\pm\sqrt{-32}}{4}$	0	n/a	0	
$f(x) = 2x^2 + 4$	$x = \frac{-0 \pm \sqrt{0^2 - (4)(2)(4)}}{2(2)}$ $= \frac{\pm\sqrt{-32}}{4}$	n/a	0	0	

Grouping

Have students complete Question 7 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 7

- How does analyzing the discriminant before you solve help you?
- What strategy did you use to solve the quadratic equation: factoring, completing the square, or the Quadratic Formula?

Every quadratic equation has 2 real roots or 0 real roots. However, if a graph of a quadratic equation has 1 x-intercept, the equation *still* has 2 real roots. In this case, the 2 real roots are considered a double root.

Therefore, you should keep in mind that there can be 0, 1, or 2 x-intercepts; there can be 0 or 2 real roots; and there can be 0, 1, or 2 real zeros.

7. Use the discriminant to determine the number of roots/zeros that each equation/function has. Then solve for the roots/zeros.

a. $y = 2x^2 + 12x - 2$

$b^2 - 4ac = 12^2 - 4(2)(-2) = 160$

The function has two zeros.

$x = \frac{-12 \pm \sqrt{160}}{4} = \frac{-12 \pm 4\sqrt{10}}{4}$

$= -3 \pm \sqrt{10} \approx 0.162, -6.162$

$(0.162, 0), (-6.162, 0)$

b. $0 = 2x^2 + 12x + 20$

$b^2 - 4ac = 12^2 - 4(2)(20) = -16$

The equation has no roots.

c. $y = x^2 + 12x + 36$

$b^2 - 4ac = 12^2 - 4(1)(36) = 0$

The function has one zero.

$x = \frac{-12 \pm \sqrt{0}}{2} = -6$

$(-6, 0)$

d. $y = 3x^2 + 7x - 20$

$b^2 - 4ac = 7^2 - 4(3)(-20) = 289$

The function has two zeros.

$x = \frac{-7 \pm \sqrt{289}}{6} = \frac{-7 \pm 17}{6} = -4, \frac{5}{3}$

$(-4, 0), \left(\frac{5}{3}, 0\right)$

e. $y = 4x^2 - 9$

$b^2 - 4ac = 0^2 - 4(4)(-9) = 144$

The function has two zeros.

$x = \frac{0 \pm \sqrt{144}}{8} = \frac{\pm 12}{8} = -\frac{3}{2}, \frac{3}{2}$

$\left(-\frac{3}{2}, 0\right)\left(\frac{3}{2}, 0\right),$

f. $0 = 9x^2 + 12x + 4$

$b^2 - 4ac = 12^2 - 4(9)(4) = 0$

The equation has one root.

$x = \frac{-12 \pm \sqrt{0}}{18} = -\frac{2}{3}$

$\left(-\frac{2}{3}, 0\right)$

14

8. Examine each of the roots/zeros of the equations/functions in Question 7. What characteristic of the discriminant determines if the roots will be rational or irrational?

If the discriminant is a perfect square, then the roots are rational; if it is not a perfect square, then they are irrational.

9. Based on the number and nature of each of the roots/zeros, decide if the discriminant is positive, negative, or zero and if the discriminant is or is not a perfect square.

a. no real roots/zeros

The discriminant is negative.

The discriminant is not a perfect square.

b. one rational root/zero

The discriminant is zero.

The discriminant is a perfect square.

c. two rational roots/zeros

The discriminant is positive.

The discriminant is a perfect square.

d. two irrational roots/zeros

The discriminant is positive.

The discriminant is not a perfect square.

PROBLEM 4 Solving Efficiently

A friend of yours is working on a project that involves the path of a kicked soccer ball. After numerous test kicks, she modeled the general path of the ball using the quadratic function $v(h) = -0.01h^2 + 0.6h$, where h is the horizontal distance the ball traveled in meters, and $v(h)$ is the vertical distance the ball traveled in meters.

1. Write an equation you can use to determine the distance the ball traveled horizontally before it hit the ground.

$0 = -0.01h^2 + 0.6h$

Problem 4

A scenario is modeled by a quadratic function. Students will write a quadratic equation and solve the equation using a method of choice. Students may not realize that multiplying each term of the equation by -100 eliminates the coefficient of the leading term.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Question 1 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 2 and 3

- Did you use the Quadratic Formula to solve this equation?
- If you used the Quadratic Formula to solve this equation, what values did you use for *a*, *b*, and *c*?
- Does this quadratic equation appear factorable? Why or why not?
- If you wanted to eliminate the coefficient of the leading term, how would you do it?
- What does each term have in common?
- If a factor of the quadratic equation is $(h - 60)$, how do you determine the root?
- If *h* is a factor of the quadratic equation, what is the value of one root?
- Which method would be most tedious? Why?

2. Determine the horizontal distance the ball traveled. Explain why you chose to either factor, complete the square, or use the Quadratic Formula.

$0 = -0.01h^2 + 0.6h$

$(-100)0 = -100(-0.01h^2 + 0.6h)$

$0 = h^2 - 60h$

$0 = h(h - 60)$

The zero that represents the distance the ball traveled is (60, 0).

I used the factoring method, but I could have used the Quadratic Formula to determine the distance the ball traveled.

3. Determine the horizontal distance the ball traveled when it reached a height of 6 meters. Explain why you chose to either factor, complete the square, or use the Quadratic Formula.

I will use the Quadratic Formula to determine the distance the ball traveled because factoring does not appear to be possible.

$6 = -0.01h^2 + 0.6h$

$0 = -0.01h^2 + 0.6h - 6$

$a = -0.01,\ b = 0.6,\ c = -6$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(0.6) \pm \sqrt{0.6^2 - 4(-0.01)(-6)}}{2(-0.01)}$$

$$x = \frac{-0.6 \pm \sqrt{0.36 - 0.24}}{-0.02}$$

$$x = \frac{-0.6 \pm \sqrt{0.12}}{-0.02}$$

$$x = \frac{-0.6 + \sqrt{0.12}}{-0.02} \approx 12.68 \quad \text{or} \quad x = \frac{-0.6 - \sqrt{0.12}}{-0.02} \approx 47.32$$

The soccer ball traveled about 12.68 meters (as it was rising) and about 47.32 meters (as it was falling) when it reached a height of 6 meters.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Consider the quadratic function: $f(x) = x^2 - 10x + 30$.

1. Write the equation for the quadratic function.

$x^2 - 10x + 30 = 0$

2. What characteristics, if any, can you identify from the equation of the function?

- The graph is a parabola
- The parabola opens upward
- The graph has an absolute minimum
- The *y*-intercept of the graph is (0, 30)
- The equation is not a perfect square
- The equation is not factorable
- The equation has 2 roots

3. Use any method of choice to solve the quadratic equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

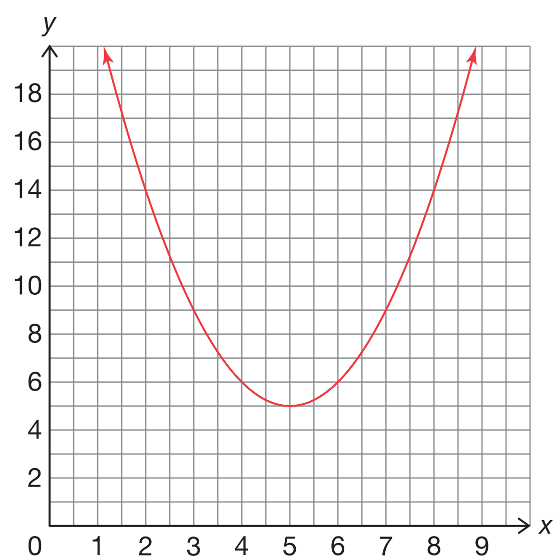
$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(30)}}{2(1)}$$

$$x = \frac{10 \pm \sqrt{100 - 120}}{2}$$

$$x = 5 \pm \frac{\sqrt{-20}}{2}$$

This quadratic equation has no real roots.

4. Graph the quadratic equation to verify your answer to Question 3.



5. How does the graph of this equation verify your answer to Question 3?

The parabola has no *x*-intercepts, so it has no real roots.

It's Watching and Tracking!

Using a Calculator-Based Ranger to Model Quadratic Movement

14.2

LEARNING GOALS

In this lesson, you will:

- Predict the graph of a ball being tossed.
- Use a calculator-based ranger (CBR) to graph the trajectory of an item.
- Attempt to replicate a trajectory that is very similar to the graph of a quadratic function.

KEY TERMS

- quadratic regression
- coefficient of determination

ESSENTIAL IDEAS

- A calculator-based ranger, or CBR, is used to model the trajectory of an object.
- A quadratic regression is a mathematical method used to determine the equation of a parabola that is the best fit for a data set.
- The correlation coefficient, r, measures the strength of the linear relationship between two variables.
- The coefficient of determination, r^2, measures the strength of the relationship between the original data and the regression equation.
- The coefficient of determination takes on values between 0 and 1. The closer the value is to 1, the better the fit of the regression equation.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-REI Reasoning with Equations and Inequalities

Solve equations and inequalities in one variable

4. Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

F-IF Interpreting Functions

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

   a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

Overview

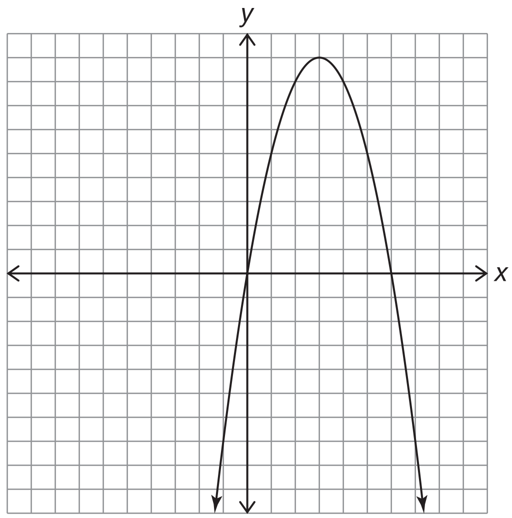
A Calculator-Based Ranger (CBR) is used to model the trajectory of an object. Students will perform an experiment first without and then with the use of a CBR and graphing calculator. The experiment involves tossing and catching a ball into the air. They will sketch graphs to represent the situation and perform a quadratic regression analyzing the coefficient of determination. Students are guided step by step through the use of all technologies. A second experiment involves dropping the ball and letting it bounce until it stops. Using a CBR and a graphing calculator, students gather data, sketch a graph, determine a quadratic regression, and then analyze the quadratic regression equation and the coefficient of determination.

