Polynomials and Quadratics

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Chapter 13 Overview

This chapter introduces operations with polynomials, including factoring quadratic trinomials. Quadratic equations are solved graphically, by factoring, and by completing the square.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
13.1	Adding and Subtracting Polynomials	A.SSE.1.a A.APR.1 A.CED.1 F.BF.1.b A.CED.2	2	This lesson introduces polynomials and uses a sorting activity for students to classify monomials, binomials, and trinomials. A problem situation presents a linear and a quadratic function. Questions ask students to model the sum of the functions using function notation, a graph, a table, and finally, using algebra.		X	X		X
13.2	Multiplying Polynomials	A.APR.1	1	This lesson introduces how to the multiply two binomials using algebra tiles, multiplication tables, and the Distributive Property. A graphing calculator is used to verify the product of the two polynomials. Questions then ask students to extend their understanding of multiplying two binomials to include multiplying a binomial by a trinomial.	X	X	X		X
13.3	Factoring Polynomials	A.SSE.3.a A.APR.1	2	This lesson focuses on writing quadratic expressions as products of factors. Students will use an area model, a multiplication table, and consider the factors of the leading coefficient and the constant term to rewrite expressions in factored form.		X	X	X	
13.4	Solving Quadratics by Factoring	A.SSE.3.a A.REI.4.b	1	This lesson introduces the Zero Product Property as a strategy to calculate the roots of a quadratic equation. Questions focus students to make the connection between the solutions to a quadratic equation using factoring to the *x*-intercepts of the equation on the graph.	X	X			

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Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
13.5	Special Products	A.SSE.2 A.SSE.3.a	1	This lesson explores the difference of two squares, perfect square trinomials, the difference of two cubes, and the sum of two cubes. Students will continue to practice solving quadratic equations and functions.			X	X	
13.6	Approximating and Rewriting Radicals	N.RN.2 A.CED.1 A.REI.4.b	1	This lesson focuses on determining square roots and extracting perfect square roots from radical expressions. Questions then ask students to solve equations containing radical expressions.		X	X		
13.7	Completing the Square	A.SSE.3.b A.REI.4.b	2	This lesson uses the knowledge of perfect square trinomials to construct a process to solve any quadratic equation that is not factorable. Questions ask students to geometrically complete a square, and then connect that skill to determining the roots of quadratic equations. Students then practice solving quadratic equations by completing the square. A worked example demonstrates how to rewrite a quadratic equation written in standard form to vertex form by completing the square.	X	X			

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Skills Practice Correlation for Chapter 13

Lesson		Problem Set	Description
13.1	Adding and Subtracting Polynomials		Vocabulary
		1 – 6	Identify terms and coefficients in expressions
		7 – 14	Determine if expressions are polynomials
		15 – 20	Determine if polynomials are monomials, binomials, or trinomials and state the degree of the polynomials
		21 – 30	Write polynomials in standard form and classify polynomials by number and degree
		31 – 40	Simplify polynomial expressions
		41 – 46	Evaluate polynomial functions using a graph
13.2	Multiplying Polynomials	1 – 6	Determine products of binomials using algebra tiles
		7 – 12	Determine products of binomials using multiplication tables
		13 – 22	Determine products of polynomials using the Distributive Property
13.3	Factoring Polynomials		Vocabulary
		1 – 10	Factor out the greatest common factor of polynomials
		11 – 16	Factor trinomials using an area model
		17 – 24	Factor trinomials using multiplication tables
		25 – 30	Factor polynomials using the trial and error method
		31 – 36	Factor trinomials
		37 – 42	Factor polynomials
13.4	Solving Quadratics by Factoring		Vocabulary
		1 – 10	Factor and solve quadratic equations
		11 – 18	Determine zeros of quadratic functions
13.5	Special Products		Vocabulary
		1 – 8	Factor binomials
		9 – 14	Factor trinomials
		15 – 20	Determine root(s) of quadratic equations
		21 – 26	Determine zero(s) of quadratic functions
13.6	Approximating and Rewriting Radicals		Vocabulary
		1 – 8	Rewrite radicals by extracting perfect squares
		9 – 14	Determine approximate values of radical expressions
		15 – 20	Solve quadratic equations and approximate roots
		21 – 26	Solve quadratic equations and rewrite roots in radical form
13.7	Completing the Square		Vocabulary
		1 – 6	Use geometric figures to complete the square for expressions
		7 – 14	Determine unknown values to make trinomials perfect squares
		15 – 20	Determine the roots of quadratic equations by completing the square

Controlling the Population

13.1

Adding and Subtracting Polynomials

LEARNING GOALS

In this lesson, you will:

- Recognize polynomial expressions.
- Identify monomials, binomials, and trinomials.
- Identify the degree of a term and the degree of a polynomial.
- Write polynomial expressions in standard form.
- Add and subtract polynomial expressions.
- Graph polynomial functions and understand the connection between the graph of the solution and the algebraic solution.

KEY TERMS

- polynomial
- term
- coefficient
- monomial
- binomial
- trinomial
- degree of a term
- degree of a polynomial

ESSENTIAL IDEAS

- A polynomial is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients.
- In a polynomial expression, each product is called a term and the number being multiplied by a power is a coefficient.
- Polynomials with one term are called monomials.
- Polynomials with two terms are called binomials.
- Polynomials with three terms are called trinomials.
- The degree of a term in a polynomial is the exponent of the term.
- The greatest exponent in a polynomial determines the degree of the polynomial.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context.

   a. Interpret parts of an expression, such as terms, factors, and coefficients.

A-APR Arithmetic with Polynomials and Rational Expressions

Perform arithmetic operations on polynomials

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

A-CED Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems

F-BF Building Functions

Build a function that models a relationship between two quantities

1. Write a function that describes a relationship between two quantities.

   b. Combine standard function types using arithmetic operations.

A-CED Creating Equations

Create equations that describe numbers or relationships

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Overview

The terms polynomial, term, coefficient, monomials, binomials, trinomials, degree of a term, and degree of a polynomial are introduced. Students will identify the coefficient, power, and exponent in several polynomials. They also explore expressions that are not polynomials. A sorting activity is used to classify monomials, binomials, and trinomials. The definition for the standard form of a polynomial is then given. A population scenario is given with functions that model the situation. One function is quadratic and the second function is linear. Students will use a graphing calculator to graph the functions, and complete a table of values to answer questions relevant to the problem situation. They also algebraically solve the functions to verify values written in the table. In the last activity, students rewrite polynomials that were written incorrectly.

Warm Up

Simplify each expression.

1. $-3x^2 + 10x^2$

   $7x^2$

2. $-(5x^2 + 4x^2)$

   $-9x^2$

3. $(2x^2 - 8) + (12x^2 + 4)$

   $14x^2 - 4$

4. $(2x^2 - 8) - (12x^2 + 4)$

   $-10x^2 - 12$

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Controlling the Population

13.1

Adding and Subtracting Polynomials

LEARNING GOALS

In this lesson, you will:

- Recognize polynomial expressions.
- Identify monomials, binomials, and trinomials.
- Identify the degree of a term and the degree of a polynomial.
- Write polynomial expressions in standard form.
- Add and subtract polynomial expressions.
- Graph polynomial functions and understand the connection between the graph of the solution and the algebraic solution.

KEY TERMS

- polynomial
- term
- coefficient
- monomial
- binomial
- trinomial
- degree of a term
- degree of a polynomial

There are all kinds of talking birds. The common crow is able to repeat a few words, while a bird called a Bedgerigar (or common parakeet) can have a vocabulary of thousands of words. A bird of this type named Puck was found in 1995 to have a vocabulary of 1728 words.

African Grey Parrots are also remarkable, not only for their knowledge of words, but also for their other mental abilities. In 2005, an African Grey Parrot named Alex was reported to have understood the concept of zero!

Problem 1

The terms polynomial, term, coefficient, monomials, binomials, trinomials, degree of a term, and degree of a polynomial are introduced. Students are given several polynomial expressions and will write each term identifying the coefficient, power, and exponent. Next, students describe why several expressions are not considered polynomials. In the next activity, students cut out several polynomial expressions and sort them into groups of monomials, binomials, and trinomials. In the last activity, students will examine polynomial expressions to determine which are written correctly in standard form and write a definition for the standard form of a polynomial.

Grouping

- Ask a student to read the information and definitions. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Which term is the leading term?
- What exponent is associated with the leading term?
- What sign is associated with the leading term?
- Which term is the constant?
- Which terms have exponents?
- Which terms are negative?
- Which terms are positive?

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PROBLEM 1 Many Terms

Previously, you worked with a number of expressions in the form $ax + b$ and $ax^2 + bx + c$. Each of these is also part of a larger group of expressions known as *polynomials*.

A **polynomial** is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. A polynomial in one variable is the sum of terms of the form ax^k, where a is any real number and k is a non-negative integer. In general, a polynomial is of the form $a_1x^k + a_2x^{k-1} + \ldots a_nx^0$. Within a polynomial, each product is a **term**, and the number being multiplied by a power is a **coefficient**.

The polynomial $m^3 + 8m^2 - 10m + 5$ has four terms.
Each term is written in the form ax^k.

First term: m^3

coefficient: $a = 1$

variable: m

power: m^3

exponent: $k = 3$

1. Write each term from the worked example and then identify the coefficient, power, and exponent. The first term has already been completed for you.

	1st	2nd	3rd	4th
Term	m^3	$8m^2$	$-10m$	5
Coefficient	1	8	-10	5
Power	m^3	m^2	m^1	m^0
Exponent	3	2	1	0

2. Analyze each polynomial. Identify the terms and coefficients in each.

   a. $-2x^2 + 100x$

   Terms: $-2x^2$ and $100x$; coefficients: -2 and 100

   b. $x^2 + 4x + 3$

   Terms: x^2, $4x$, and 3; coefficients: 1, 4, and 3

   c. $4m^3 - 2m^2 + 5$

   Terms: $4m^3$, $-2m^2$, and 5; coefficients: 4, -2, and 5

Grouping

- Ask a student to read the information and definitions. Discuss as a class.
- Have students complete Question 3 as a class.
- Have students complete Question 4 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- What is the difference between a term with an exponent of 2 and a term with an exponent of -2?
- Can the term with the -2 exponent be rewritten with a positive exponent?
- Do all of the exponents in a polynomial have to be non-negative?
- Can a polynomial have a term with a negative exponent? Why not?
- Can a polynomial contain an absolute value sign? Why not?
- How is a radical term rewritten as an exponential term?

Polynomials are named according to the number of terms they have. Polynomials with only one term are **monomials**. Polynomials with exactly two terms are **binomials**. Polynomials with exactly three terms are **trinomials**.

The **degree of a term** in a polynomial is the exponent of the term. The greatest exponent in a polynomial determines the **degree of the polynomial**. In the polynomial $4x + 3$, the greatest exponent is 1, so the degree of the polynomial is 1.

3. Khalil says that $3x^{-2} + 4x - 1$ is a polynomial with a degree of 1 because 1 is the greatest exponent and it is a trinomial because it has 3 terms. Jazmin disagrees and says that this is not a polynomial at all because the power on the first term is not a whole number. Who is correct? Explain your reasoning.

   Jazmin is correct. The exponent of -2 means the first term is not in the form ax^k, where k is a non-negative integer. Because the expression is not a polynomial, its degree cannot be identified nor can it be named as a trinomial.

4. Describe why each expression is not a polynomial.

   a. $\frac{4}{x}$

   This expression can be rewritten as $4x^{-1}$. This expression is not a polynomial because the exponent is not a non-negative integer.

   b. $\sqrt{x}$

   This expression can be rewritten as $x^{1/2}$. This expression is not a polynomial because the exponent is not a non-negative integer.

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Guiding Questions for Share Phase, Question 5

- Which of the polynomials is of the greatest degree?
- Do all polynomials contain variables?
- Do all polynomials contain constants?
- How many of the polynomials are considered monomials?
- How many of the polynomials are considered binomials?
- How many of the polynomials are considered trinomials?
- Are there any polynomials in the sorting activity that are not considered monomials, binomials, or trinomials?

5. Cut out each polynomial.

Identify the degree of each polynomial and then analyze and sort according to the number of terms of the polynomial. Finally, glue the sorted polynomials in the appropriate column of the table.

$4x - 6x^2$ $-6x^2 + 4x$ binomial Degree: 2	$125p$ monomial Degree: 1	$\frac{4}{5}r^3 + \frac{2}{5}r - 1$ trinomial Degree: 3
$-\frac{2}{3}$ monomial Degree: 1	$y^2 - 4y + 10$ trinomial Degree: 2	$5 - 7h$ $-7h + 5$ binomial Degree: 1
$-3 + 7n + n^2$ $n^2 + 7n - 3$ trinomial Degree: 2	-6 monomial Degree: 1	$-13s + 6$ binomial Degree: 1
$12.5t^3$ monomial Degree: 3	$78j^3 - 3j$ binomial Degree: 3	$25 - 18m^2$ $-18m^2 + 25$ binomial Degree: 2

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Monomial	Binomial	Trinomial
$125p$ monomial Degree: 1	$-6x^2 + 4x$ binomial Degree: 2	$\frac{4}{5}r^3 + \frac{2}{5}r - 1$ trinomial Degree: 3
$-\frac{2}{3}$ monomial Degree: 1	$-7h + 5$ binomial Degree: 1	$y^2 - 4y + 10$ trinomial Degree: 2
$12.5t^3$ monomial Degree: 3	$-13s + 6$ binomial Degree: 1	$n^2 + 7n - 3$ trinomial Degree: 2
-6 monomial Degree: 1	$78j^3 - 3j$ binomial Degree: 3	
	$-18m^2 + 25$ binomial Degree: 2	

A polynomial is written in standard form when the terms are in descending order, starting with the term with the greatest degree and ending with the term with the least degree.

6. Analyze the polynomials you just sorted. Identify the polynomials not written in standard form and write them in standard form.
See table.

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Problem 2

A scenario is used that is represented by a quadratic function and a linear function. Models of these functions are given. Students will model the sum of the two functions using function notation, a graph, a table, and then using algebra. They then compare the graphical representation to the algebraic representation. Next, students will model the difference between the two functions in a similar way.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- Is the function describing the population of the Orange-bellied Parrot a linear function or a quadratic function?
- Is the function describing the population of the Yellow-headed Parrot a linear function or a quadratic function?
- How would you describe all possible intersections of the graph of a quadratic function and the graph of a linear function?