Warm Up

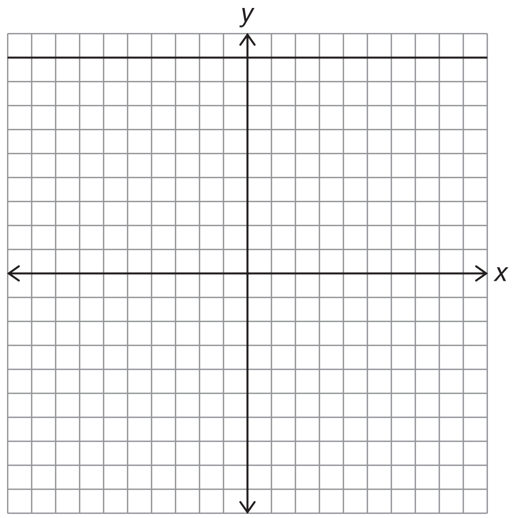
A quarterback throws a 40-yard pass to a receiver.

1. Which graph of the ball's path in terms of distance and time traveled best fits this situation?

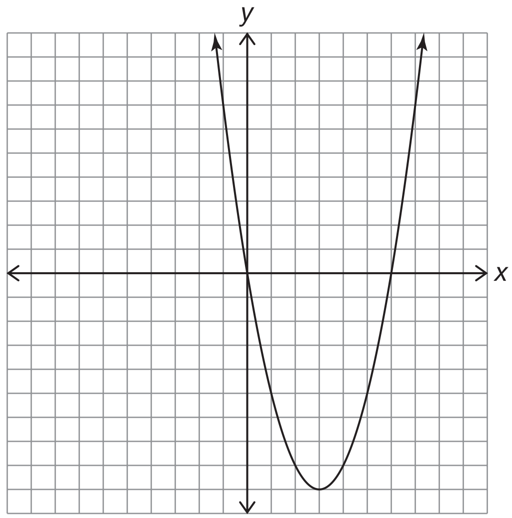
Graph I



Graph II



Graph III



2. What reasoning was used to select the appropriate graph?

The situation should be a quadratic function because the ball spirals upward and then downward.

It's Watching and Tracking!

Using a Calculator-Based Ranger to Model Quadratic Motion

14.2

LEARNING GOALS

In this lesson, you will:

- Predict the graph of a ball being tossed.
- Use a calculator-based ranger (CBR) to graph the trajectory of an item.
- Attempt to replicate a trajectory that is very similar to the graph of a quadratic function.

KEY TERMS

- quadratic regression
- coefficient of determination

If you live in a city—no matter how large or small—there is a strong chance you will encounter traffic at some point during a typical week or month. For some commuters, traffic is a daily headache. And for many years, commuters relied on traffic reports during rush hour.

However, with new technology, routine radio traffic reports are becoming obsolete. The reason? Simple: traffic cameras and traffic sensors. On many major cable providers, a section of channels is dedicated to traffic cameras—so you can see typical traffic spots 24 hours a day—something useful in a city like New York City!

And what about these traffic sensors? In many city highway systems, sensors were installed to measure the speed of cars as the traveled. Then these sensors reported average car speeds onto maps. For many years, these maps were only available to news agencies; however, these maps are now available by simply typing "traffic" and the city name into an Internet search. An interesting thought: you can learn about a stalled vehicle on the Dan Ryan Expressway in Chicago at the same time you learn about a delay-free ride on the Julia Tuttle Causeway in Miami.

Can you think of other devices that routinely track movements of people or animals? What benefits and disadvantages are there to technological devices that watch and record human movements?

Problem 1

Working with a partner, one student tosses a ball into the air and catches it while the other student uses a CBR to gather data on the distance and time the ball travels. First students will conduct the experiment without the CBR and make predictions. Next, they will use the CBR to record additional data, and graph the function, and finally, they will conduct another trial and use the graph to analyze the data. Tossing the ball the same height and same velocity during each trial ensures reliable data. Students use their data and graphing calculator to perform a quadratic regression and analyze the coefficient of determination. Step by step instructions for using the CBR and the graphing calculator are provided.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Question 1 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 1

- A trajectory produces what type of function?
- What quantity is labeled on the *x*-axis? The *y*-axis?
- As the time increases does the distance increase?
- As the time increases does the distance decrease?
- Will the distance reach an absolute maximum or an absolute minimum? Why?
- Would you describe the graph as being parabolic? Why?

PROBLEM 1 What's Your Prediction?

In this lesson, you will conduct an experiment using a calculator-based ranger (CBR) to model the trajectory of an object. Working with a partner, you will toss a ball into the air and catch it while the CBR gathers data on distance and time.

You and your partner will have the opportunity to toss the ball a number of times. First, you will toss the ball without the CBR in order to make predictions about the data. Then, you will complete three trial tosses with the CBR. Finally, you will make a fourth toss and use this graph to analyze the data. **It is important that you try to toss the ball at the same velocity and height each time**.

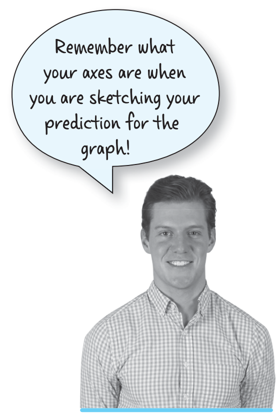
1. Gently toss the ball into the air and catch it in your hands.

a. Describe the type of function that best models this situation. Explain your reasoning.

This situation should be a quadratic function because the ball is being tossed, which is a trajectory.

b. Sketch a graph of the ball's path in terms of distance and time traveled.

Graphs may vary. Graphs should be parabolas.

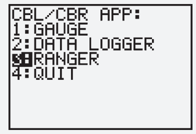


Grouping

Ask a student to read the information and discuss the steps for using the CBR and graphing calculator to collect data.

Now you will use the CBR to record the tossing of the ball into the air. The CBR records the distance of the ball from the CBR over time. However, before you begin, you need to set up your calculator to receive the data from the CBR.

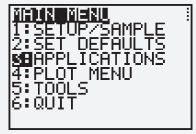
Step 1: Attach the CBR to your graphing calculator with the unit-to-unit link cable that comes with the CBR.



Step 2: Press **APPS** and choose **CBL/CBR**.

Step 3: Press any key to advance past the introduction screen.

Step 4: Select **3:Ranger**.

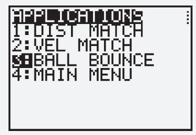


Step 5: Press **ENTER** to advance past the introduction screen.

Step 6: Select **3:APPLICATIONS**.

Step 7: For this experiment, we will be using customary units of measure. Select **2:FEET**.

Step 8: Open the CBR. Make sure that the indicator below the CBR scanner (that looks like a metallic metal mesh circle) is selected on the ball icon.



Step 9: Select **3:BALL BOUNCE**. Follow the instructions with the **BALL BOUNCE** program.

Step 10: At this point, you can detach the CBR from the calculator. Press **ENTER**.

Step 11: Press the **TRIGGER** when you are ready to record the data. You should hear a faint tapping sound from the CBR when it is recording data.

14

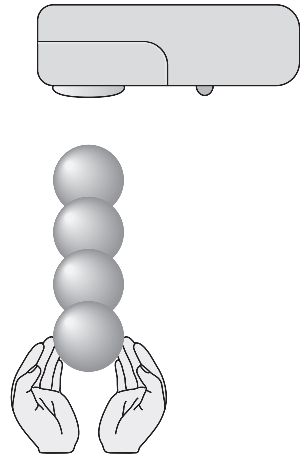
Grouping

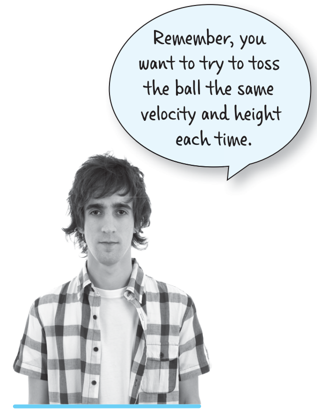
Have students complete Questions 2 through 5 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 2 through 5

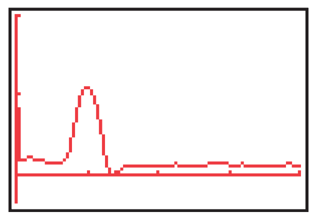
- How does the graph produced by using the data from the CBR compare to the graph produced by using the data from the first trial?
- How does the graph produced by using the data from the CBR the second time compare to the graph produced by using the data from the CBR the first time?
- Is there a relationship between the absolute maximums on the graphs?
- How do the intervals of increase compare to each other on the graphs?
- How do the intervals of decrease compare to each other on the graphs?
- How does the duration of the trials compare to each other?
- Is your ball perfectly round?
- Did you toss the ball the same height throughout all trials?
- Did you toss the ball with the same velocity throughout all trials?
- Did the person holding the CBR remain motionless throughout all trials?
- What does quadratic data look like?