PROBLEM 2 Something to Squawk About

You are playing a new virtual reality game called "Species." You are an environmental scientist who is responsible for tracking two species of endangered parrots, the Orange-bellied Parrot and the Yellow-headed Parrot. Suppose the Orange-bellied Parrots' population can be modeled by the function

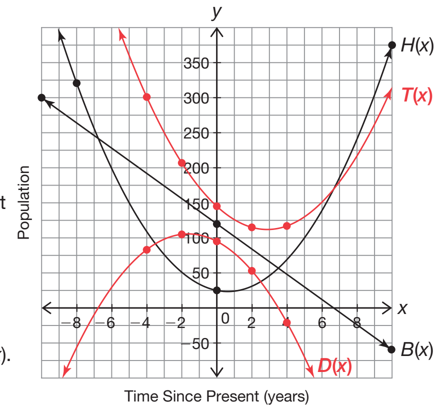
$$B(x) = -18x + 120,$$

where x represents the number of years since the current year. Then suppose that the population of the Yellow-headed Parrot can be modeled by the function

$$H(x) = 4x^2 - 5x + 25.$$

The graphs of the two polynomial functions are shown.

Your new task in the game is to determine the total number of these endangered parrots over a six-year span. You can calculate the total population of parrots using the two graphed functions.



1. Use the graphs of $B(x)$ and $H(x)$ to determine the function, $T(x)$, to represent the total population of parrots.

   a. Write $T(x)$ in terms of $B(x)$ and $H(x)$.

   $T(x) = B(x) + H(x)$

   b. Predict the shape of the graph of $T(x)$.

   Answers will vary.

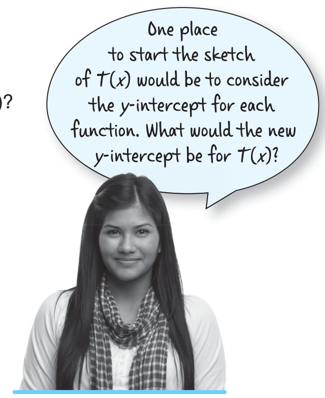
   I think the shape of the graph will be a parabola.

   c. Sketch a graph of $T(x)$ on the coordinate plane shown. First choose any 5 x-values and add their corresponding y-values to create a new point on the graph of $T(x)$. Then connect the points with a smooth curve.

   See graph.

   d. Did your sketch match your prediction in part (b)? Describe the function family $T(x)$ belongs to.

   The graph of $T(x)$ is a quadratic function.



- What would you predict the intersection of these two functions to look like? Why?
- Can time be negative?
- How can you have −3 years? What does −3 years mean?
- Did you use the table function on the graphing calculator to determine any values when completing the table of values in Question 2?

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Grouping

Have students complete Questions 5 and 6 with a partner. Then share the responses as a class.

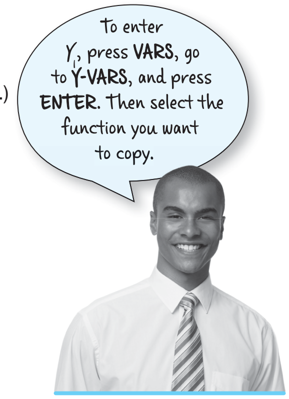
2. Use a graphing calculator to check your sketch by graphing $B(x)$ and $H(x)$ and then the sum of the two functions.

a. In Y_1, enter the function for $B(x)$.

b. In Y_2, enter the function for $H(x)$.

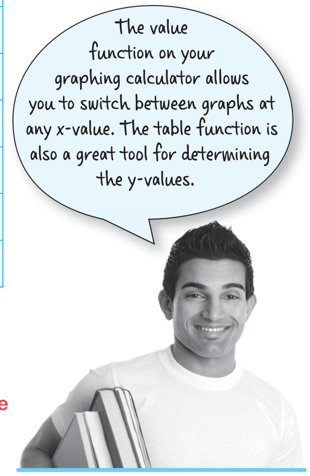
c. In Y_3, enter the sum of the functions $B(x)$ and $H(x)$.

(Since $Y_1 = B(x)$ and $Y_2 = H(x)$, you can enter $Y_1 + Y_2$ in Y_3 to represent the sum of $B(x)$ and $H(x)$.)



3. Complete the table.

Time Since Present (years)	Number of Orange-bellied Parrots	Number of Yellow-headed Parrots	Total Number of Parrots
−3	174	76	250
−2	156	51	207
−1	138	34	172
0	120	25	145
1	102	24	126
2	84	31	115
3	66	46	112



4. How did you calculate the total number of parrots?

I added the number of Orange-bellied Parrots to the number of Yellow-headed Parrots for each given time.

5. Write a function, $T(x)$, in terms of x that can be used to calculate the total number of parrots at any time.

$T(x) = B(x) + H(x)$

$T(x) = (-18x + 120) + (4x^2 - 5x + 25)$

Guiding Questions for Share Phase, Question 6

- When you combined the linear equation with the quadratic equation, which terms could be combined because they were like terms?
- Was there any term that could not be combined with another term? Which one?

Grouping

Have students complete Questions 7 and 8 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 7 and 8

- What is the difference between three years ago and three years from now?
- How is three years ago represented in the table of values?
- How is "currently" represented in the table of values?
- How is three years from now represented in the table of values?
- When will the Orange-bellied Parrot be extinct?
- How can the graph be used to determine when the Orange-bellied Parrot will be extinct?
- How can the equation be used to determine when the Orange-bellied Parrot will be extinct?

6. Analyze the function you wrote in Question 5.

a. Identify the like terms of the polynomial functions.

The like terms are $-18x$ and $-5x$, and 120 and 25. $4x^2$ does not have a like term.

b. Use the Associative Property to group the like terms together.

$$T(x) = (-18x + 120) + (4x^2 - 5x + 25)$$
$$= 4x^2 + (-18x + (-5x)) + (120 + 25)$$

c. Combine like terms to simplify the expression.

$$T(x) = 4x^2 + (-18x + (-5x)) + (120 + 25)$$
$$= 4x^2 - 23x + 145$$

d. Graph the function you wrote in part (c) to verify that it matches the graph of the sum of the two functions in Y_3.

7. Use your new polynomial function to confirm that your solution in the table for each time is correct.

a. 3 years ago

$$T(x) = 4x^2 - 23x + 145$$
$$= 4(-3)^2 - 23(-3) + 145$$
$$= 36 + 69 + 145$$
$$T(x) = 250$$

b. currently

$$T(x) = 4x^2 - 23x + 145$$
$$= 4(0)2 - 23(0) + 145$$
$$T(x) = 145$$

c. 3 years from now

$$T(x) = 4x^2 - 23x + 145$$
$$= 4(3)^2 - 23(3) + 145$$
$$= 36 - 69 + 145$$
$$T(x) = 112$$

8. Zoe says that using $T(x)$ will not work for any time after 6 years from now because by that point the Orange-bellied Parrot will be extinct. Is Zoe's statement correct? Why or why not?

Algebraically the method still works. However, based on the problem situation the answer no longer makes sense after 6 years because you cannot have a negative number of parrots. So, the total population will be equal to the number of Yellow-headed Parrots.

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Grouping

Have students complete Questions 9 and 10 with a partner. Then share the responses as a class.

Throughout the game "Species," you must always keep track of the difference between the populations of each type of species. If the difference gets to be too great, you lose the game.

9. Use the graphs of $B(x)$ and $H(x)$ to determine the function, $D(x)$, to represent the difference between the populations of each type of species.

a. Write $D(x)$ in terms of $B(x)$ and $H(x)$.

$D(x) = B(x) - H(x)$

b. Predict the shape of the graph of $D(x)$.

Answers will vary.

I think the shape of the graph will be a parabola.

c. Sketch a graph of $D(x)$ on the coordinate plane in Question 1. Choose any 5 x-values and subtract their corresponding y-values to create a new point to form the graph of $D(x)$. Then connect the points with a smooth curve.

See graph.

d. Did your sketch match your prediction in part (b)? Describe the function family $D(x)$ belongs to.

The graph of $D(x)$ is a quadratic function.

10. Use a graphing calculator to check your sketch by graphing $B(x)$ and $H(x)$ and then the difference of the two functions.

a. For Y_1, enter the function for $B(x)$.

b. For Y_2, enter the function for $H(x)$.

c. For Y_3, enter the difference of the functions $B(x)$ and $H(x)$.

(Since $Y_1 = B(x)$ and $Y_2 = H(x)$, you can enter $Y_1 - Y_2$ in Y_3 to represent $B(x) - H(x)$.)

Grouping

Have students complete Questions 11 through 13 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 11 and 12

- When you combined the linear equation with the quadratic equation, what operation did you use?
- If you want to write an equation that describes the difference between the two functions, what operation will need to be used?
- If you are subtracting an entire equation from another equation, what happens to the signs of each term in the equation that is being subtracted?
- When you subtracted the linear equation from the quadratic equation, which terms could be combined because they were like terms?
- How many years from now will the Orange-bellied Parrot population become extinct?
- Does it make sense to have negative number of Orange-bellied Parrots?
- How many Yellow-headed Parrots will there be in 7 years?

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11. Consider the original functions, $B(x) = -18x + 120$ and $H(x) = 4x^2 - 5x + 25$.

a. Write a new function, $D(x)$, in terms of x to represent the difference between the population of Orange-bellied Parrots and the population of Yellow-headed Parrots.

$D(x) = (-18x + 120) - (4x^2 - 5x + 25)$

b. Because you are subtracting one function from another function, you must subtract each term of the second function from the first function. To do this, use the Distributive Property to distribute the negative sign to each term in the second function.

$D(x) = (-18x + 120) - (4x^2 - 5x + 25)$

$D(x) = (-18x + 120) + (-4x + 5x - 25)$

c. Now, combine the like terms to simplify the expression.

$D(x) = -4x^2 + (-18x + 5x) + (120 - 25)$

$= -4x^2 - 13x + 95$

d. Graph the function you wrote in part (c) to verify that it matches the graph of the difference of the two functions in Y_3.

12. For each time given, use the table in Question 2 to determine the difference between the population of Orange-bellied Parrots and the population of Yellow-headed Parrots. Then, use the function you wrote in Question 8, part (c) to verify your solutions.

a. 3 years ago

$174 - 76 = 98$

$D(x) = -4(-3)^2 - 13(-3) + 95$

$= -36 + 39 + 95$

$D(x) = 98$

b. currently

$120 - 25 = 95$

$D(x) = -4(0)^2 - 13(0) + 95$

$D(x) = 95$

c. 3 years from now

$66 - 46 = 20$

$D(x) = -4(3)^2 - 13(3) + 95$

$= -36 - 39 + 95$

$D(x) = 20$

13. Eric uses his function function, $D(x) = -4x^2 - 13x + 95$, to determine that the difference between the number of Orange-bellied Parrots and the number of Yellow-headed Parrots 7 years from now will be 192. Is Eric correct or incorrect? If he is correct, explain to him what his answer means in terms of the problem situation. If he is incorrect, explain where he made his error and how to correct it.

$$D(x) = -4(7)^2 - 13(7) + 95$$
$$= -196 - 91 + 95$$
$$D(x) = -192$$

Algebraically, Eric is correct. However, contextually this answer does not make sense. From the graph, I can see that 7 years from now there will be no more Orange-bellied Parrots in the world, not a negative number of them. Because of this the difference is just the number of Yellow-headed Parrots there are, 186. There will be 186 fewer Orange-bellied than Yellow-headed Parrots 7 years from now.

Problem 3

Students will examine polynomials that are incorrectly written and rewrite them correctly.

Grouping

Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 1

- Are $3x^2$ and $5x^2$ like terms?
- When combining $3x^2$ and $5x^2$, do you add the coefficients?
- When combining $3x^2$ and $5x^2$, do you add the exponents?
- When distributing a negative sign across two terms, what happens to the sign of each term?

PROBLEM 3 Just the Math

1. Analyze the work. Determine the error and make the necessary corrections.

a.

Marco

$3x^2 + 5x^2 = 8x^4$

Marco incorrectly added the exponents. When combining like terms, add the coefficients, not the exponents.

$3x^2 + 5x^2 = 8x^2$

b.

Kamiah

$2x - (4x + 5)$
$2x - 4x + 5$
$-2x + 5$

Kamiah did not correctly distribute the negative sign.

$2x - (4x + 5)$
$2x - 4x - 5$
$-2x - 5$

Guiding Questions for Share Phase, Questions 1 and 2

- Why is $-5 + 7$ not equal to $-(5 + 7)$?
- If an addition sign is located between the two expressions, does this change the sign of each term in the second expression?
- If a subtraction sign is located between the two expressions, does this change the sign of each term in the second expression?
- How did you decide which sets of terms were like terms?
- Were there any terms that could not be combined with other terms? Which ones?

c.

Alexis

$(4x^2 - 2x - 5) + (3x^2 + 7)$
$(4x^2 + 3x^2) - (2x) - (5 + 7)$
$x^2 - 2x - 12$

Alexis incorrectly combined the last set of like terms. The negative sign in front of the 5 in the first polynomial should not be distributed to both constant terms.

$(4x^2 - 2x - 5) + (3x^2 + 7)$
$(4x^2 + 3x^2) + (-2x) + (-5 + 7)$
$x^2 - 2x + 2$

2. Determine the sum or difference of each. Show your work.

a. $(x^2 - 2x - 3) + (2x + 1)$

$x^2 + (-2x + 2x) + (-3 + 1)$
$= x^2 - 2$

b. $(2x^2 + 3x - 4) - (2x^2 + 5x - 6)$

$(2x^2 - 2x^2) + (3x - 5x) + (-4 + 6)$
$= -2x + 2$

c. $(4x^3 + 5x - 2) + (7x^2 - 8x + 9)$

$4x^3 + 7x^2 + (5x - 8x) + (-2 + 9)$
$= 4x^3 + 7x^2 - 3x + 7$

d. $(9x^4 - 5) - (8x^4 - 2x^3 + x)$

$(9x^4 - 8x^4) + 2x^3 - x - 5$
$= x^4 + 2x^3 - x - 5$

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Check for Students' Understanding

Match each expression with the equivalent polynomial.

Expressions		Polynomials
1. $(x^2 - 3) + (x^2 + 2)$	(D)	A. -1
2. $(x^2 - 3) - (x^2 + 2)$	(F)	B. $-2x^2 - 1$
3. $(x^2 - 3) - (x^2 - 2)$	(A)	C. $-2x^2 - 5$
4. $(x^2 - 3) + (x^2 - 2)$	(E)	D. $2x^2 - 1$
5. $-(x^2 - 3) - (x^2 - 2)$	(B)	E. $2x^2 - 5$
6. $-(x^2 - 3) - (x^2 - 2)$	(C)	F. -5

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They're Multiplying—Like Polynomials!

13.2

Multiplying Polynomials

LEARNING GOALS

In this lesson, you will:

- Model the multiplication of a binomial by a binomial using algebra tiles.
- Use multiplication tables to multiply binomials.
- Use the Distributive Property to multiply polynomials.

ESSENTIAL IDEAS

- Algebra tiles are used to model the product of two binomials.
- Multiplication tables are used to model the product of two binomials.
- The graphing calculator is used to verify the product of two binomials.
- The FOIL method, (First, Outer, Inner, Last) is a method used to multiply two binomials.
- The distributive property is used to multiply polynomials.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-APR Arithmetic with Polynomials and Rational Expressions

Perform arithmetic operations on polynomials

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Overview

Algebra tiles are used to model the multiplication of two binomials. Students are given an example and will use the example to perform multiplication on other binomials. They then use a graphing calculator to verify that an equation written in the form of two binomials is equal to the equation written as the product of the two binomials, or a trinomial. In the next activity, students use a second method for determining the product of two binomials. A multiplication table in which the terms of one binomial are written horizontally and the other vertically is given. Students then use the distributive property and the multiplication table to compute the product of the binomials. The FOIL method is introduced and an example provided. Students use the FOIL method to solve problems. In the last activity, students will use the distributive property to determine the products of a binomial and a trinomial.

Warm Up

Determine the product.

1. $4x \cdot x$

   $4x^2$

2. $-3x \cdot 10x$

   $-30x^2$

3. $-x(5 + 4x)$

   $-4x^2 - 5x$

4. $(2x - 8) \cdot (12x)$

   $24x^2 - 96x$

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They're Multiplying—Like Polynomials!

Multiplying Polynomials

13.2

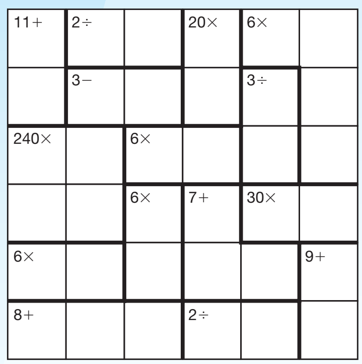
LEARNING GOALS

In this lesson, you will:

- Model the multiplication of a binomial by a binomial using algebra tiles.
- Use multiplication tables to multiply binomials.
- Use the Distributive Property to multiply polynomials.

KenKen has been a popular mathematics puzzle game around the world since at least 2004. The goal is to fill in the board with the digits 1 to whatever, depending on the size of the board. If it's a 5×5 board, only the digits 1 through 5 can be used. If it's a 6×6 board, only the digits 1 through 6 can be used. Each row and column must contain the numbers 1 through whatever without repeating numbers.

Many KenKen puzzles have regions called "cages" outlined by dark bold lines. In each cage, you must determine a certain number of digits that satisfy the rule. For example, in the cage "2÷" shown, you have to determine two digits that divide to result in 2.



Can you solve this KenKen?

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Problem 1

Algebra tiles are used to model the product of two binomials. An example of setting up the tiles is provided and students perform the actual multiplication. A 'Who's Correct?' scenario focuses on the Commutative Property of Multiplication using the two binomials. Students will use a graphing calculator to graph quadratic functions written as the multiplication of two binomials and the product of the two binomials (a trinomial) and conclude that it is the same graph, therefore they are equal equations.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- When you multiplied x by $(x + 2)$, how many terms were multiplied by x?
- When you multiplied 1 by $(x + 2)$, how many terms were multiplied by 1?
- Are there any like terms?
- Which terms can be combined?

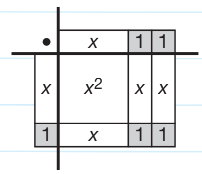
PROBLEM 1 **Modeling Binomials**

So far, you have learned how to add and subtract polynomials. But what about multiplying polynomials?

Let's consider the binomials $x + 1$ and $x + 2$. You can use algebra tiles to model the two binomials and determine their product.

Represent each binomial with algebra tiles.

Create an area model using each binomial.



1. What is the product of $(x + 1)(x + 2)$?

   $(x + 1)(x + 2) = x^2 + x + x + x + 1 + 1$

   $= x^2 + 3x + 2$

2. How would the model change if the binomial $x + 2$ was changed to $x + 4$. What is the new product of $x + 1$ and $x + 4$?

   I would add two more 1s to $(x + 2)$ and two x's and two 1s to the area model.

   $(x + 1)(x + 4) = x^2 + 5x + 4$.

- Is the product of 2 times 3 equal to the product of 3 times 2? This is an example of which property?

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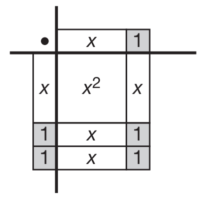
3. Jamaal represented the product of $(x + 1)$ and $(x + 2)$ as shown.



Natalie looked at the area model and told Jamaal that he incorrectly represented the area model because it does not look like the model in the example. Jamaal replied that it doesn't matter how the binomials are arranged in the model.

Determine who's correct and use mathematical principles or properties to support your answer.

Jamaal is correct because of the Commutative Property of Multiplication, which states that the factors in a multiplication expression can be multiplied in any order. Therefore, Jamaal is correct in stating that it does not matter how the binomials are arranged in the model.

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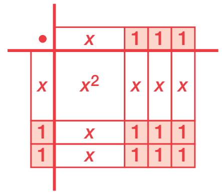
Grouping

Have students complete Questions 4 through 5 with a partner. Then share the responses as a class.

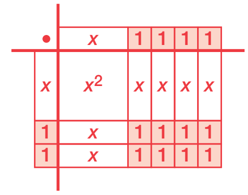
4. Use algebra tiles to determine the product of the binomials in each.

a. $x + 2$ and $x + 3$



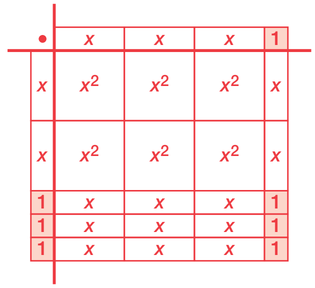
$x^2 + 5x + 6$

b. $x + 2$ and $x + 4$



$x^2 + 6x + 8$

c. $2x + 3$ and $3x + 1$



$6x^2 + 11x + 3$

You can use a graphing calculator to check if the product of two binomials is correct.

Step 1: Press **Y=**. Enter the two binomials multiplied next to Y_1. Then enter their product next to Y_2.

To distinguish between the graphs of Y_1 and Y_2, move your cursor to the left of Y_2 until the \ flashes. Press **ENTER** one time to select the bold \.

Step 2: Press **WINDOW** to set the bounds and intervals for the graph.

Step 3: Press **GRAPH**.

Guiding Questions for Share Phase, Question 5

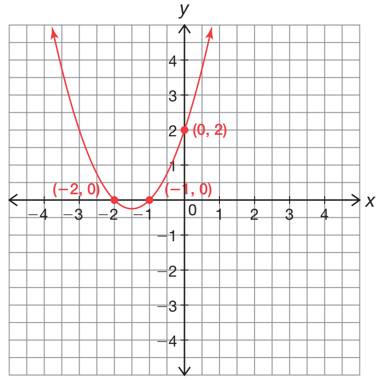
- Looking at the graph for $y = (x + 1)(x + 2)$, what are the x-intercepts?
- Looking at the graph for $y = (x + 1)(x + 2)$, what are the zeros of the function? Is this something you could conclude from the equation for the function?
- Looking at the graph for $y = x^2 + 3x + 2$, what is the y-intercept? Is this something you could conclude from the equation for the function?

Grouping

Have students complete Questions 6 and 7 with a partner. Then share the responses as a class.

5. Use a graphing calculator to verify the product from the worked example: $(x + 1)(x + 2) = x^2 + 3x + 2$.

a. Sketch both graphs on the coordinate plane.



b. How do the graphs verify that $(x + 1)(x + 2)$ and $x^2 + 3x + 2$ are equivalent?

The graphs are the same.

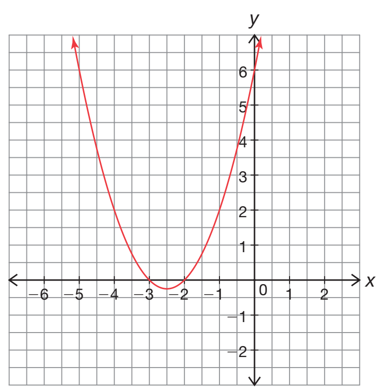
c. Plot and label the x-intercepts and the y-intercept on your graph. How do the forms of each expression help you identify these points?

The factored form of a function is $f(x) = a(x - r_1)(x - r_2)$. So, I know from the expression $(x + 1)(x + 2)$ that r_1 is -1 and r_2 is -2.

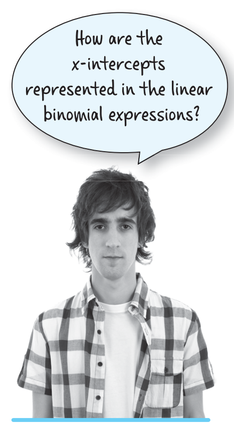
The standard form of a function is $f(x) = ax^2 + bx + c$. So, I know from $x^2 + 3x + 2$ that the y-intercept is (0, 2).

6. Verify that the products you determined in Question 5, part (a) through part (c) are correct using your graphing calculator. Write each pair of factors and the product. Then sketch each graph on the coordinate plane.

a.



$(x + 2)(x + 3) = x^2 + 5x + 6$



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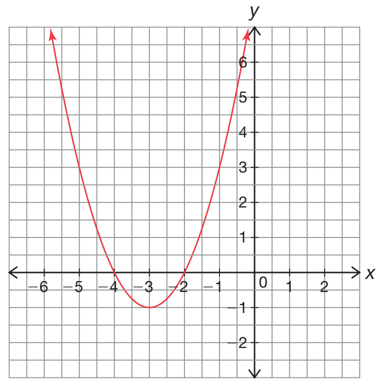
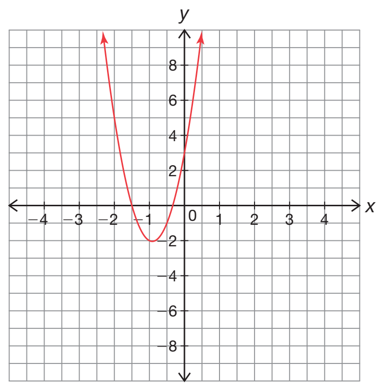
b.



$(x + 2)(x + 4) = x^2 + 6x + 8$

c.



$(2x + 3)(3x + 1) = 6x^2 + 11x + 3$

Recall that r_1 and r_2 are the x-intercepts of a function written in factored form, $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$.

7. How can you determine whether the products in Question 5, part (a) through part (c) are correct using factored form? Explain your reasoning.

I can look at the multiplication of the two binomials as a function in factored form to determine the x-intercepts of those graphs. If the graphs of the products are the same, they should have the same x-intercepts. I can substitute the x-intercepts into the product expressions. If the x-intercepts are the same, those expressions should simplify to 0, because the x-intercepts of a function are the points at which the function is equal to 0.

Finally, I have to consider the a values. These must be the same for the products to be correct.

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Problem 2

Rather than using actual algebra tiles to model the multiplication of two large binomials, students are given an example of how they can draw the model using a multiplication table and will use the distributive property to compute the product. Students then practice using this method to compute the product of different binomials.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.
- Ask a student to read the information and complete Question 3 as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Could Todd have placed the $4x$ and 7 in the top row and the $5x$ and -3 down the first column and gotten the same product? Why?
- Which terms are the like terms in the table?
- What is the sum of $35x$ and $-12x$?
- Do both methods use the distributive property?

PROBLEM 2 I'm Running Out of Algebra Tiles!

While using algebra tiles is one way to determine the product of polynomials, they can also become difficult to use when the terms of the polynomials become more complex.

Todd was calculating the product of the binomials $4x + 7$ and $5x - 3$. He thought he didn't have enough algebra tiles to determine the product. Instead, he performed the calculation using the model shown.

Todd

$\cdot$	$5x$	-3
$4x$	$20x^2$	$-12x$
7	$35x$	-21

$20x^2 + 23x - 21$

1. Describe how Todd calculated the product of $4x + 7$ and $5x - 3$.

   Todd put a term of each binomial in either a column head or a row head. Then, for each cell, Todd wrote the product of the corresponding row head and column head. Finally, Todd wrote the sum of all of the products, combining like terms.

2. How is Todd's method similar to and different from using the algebra tiles method?

   The two methods are similar in that both use the Distributive Property to determine the product of the two binomials.

   The methods are different in that using algebra tiles would be difficult to represent the product, while Todd's method organizes the terms in the table and then he can combine like terms more efficiently.

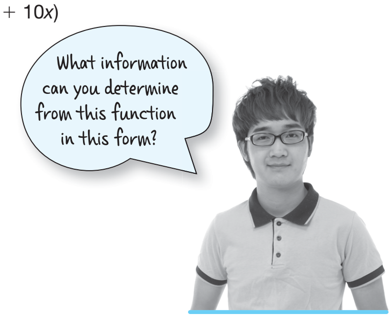
Todd used a multiplication table to calculate the product of the two binomials. By using a multiplication table, you can organize the terms of the binomials as factors of multiplication expressions. You can then use the Distributive Property of Multiplication to multiply each term of the first polynomial with each term of the second polynomial.

Recall the problem *Making the Most of the Ghosts* in Chapter 11. In it, you wrote the function $r(x) = (50 - x)(100 + 10x)$, where the first binomial represented the possible price reduction of a ghost tour, and the second binomial represented the number of tours booked if the price decrease was x dollars per tour.

3. Determine the product of $(50 - x)$ and $(100 + 10x)$ using a multiplication table.

$\cdot$	100	$10x$
50	5000	$500x$
$-x$	$-100x$	$-10x^2$

$-10x^2 + 500x - 100x + 5000$
$-10x^2 + 400x + 5000$



Guiding Questions for Discuss Phase, Question 3

- What is the product of $-x$ and $10x$?
- Which terms are like terms?
- What does the 5000 in the equation tell you about the parabola?
- What does the coefficient of the x^2 term tell you about the parabola?

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Grouping

Have students complete Questions 4 and 5 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 4 and 5

- Did you place the terms from the expression $3u + 17$ in the multiplication table horizontally or vertically?
- Did you place the terms from the expression $4u - 6$ in the multiplication table horizontally or vertically?
- What are the like terms?
- Looking at the equation containing the trinomial, what is the y-intercept?
- Looking at the equation containing the trinomial, does the parabola open up or open down?