2. Have one partner hold the CBR above the ball. The CBR scanner should be facing down over the ball. The other partner will gently toss the ball and catch it in his or her hands, just as you did to make your prediction.





a. Reattach the CBR to your calculator and press **ENTER** to transfer the data, and then sketch the graph from the CBR.



b. Describe what each section of the graph represents.

Answers may vary.

Students should be able to recognize the data prior to tossing the ball, the data of the actual ball toss, and the data after catching the ball.

3. Press **ENTER** to return to the Plot Menu. Choose **5:REPEAT SAMPLE** to run another trial.

a. Sketch the graph of your data from your second trial.

b. Sketch the graph of your data from your third trial.

4. Let's analyze the graph of each trial.

a. Compare the graphs from the CBR to the predicted graph you sketched. Was your prediction accurate? Explain why or why not.

Answers may vary.

My prediction was accurate because the graph is similar.

b. Do any of your graphs appear to show a perfect parabola? Explain your reasoning.

No. There are extra data and the parabola of the toss is not perfectly symmetrical in any of the graphs.

c. What factors may affect your data collection?

Answers may vary.

The ball may not be perfectly round.

The person tossing the ball may not throw it straight up and down.

The person holding the CBR may move.

5. Perform a fourth toss of the ball. You will use this trial toss for the rest of the problem, so make it a good one! Sketch the graph of the data the CBR collected.

Grouping

- Ask a student to read the information and follow the steps for using the graphing calculator to perform a quadratic regression. Discuss as a class.
- Have students complete Questions 6 and 7 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 6 and 7

- Did you turn on 'diagnostics' on your graphing calculator?
- Which lists did you use on your graphing calculator to determine the regression equation?
- How many data points on your scatterplot actually lie on the parabola?
- Should all of the data points lie on the parabola?
- What does it mean if the data points are not on the parabola?

You can determine how closely a graph is to being a quadratic function not only visually, but also mathematically. The **quadratic regression** is a mathematical method used to determine the equation of a "parabola of best fit" for a data set.

To display the r^2 value, press **2ND CATALOG**. Press **ALPHA D**. Scroll to **DiagnosticOn** and press **ENTER** twice.

You can use a graphing calculator to determine the quadratic regression of a set of data.

Step 1: Press **ENTER** to return to the Plot Menu. Scroll down to **7:QUIT**. Press **ENTER**.

Step 2: You will see on the screen Lists 1 through 4(**L1-L4**). The data needed to determine the quadratic regression is located in Lists 1 and 2 which measures time and distance.

Step 3: Press **STAT**. Scroll right to **CALC**. Choose **5:QuadReg**. Press **ENTER**.

Step 4: Select **L1** , **L2**. Press **ENTER**.

Step 5: The information shows you the standard form of a quadratic equation and the values of a, b, c, and r2.

6. Graph the quadratic regression equation to see the "parabola of best fit."

7. Do you think this quadratic regression equation shows a good fit? Why or why not?

Answers may vary.

No. This quadratic regression equation is not a good fit because the points of the data do not match with the parabola.

Grouping

Ask a student to read the information and definition. Complete Question 8 as a class.

Guiding Questions for Discuss Phase

What is meant by extraneous data?

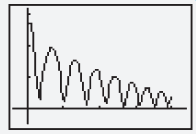
When dealing with linear regressions, you can use the correlation coefficient (r) to determine the accuracy of a line of best fit. The correlation coefficient represents the linear relationship between two variables. The correlation coefficient can then be squared to become the *coefficient of determination*. The **coefficient of determination** measures the "strength" of the relationship between the original data and the regression equation. The coefficient of determination (r^2) is used when dealing with quadratics because we are no longer looking at a linear relationship; therefore, the correlation coefficient cannot be used. The coefficient of determination can take on values between 0 and 1. The closer the value is to 1, the better the fit of the regression equation.

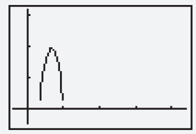
8. What is the coefficient of determination of your current data? Does this indicate a good fit? Explain your reasoning.

Answers will vary.

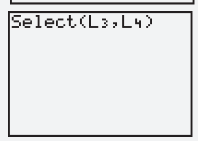
Because the CBR records *all* movement of the ball you tossed, there is often extraneous data on your graph. Luckily, you can select just the data that best resembles a quadratic function using your graphing calculator.

Step 1: Use the built-in **Select(** feature of the TI-83 to select the first "good" toss that is collected.

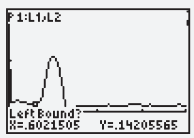


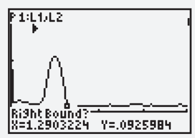


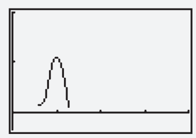
Step 2: With the plot displayed, press **2nd LIST**. Scroll to **OPS**, and then chose **8:select(** to enter where you want to store the selected data. To use **L3** and **L4**, press **2nd L3 , 2nd L4) ENTER**.



Step 3: To actually select a part of the graph you will use, press ▶ to move to the left end of the data you want to keep. Press **ENTER**. This sets the left bound. Press ▶ to move to the right end of the data you want. Press **ENTER**. The selected data will be placed in **L3** and **L4**, and then this data will be displayed.







Grouping

Have students complete Questions 9 through 11 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 9 through 11

- How does this graph of the quadratic regression equation compare to the graph from the selected experimental data?
- Which graphs are most alike?
- What is the coefficient of determination? Is the value closer to 1?
- What data set resulted in the r^2 value closest to 1?
- What times are associated with the interval of increase on the graph?
- What times are associated with the interval of decrease on the graph?
- What heights are associated with two different values for the time?
- What height is associated with only one time value?
- How long did it take for the ball to be caught?
- Using the graph, how do you know when the ball is first tossed?
- Using the graph, how do you know when the ball is caught?

Now that you have selected a certain section of data, repeat the graphing calculator steps to determine the coefficient of determination, r^2.

9. Graph the quadratic regression equation of your selected data. Sketch the graph. Compare this graph to the graph you sketched in Question 6. What do you notice?

The graph of the quadratic regression equation seems to fall almost exactly on top of the graph from the selected experimental data. It fits much better than the graph of the regression I sketched in Question 6.

10. What do you notice about the coefficient of determination from the selection you made?

Answers may vary.

The r^2 value is now closer to 1 than it was with the original data set.

11. Analyze your data.

a. What was the maximum height of the ball?

Answers will vary.

b. How long did it take for the ball to reach its maximum height?

Answers will vary.

c. Determine the height of the ball after 0.25 second.

Answers will vary.

d. When else will the ball be at this height?

Answers will vary.

Problem 2

This problem is similar to Problem 1 however in this experiment, the ball is bounced rather than tossed into the air. One person drops the ball allowing it to bounce until it stops while the second person holds the CBR in a position above the bouncing ball. Students will sketch a prediction of the graph and describe the type of function that best models the situation. They then use the data collected to perform a quadratic regression and analyze the graph and coefficient of determination.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 5 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- How is this experiment similar to the first experiment?
- How is this experiment different that the first experiment?
- How is the predicted path of the bouncing ball different than the path of the ball that was tossed into the air?

PROBLEM 2 That's the Way the Ball Bounces . . .

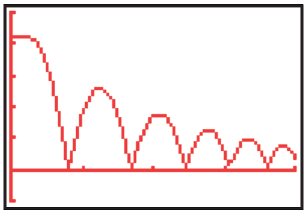
Now that you have determined the quadratic regression for tossing a ball, you will bounce a ball and determine the best regression.

For this problem, one partner is going to hold the CBR above the ball. The other partner will drop the ball, allowing it to bounce until it stops.

1. Do a test run of the ball bounce experiment without the CBR. Drop the ball, allowing it to bounce until it stops.
   a. Sketch a prediction of the ball's path in terms of distance and height.
   b. Describe the type of function that best models this situation. Explain your reasoning.
   Answers may vary.
   A quadratic equation for each bounce would model the situation.

This time, have one partner hold the CBR above the ball. The other partner should hold the ball about 1.5 feet below the CBR. The partner holding the ball will drop it and allow it to bounce until it stops.

2. Sketch the graph from the CBR. How does your predicted graph and the graph of the CBR compare?



Answers may vary.

3. Describe what each section of the graph represents.
   Answers may vary.
   The graph shows each bounce as a close ended parabola, with the vertex of each parabola decreasing with each bounce because the height of the ball decreases with each bounce.

- Does a quadratic function model the behavior of the bouncing ball?
- How would you describe the graph on the CBR that models the situation?
- Does the graph contain more than one parabola? How many?
- Do the absolute maximums of each parabola appear linear?
- Do the vertices of the consecutive parabolas appear to increase or decrease in height?

Guiding Questions for Share Phase, Question 4

- Why isn't the quadratic regression equation a good fit?
- Does the quadratic regression equation ever produce the correct solution?
- Are there any points on the graph where the regression equation and the original data overlap? Where?

4. Remember that the coefficient of determination measures how well a quadratic equation fits the data.

a. Do you think the coefficient of determination for these data will be closer to 0 or to 1? Explain your reasoning.

Answers may vary.

I think the coefficient of determination will be closer to zero because while this graph is made up of a number of parabolas, one quadratic equation cannot accurately follow all the data.

b. Determine the quadratic regression equation and the coefficient of determination.

Answers will vary.

The coefficient of determination should not show a very good fit.

c. Graph the quadratic regression equation. What do you notice?

The quadratic regression equation graph follows the decreasing height of each bounce, but then at the end, the graph starts to go back up away from the bounces to form a parabola.

Be prepared to share your solutions and methods.

14

Check for Students' Understanding

1. Describe different experiments that would produce a graph similar to the ball that was tossed into the air in Problem 1.

   Answers will vary.