4. Determine the product of the binomials using multiplication tables. Write the product in standard form.

a. $3u + 17$ and $4u - 6$

$12u^2 - 18u + 68u - 102$

$12u^2 + 50u - 102$

$\cdot$	$4u$	-6
$3u$	$12u^2$	$-18u$
17	$68u$	-102

b. $8x + 16$ and $6x + 3$

$48x^2 + 24x + 96x + 48$

$48x^2 + 120x + 48$

$\cdot$	$6x$	3
$8x$	$48x^2$	$24x$
16	$96x$	48

c. $7y - 14$ and $8y - 4$

$56y^2 - 28y - 112y + 56$

$56y^2 - 140y + 56$

$\cdot$	$8y$	-4
$7y$	$56y^2$	$-28y$
-14	$-112y$	56

d. $9y - 4$ and $y + 5$

$9y^2 + 45y - 4y - 20$

$9y^2 + 41y - 20$

$\cdot$	y	5
$9y$	$9y^2$	$45y$
-4	$-4y$	-20

5. Describe the degree of the product when you multiply two binomials with a degree of 1.

It appears that the multiplication of two binomials with a degree of 1 will result in a polynomial with a degree of 2, or a quadratic product.

Problem 3

Students begin by using the distributive property to multiply a monomial and a binomial. They then use the distributive property to multiply two binomials. The FOIL method is introduced and an example is provided. Students will use the FOIL method to determine the product of two binomials.

Grouping

- Ask a student to read the information and complete Question 1 as a class.
- Have students complete Question 2 with a partner. Then share the responses as a class.

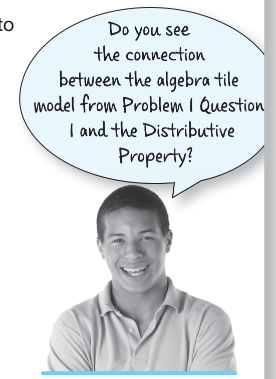
Guiding Questions for Share Phase, Question 2

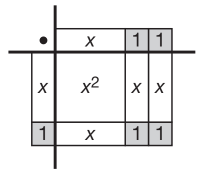
- Which term is distributed?
- Are there any like terms?
- What is the product of the "first" terms?
- What is the product of the "outer" terms?
- What is the product of the "inner" terms?
- What is the product of the "last" terms?
- Which terms can be combined?
- Is the product of two binomials always a trinomial? Why or why not?
- How is it possible that the product of two binomials could result in a binomial?

PROBLEM 3 You *Have Been* Distributing the Whole Time!

So far, you have used both algebra tiles and multiplication tables to determine the product of two polynomials.

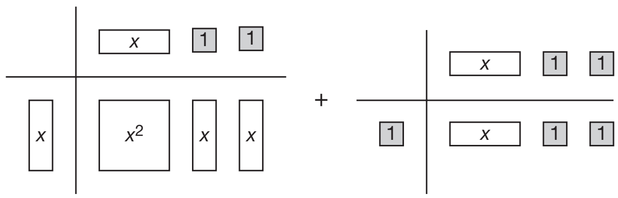
Let's look at the original area model and think about multiplying a different way. The factors and equivalent product for this model are:





$(x + 1)(x + 2) = x^2 + 3x + 2$

The model can also be shown as the sum of each row.



$x(x + 1) = x^2 + x$ $1(x + 2) = x + 2$

1. Write the factors and the equivalent product for each row represented in the model.

2. Use your answers to Question 1 to rewrite $(x + 1)(x + 2)$.
   a. Complete the first equivalent statement using the factors from each row.
   b. Next, write an equivalent statement using the products of each row.

   $(x + 1)(x + 2) = x(x + 1) + 1(x + 2)$ Distributive Property

   $= x^2 + x + x + x + 2$ Distributive Property

   $= x^2 + 3x + 2$ Combine like terms

   c. Write the justification for each step.

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The Distributive Property can be used to multiply polynomials. The number of times that you need to use the Distributive Property depends on the number of terms in the polynomials.

3. How many times was the Distributive Property used in Question 2?

The Distributive Property was used twice.

4. Use the Distributive Property to multiply a monomial by a binomial.

$$(3x)(4x + 1) = (\underline{\ 3x\ })(\underline{\ 4x\ }) + (\underline{\ 3x\ })(\underline{\ 1\ })$$

$$= \underline{\ 12x^2 + 3x\ }$$

To multiply the polynomials $x + 5$ and $x - 2$, you can use the Distributive Property.

First, use the Distributive Property to multiply each term of $x + 5$ by the entire binomial $x - 2$.

$$(x + 5)(x - 2) = (x)(x - 2) + (5)(x - 2)$$

Now, distribute x to each term of $x - 2$ and distribute 5 to each term of $x - 2$.

$$x^2 - 2x + 5x - 2$$

Finally, collect the like terms and write the solution in standard form.

$$x^2 + 3x - 2$$

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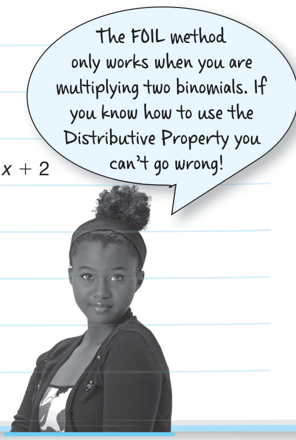
Another method that can be used to multiply polynomials is called the FOIL method. The word FOIL indicates the order in which you multiply the terms. You multiply the First terms, then the Outer Terms, then the Inner terms, and then the Last terms. FOIL stands for First, Outer, Inner, Last.

You can use the FOIL method to determine the product of $(x + 1)$ and $(x + 2)$.

$$\left.\begin{array}{c} \text{First} \\ (x + 1)(x + 2) = x^2 \\ \text{Outer} \\ (x + 1)(x + 2) = 2x \\ \text{Inner} \\ (x + 1)(x + 2) = x \\ \text{Last} \\ (x + 1)(x + 2) = 2 \end{array}\right\} x^2 + 2x + x + 2$$



Collect the like terms and write the solution in standard form.

$$x^2 + 3x + 2$$

5. Determine each product.

a. $2x(x + 3)$

$2x(x) + 2x(3)$

$2x^2 + 6x$

b. $5x(7x - 1)$

$(5x)(7x) - (5x)(1)$

$35x^2 - 5x$

c. $(x + 1)(x + 3)$

$(x + 1)(x) + (x + 1)(3)$

$x(x) + 1(x) + x(3) + 1(3)$

$x^2 + x + 3x + 3$

$x^2 + 4x + 3$

d. $(x - 4)(2x + 3)$

$(x - 4)(2x) + (x - 4)(3)$

$(x)(2x) - 4(2x) + (x)(3) - 4(3)$

$x^2 - 8x + 3x - 12$

$x^2 - 5x - 12$

Problem 4

An example of using the distributive property to determine the product of a binomial and a trinomial is provided. Students will complete the example and model the same problem using the multiplication table method. They then use this method to solve additional problems.

Grouping

Ask a student to read the information and complete Questions 1 through 3 as a class.

Guiding Questions for Discuss Phase, Questions 1 and 2

- Why is it difficult to use algebra tiles when multiplying three binomials?
- How many additional columns/rows would you need to set up a multiplication table to multiply three binomials?
- When using the distributive property to multiply a binomial by a trinomial, how do you know what sign to place between each term?

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PROBLEM 4 Moving Beyond Binomials

1. Can you use algebra tiles to multiply three binomials? Explain why or why not.

   Yes. I can use algebra tiles to multiply three binomials, but it would be very difficult because I would need to perform extra steps. I could represent two of the three binomials with algebra tiles and determine the product of the modeled binomials. Then, I could use that product and the third polynomial and create another area model. From here, I could determine the product of all three binomials.

2. Can you use multiplication tables to multiply three binomials? Explain why or why not.

   Yes. I can use multiplication tables to multiply three binomials. The rows and columns can be extended on the multiplication table to include the third binomial. Then I could use the Distributive Property to multiply the first binomial by the other two binomials and the second binomial by the third binomial. Finally, I would need to combine like terms.

You can use the Distributive Property to determine the product of a binomial and a trinomial.

Consider the polynomials $x + 1$ and $x^2 - 3x + 2$. You need to use the Distributive Property twice to determine the product.

First, use the Distributive Property to multiply each term of $x + 1$ by the polynomial $x^2 - 3x + 2$.

$$(x + 1)(x^2 - 3x + 2) = (x)(x^2 - 3x + 2) + (1)(x^2 - 3x + 2)$$

Now, distribute x to each term of $x^2 - 3x + 2$, and distribute 1 to each term of $x^2 - 3x + 2$.

$$(x + 1)(x^2 - 3x + 2) = (x)(x^2) + (x)(-3x) + (x)(2) + (1)(x^2) + (1)(-3x) + (1)(2)$$

Finally, multiply and collect the like terms and write the solution in standard form.

$$x^3 - 3x^2 + 2x + x^2 - 3x + 2$$

$$x^3 - 2x^2 - x + 2$$

Guiding Questions for Share Phase, Question 3

- Which terms in the multiplication table can be combined?
- Is the product determined by using the multiplication table the same as the product determined in the example?
- Is the product a trinomial? Why not? How is it classified?

Grouping

Have students complete Question 4 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 4

- Are the answers to these problems in the form of a trinomial or a polynomial?
- How could you use the graphing calculator to verify your polynomial answer?

3. You can also use a multiplication table to multiply a binomial by a trinomial. Complete the table to determine the product.

$\cdot$	x^2	$-3x$	2
x	x^3	$-3x^2$	$2x$
1	x^2	$-3x$	2

$x^3 - 3x^2 + 2x + x^2 - 3x + 2$
$x^3 - 2x^2 - x + 2$

4. Determine each product.

a. $(x - 5)(x^2 + 3x + 1)$

$(x - 5)(x^2) + (x - 5)(3x) + (x - 5)(1)$

$= x(x^2) - 5(x^2) + x(3x) - 5(3x) + x(1) - 5(1)$

$= x^3 - 5x^2 + 3x^2 - 15x + x - 5$

$= x^3 - 2x^2 - 14x - 5$

b. $(x + 5)(2x^2 - 3x - 4)$

$(x)(2x^2) - x(3x) - x(4) + 5(2x^2) - 5(3x) - 5(4)$

$= 2x^3 - 3x^2 - 4x + 10x^2 - 15x - 20$

$= 2x^3 + 7x^2 - 19x - 20$

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c. $(x - 4)(x^2 - 8x + 16)$

$(x)(x^2) - x(8x) + x(16) - 4(x^2) + 4(8x) - 4(16)$

$= x^3 - 8x^2 + 16x - 4x^2 + 32x - 64$

$= x^3 - 12x^2 + 48x - 64$

Be prepared to share your solutions and methods.

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Check for Students' Understanding

Match each expression with the equivalent polynomial.

Expressions		Polynomials
1. $(x - 3) \cdot (x + 2)$	(C)	**A.** $x^2 + 5x + 6$
2. $(x + 3) \cdot (x - 2)$	(D)	**B.** $x^2 - 5x + 6$
3. $(x + 3) \cdot (x + 2)$	(A)	**C.** $x^2 - x - 6$
4. $(x - 3) \cdot (x - 2)$	(B)	**D.** $x^2 + x - 6$

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13.3

What *Factored* Into It?

Factoring Polynomials

LEARNING GOALS

In this lesson, you will:

- Factor polynomials by determining the greatest common factor.
- Factor polynomials by using multiplication tables.

ESSENTIAL IDEAS

- To factor an expression means to rewrite the expression as a product of factors.
- Using the Symmetric Property of Equality we can write $a(b + c) = ab + ac$ as $ab + ac = a(b + c)$.
- The product of two linear expressions produces a quadratic expression.
- An area model can be used to factor a quadratic trinomial when the expression contains small numbers.
- A multiplication table can be used to factor a quadratic trinomial when the expression contains large numbers.
- Factoring the lead term and the constant term in conjunction with trial and error can be used to factor some quadratic trinomials.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Write expressions in equivalent forms to solve problems.

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

a. Factor a quadratic expression to reveal the zeros of the function it defines.

A-APR Arithmetic with Polynomials and Rational Expressions

Perform arithmetic operations on polynomials

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Overview

This lesson focuses on writing a quadratic expression as a product of factors. Students will use the Distributive Property in reverse to determine the greatest common factor in simple algebraic expressions. Three different methods of factoring quadratic trinomials are then introduced: an area model, using a multiplication table, and using the factors of the lead term and the constant term. Students will practice each method.

Warm Up

Determine the product for each.

1. $(x + 5)(x + 7)$
   $x^2 + 12x + 35$

2. $(x - 5)(x - 7)$
   $x^2 - 12x + 35$

3. $(x - 5)(x + 7)$
   $x^2 + 2x + 35$

4. $(x + 5)(x - 7)$
   $x^2 - 2x + 35$

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What *Factored* Into It?

Factoring Polynomials

13.3

LEARNING GOALS

In this lesson, you will:

- Factor polynomials by determining the greatest common factor.
- Factor polynomials by using multiplication tables.

The history of the word "multiply" is an interesting one. You probably know that "multi-" means "many." But did you know that the rest of the word, based on the Latin "plicare," means "to fold"?

This is why you might read in older texts that someone increased their money "twofold" or "tenfold."

Multiplication is closely related to folding. Can you see how?

Problem 1

Students will determine the greatest common factor in several algebraic expressions and rewrite the expression as a product of factors.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- What is the greatest common factor of 4 and 12?
- Does each term in the expression contain one or more x's?
- Can x be factored out of each term in the expression?
- Must x be in each term of the expression to factor it out?
- If both terms in an expression are negative, is -1 considered a common factor?
- If two out of the three terms in the expression contain a factor of 5, can 5 be factored out of the expression? Why not?
- How can the distributive property be used to determine if the expression is completely factored?

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PROBLEM 1 What About the Other Way Around?

In the previous lesson, you multiplied polynomials. More specifically, you multiplied two linear expressions to determine a quadratic expression. In this lesson, you will go in reverse and think about how to take a polynomial represented as the sum of terms and write an equivalent expression in factored form, if it is possible. To factor an expression means to rewrite the expression as a product of factors.

One way to factor an expression is to factor out the greatest common factor first.

Consider the polynomial $3x + 15$.

The greatest common factor is 3.

$$3x + 15 = 3x + 3(5)$$
$$= 3(x + 5)$$

Therefore, $3x + 15 = 3(x + 5)$.

In order to factor out a greatest common factor, use the Distributive Property in reverse. Recall that, using the Distributive Property, you can rewrite the expression $a(b + c)$ as $ab + ac$.

1. Factor out the greatest common factor for each polynomial, if possible.

   a. $4x + 12$

   greatest common factor = 4
   $4x + 12 = 4(x) + 4(3)$
   $= 4(x + 3)$

   b. $x^3 - 5x$

   greatest common factor = x
   $x^3 - 5x = x(x^2) - x(5)$
   $= x(x^2 - 5)$

   c. $3x^2 - 9x - 3$

   greatest common factor = 3
   $3x^2 - 9x - 3 = 3(x^2) - 3(3x) - 3(1)$
   $= 3(x^2 - 3x - 1)$

   d. $-x - 7$

   greatest common factor = -1
   $-x - 7 = -1(x) - 1(7)$
   $= -1(x + 7)$

   e. $2x - 11$

   no greatest common factor

   f. $5x^2 - 10x + 5$

   greatest common factor = 5
   $5x^2 - 10x + 5 = 5(x^2) - 5(2x) + 5(1)$
   $= 5(x^2 - 2x + 1)$

Problem 2

Three different methods for factoring a quadratic trinomial are given. Each method is introduced and includes a worked example. The first method is using an area model, the second method uses a multiplication table, and the third method involves a degree of trial and error where students use the factors of the leading term and the constant term. Students will practice factoring quadratic trinomials using each method.