   - Tracing the path of a tennis ball
   - Tracing the path of shooting a basketball into a hoop

2. Describe different experiments that would produce a graph similar to the ball that was tossed into the air in Problem 2.

   Answers will vary.

   - Tracing the path of a yo-yo
   - Tracing the path of someone using a trampoline
   - Tracing the path of a dribbling basketball

14.3

They're A Lot More Than Just Sparklers!

Solving Quadratic Inequalities

LEARNING GOALS

In this lesson, you will:

- Use the Quadratic Formula to solve quadratic inequalities.

ESSENTIAL IDEAS

- Quadratic functions are used as models for real world situations.
- The solution set of a quadratic inequality is determined by first solving for the roots of the quadratic equation, then using the roots to divide the graph into intervals, and finally testing each interval to determine which interval(s) satisfy the inequality.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-CED Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

A-REI Reasoning with Equations and Inequalities

Solve equations and inequalities in one variable

4. Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Overview

The scenarios used are trajectory problems. Students will identify variables, and write the function that represents each situation. Questions related to the situations require students to write and solve a quadratic inequality related to the solution set of the quadratic function. A worked example of solving a quadratic inequality is provided. Students will determine the solution set of the inequality by dividing the graph into intervals defined by the roots of the quadratic equation, and then test values in each interval to determine which intervals satisfy the inequality.

Warm Up

A bottle rocket is launched straight up into the air with an initial velocity of 300 feet per second from a platform that is 3 feet off the ground.

1. Which equation best models this situation?

- $h(t) = -16t^2 + 300t - 3$
- $h(t) = 16t^2 + 300t + 3$
- $h(t) = -16t^2 + 300t + 3$
- $h(t) = 16t^2 + 300t - 3$

The correct equation is $h(t) = -16t^2 + 300t + 3$.

2. Describe the reasoning you used to select the appropriate equation.

The general form of a trajectory equation is $h(t) = -16t^2 + v_0t + h$, where v_0 is the velocity of the object and h is the initial height of the object. If the initial height of the bottle rocket is 3 feet above the ground, then $h = 3$, so the correct equation is $h(t) = -16t^2 + 300t + 3$.

They're a Lot More Than Just Sparklers!

Solving Quadratic Inequalities

14.3

LEARNING GOALS

In this lesson, you will:

- Use the Quadratic Formula to solve quadratic inequalities.

Many historians believe fireworks were first created and used in China during the 7th century. To these ancient people, fireworks were a way to scare spirits away—a tradition that is still used for Chinese New Year festivities. Also in this modern time, fireworks are often used during celebrations, and they come in many different shapes and colors. Many aerial fireworks burst at different heights and can travel anywhere from 300 to about 1300 feet in the air!

Safety is a major concern when planning a fireworks display. In order to keep spectators safe, many states have regulations stating how far away audiences should be from the launch sites. Depending on the size of the firework, audiences may have to be as far as 1500 feet from the launch site. How do you think safety officials determine the safety distances for the audience? Why might different fireworks have different safety distances? What other factors, besides size of the firework, may come into play when determining these distances?

Problem 1

A scenario modeled by a vertical motion function is given. Students will identify the variables and write the quadratic function to represent the situation. They sketch a graph and determine when the object hits the ground by using a solution method of choice. They then answer several questions about the situation and write an inequality that represents times when the object is a specified height. A worked example of solving an inequality is provided. Students will analyze the solution set of the inequality graphically by using the roots of the quadratic equation to divide the graph into 3 regions and choosing values that satisfy the inequality to determine the intervals. They also describe an error in the solution set of student work provided.

Grouping

- Ask a student to read the information and complete Question 1 as a class.
- Have students complete Questions 2 through 5 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- A trajectory produces what type of function?
- What quantity is labeled on the x-axis? The y-axis?
- As the time increases does the distance increase?
- As the time increases does the distance decrease?

PROBLEM 1 Fireworks Go Boom!

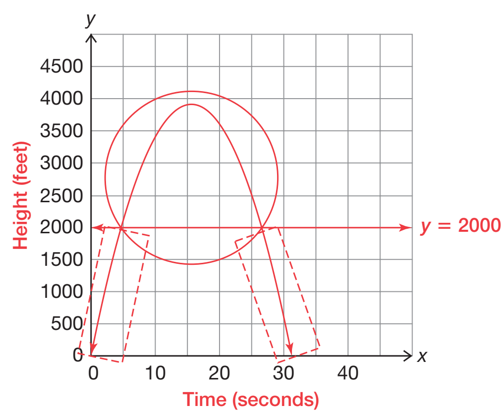
A firework is shot straight up into the air with an initial velocity of 500 feet per second from 5 feet off the ground.

1. Identify the variables and write a quadratic function to represent this situation.

   Let t represent the time in seconds the firework is in the air, and let h represent the height of the firework in feet.

   $h(t) = -16t^2 + 500t + 5$

2. Sketch a graph of the path of the firework on the coordinate plane.



3. Determine when the firework will return to the ground.

   Methods may vary.

   $-16t^2 + 500t + 5 = 0$

   $a = -16;\ b = 500;\ c = 5$

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

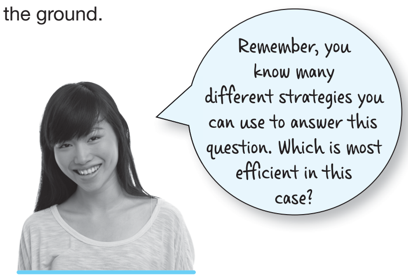
   $$x = \frac{-500 \pm \sqrt{500^2 - 4(-16)(5)}}{2(-16)}$$

   $$x = \frac{-500 \pm \sqrt{250{,}000 + 320}}{-32}$$

   $$x = \frac{-500 \pm \sqrt{250{,}320}}{-32}$$

   $$x = \frac{-500 + \sqrt{250{,}320}}{-32} \approx -0.01 \quad \text{or} \quad x = \frac{-500 - \sqrt{250{,}320}}{-32} \approx 31.26$$

   The firework will return to the ground after about 31.26 seconds.



- Will the distance reach an absolute maximum or an absolute minimum? Why?
- Would you describe the graph as being parabolic? Why?
- Was the method you chose to determine when the firework will return to the ground an algebraic or graphic method?
- What method did you choose to determine when the firework will return to the ground?
- Is the quadratic equation representing the situation factorable?
- Is your solution an exact solution or an approximate solution? Why?

Guiding Questions for Share Phase, Questions 4 and 5

- Did you use the same method to determine when the firework will be 2000 feet from the ground? Why or why not?
- In how many instances will the firework will be 2000 feet from the ground? Why?
- For how many seconds is the firework higher than 2000 feet?
- For how many seconds is the firework below 2000 feet?
- What inequality sign is associated with the inequality representing the times when the firework is below 2000 feet?
- What inequality sign is associated with the inequality representing the times when the firework is higher than 2000 feet?

4. Determine when the firework will be 2000 feet off the ground.

$2000 = -16t^2 + 500t + 5$

$0 = -16t^2 + 500t - 1995$

$a = -16$; $b = 500$; $c = -1995$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-500 \pm \sqrt{500^2 - 4(-16)(-1995)}}{2(-16)}$$

$$x = \frac{-500 \pm \sqrt{250{,}000 - 127{,}680}}{-32}$$

$$x = \frac{-500 + \sqrt{122{,}320}}{-32}$$

$$x \approx \frac{-500 + 349.74}{-32} \approx 4.7 \quad \text{or} \quad x \approx \frac{-500 - 349.74}{-32} \approx 26.55$$

The firework is 2000 feet off the ground at 4.7 seconds and at 26.55 seconds.

5. Analyze your solution in Question 4.

a. When is the firework higher than 2000 feet? Circle this portion of the graph.

The firework is higher than 2000 feet for times between 4.7 seconds and 26.55 seconds.

b. When is the firework below 2000 feet? Draw a box around this portion of the graph.

The firework is below 2000 feet at times less than 4.7 seconds and between 26.55 and 31.26 seconds.

c. Write a quadratic inequality that represents the times when the firework is below 2000 feet.

$-16t^2 + 500t + 5 < 2000$

Grouping

Ask a student to read the worked example. Discuss as a class.

Guiding Questions for Discuss Phase, Worked Example

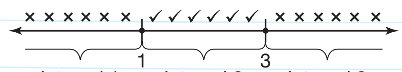
- In the example, how was the quadratic inequality changed into a quadratic equation?
- In the example, how were the roots of the quadratic equation determined?
- Is the quadratic equation a perfect square? Why not?
- In the example, how were the possible solution intervals determined?
- In the example, how was the solution interval determined?
- Under what circumstance would the inequality include 1 and 3 in the solution set?
- Under what circumstance would an inequality include negative numbers in the solution set?
- Under what circumstance would an inequality include positive numbers greater than 3 in the solution set?

The solution to a quadratic inequality is the set of values that satisfy the inequality.

Let's determine the solution of the quadratic inequality $x^2 - 4x + 3 < 0$.

Write the corresponding quadratic equation.	$x^2 - 4x + 3 = 0$
Calculate the roots of the quadratic equation using an appropriate method.	$(x - 3)(x - 1) = 0$ $(x - 3) = 0$ or $(x - 1) = 0$ $x = 3$ or $x = 1$

Plot the roots to divide the number line into three regions.



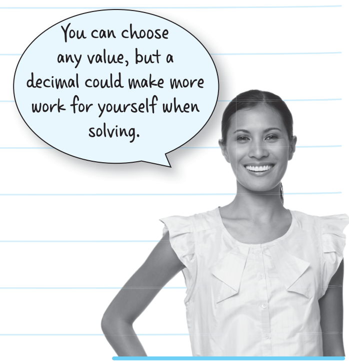
Interval 1	Interval 2	Interval 3
Try $x = 0$	Try $x = 2$	Try $x = 4$
$0^2 - 4(0) + 3 < 0$	$2^2 - 4(2) + 3 < 0$	$4^2 - 4(4) + 3 < 0$
$3 < 0$ ×	$4 - 8 + 3 < 0$	$16 - 16 + 3 < 0$
	$-1 < 0$ ✓	$3 < 0$ ×

Choose a value from each interval to test in the original inequality.

Identify the solution set as the interval(s) in which your test value satisfies the inequality.

Interval 2 satisfies the original inequality, so the solution includes all numbers between 1 and 3.

Solution: $x \in (1, 3)$



Grouping

Have students complete Questions 6 and 7 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 7

- What graphic key characteristics are obvious from the quadratic equation?
- The graph of the quadratic function is divided into how many intervals?
- How many intervals on the graph are used to describe the solution set?
- Which axis represents the number line?
- How many ovals did you draw on the graph of the quadratic function? What do they indicate?
- How many boxes did you draw on the graph of the quadratic function? What do they indicate?

6. Analyze the worked example.

a. How would the solution set change if the inequality was less than or equal to? Explain your reasoning.

Solution: $x \in [1, 3]$

The solution would now include the numbers 1 and 3 in the solution.

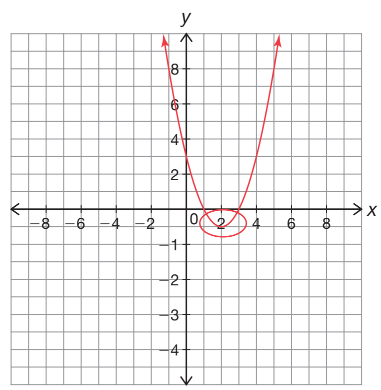
b. How would the solution set change if the inequality was greater than or equal to? Explain your reasoning.

Solution: $x \in [-\infty, 1]$ or $x \in [3, \infty]$

The solution would include all numbers less than or equal to 1 and greater than or equal to 3.

7. Think about the graph of $x^2 - 4x + 3 < 0$.

a. First, graph the function $f(x) = x^2 - 4x + 3$ on the coordinate plane shown.



b. Identify the portion of the graph that is a part of the solution of $x^2 - 4x + 3 < 0$.

See graph.

c. How does this graph represent what is shown on the number line in the worked example?

The x-axis on the graph represents the number line. The interval that I circled represents the solution because these are the values that are between 1 and 3. The boxes represent the intervals that are not solutions because they are less than 1 and greater than 3.

Grouping

Have students complete Question 8 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 8

- If the roots of a quadratic equation are correct, can the solution set be incorrect?
- Is there a difference between the roots of a quadratic equation and the solution set of a quadratic equation?
- What is the difference between the roots of a quadratic equation and the solution set of a quadratic equation?
- After determining the roots of a quadratic equation, how is the solution set determined?

8. Jeff correctly determined the roots of the quadratic inequality $2x^2 - 14x + 27 \geq 7$ to be $x = 5$ and $x = 2$. However, he incorrectly determined the solution set. His work is shown.

Jeff

$2x^2 - 14x + 27 = 7$

$2x^2 - 14x + 20 = 0$

$2(x^2 - 7x + 10) = 0$

$2(x - 5)(x - 2) = 0$

$x = 5$ or $x = 2$

$x = 1$	$x = 3$	$x = 6$
$2(1)^2 - 14(1) + 27 \geq 7$	$2(3)^2 - 14(3) + 27 \geq 7$	$2(6)^2 - 14(6) + 27 \geq 7$
$2 - 14 + 27 \geq 7$	$18 - 42 + 27 \geq 7$	$72 - 84 + 27 \geq 7$
$15 \geq 7$ ✓	$3 \geq 7$ ✗	$15 \geq 7$ ✓

Solution: $x \in (-\infty, 1]$ or $x \in [6, \infty)$

Describe Jeff's error determining the solution set of the quadratic inequality. Then, determine the correct solution set for the inequality.

Jeff based his solution set on his chosen values instead of on the roots of the quadratic equation. The correct solution set is x [$(-\infty, 2]$ or x [$[5, \infty)$.

14

Problem 2

A scenario modeled by a vertical motion function is given. Students will identify the variables and write the quadratic function to represent the situation. They then determine when the object hits the ground by using a solution method of choice. They also write and solve an inequality that represents times when the object is specified heights.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- A trajectory produces what type of function?
- As the time increases does the distance increase?
- As the time increases does the distance decrease?
- Will the distance reach an absolute maximum or an absolute minimum? Why?
- What method did you choose to determine when the balloon was above 30 feet?

PROBLEM 2 Making a Splash

A water balloon is launched from a machine upward from a height of 10 feet with an initial velocity of 46 feet per second.

1. Identify the variables and write a quadratic function representing this situation.

   Let t represent the time in seconds, and let h represent the height of the balloon in feet f seconds after it is thrown.

   $h(t) = -16t^2 + 46t + 10$

2. How long does it take for the balloon to reach the ground? Show your work.

   $-16t^2 + 46t + 10 = 0$

   $a = -16;\ b = 46;\ c = 10$

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   $$x = \frac{-46 \pm \sqrt{46^2 - 4(-16)(10)}}{2(-16)}$$

   $$x = \frac{-46 \pm \sqrt{2116 - 640}}{-32}$$

   $$x = \frac{-46 \pm \sqrt{2756}}{-32}$$

   $$x = \frac{-46 \pm 2\sqrt{689}}{-32}$$

   $$x = \frac{-46 + 2\sqrt{689}}{-32} \approx -0.203 \quad \text{or} \quad x = \frac{-46 - 2\sqrt{689}}{-32} \approx 3.078$$

   It takes the balloon about 3 seconds to reach the ground.

3. Determine when the balloon is above 30 feet.

   a. Write a quadratic inequality to represent this situation.

   $-16t^2 + 46t + 10 > 30$

   b. Use the inequality to determine when the balloon is above 30 feet. Show your work.

   $-16t^2 + 46t + 10 > 30$

   $-16t^2 + 46t + 10 = 30$

   $-16t^2 + 46t - 20 = 0$

   $a = -16,\ b = 46,\ c = -20$

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   $$x = \frac{-46 \pm \sqrt{46^2 - 4(-16)(-20)}}{2(-16)}$$

   $$x = \frac{-46 \pm \sqrt{2116 - 1280}}{-32}$$

   $$x = \frac{-46 \pm \sqrt{836}}{-32}$$

   $$x = \frac{-46 \pm 2\sqrt{209}}{-32}$$

- Is the quadratic equation representing the situation factorable?
- Is your solution an exact solution or an approximate solution? Why?

14

Guiding Questions for Share Phase, Question 4

- Did you use the same method to determine when the height was less than or equal to 43 feet?
- What inequality sign is associated with the inequality representing the times when the balloon is lower than or equal to 43 feet?
- How do you know where each interval on the graph begins and ends?
- How many intervals is the graph divided into?
- Which interval(s) satisfied the inequality?

$x = \frac{-46 + 2\sqrt{209}}{-32} = 0.534$ or $x = \frac{-46 - 2\sqrt{209}}{-32} = 2.341$

$x = 0$	$x = 1$	$x = 3$
$30 < -16(0)^2 + 46(0) + 10$	$30 < -16(1)^2 + 46(1) + 10$	$30 < -16(3)^2 + 46(3) + 10$
$30 < 10$ ×	$30 < -16 + 46 + 10$	$30 < -144 + 138 + 10$
	$30 < 40$ ✓	$30 < 4$ ×

Solution: $x \in (0.534, 2.341)$

The balloon is above 30 feet between 0.534 and 2.341 seconds.

4. Determine when the balloon is at or below 43 feet.

$-16t^2 + 46t + 10 \le 43$

$-16t^2 + 46t + 10 = 43$

$-16t^2 + 46t - 33 = 0$

$a = -16;\ b = 46,\ c = -33$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-46 \pm \sqrt{46^2 - 4(-16)(-33)}}{2(-16)}$

$x = \frac{-46 \pm \sqrt{2116 - 2112}}{-32}$

$x = \frac{-46 \pm \sqrt{4}}{-32}$

$x = \frac{-46 + 2}{-32} = 1.375$ or $x = \frac{-46 - 2}{-32} = 1.5$

$x = 1$	$x = 1.4$	$x = 2$
$43 \ge -16(1)^2 + 46(1) + 10$	$43 \ge -16(1.4)^2 + 46(1.4) + 10$	$43 \ge -16(2)^2 + 46(2) + 10$
$43 \ge -16 + 46 + 10$	$43 \ge -31.36 + 64.4 + 10$	$43 \ge -64 + 92 + 10$
$43 \ge 40$ ✓	$43 \ge 43.04$ ×	$43 \ge 38$ ✓

Solution: $x \in [0, 1.375]$ or $x \in [1.5, 3.078]$

The balloon is less than 43 feet between 0 and 1.375 seconds and between 1.5 and 3.078 seconds.

Be prepared to share your solutions and methods.

14

Check for Students' Understanding

In Problem 1, a firework was shot straight up into the air with an initial velocity of 500 feet per second from 5 feet off the ground. The function representing the situation was identified as $h(t) = -16t^2 + 500t + 5$. You determined the firework will return to the ground after approximately 31.26 seconds.

Suppose a second firework was shot straight up into the air with an initial velocity of 500 feet per second from the ground.

1. Predict when the second firework will return to the ground. Will it return to the ground sooner, later, or in the same amount of time when compared to the first firework?

   Answers will vary.

   I predict the second firework will return to the ground sooner because it didn't travel as high as the first firework.