2. How can you check to see if you factored out the greatest common factor of each correctly?

Answers will vary.

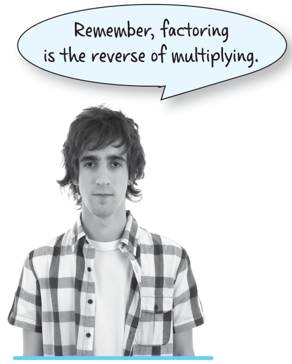
I can use the Distributive Property and multiply my factored form to see if I get the given polynomial.

I can use my graphing calculator to graph the given polynomial and the factored form of the polynomial at the same time. If the graphs are the same, then the two forms of the polynomial are equivalent.

PROBLEM 2 Factoring Trinomials

In the previous chapter, you used a graphing calculator to rewrite a quadratic expression in factored form, $ax^2 + bx + c = a(x - r_1)(x - r_2)$.

Now, let's consider a strategy for factoring quadratic expressions without the use of technology. Understanding that the product of two linear expressions produces a quadratic expression is necessary for understanding how to factor a quadratic expression.



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Grouping

- Ask a student to read the information and complete Question 1 as a class.
- Have students complete Questions 2 and 3 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Are the numbers in both binomials small or large?
- If the numbers in the binomials were very large, how would this affect the use of the area model for determining the terms of the quadratic trinomial?
- Why would using the area model to determine the terms of the quadratic trinomial be tedious if the numbers in the binomials were very large?
- How did you represent the leading term in the area model?
- How did you represent the constant term in the area model?
- How did you represent the middle term in the area model?
- Do all of the pieces form a rectangle?
- How many different ways can the parts be arranged to form a rectangle?

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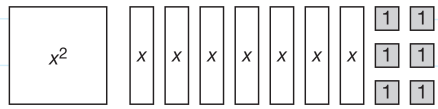
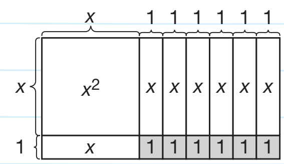
An area model can be used to factor $x^2 + 7x + 6$.

First, represent each part of the trinomial as a piece of the area model. In this problem, $x^2 + 7x + 6$ consists of one x^2 block, seven x blocks, and six constant blocks.



Second, use all of the pieces to form a rectangle. In this problem, the parts can only be arranged in one way.

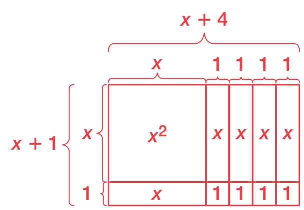


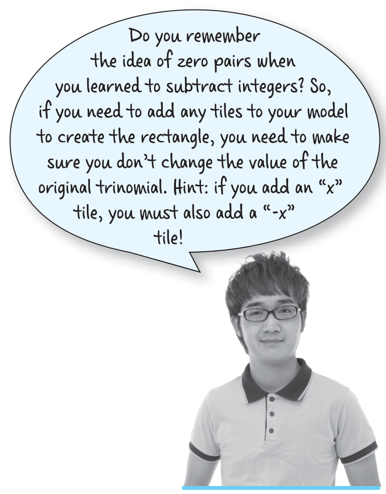
1. Write the trinomial as the product of the two factors.

   $x^2 + 7x + 6 = (x + 1)(x + 6)$

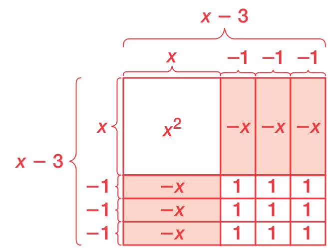
2. Factor each trinomial using an area model.

   a. $x^2 + 5x + 4 = (x + 4)(x + 1)$

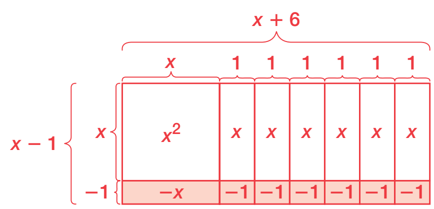




b. $x^2 - 6x + 9 = (x - 3)(x - 3)$



c. $x^2 + 5x - 6 = (x + 6)(x - 1)$



3. Look back at the quadratic trinomials you just factored using area models. What do you notice about the constant term of the trinomial, c, and the constants of the binomial factors r_1 and r_2?

The product of r_1 and r_2 is equal to c.

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Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Question 4 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 4

- Did you begin by writing the lead term and the constant in the multiplication table?
- Is there more than one way to factor the leading term?
- Is there more than one way to factor the constant?
- Do the factors of the constant you used determine the correct middle term?

Let's consider using a multiplication table model to factor trinomials.

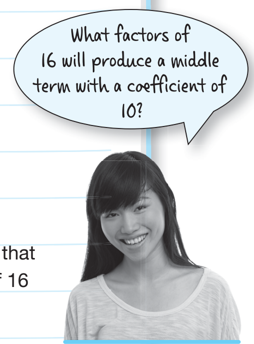
Factor the trinomial $x^2 + 10x + 16$.

Start by writing the leading term (x^2) and the constant term (16) in the table.

$\cdot$		
	x^2	
		16

Factor the leading term.

$\cdot$	x	
x	x^2	
		16



Experiment with factors of the constant term to determine the pair that produces the correct coefficient for the middle term. The factors of 16 are (1)(16), (2)(8), and (4)(4).

$\cdot$	x	8
x	x^2	$8x$
2	$2x$	16

The sum of $2x$ and $8x$ is $10x$.

So, $x^2 + 10x + 16 = (x + 2)(x + 8)$.

4. Explain why the other factor pairs for $c = 16$ do not work.
   a. (1)(16)
      The factor pair (1)(16) produces a middle term of $17x$.
   b. (4)(4)
      The factor pair (4)(4) produces a middle term of $8x$.
5. Use multiplication tables to factor each trinomial.
   a. $x^2 + 9x + 20$
      $(x + 4)(x + 5)$

	x	5
x	x^2	$5x$
4	$4x$	20

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Grouping

Ask a student to read the information. Discuss as a class.

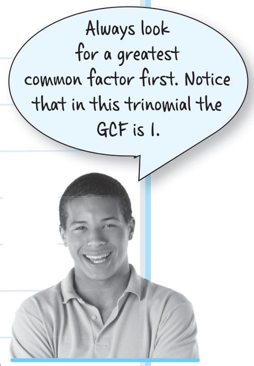
b. $x^2 + 11x + 18$

$(x + 9)(x + 2)$

	x	2
x	x^2	$2x$
9	$9x$	18

Another method for factoring a trinomial is trial and error, using the factors of the leading term, ax^2, and the constant term, c.

Factor the trinomial $x^2 - 10x - 24$.



To factor the quadratic expression, you must consider the factors of ax^2 and c.

Factors of the leading term, x^2, are x and x.

Factors of the constant term, -24, are: -1, 24 and 1, -24
-2, 12 and 2, -12
-3, 8 and 3, -8
-4, 6 and 4, -6.

Use these factor pairs to write binomial products. Determine which binomials produce the correct middle term.

$(x - 1)(x + 24) = x^2 + 23x - 24$ not correct
$(x + 1)(x - 24) = x^2 - 23x - 24$ not correct

$(x - 2)(x + 12) = x^2 + 10x - 24$ not correct
$(x + 2)(x - 12) = x^2 - 10x - 24$ correct

$(x - 3)(x + 8) = x^2 + 5x - 24$ not correct
$(x + 3)(x - 8) = x^2 - 5x - 24$ not correct

$(x - 4)(x + 6) = x^2 + 2x - 24$ not correct
$(x + 4)(x - 6) = x^2 - 2x - 24$ not correct

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Grouping

Have students complete Questions 6 and 7 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 6 and 7

- What are the factors of the leading term?
- Is there more than one way to factor the leading term?
- What are the factors of the constant term?
- Is there more than one way to factor the constant?
- Do the factors of the constant you used determine the correct middle term?
- How do the *a* and *c* values compare in both trinomials?
- How do the *b* values compare in both trinomials?
- What do you notice about the signs of the two binomials?

6. Factor each trinomial using the method from the worked example. List the factor pairs.

a. $x^2 + 5x - 24$

$-1, 24$ $\quad 1, -24$
$-2, 12$ $\quad 2, -12$
$-3, 8$ $\quad 3, -8$
$-4, 6$ $\quad 4, -6$

$x^2 + 5x - 24 = (x + 8)(x - 3)$

b. $x^2 - 3x - 28$

$-1, 28$ $\quad 1, -28$
$-2, 14$ $\quad 2, -14$
$-4, 7$ $\quad 4, -7$

$x^2 - 3x - 28 = (x - 7)(x + 4)$

7. Consider the two examples shown.

$2x^2 - 3x - 5$

$\cdot$	x	1
$2x$	$2x^2$	$2x$
-5	$-5x$	-5

$2x^2 - 3x - 5 = (2x - 5)(x + 1)$

$2x^2 + 3x - 5$

$\cdot$	x	-1
$2x$	x^2	$-2x$
5	$5x$	-5

$2x^2 + 3x - 5 = (2x + 5)(x - 1)$

a. Compare the two given trinomials. What is the same and what is different about the *a*, *b*, and *c* values?

The *a* and *c* values are both the same. The *b* values are opposites of each other. The *a*, *b*, and *c* values are all prime numbers.

b. Compare the factored form of each trinomial. What do you notice?

The operations in the binomials are reversed.

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Grouping

- Have students complete Questions 8 and 9 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 8 and 9

- If the middle term of the quadratic trinomial is $5x$, and the constant is 4, what are the signs in each binomial factor?
- If the middle term of the quadratic trinomial is $-5x$, and the constant is 4, what are the signs in each binomial factor?
- If the middle term of the quadratic trinomial is $3x$, and the constant is -4, what are the signs in each binomial factor?
- If the middle term of the quadratic trinomial is $-3x$, and the constant is -4, what are the signs in each binomial factor?
- If the constant term is positive, can the signs in the binomial factors both be negative?
- If the constant term is positive, can the signs in the binomial factors both be positive?
- If the constant term is positive, how do you know when the signs in the binomial factors are both positive or both negative?

8. Choose from the list to write the correct factored form for each trinomial.

a. $x^2 + 5x + 4 =$ $(x + 1)(x + 4)$

$x^2 - 5x + 4 =$ $(x - 1)(x - 4)$

$x^2 + 3x - 4 =$ $(x - 1)(x + 4)$

$x^2 - 3x - 4 =$ $(x + 1)(x - 4)$

- $(x + 1)(x - 4)$
- $(x + 1)(x + 4)$
- $(x - 1)(x + 4)$
- $(x - 1)(x - 4)$

b. $2x^2 + 7x + 3 =$ $(2x + 1)(x + 3)$

$2x^2 - 7x + 3 =$ $(2x - 1)(x - 3)$

$2x^2 - 5x - 3 =$ $(2x + 1)(x - 3)$

$2x^2 + 5x - 3 =$ $(2x - 1)(x + 3)$

- $(2x - 1)(x - 3)$
- $(2x - 1)(x + 3)$
- $(2x + 1)(x + 3)$
- $(2x + 1)(x - 3)$

c. $x^2 + 7x + 10 =$ $(x + 2)(x + 5)$

$x^2 - 7x + 10 =$ $(x - 2)(x - 5)$

$x^2 - 3x - 10 =$ $(x + 2)(x - 5)$

$x^2 + 3x - 10 =$ $(x - 2)(x + 5)$

- $(x - 2)(x + 5)$
- $(x + 2)(x + 5)$
- $(x - 2)(x - 5)$
- $(x + 2)(x - 5)$

9. Analyze the signs of each quadratic expression written in standard form and the operations in the binomial factors in Question 8. Then complete each sentence.

the same	both positive	one positive and one negative
different	both negative	

a. If the constant term is positive, then the operations in the binomial factors are the same.

b. If the constant term is positive and the middle term is positive, then the operations in the binomial factors are both positive.

c. If the constant term is positive and the middle term is negative, then the operations in the binomial factors are both negative.

d. If the constant term is negative, then the operations in the binomial factors are different.

e. If the constant term is negative and the middle term is positive, then the operations in the binomial factors are one positive and one negative.

f. If the constant term is negative and the middle term is negative, then the operations in the binomial factors are one positive and one negative.

- If the constant term is positive and the middle term is negative, what does that tell you about the signs in the binomial factors?
- If both the constant term and the middle term are negative, do both signs in the binomial factors have to be negative?

Guiding Questions for Share Phase, Questions 10 and 11

- What is the sign of the constant and the sign of the middle term?
- What does this tell you about the sign in the factors of the binomials?
- What are the possible factors of the constant?
- Which factors of the constant will give you a product that is the same as the middle term?
- What is the sign of the leading term?
- Will Grace's solution provide the correct sign for the leading term?
- Will Elaine's solution provide the correct sign for the leading term?
- Will Maggie's solution provide the correct sign for the leading term?
- What is the sign of the constant term?
- Will Grace's solution provide the correct sign for the constant term?
- Will Elaine's solution provide the correct sign for the constant term?
- Will Maggie's solution provide the correct sign for the constant term?
- What is the sign of the middle term?
- Will Grace's solution provide the correct sign for the middle term?
- Will Elaine's solution provide the correct sign for the middle term?
- Will Maggie's solution provide the correct sign for the middle term?

10. Factor each quadratic expression.

a. $x^2 + 8x + 15 = (x + 3)(x + 5)$

$x^2 - 8x + 15 = (x - 3)(x - 5)$

$x^2 + 2x - 15 = (x + 5)(x - 3)$

$x^2 - 2x - 15 = (x - 5)(x + 3)$

b. $x^2 + 10x + 24 = (x + 6)(x + 4)$

$x^2 - 10x + 24 = (x - 6)(x - 4)$

$x^2 + 2x - 24 = (x + 6)(x - 4)$

$x^2 - 2x - 24 = (x - 6)(x + 4)$

11. Elaine, Maggie, and Grace were asked to factor the trinomial $15 + 2x - x^2$.

Grace

$15 + 2x - x^2$

$(5 - x)(3 + x)$

Elaine

$15 + 2x - x^2$

$(5 - x)(3 + x)$

$(x - 5)(x + 3)$

Maggie

$15 + 2x - x^2$

$-x^2 + 2x + 15$

$-(x^2 - 2x - 15)$

$-(x - 5)(x + 3)$

Who's correct? Determine which student is correct and explain how that student determined the factored form. If a student is not correct, state why and make the correction.