2. Write a quadratic function to represent the path of the second firework.

   $h(t) = -16t^2 + 500t$

3. Determine when the second firework will return to the ground.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-500 \pm \sqrt{500^2 - 4(-16)0}}{2(-16)}$$

$$x = \frac{-500 \pm \sqrt{250{,}000 - 0}}{-32}$$

$$x = \frac{-500 \pm \sqrt{250{,}000}}{-32}$$

$$x = \frac{-500 + \sqrt{250{,}000}}{-32} \approx 0$$

$$x = \frac{-500 - \sqrt{250{,}000}}{-32} \approx 31.26$$

   The second firework will return to the ground in approximately 31.26 seconds.

4. Was the prediction made in Question 1 correct?

   Answers vary.

5. Compare the graph of the path of the first firework to the graph of the path of the second firework. What do you notice?

   The graphs appear to be the same graph but they are not exactly the same. The *y*-intercepts are different, but the difference is only 5 feet on a scale of 5,000 feet so the difference is too small to distinguish the difference between the two functions.

14.4

You Must Have a System

Systems of Quadratic Equations

LEARNING GOALS

In this lesson, you will:

- Solve systems of a linear equation and a quadratic equation.
- Solve systems of two quadratic equations.

ESSENTIAL IDEAS

- A system of equations containing a linear equation and a quadratic equation are solved both algebraically and graphically.
- A system of equations containing a linear equation and a quadratic equation may have one solution, two solutions, or no solutions.
- A system of equations containing two quadratic equations are solved both algebraically and graphically.
- A system of equations containing two quadratic equations may have one solution, two solutions, or no solutions.
- The quadratic formula, substitution, and factoring are used to algebraically solve systems of equations.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-REI Reasoning with Equations and Inequalities

Solve systems of equations

7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.

A-CED Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Overview

Students will solve systems of equations composed a linear equation and a quadratic equation. They solve the systems algebraically using the quadratic formula, substitution, and factoring. They then verify their algebraic solutions graphically by determining the coordinates of the points of intersection. In the second activity, students will solve a system composed of two quadratic equations using substitution and factoring. The systems of equations used in this lesson have one solution, two solutions, or no solutions.

Warm Up

1. Use an algebraic method to solve the system of linear equations.

$$\begin{cases} y = 4 - 5x \\ y = 3x - 8 \end{cases}$$

$4 - 5x = 3x - 8$

$8x = 12$

$x = 1.5$

$3(1.5) - 8 = -3.5$

The solution to the system of equations is $(1.5, -3.5)$.

2. What method did you use to solve the system of equations?

Answers vary.

I used the substitution method.

3. Without graphing, describe the graphical representation of the system of linear equations.

The graph of the line $y = 4 - 5x$ has a negative slope and is drawn from the upper left portion of the graph to the lower right. The graph of the line $y = 3x - 8$ has a positive slope and is drawn from the upper right portion of the graph to the lower left. The graph of the line $y = 4 - 5x$ is steeper than the graph of the line $y = 3x - 8$. The graphs of the two lines intersect at the point $(1.5, -3.5)$.

You Must Have a System

Systems of Quadratic Equations

14.4

LEARNING GOALS

In this lesson, you will:

- Solve systems of a linear equation and a quadratic equation.
- Solve systems of two quadratic equations.

Your body is an amazing collection of different systems. Your cardiovascular system pumps blood throughout your body, your skeletal system provides shape and support, and your nervous system controls communication between your senses and your brain. Your skin, including your hair and fingernails, is a system all by itself—the integumentary system—and it protects all of your body's other systems. You also have a digestive system, endocrine system, excretory system, immune system, muscular system, reproductive system, and respiratory system.

Why do we call these systems "systems"? What do you think makes up a system?

Problem 1

Students will solve the systems of equations algebraically using methods such as the quadratic formula, substitution, and factoring. The results are verified graphically by identifying the coordinates of the points of intersection of the two graphs. Systems of equations have one, two, or no solutions.

Grouping

Have students complete Question 1 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 1

- Why does it make sense to substitute $2x + 7$ in for y in the quadratic equation?
- After the substitution, which terms are considered to be like terms?
- What operation is used to combine the like terms?
- Why is it important to have all of the terms on one side of the equation and 0 on the other side of the equation?
- Is the resulting quadratic equation factorable?
- Once the equation is factored, how are the roots determined?
- Once the values for x have been determined, how do you solve for the values of y?
- Why is it important to solve for the values of y?

PROBLEM 1 Solving a System With a Linear Equation and a Quadratic Equation

A system of equations can also involve non-linear equations, such as quadratic equations. Luckily, methods for solving a system of non-linear equations can be similar to methods for solving a system of linear equations.

1. Consider the system of a linear equation and a quadratic equation: $\begin{cases} y = 2x + 7 \\ y = x^2 + 4 \end{cases}$.

   a. Write a new equation you can use to solve this system.

   $2x + 7 = x^2 + 4$

   b. Solve the resulting equation for x.

   $2x + 7 = x^2 + 4$

   $0 = x^2 - 2x - 3$

   $0 = (x - 3)(x + 1)$

   $x - 3 = 0$ or $x + 1 = 0$

   $x = 3$ or $x = -1$

   c. Calculate the corresponding values for y.

   Substitute $x = 3$ into the linear equation.
   $y = 2(3) + 7 = 6 + 7 = 13$
   Substitute $x = -1$ into the linear equation.
   $y = 2(-1) + 7 = -2 + 7 = 5$

   d. What is/are the solution(s) to the system of equations?

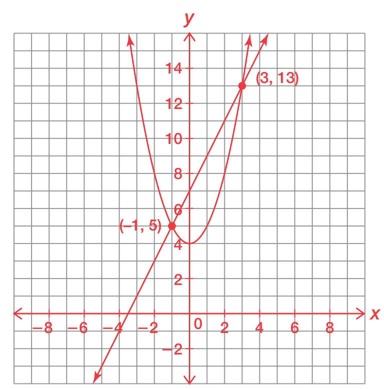
   The system has two solutions: (3, 13) and (−1, 5).

   e. Graph each equation of the system and calculate the points of intersection.

   The system has two solutions: (3, 13) and (−1, 5).

   f. What do you notice about the solutions you calculated algebraically and graphically?

   The solutions are the same.



- If the algebraic solutions to the system of equations are (3, 13) and (−1, 5), what implications does this have graphically?

14

Guiding Questions for Share Phase, Question 2

- Is it possible for the graph of a quadratic equation and the graph of a linear equation to intersect in exactly one point?
- Is it possible for the graph of a quadratic equation and the graph of a linear equation to intersect in exactly two points?
- Is it possible for the graph of a quadratic equation and the graph of a linear equation to intersect in exactly three points? Why not?
- Is it possible for the graph of a quadratic equation and the graph of a linear equation to intersect in an infinite number of points? Why not?
- Is it possible for the graph of a quadratic equation and the graph of a linear equation to intersect in exactly no points?

Grouping

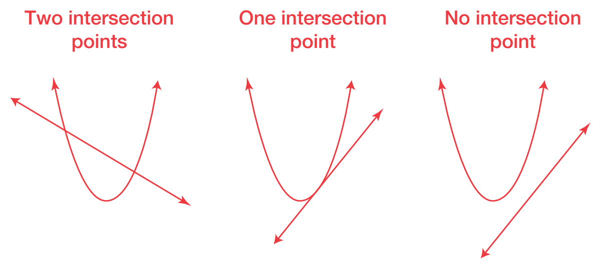
Have students complete Question 3 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 3

- Looking only at the equations of the system, are there any hints to help you determine how many solutions the system will have?

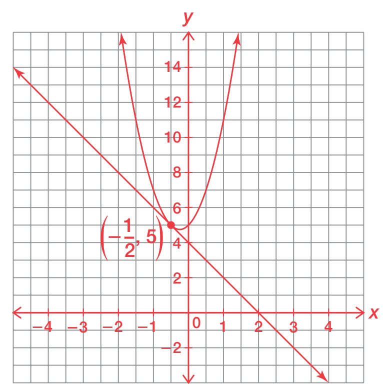
2. Think about the graphs of a linear equation and a quadratic equation. Describe the different ways in which the two graphs can intersect, and provide a sketch of each case.

The graph of a linear equation can intersect the graph of a quadratic equation in three ways. The graphs can intersect at two points, at one point, or not at all.



3. Solve each system of equations algebraically over the set of real numbers. Then verify the solution graphically.

a. $\begin{cases} y = -2x + 4 \\ y = 4x^2 + 2x + 5 \end{cases}$



$-2x + 4 = 4x^2 + 2x + 5$

$0 = 4x^2 + 4x + 1$

$0 = (2x + 1)(2x + 1)$

$2x + 1 = 0$

$2x = -1$

$x = -\frac{1}{2}$

Substitute $x = -\frac{1}{2}$ into the linear equation.

$y = -2\left(-\frac{1}{2}\right) + 4 = 1 + 4 = 5$

The system has one solution: $\left(-\frac{1}{2}, 5\right)$.

- Looking only at the equations of the system, are there any hints to help you determine that the system will have no solutions?
- How will you know when the quadratic formula is needed to solve the system of equations?
- How will you know when the system of equations has no solutions?

14

Problem 2

Students will solve the systems of equations algebraically using methods such as the quadratic formula, substitution, and factoring. The results are then verified graphically by identifying the coordinates of the points of intersection of the two graphs. Systems of equations have one, two, or no solutions.

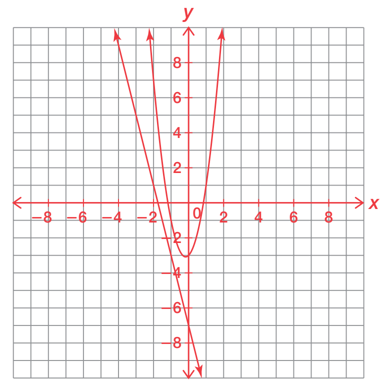
Grouping

Have students complete Questions 1 through 3 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 1

- Why does it make sense to substitute $x^2 + 3x - 5$ in for y in the other quadratic equation?
- After the substitution, which terms are considered to be like terms?
- What operation is used to combine the like terms?
- Why is it important to have all of the terms on one side of the equation and 0 on the other side of the equation?
- Is the resulting quadratic equation factorable?
- Once the equation is factored, how are the roots determined?
- Once the values for x have been determined, how do you solve for the values of y?

b. $\begin{cases} y = -4x - 7 \\ y = 3x^2 + x - 3 \end{cases}$



$-4x - 7 = 3x^2 + x - 3$

$0 = 3x^2 + 5x + 4$

$x = \dfrac{-5 \pm \sqrt{5^2 - 4(3)(4)}}{2(3)}$

$x = \dfrac{-5 \pm \sqrt{-23}}{6}$

The system has no real solutions.

PROBLEM 2 Solving a System of Two Quadratic Equations

1. Consider the system of two quadratic equations: $\begin{cases} y = x^2 + 3x - 5 \\ y = -x^2 + 10x - 1 \end{cases}$

   a. Set the expressions equal to each other.

   $x^2 + 3x - 5 = -x^2 + 10x - 1$

   b. Solve the resulting equation for x.

   $x^2 + 3x - 5 = -x^2 + 10x - 1$

   $2x^2 - 7x - 4 = 0$

   $(2x + 1)(x - 4) = 0$

   $2x + 1 = 0 \quad \text{or} \quad x - 4 = 0$

   $x = -\frac{1}{2} \quad \text{or} \quad x = 4$

   c. Calculate the corresponding values for y.

   Substitute $x = -\frac{1}{2}$ into the first quadratic equation.

   $y = \left(-\frac{1}{2}\right)^2 + 3\left(-\frac{1}{2}\right) - 5 = \frac{1}{4} - \frac{3}{2} - 5 = -6\frac{1}{4}$

   Substitute $x = 4$ into the first quadratic equation.

   $y = (4)^2 + 3(4) - 5 = 16 + 12 - 5 = 23$

- Why is it important to solve for the values of y?
- If the algebraic solutions to the system of equations are $(-\frac{1}{2}, -6\frac{1}{4})$ and $(4, 23)$, what implications does this have graphically?
- Is it possible for the graphs of two quadratic equations to intersect in exactly one point?
- Is it possible for the graphs of two quadratic equations to intersect in exactly two points?

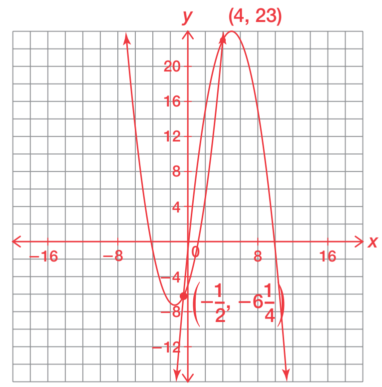
Guiding Questions for Share Phase, Questions 1 and 2

- Is it possible for the graphs of two quadratic equations to intersect in exactly three points? Why not?
- Is it possible for the graphs of two quadratic equations to intersect in an infinite number of points?
- Is it possible for the graphs of two quadratic equations to intersect in exactly no points?

d. What is the solution to the system of equations?

The system has two solutions: $\left(-\frac{1}{2}, -6\frac{1}{4}\right)$ and (4, 23).

e. Graph each equation of the system and calculate the points of intersection.

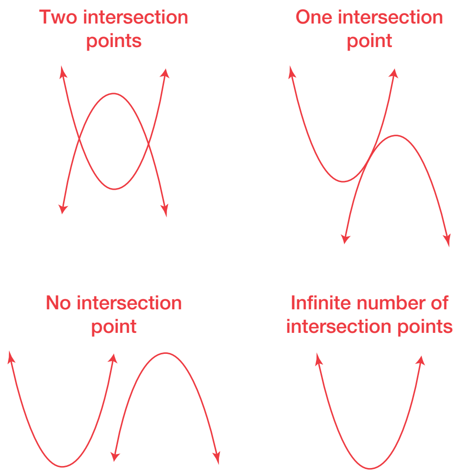


f. What do you notice about the solutions you calculated algebraically and graphically?

The solutions are the same.

2. Think about the graphs of two quadratic equations. Describe the different ways in which the two graphs can intersect and provide a sketch of each case.

The graphs of two quadratic equations can intersect in four ways. The graphs can intersect at two points, at one point, at an infinite number of points, or not at all.



Grouping

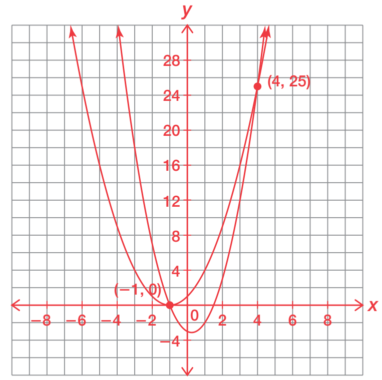
Have students work with a partner to complete Question 3. Then share the responses as a class.

Guiding Questions for Share Phase, Question 3

- Looking only at the equations of the system, are there any hints to help you determine how many solutions the system will have?
- Looking only at the equations of the system, are there any hints to help you determine that the system will have no solutions?
- Looking only at the equations of the system, are there any hints to help you determine that the system will have an infinite number of solutions?
- How will you know when the quadratic formula is needed to solve the system of equations?
- How will you know when the system of equations has no solutions?
- How will you know when the system of equations has an infinite number of solutions?

3. Solve each system of equations algebraically over the set of real numbers. Then verify the solution graphically.

a. $\begin{cases} y = x^2 + 2x + 1 \\ y = 2x^2 - x - 3 \end{cases}$



$x^2 + 2x + 1 = 2x^2 - x - 3$

$0 = x^2 - 3x - 4$

$0 = (x - 4)(x + 1)$

$x - 4 = 0$ or $x + 1 = 0$

$x = 4$ or $x = -1$

Substitute $x = 4$ into the first equation.

$y = (4)^2 + 2(4) + 1 = 16 + 8 + 1 = 25$

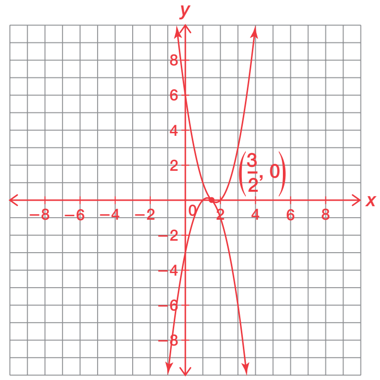
Substitute $x = -1$ into the first equation.

$y = (-1)^2 + 2(-1) + 1 = 1 - 2 + 1 = 0$

The system has two solutions: (4, 25) and (−1, 0).

14

b. $\begin{cases} y = 2x^2 - 7x + 6 \\ y = -2x^2 + 5x - 3 \end{cases}$



$2x^2 - 7x + 6 = -2x^2 + 5x - 3$

$4x^2 - 12x + 9 = 0$

$(2x - 3)(2x - 3) = 0$

$2x - 3 = 0$

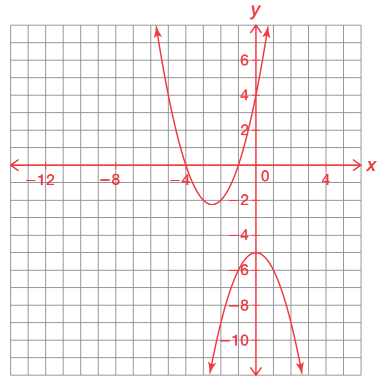
$x = \frac{3}{2}$

Substitute $x = \frac{3}{2}$ into the first equation.

$y = 2\left(\frac{3}{2}\right)^2 - 7\left(\frac{3}{2}\right) + 6 = \frac{9}{2} - \frac{21}{2} + 6 = 0$

The system has one solution: $\left(\frac{3}{2}, 0\right)$.

c. $\begin{cases} y = x^2 + 5x + 4 \\ y = -x^2 - 5 \end{cases}$



$x^2 + 5x + 4 = -x^2 - 5$

$2x^2 + 5x + 9 = 0$

$x = \dfrac{-5 \pm \sqrt{5^2 - 4(2)(9)}}{2(2)}$

$x = \dfrac{-5 \pm \sqrt{-47}}{4}$

The system has no real solutions.

Be prepared to share your solutions and methods.

14

Check for Students' Understanding

1. How is a linear equation distinguished from a quadratic equation?

   A linear equation has a degree of 1, and a quadratic equation has a degree of 2. There is always an x^2 term in a quadratic equation.

2. A system of equations consisting of two linear equations has how many possible solutions?

   A system of equations consisting of two linear equations can have one solution, no solutions, or an infinite number of solutions.

3. A system of equations consisting of two quadratic equations has how many possible solutions?

   A system of equations consisting of two quadratic equations can have one solution, two solutions, no solutions, or an infinite number of solutions.

4. A system of equations consisting of a linear equation and a quadratic equation has how many possible solutions?

   A system of equations consisting of a linear equation and a quadratic equation can have one solution, two solutions, or no solutions.

Chapter 14 Summary

KEY TERMS

- Quadratic Formula (14.1)
- discriminant (14.1)
- quadratic regression (14.2)
- coefficient of determination (14.2)

14.1 Using the Quadratic Formula to Determine the Zeros of a Quadratic Function or the Roots of a Quadratic Equation

Calculating the x-intercepts for a quadratic function can be difficult if the quadratic function cannot be factored. When a quadratic function cannot be factored, the Quadratic Formula can be used to determine the solutions. The Quadratic Formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

$f(x) = 2x^2 - 4x - 3$

$a = 2;\ b = -4;\ c = -3$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{4 \pm \sqrt{16 + 24}}{4}$$

$$x = \frac{4 \pm \sqrt{40}}{4}$$

$$x \approx \frac{4 + 6.325}{4} \approx 2.581 \quad \text{or} \quad x \approx \frac{4 - 6.325}{4} \approx -0.581$$

The roots are approximately 2.581 and -0.581.