Grace is correct. She kept the form of the trinomial and factored it.

Elaine is not correct. She made an error in the final step. She should have factored out the −1 from the first binomial factor.

Maggie is correct. Maggie rewrote the trinomial in standard form. Then she factored out the negative to make the leading coefficient positive and finally factored the quadratic expression.

Guiding Questions for Share Phase, Question 12

- When Marilyn and Jake factored out the greatest common denominator, were they correct?
- If Marilyn and Jake factored out the greatest common denominator, could either resulting binomial or linear expression contain common factor?

12. Marilyn and Jake were working together to factor the trinomial $4x^2 + 22x + 24$. They first noticed that there was a greatest common factor and rewrote the trinomial as

$$2(2x^2 + 11x + 12).$$

Next, they considered the factor pairs for $2x^2$ and the factor pairs for 12.

$2x^2$: $(2x)(x)$

12: (1) (12)
(2) (6)
(3) (4)

Marilyn listed all out all the possible combinations.

$2(2x + 1)(x + 12)$
$2(2x + 12)(x + 1)$

$2(2x + 2)(x + 6)$
$2(2x + 6)(x + 2)$

$2(2x + 3)(x + 4)$
$2(2x + 4)(x + 3)$

Jake immediately eliminated four out of the six possible combinations because the terms of one of the linear expressions contained common factors.

$2(2x + 1)(x + 12)$
$\cancel{2(2x + 12)(x + 1)}$

$\cancel{2(2x + 2)(x + 6)}$
$\cancel{2(2x + 6)(x + 2)}$

$2(2x + 3)(x + 4)$
$\cancel{2(2x + 4)(x + 3)}$

Explain Jake's reasoning. Then circle the correct factored form of $4x^2 + 22x + 24$.

They had already factored out a GCF as their initial step, so none of the linear expressions should contain an additional GCF.

The correct factored form is $2(2x + 3)(x + 4)$.

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Talk the Talk

Students will factor polynomials and complete statements describing the factors of a quadratic expression.

Grouping

Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Which if any polynomials are not written in standard form?
- Does a polynomial need to be rewritten in standard form to factor the polynomial?
- Which if any polynomials contain a greatest common factor?
- If the middle term is negative and the constant is negative, what are the signs in the binomial factors?
- What are the possible factors of the constant term?
- Which factors of the constant will give you a product that is the same as the middle term?
- If the middle term is negative and the constant is positive, what are the signs in the binomial factors?

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Talk the Talk

1. Factor each polynomial completely. First, determine if there is a greatest common factor, and then write the polynomial in factored form.

a. $x^2 - 9x - 10$

$(x + 1)(x - 10)$

b. $4x^2 - 20x + 16$

$4(x^2 - 5x + 4)$

$4(x - 4)(x - 1)$

c. $-20 + 9b - b^2$

$-b^2 + 9b - 20$

$-(b^2 - 9b + 20)$

$-(b - 5)(b - 4)$

d. $3y^2 - 8y - 3$

$(3x + 1)(x - 3)$

e. $7x^2 - 7x - 56$

$7(x^2 - x - 6)$

$7\ (x - 3)(x + 2)$

f. $3y^3 - 27y^2 - 30y$

$3y(y^2 - 9y - 10)$

$3y(y - 10)(y + 1)$

2. Use the word bank to complete each sentence. Then explain your reasoning.

always	sometimes	never

a. The product of two linear expressions will ___never___ be a trinomial with a degree of 3.

If the greatest degree of a binomial is 1, then there is no chance of the product of two binomials resulting in a trinomial with a degree of 3.

b. The two binomial factors of a quadratic expression will ___always___ have a degree of one.

Because a quadratic is a polynomial with a degree of 2, the factors will be binomials with a degree of 1.

c. The factoring of a quadratic expression will ___sometimes___ result in two binomials.

The quadratic expression may have contained a greatest common factor which would be a third factor.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Match each polynomial expression with the equivalent polynomial in factored form.

Expressions		Polynomials
1. $x^2 - 14x + 48$	(C)	**A.** $(x + 6)(x - 8)$
2. $x^2 - 2x - 48$	(D)	**B.** $(x + 6)(x + 8)$
3. $x^2 - 2x - 48$	(A)	**C.** $(x - 6)(x - 8)$
4. $x^2 + 14x + 48$	(B)	**D.** $(x - 6)(x + 8)$

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13.4

Zeroing In

Solving Quadratics by Factoring

LEARNING GOALS

In this lesson, you will:

- Solve quadratic equations and functions using factoring.
- Connect the zeros of a function to the *x*-intercepts of a graph.
- Determine the roots of quadratic equations.

KEY TERMS

- Zero Product Property
- Converse of Multiplication Property of Zero
- roots

ESSENTIAL IDEAS

- The Zero Product Property states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero. If $ab = 0$, then $a = 0$ or $b = 0$.
- The roots of a quadratic function are calculated by writing the quadratic equation in factored form, writing two linear equations by setting each factor equal to zero, and solving each linear equation.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Write expressions in equivalent forms to solve problems.

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

a. Factor a quadratic expression to reveal the zeros of the function it defines.

A-REI Reasoning with Equations and Inequalities

Solve equations and inequalities in one variable

4. Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Overview

The three methods of factoring polynomials from the previous lesson are reviewed and the Zero Product Property is introduced. Students will calculate the roots for quadratic equations by factoring the quadratic expression, setting each factor equal to zero, and solving the resulting equations for the roots. Next, students use a graphing calculator to graph a quadratic function, and identify the vertex, the x- and y-intercepts, and the line of symmetry. They also algebraically calculate the zeros of different quadratic functions where possible.

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Warm Up

Factor each quadratic expression.

1. $-x^2 - 49x$

   $-x(x + 49)$

2. $3x^2 + 6x$

   $3x(x + 2)$

3. $3x^2 - 11x - 4$

   $(3x + 1)(x - 4)$

4. $8x^2 - 16x + 6$

   $(2x - 3)(2x - 1)$

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Zeroing In

Solving Quadratics by Factoring

13.4

LEARNING GOALS

In this lesson, you will:

- Solve quadratic equations and functions using factoring.
- Connect the zeros of a function to the *x*-intercepts of a graph.
- Determine the roots of quadratic equations.

KEY TERMS

- Zero Product Property
- Converse of Multiplication Property of Zero
- roots

The word *zero* has had a long and interesting history. The word comes from the Hindu word *sunya*, which meant "void" or "emptiness." In Arabic, this word became *sifr*, which is also where the word *cipher* comes from. In Latin, it was changed to *cephirum*, and finally, in Italian it became *zevero* or *zefiro*, which was shortened to *zero*.

The ancient Greeks, who were responsible for creating much of modern formal mathematics, did not even believe zero was a number!

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Problem 1

The three strategies for factoring a quadratic expression are reviewed. The Zero Product Property is then introduced, and students will use this property to determine solutions to the given quadratic equation. Students then graph the quadratic function on the coordinate plane and identify the vertex, x- and y-intercepts, and the line of symmetry. Next, they will compare the zeros of the graph of the function to the solutions determined algebraically. They then calculate the zeros of different quadratic functions where possible.

Grouping

Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 1

- Did you use a graphing calculator to graph the quadratic function?
- Does the quadratic function contain an absolute minimum or an absolute maximum?
- Are the coordinates of the y-intercept obvious from the equation of the quadratic function? Does the of the function graph verify this?

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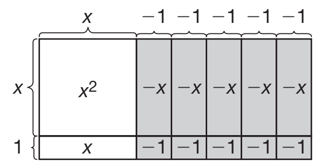
PROBLEM 1 Roots of Quadratic Equations

Recall that a quadratic expression of the form $x^2 + bx + c$ can be factored using an area model, a multiplication table, or the factors of the constant term c. The quadratic expression $x^2 - 4x - 5$ is factored using each method as shown.

- Area model



$x^2 - 4x - 5 = (x - 5)(x + 1)$

- Multiplication table

$\cdot$	x	-5
x	x^2	$-5x$
1	x	-5

$x^2 - 4x - 5 = (x - 5)(x + 1)$

- Factors of the constant term c

Factors of -5: $-5, 1$ $\quad -1, 5$

Sums: $-5 + 1 = -4$ $\quad 5 + (-1) = 4$

$x^2 - 4x - 5 = (x - 5)(x + 1)$

The **Zero Product Property** states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.

If $ab = 0$, then $a = 0$ or $b = 0$.

This is also referred to as the **Converse of the Multiplication Property of Zero**.

1. Use the Zero Product Property to determine the solutions of the quadratic equation $x^2 - 4x - 5 = 0$. Then, check your solutions by substituting back into the original equation.

$(x - 5)(x + 1) = 0$

$x - 5 = 0$ or $x + 1 = 0$

$x = 5$ $\qquad x = -1$

Check: $(5)^2 - 4(5) - 5 = 25 - 20 - 5 = 0$

$(-1)^2 - 4(-1) - 5 = 1 + 4 - 5 = 0$

- Are the coordinates of the x-intercepts obvious from the equation of the quadratic function? Why not?
- Are the coordinates of the x-intercepts obvious from the factored equation of the quadratic function?
- How is solving for the zeros of a quadratic function different than solving for the roots of a quadratic function?

2. Let's examine the quadratic equation $0 = x^2 - 4x - 5$.

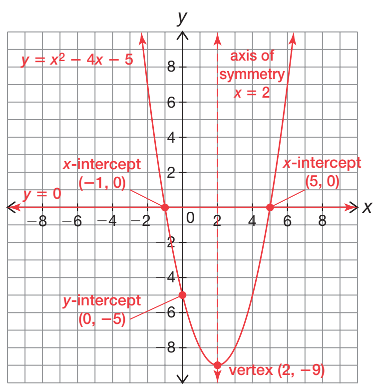
   a. Graph both sides of the quadratic equation on the coordinate plane shown.

   b. Rewrite the equation in factored form.

      $0 = (x - 5)(x + 1)$

   c. Identify the vertex, x- and y-intercepts, and the axis of symmetry.

      - y-intercept: $(0, -5)$
      - x-intercept(s): $(-1, 0)$ and $(5, 0)$
      - axis of symmetry: $x = 2$
      - vertex: $(2, -9)$



   d. Compare the x-intercepts of the equation $y = x^2 - 4x - 5$ to the solutions to Question 1. What do you notice?

      The x-intercepts of the equation $y = x^2 - 4x - 5$ are the same as the solutions in Question 1.

   e. Compare the intersections of the two equations you graphed to the solutions to Question 1. What do you notice?

      The points of intersection of the two equations graphed $y = 0$ and $y = x^2 - 4x - 5$ are the same as the solutions in Question 1.

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Grouping

Have students complete Questions 3 through 10 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 3 through 10

- What are the factors of the leading term?
- Is there more than one way to factor the leading term?
- What are the factors of the constant term?
- Is there more than one way to factor the constant?
- Do the factors of the constant you used determine the correct middle term?
- Did you set each factor of the quadratic trinomial equal to zero?
- Can you determine the sign of each root from the equations used to solve for the root? How?
- What operation is used to solve for each root?

The solutions to a quadratic equation are called *roots*. The **roots** indicate where the graph of a quadratic equation crosses the x-axis. So, roots, zeros, and x-intercepts are all related.

To calculate the roots of a quadratic equation using factoring:

- Perform transformations so that one side of the equation is equal to zero.
- Factor the quadratic expression on the other side of the equation.
- Set each factor equal to zero.
- Solve the resulting equations for the roots. Check each solution in the original equation.

You can calculate the roots for the quadratic equation $x^2 - 4x = -3$.

$$x^2 - 4x = -3$$
$$x^2 - 4x + 3 = -3 + 3$$
$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$
$$(x - 3) = 0 \quad \text{or} \quad (x - 1) = 0$$
$$x - 3 + 3 = 0 + 3 \quad \text{or} \quad x - 1 + 1 = 0 + 1$$
$$x = 3 \quad \text{or} \quad x = 1$$

Check:

$$x^2 - 4x = (3)^2 - 4(3) = 9 - 12 = -3$$
$$x^2 - 4x = (1)^2 - 4(1) = 1 - 4 = -3$$

Determine the roots of each quadratic equation.

3. $x^2 - 8x + 12 = 0$

$$x^2 - 8x + 12 = 0$$
$$(x - 6)(x - 2) = 0$$
$$x - 6 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = 6 \qquad x = 2$$

Check: $(6)^2 - 8(6) + 12 = 36 - 48 + 12 = 0$

$(2)^2 - 8(2) + 12 = 4 - 16 + 12 = 0$

4. $x^2 - 5x - 24 = 0$

$$x^2 - 5x - 24 = 0$$
$$(x - 8)(x + 3) = 0$$
$$x - 8 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = 8 \qquad x = -3$$

Check: $(8)^2 - 5(8) - 24 = 64 - 40 - 24 = 0$

$(-3)^2 - 5(-3) - 24 = 9 + 15 - 24 = 0$

5. $x^2 + 10x - 75 = 0$

$$x^2 + 10x - 75 = 0$$
$$(x - 5)(x + 15) = 0$$
$$x - 5 = 0 \quad \text{or} \quad x + 15 = 0$$
$$x = 5 \qquad\qquad x = -15$$

Check: $(5)^2 + 10(5) - 75 = 25 + 50 - 75 = 0$

$(-15)^2 + 10(-15) - 75 = 225 - 150 - 75 = 0$

6. $x^2 - 11x = 0$

$$x^2 - 11x = 0$$
$$x(x - 11) = 0$$
$$x = 0 \quad \text{or} \quad x - 11 = 0$$
$$x = 0 \qquad\qquad x = 11$$

Check: $(0)^2 - 11(0) = 0$

$(11)^2 - 11(11) = 121 - 121 = 0$

7. $x^2 + 8x = -7$

$$x^2 + 8x = -7$$
$$x^2 + 8x + 7 = -7 + 7$$
$$x^2 + 8x + 7 = 0$$
$$(x + 7)(x + 1) = 0$$
$$x + 7 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x = -7 \quad \text{or} \quad x = -1$$

Check: $x^2 + 8x = (-7)^2 + 8(-7) = 49 - 56 = -7$

$x^2 + 8x = (-1)^2 + 8(-1) = 1 - 8 = -7$

8. $x^2 - 5x = 13x - 81$

$$x^2 - 5x = 13x - 81$$
$$x^2 - 5x - 13x + 81 = 13x - 81 - 13x + 81$$
$$x^2 - 18x + 81 = 0$$
$$(x - 9)(x - 9) = 0$$
$$x - 9 = 0 \quad \text{or} \quad x - 9 = 0$$
$$x = 9 \quad \text{or} \quad x = 9$$

Check: $x^2 - 5x = 13x - 81$

$(9)^2 - 5(9) = 13(9) - 81$

$81 - 45 = 117 - 81$

$36 = 36$

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9. $3x^2 - 22x + 7 = 0$

$$3x^2 - 22x + 7 = 0$$
$$(3x - 1)(x - 7) = 0$$
$$3x - 1 = 0 \quad \text{or} \quad x - 7 = 0$$
$$\frac{3x}{3} = \frac{1}{3} \quad \text{or} \quad x = 7$$
$$x = \frac{1}{3} \quad \text{or} \quad x = 7$$

Check: $3\left(\frac{1}{3}\right)^2 - 22\left(\frac{1}{3}\right) + 7 = \frac{1}{3} - \frac{22}{3} + 7 = -\frac{21}{3} + 7 = 0$

$3(7)^2 - 22(7) + 7 = 147 - 154 + 7 = 0$

10. $8x^2 + 2x - 21 = 0$

$$8x^2 + 2x - 21 = 0$$
$$(4x + 7)(2x - 3) = 0$$
$$4x + 7 = 0 \quad \text{or} \quad 2x - 3 = 0$$
$$4x = -7 \quad \text{or} \quad 2x = 3$$
$$\frac{4x}{4} = -\frac{7}{4} \quad \text{or} \quad \frac{2x}{2} = \frac{3}{2}$$
$$x = -\frac{7}{4} \quad \text{or} \quad x = \frac{3}{2}$$

Check: $8\left(-\frac{7}{4}\right)^2 + 2\left(-\frac{7}{4}\right) - 21 = \frac{49}{2} - \frac{7}{2} - 21 = \frac{42}{2} - 21 = 0$

$8\left(\frac{3}{2}\right)^2 + 2\left(\frac{3}{2}\right) - 21 = 18 + 3 - 21 = 0$

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Problem 2

Students will calculate the zeros and roots for quadratic functions and equations by factoring the quadratic expression, setting each factor equal to zero, and solving the resulting equation.