14.1 Determining the Number of Roots for a Quadratic Function

The portion of the Quadratic Formula under the radical determines the number of roots a quadratic function has. Because this portion of the formula "discriminates" the number of the roots, it is called the discriminant. If $b^2 - 4ac < 0$, there are no real roots. If $b^2 - 4ac = 0$, there is one real root. If $b^2 - 4ac > 0$, there are two real roots. You can also use the discriminant to describe the "nature" of the roots. If the discriminant is a perfect square, then the roots are rational. If the discriminant is not a perfect square, then the roots are irrational.

Example

$y = x^2 - 6x + 9$

$$\begin{aligned} b^2 - 4ac &= (-6)^2 - 4(1)(9) \\ &= 36 - 36 \\ &= 0 \end{aligned}$$

The function has one rational root.

$x = \dfrac{-(-6) \pm \sqrt{0}}{2} = 3$

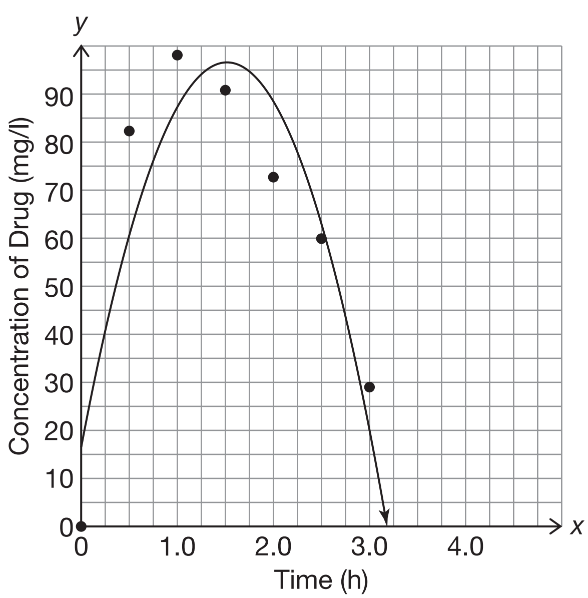
The zero of the function is at (3, 0).

14.2 Determining the Quadratic Regression on a Data Set

The quadratic regression is a mathematical method to determine the equation of a parabola of "best fit" for a data set. A graphing calculator can be used to determine the quadratic regression of a data set.

Example

Time (h)	Concentration of drug (mg/l)
0	0
0.5	82.3
1.0	98.1
1.5	90.8
2.0	72.7
2.5	59.9
3.0	29.0



The quadratic regression equation for the data is $y = -34.790x^2 + 105.57x + 16.5$.

14.2 Determining the Fit of a Quadratic Equation

When dealing with linear regressions, you can use the correlation coefficient (r) to determine the accuracy of a line of best fit. The correlation coefficient can then be squared to become the coefficient of determination. The coefficient of determination measures the "strength" of the relationship between the original data and the regression equation. The coefficient of determination (r^2) is used when dealing with quadratics because the relationship is not linear; therefore, the correlation coefficient cannot be used. The coefficient of determination can take on values between 0 and 1. The closer the value is to 1, the better the fit of the regression equation.

Example

Time (h)	Concentration of drug (mg/l)
0	0
0.5	82.3
1.0	98.1
1.5	90.8
2.0	72.7
2.5	59.9
3.0	29.0

The quadratic regression equation for the data is

$y = -34.790x^2 + 105.57x + 16.5.$

The coefficient of determination is $r^2 \approx 0.8378$. Because the r^2 value is close to 1, the equation is a pretty good fit for the data.

14.3 Writing a Quadratic Inequality

To write a quadratic inequality, define the variables and write the corresponding quadratic equation. Then use the problem situation and inequality symbols to compare the equation to the given limit.

Example

An object is launched upwards from a 10-foot high platform with an initial velocity of 60 feet per second. Write a quadratic inequality that can be used to determine when the object is above 50 feet.

Let t represent time in seconds, and h represent height in feet.

$h(t) = -16t^2 + 60t + 10$

$50 < -16t^2 + 60t + 10$

14.3 Determining a Solution Set to a Quadratic Inequality

The interval method is used to determine the solution to a quadratic inequality. The solution is the set of values that satisfy the inequality. To use the interval method, write the inequality as an equation and calculate the roots. Use the roots to create three intervals and choose a test value that falls within each interval. Identify the solution set as the interval(s) in which the test value satisfies the inequality.

Example

$-16t^2 + 60t + 10 > 50$

$-16t^2 + 60t + 10 = 50$

$-16t^2 + 60t - 40 = 0$

$a = -16;\ b = 60;\ c = -40$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-60 \pm \sqrt{60^2 - 4(-16)(-40)}}{2(-16)}$$

$$x = \frac{-60 \pm \sqrt{1040}}{-32}$$

$$x = \frac{-60 \pm 4\sqrt{65}}{-32}$$

$$x = \frac{-60 + 4\sqrt{65}}{-32} \approx 0.867 \quad \text{or} \quad x = \frac{-60 - 4\sqrt{65}}{-32} \approx 2.883$$

$x = 0$	$x = 2$	$x = 3$
$50 < -16(0)^2 + 60(0) + 10$	$50 < -16(2)^2 + 60(2) + 10$	$50 < -16(3)^2 + 60(3) + 10$
$50 < 10$ ✗	$50 < 66$ ✓	$50 < 46$ ✗

The solution set is between 0.867 and 2.883 seconds.

14.3 Determining and Graphing the Solutions of a System of Linear and Quadratic Equations

The graph of a linear equation can intersect the graph of a quadratic equation in three ways. The graphs can intersect at two points, at one point, or not at all. The method to determine the solution or solutions of a system of linear and quadratic equations is similar to solving a system of linear equations. First, substitute the linear equation into the quadratic equation. Then solve the resulting equation for x and calculate the corresponding values for y. These values represent the point(s) of intersection. Finally, graph each equation of the system to verify the points of intersection.

Example

$$\begin{cases} y = 6x + 15 \\ y = x^2 + 5x + 3 \end{cases}$$

$$6x + 15 = x^2 + 5x + 3$$

$$0 = x^2 - x - 12$$

$$0 = (x - 4)(x + 3)$$

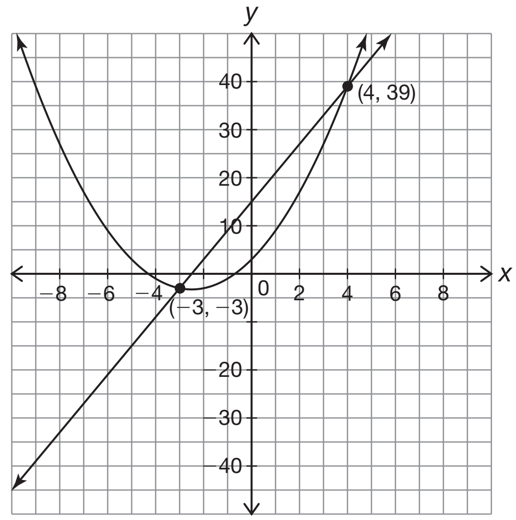
$$x - 4 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = 4 \quad \text{or} \quad x = -3$$

$$y = 6(4) + 15 = 39$$

$$y = 6(-3) + 15 = -3$$

The system has two solutions: (4, 39) and (−3, −3).



14.4 Determining and Graphing the Solutions of a System of Two Quadratic Equations

The graphs of two quadratic equations can intersect in four ways. The graphs can intersect at two points, at one point, at an infinite number of points, or not at all. To determine the intersection point(s), substitute the first quadratic equation into the second quadratic equation. Then solve the resulting equation for x and calculate the corresponding values for y. These values represent the point(s) of intersection. Finally, graph each equation of the system to verify the point(s) of intersection.

Example

$$\begin{cases} y = 5x^2 + 8x + 6 \\ y = x^2 - 4x - 3 \end{cases}$$

$$x^2 - 4x - 3 = 5x^2 + 8x + 6$$

$$0 = 4x^2 + 12x + 9$$

$$0 = (2x + 3)(2x + 3)$$

$$2x + 3 = 0$$

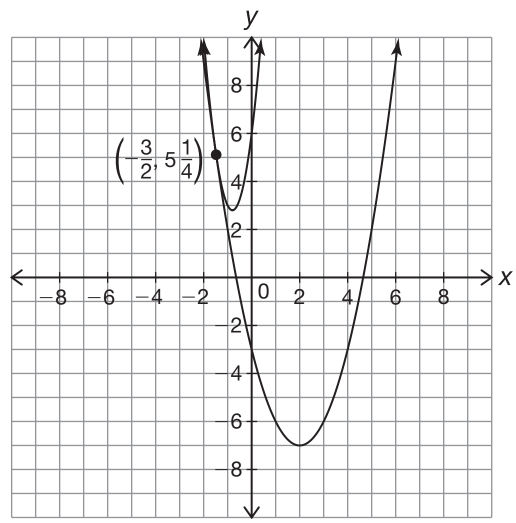
$$2x = -3$$

$$x = -\frac{3}{2}$$

$$y = \left(-\frac{3}{2}\right)^2 - 4\left(-\frac{3}{2}\right) - 3$$

$$= 5\frac{1}{4}$$

The system has one solution: $\left(-\frac{3}{2}, 5\frac{1}{4}\right)$.



14