Grouping

Have students complete Questions 1 through 5 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- What are the factors of the leading term?
- Is there more than one way to factor the leading term?
- What are the factors of the constant term?
- Is there more than one way to factor the constant?
- Do the factors of the constant you used determine the correct middle term?
- Did you set each factor of the quadratic trinomial equal to zero?
- Can you determine the sign of each root from the equations used to solve for the root? How?
- What operation is used to solve for each root?

PROBLEM 2 More Practice

Calculate the zeros of each quadratic function, or the roots of each quadratic equation, if possible.

1. $f(x) = x^2 - 7x - 18$

$x^2 - 7x - 18 = 0$

$(x - 9)(x + 2) = 0$

$x - 9 = 0$ or $x + 2 = 0$

$x = 9$ $\quad x = -2$

Check: $(9)^2 - 7(9) - 18 = 81 - 63 - 18 = 0$

$(-2)^2 - 7(-2) - 18 = 4 + 14 - 18 = 0$

2. $f(x) = x^2 - 11x + 12$

No real zeros

3. $f(x) = x^2 + 10x - 39$

$x^2 + 10x - 39 = 0$

$(x - 3)(x + 13) = 0$

$x - 3 = 0$ or $x + 13 = 0$

$x = 3$ $\quad x = -13$

Check: $(3)^2 + 10(3) - 39 = 9 + 30 - 39 = 0$

$(-13)^2 + 10(-13) - 39 = 169 - 130 - 39 = 0$

4. $2x^2 + 4x = 0$

$2x^2 + 4x = 0$

$2x(x + 2) = 0$

$2x = 0$ or $x + 2 = 0$

$x = 0$ or $x = -2$

Check: $2(0)^2 + 4(0) = 0$

$2(-2)^2 + 4(-2) = 8 - 8 = 0$

5. $\frac{2}{3}x^2 - \frac{5}{6}x = 0$

$\frac{2}{3}x^2 - \frac{5}{6}x = 0$

$6\left(\frac{2}{3}x^2 - \frac{5}{6}x = 0\right)$

$4x^2 - 5x = 0$

$x(4x - 5) = 0$

$x = 0$ or $4x - 5 = 0$

$x = 0$ or $x = \frac{5}{4}$

Check: $\frac{2}{3}(0)^2 - \frac{5}{6}(0) = 0$

$\frac{2}{3}\left(\frac{5}{4}\right)^2 - \frac{5}{6}\left(\frac{5}{4}\right) = \frac{2}{3}\left(\frac{25}{16}\right) - \frac{25}{24} = \frac{25}{24} - \frac{25}{24} = 0$

Be prepared to share your solutions and methods.

13

Check for Students' Understanding

Write a quadratic function for each description.

1. A quadratic function with x-intercepts $(-5, 0)$ and $(4, 0)$.

   $(x + 5)(x - 4) = 0$

   $x^2 + x - 20 = 0$

   $f(x) = x^2 + x - 20$

2. A quadratic function with roots $x = 8$ and $x = -2$.

   $(x + 2)(x - 8) = 0$

   $x^2 - 6x - 16 = 0$

   $f(x) = x^2 - 6x - 16$

13

13.5

What Makes You So Special?

Special Products

LEARNING GOALS

In this lesson, you will:

- Identify and factor the difference of two squares.
- Identify and factor perfect square trinomials.
- Solve quadratic equations and functions using factoring.
- Identify and factor the difference of two cubes.
- Identify and factor the sum of cubes.

KEY TERMS

- difference of two squares
- perfect square trinomial
- difference of two cubes
- sum of two cubes

ESSENTIAL IDEAS

- The difference of two squares in an expression in the form $a^2 - b^2$ that can be factored as $(a + b)(a - b)$.
- A perfect square trinomial is an expression in the form $a^2 + 2ab + b^2$ which can be written as $(a + b)^2$, the square of a binomial.
- The difference of two cubes in an expression in the form $a^3 - b^3$ that can be factored as $(a - b)(a^2 + ab + b^2)$.
- The sum of two cubes in an expression in the form $a^3 + b^3$ that can be factored as $(a + b)(a^2 - ab + b^2)$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions.

2. Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems.

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
   a. Factor a quadratic expression to reveal the zeros of the function it defines.

Overview

The difference of two squares is introduced. The general form of a perfect square trinomial expression is given and written as the square of a binomial. Students will write the unfactored and factored forms of several expressions, and then calculate the roots of quadratic equations, and zeros of quadratic functions. Next, students explore the difference of two cubes and the sum of two cubes. General formulas for each are provided and students factor several polynomials using the one of these forms. In the last activity, students will summarize all of the special products reviewed in this lesson.

13

Warm Up

Determine each product.

1. $(x + 3)(x + 3)$
   $x^2 + 6x + 9$

2. $(x - 3)(x - 3)$
   $x^2 - 6x + 9$

3. $(x + 3)(x - 3)$
   $x^2 - 9$

4. $(x - 3)(x + 3)$
   $x^2 - 9$

13

13.5

What Makes You So Special?

Special Products

LEARNING GOALS

In this lesson, you will:

- Identify and factor the difference of two squares.
- Identify and factor perfect square trinomials.
- Solve quadratic equations and functions using factoring.
- Identify and factor the difference of two cubes.
- Identify and factor the sum of cubes.

KEY TERMS

- difference of two squares
- perfect square trinomial
- difference of two cubes
- sum of two cubes

There are a number of rare elements on Earth. Precious gems are relatively rare, which is why they're so valuable.

Some blood types are rare too. The O blood type is the most common, with about 37% of the population having it. The least common is the blood type AB, with only about 4% of the population having it.

People with rare blood types are strongly encouraged to donate blood if they can, since it is more difficult to find rare blood types in cases of emergency.

What is your blood type?

Problem 1

The difference of two squares is introduced and students will calculate the difference of two squares. The general form of a perfect square trinomial expression is given and written as the square of a binomial. Students then write the unfactored and factored forms of several expressions. A "Who's Correct?" problem focuses incorrect ways of writing the sum of two squares. Students factor polynomials and calculate the roots of quadratic equations and the zeros of quadratic functions.

Grouping

- Have students complete Questions 1 through 3 with a partner. Then share the responses as a class.
- Ask a student to read the information and definitions. Discuss as a class.
- Have students complete Questions 4 and 5 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

13

- What is the product of the first terms of the two binomials?
- What is the product of the outer terms of the two binomials?
- What is the product of the inner terms of the two binomials?

PROBLEM 1 Special Products

1. Multiply the binomials.

a. $(x - 4)(x + 4) =$ $x^2 - 16$

$(x + 4)(x + 4) =$ $x^2 + 8x + 16$

$(x - 4)(x - 4) =$ $x^2 - 8x + 16$

b. $(x - 3)(x + 3) =$ $x^2 - 9$

$(x + 3)(x + 3) =$ $x^2 + 6x + 9$

$(x - 3)(x - 3) =$ $x^2 - 6x + 9$

c. $(3x - 1)(3x + 1) =$ $9x^2 - 1$

$(3x + 1)(3x + 1) =$ $9x^2 + 6x + 1$

$(3x - 1)(3x - 1) =$ $9x^2 - 6x + 1$

d. $(2x - 1)(2x + 1) =$ $4x^2 - 1$

$(2x + 1)(2x + 1) =$ $4x^2 + 4x + 1$

$(2x - 1)(2x - 1) =$ $4x^2 - 4x + 1$

2. What patterns do you notice between the factors and the products?

Answers will vary.

When the operations in the binomials are one positive and one negative, the resulting quadratic has 2 perfect square terms.

3. Multiply these binomials.

$(ax - b)(ax + b) =$ $a^2x^2 - b^2$

$(ax + b)(ax + b) =$ $a^2x^2 + 2abx + b^2$

$(ax - b)(ax - b) =$ $a^2x^2 - 2abx + b^2$

In Questions 1 and 3, you should have observed a few special products. The first type of special product is called the *difference of two squares*. The **difference of two squares** is an expression in the form $a^2 - b^2$ that can be factored as $(a + b)(a - b)$.

The second type of special product is called a *perfect square trinomial.* A **perfect square trinomial** is an expression in the form $a^2 + 2ab + b^2$ or in the form $a^2 - 2ab + b^2$. A perfect square trinomial can be written as the square of a binomial.

4. Identify the expressions in Questions 1 and 3 that are examples of the difference of two squares. Write both the unfactored and factored forms of each expression.

$(x - 4)(x + 4) = x^2 - 16$

$(x - 3)(x + 3) = x^2 - 9$

$(3x - 1)(3x + 1) = 9x^2 - 1$

$(2x - 1)(2x + 1) = 4x^2 - 1$

$(ax - b)(ax + b) = a^2x^2 - b^2$

- What is the sum of the products of the outer and inner terms?
- What is the product of the last terms of the two binomials?
- In which situation does the sum of the products of the outer terms and the inner terms equal zero?
- Is the quadratic expression resulting from this type of situation a trinomial or a binomial?

Guiding Questions for Share Phase, Questions 4 and 5

- Does the difference of two squares result in a binomial or a trinomial expression?
- How would you describe the binomials when the product of the binomials results in the difference of two squares?
- How would you describe the binomials used when the product of the binomials result in a trinomial that is a perfect square trinomial?
- What is the sign used in the binomials when the product of the binomials result in an expression of the form $ax^2 + 2ab + b^2$?
- What is the sign used in the binomials when the product of the binomials result in an expression of the form $ax^2 - 2ab + b^2$?

Grouping

Have students complete Questions 6 and 7 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 6 and 7

- Using Tizeh's factors and the FOIL method, does it result in the quadratic expression $x^2 + 16$? Why not?
- Using Cheyanne's factors and the FOIL method, does it result in the quadratic expression $x^2 + 16$? Why not?

5. Identify the expressions in Questions 1 and 3 that are examples of perfect square trinomials. Write both the unfactored and factored forms of each expression.

a. Of the form $ax^2 + 2ab + b^2$:

$(x + 4)(x + 4) = x^2 + 8x + 16$
$(x + 3)(x + 3) = x^2 + 6x + 9$
$(3x + 1)(3x + 1) = 9x^2 + 6x + 1$
$(2x + 1)(2x + 1) = 4x^2 + 4x + 1$
$(ax + b)(ax + b) = a^2x^2 + 2abx + b^2$

b. Of the form $ax^2 - 2ab + b^2$:

$(x - 4)(x - 4) = x^2 - 8x + 16$
$(x - 3)(x - 3) = x^2 - 6x + 9$
$(3x - 1)(3x - 1) = 9x^2 - 6x + 1$
$(2x - 1)(2x - 1) = 4x^2 - 4x + 1$
$(ax - b)(ax - b) = a^2x^2 - 2abx + b^2$

6. Tizeh says that he can factor the *sum* of two squares in the same way as he factors the difference of two squares. It's just that addition will be used in both binomials. His work is shown.

Tizeh

$x^2 + 16$
$(x + 4)(x + 4)$

Cheyanne disagrees and says that to factor the sum of two squares, you should use subtraction in each binomial. Her work is shown.

Cheyanne

$x^2 + 16$
$(x - 4)(x - 4)$

Who is correct? Explain your reasoning.

Neither Tizeh nor Cheyanne are correct. If you were to multiply the binomials out, you would get $x^2 + 8x + 16$ for Tizeh's answer or $x^2 - 8x + 16$ for Cheyanne's answer. Factoring the sum of two squares doesn't work in the same way because adding the two x terms in the product will not result in zero.

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Guiding Questions for Share Phase, Question 7

- Can the difference of any two roots squared be determined the same way?
- Which equations in factored form in the table can be factored even further? How do you know?

Grouping

Have students complete Questions 8 through 10 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 8 and 9

- Were there any polynomials that you were not able to factor? Which one(s)?
- How many of the polynomials are difference of two squares? Which one(s)?
- How many of the polynomials are a perfect square trinomial? Which one(s)?
- What is the first step when calculating the roots of the quadratic equation?
- After writing the quadratic equation in factored form, how do you determine the roots?

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7. Complete the table to represent each difference of squares.

a	*b*	$a^2 - b^2$	Factored Form
x	2	$x^2 - 4$	$(x + 2)(x - 2)$
$2x$	3	$4x^2 - 9$	$(2x + 3)(2x - 3)$
x^2	4	$x^4 - 16$	$(x^2 + 4)(x^2 - 4)$
x^2	y^2	$x^4 - y^4$	$(x^2 + y^2)(x^2 - y^2)$

a. What does the information in this table show?

This information shows that there is the same pattern in the factored form of the difference of any two squares.

b. Can any of the expressions in factored form be factored further? If so, factor them further.

$(x^2 + 4)(x^2 - 4) = (x^2 + 4)(x + 2)(x - 2)$

$(x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y)$

8. Factor each polynomial, if possible.

a. $x^2 + 10x + 25$

$(x + 5)(x + 5)$

b. $4x^2 + 20x + 25$

$(2x + 5)(2x + 5)$

c. $x^2 - 24x + 144$

$(x - 12)(x - 12)$

d. $36x^2 - 36x + 9$

$9(4x^2 - 4x + 1)$

$9(2x - 1)^2$

e. $x^2 + 25$

Not factorable

f. $16x^4 - 1$

$(4x^2 - 1)(4x^2 + 1)$

$(2x - 1)(2x + 1)(4x^2 + 1)$

9. Calculate the roots of each quadratic equation.

a. $x^2 - 12x + 36 = 0$

$(x - 6)^2 = 0$

$x - 6 = 0$

$x = 6$

The double root is 6.

b. $9x^2 - 25 = 0$

$(3x - 5)(3x + 5) = 0$

$3x - 5 = 0$ or $3x + 5 = 0$

$3x = 5$ or $3x = -5$

$x = \frac{5}{3}$ or $x = -\frac{5}{3}$

The roots are $+\frac{5}{3}$ and $-\frac{5}{3}$.

Guiding Questions for Share Phase, Question 10

- What is the first step when calculating the zeros of the quadratic function?
- After writing the quadratic function in factored form, how do you determine the zeros?
- What is the difference between calculating the roots of a quadratic equation and calculating the zeros of a quadratic function?

Problem 2

Students will explore the difference of two cubes and the sum of two cubes. They then begin by using a multiplication table to determine the product of a quadratic expression and a linear expression. They then conclude that the difference of two cubes is the difference of the roots multiplied by the sum of the squares of the roots and the product of the roots. Students use the general forms of the formulas to solve related problems.

Grouping

Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

10. Calculate the zeros of each function.

a. $f(x) = 25x^2 + 20x + 4$

$(5x + 2)(5x + 2) = 0$

$(5x + 2) = 0$

$5x = -2$

$x = -\frac{2}{5}$

b. $f(x) = 9x^2 + 1$

no real zeros

c. $f(x) = 9 - 24x + 16x^2$

$(3 - 4x)^2 = 0$

$3 - 4x = 0$

$3 = 4x$

$\frac{3}{4} = x$

d. $f(x) = \frac{1}{4}x^2 - 1$

$\left(\frac{1}{2}x - 1\right)\left(\frac{1}{2}x + 1\right) = 0$

$\frac{1}{2}x - 1 = 0$ or $\frac{1}{2}x + 1 = 0$

$\frac{1}{2}x = 1$ or $\frac{1}{2}x = -1$

$x = 2$ or $x = -2$

PROBLEM 2 Are Cubes Perfect Too?

In Problem 1, you dealt with special products that had degrees of 2. There are also special products with degrees of 3.

1. Use a multiplication table to determine $(x - 2)(x^2 + 2x + 4)$.

$\cdot$	x	-2
x^2	x^3	$-2x^2$
$2x$	$2x^2$	$-4x$
4	$4x$	-8

$x^3 - 2x^2 + 2x^2 - 4x + 4x - 8$

$x^3 + (-2x^2 + 2x^2) + (-4x + 4x) - 8$

$x^3 - 8$

2. Use a multiplication table to determine $(x - y)(x^2 + xy + y^2)$.

$\cdot$	x	$-y$
x^2	x^3	$-x^2y$
xy	x^2y	$-xy^2$
y^2	xy^2	$-y^3$

$x^3 - x^2y + x^2y - xy^2 + xy^2 - y^3$

$x^3 + (-x^2y + x^2y) + (xy^2 - xy^2) - y^3$

$x^3 - y^3$

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Guiding Questions for Share Phase, Questions 1 and 2

- What is x times x^2?
- Which terms are like terms?
- What is the sum of $-2x^2 + 2x^2$?
- What is the sum of $-4x + 4x$?
- How can the number 8 be rewritten as a cubic number?
- What is the cube root of 8?

Grouping

- Ask a student to read the definition and complete Questions 4 and 5 as a class.
- Have students complete Questions 6 through 11 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 4 and 5

- How is the difference of two squares similar to the difference of two cubes?
- How is the difference of two squares different from the difference of two cubes?
- Which part of the factored form represents the difference of the cube roots?
- Which part of the factored form represents the product of the cube roots?
- Which part of the factored form represents the squares of the cube root?

Guiding Questions for Share Phase, Question 6

- How can the number 27 be rewritten as a cubic number?
- What is the cube root of 27?
- How will the cube root of 27 help to factor this polynomial?
- Which terms are like terms?

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3. Analyze your solution in Question 2.

a. What happened to the original terms x and y?

My solution is the original terms, x and $-y$, cubed. The other terms cancelled out.

b. Does the solution in Question 2 follow the same pattern as the solution in Question 1? Explain your reasoning.

Yes. The result in Question 1 shows the terms x and -2 cubed.

Each expression and product in Questions 1 and 2 represents the *difference of two cubes*. The **difference of two cubes** is an expression in the form $a^3 - b^3$ that can be factored as $(a - b)(a^2 + ab + b^2)$.

4. Each part of the factored form is related to the cube root of each term in the original expression. Identify each part of the factored form as it relates to the cube root of one of the original terms.

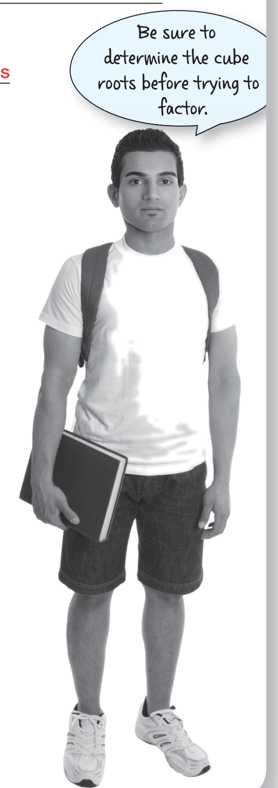
product of the cube roots

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

difference of the cube roots　　squares of the cube roots

5. Using the parts you just identified, explain the formula for the difference of two cubes.

The difference of two cubes is the difference of the cube roots multiplied by the sum of the squares of the cube roots and the product of the cube roots.



6. Factor the difference of the two cubes: $x^3 - 27$.

$$\begin{aligned} x^3 - 27 &= x^3 - 3^3 \\ &= (x - 3)(x^2 + 3x + 3^2) \\ &= (x - 3)(x^2 + 3x + 9) \end{aligned}$$

Guiding Questions for Share Phase, Questions 7 and 8

- What is the sum of $5x^2 - 5x^2$?
- What is the sum of $-25x + 25x$?
- How can 125 be rewritten as a cubic number?
- How is the sum of two cubes similar to the sum of two squares?
- How is the sum of two cubes different than the sum of two squares?
- How do you determine the linear factor when writing the sum of two cubes in factor form?
- How do you determine the middle term of the trinomial factor when writing the sum of two cubes in factor form?

Let's consider the products of two more polynomials in factored form.

7. Use multiplication tables to determine each product.

a. $(x + 5)(x^2 - 5x + 25)$

$\cdot$	x	5
x^2	x^3	$5x^2$
$-5x$	$-5x^2$	$-25x$
25	$25x$	125

$x^3 + 5x^2 - 5x^2 - 25x + 25x + 125$
$x^3 + (5x^2 - 5x^2) + (-25x + 25x) + 125$
$x^3 + 125$
$x^3 + 5^3$

b. $(x + y)(x^2 - xy + y^2)$

$\cdot$	x	y
x^2	x^3	x^2y
$-xy$	$-x^2y$	$-xy^2$
y^2	xy^2	y^3

$x^3 + x^2y - x^2y - xy^2 + xy^2 + y^3$
$x^3 + (x^2y - x^2y) - (xy^2 + xy^2) + y^3$
$x^3 + y^3$

Each product represents the *sum of two cubes*. Previously you determined that you cannot factor the sum of two squares. Based on your products in Question 7, you can factor the sum of two cubes. The **sum of two cubes** is an expression in the form $a^3 + b^3$ that can be factored as $(a + b)(a^2 - ab + b^2)$.

8. Factor the sum of the two cubes: $u^3 + 8$.

$u^3 + 8 = u^3 + 2^3$
$= (u + 2)(u^2 - 2u + 2^2)$
$= (u + 2)(u^2 - 2u + 4)$

Guiding Questions for Share Phase, Questions 9 through 11

- One term in Emilio's trinomial factor is incorrect, which is it?
- Looking at the third line of Emilio's work, is $4x^2$ times $4x$ equal to $64x^3$?
- What is the greatest common factor of $250x^4$ and $128x$?
- There is a sign incorrect in Sophie's factored form, which sign?

9. Analyze the expression Emilio factored.

Emilio

$64x^3 + 125$
$(4x)^3 + (5)^3$
$(4x + 5)(4x^2 - (4x)(5) + 5^2)$
$(4x + 5)(4x^2 - 20x + 25)$

Explain to Emilio what he did wrong and correctly write the expression in factored form.

Emilio's mistake is on the square of the first cube root. The first cube root, $4x$, should be squared to equal $16x^2$ but he did not square the coefficient.

The correct factored expression is $(4x + 5)(16x^2 - 20x + 25)$.

10. Sophie factored the expression shown.

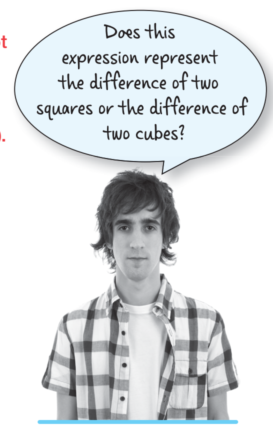
Sophie

$250x^4 + 128x$
$2(125x^3 + 64)$
$2(5x + 4)((5x)^2 - (5x)(4) + 4^2)$
$2(5x + 4)(25x^2 - 20x + 16)$

Explain Sophie's mistake and correctly write the expression in factored form.

Sophie's error occurred when she determined the greatest common factor. The greatest common factor here is $2x$, not just 2, which means the expression is equivalent to $2x(125x^3 + 64x)$.

The correct factored expression is $2x(5x + 4)(25x^2 - 20x + 16)$.



11. Completely factor the expression $x^6 - y^6$.

$x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$

$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$

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Talk the Talk

Students will complete a table that list the key characteristics of the special products reviewed in this lesson; Perfect Square Trinomial, Difference of Squares, Sum of Cubes, and Difference of Cubes. They then factor polynomial expressions.

Grouping

Have students complete the Questions 1 and 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- $(x + 2)(x + 2)$ is an example of which special product?
- $(x - 2)(x - 2)$ is an example of which special product?
- $(x + 2)(x - 2)$ is an example of which special product?
- $(2x + 3)(4x^2 + 6x + 9)$ is an example of which special product?
- $(5 - x)(25 + 5x + x^2)$ is an example of which special product?
- What is the greatest common factor of $2x^2 + 18$?
- How can the term $9x^6$ be written as a squared term?
- How can the term y^8 be written as a squared term?
- What is the greatest common factor of $2x^3 - 16$?
- What is the cube root of 125?
- What is the cube root of 27?

Talk the Talk

1. Complete the table to define each special product.

	Formula	Factors	Definition	Example
Perfect Square Trinomial	$ax^2 + 2ab + b^2$	$(a + b)^2$	The square of a binomial	$(x + 2)(x + 2)$ $x^2 + 4x + 4$ or $(x - 2)(x - 2)$ $x^2 - 4x + 4$
	$ax^2 - 2ab + b^2$	$(a - b)^2$		
Difference of squares	$a^2 - b^2$	$(a - b)(a + b)$	A number squared is subtracted from another squared number.	$(x + 5)(x - 5)$ $x^2 + 5x - 5x - 25$ $x^2 - 25$
Sum of Cubes	$a^3 + b^3$	$(a + b)(a^2 - ab + b^2)$	A number cubed added to another cubed number	$8x^3 + 27$ $(2x)^3 + 3^3$ $(2x + 3)((2x)^2 + (2x)(3) + (3)^2)$ $(2x + 3)(4x^2 + 6x + 9)$
Difference of Cubes	$a^3 - b^3$	$(a - b)(a^2 + ab + b^2)$	A number cubed is subtracted from another cubed number	$125 - x^3$ $5^3 - x^3$ $(5 - x)(5^2 + (5)(x) + x^2)$ $(5 - x)(25 + 5x + x^2)$

2. Factor each expression.

a. $2x^2 + 18$

$2(x^2 + 9)$

b. $9x^6 - y^8$

$(3x^3)^2 - (y^4)^2$

$(3x^3 - y^4)(3x^3 + y^4)$

c. $2x^3 - 16$

$2(x^3 - 8)$

$2(x^3 - 2^3)$

$2(x - 2)(x^2 + 2x + 2^2)$

$2(x - 2)(x^2 + 2x + 4)$

d. $125a^3 + 27$

$(5a + 3)(25a^2 - 15a + 9)$

Be prepared to share your solutions and methods.

Check for Students' Understanding

Match each polynomial with the equivalent polynomial in factored form.

Expressions		Polynomials
1. $x^3 + 216$	(B)	**A.** $(x - 6)^2$
2. $x^2 - 36$	(C)	**B.** $(x + 6)(x^2 - 6x + 36)$
3. $x^3 - 216$	(D)	**C.** $(x - 6)(x + 6)$
4. $x^2 - 12x + 36$	(A)	**D.** $(x - 6)(x^2 + 6x + 36)$

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Could It Be Groovy to Be a Square?

13.6

Approximating and Rewriting Radicals

LEARNING GOALS

In this lesson, you will:

- Determine the square root of perfect squares.
- Determine the approximate square root of given values.
- Determine the exact value of a square root of given values.
- Rewrite radicals by extracting perfect squares.

KEY TERMS

- square root
- positive square root
- principal square root
- negative square root
- extract the square root
- radical expression
- radicand

ESSENTIAL IDEAS

- A number b is a square root of a if $b^2 = a$.
- Every whole number is associated with a principal or positive square root, and a negative square root.
- When solving an equation in the form $b^2 = a$ for the value of b, extracting a square root in the equation is accomplished by square rooting both sides of the equation.
- A radical expression is an expression that contains a radical or square root sign.
- The radicand is the expression enclosed within the radical symbol.
- When determining the square root of a number that is not a perfect square, a truncated decimal is an approximate answer.
- When determining the square root of a number that is not a perfect square, an answer containing a radical where the radicand consists of only prime numbers is an exact answer.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

N-RN The Real Number System

Extend the properties of exponents to rational exponents.

2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

A-CED Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems

A-REI Reasoning with Equations and Inequalities

Solve equations and inequalities in one variable

4. Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Overview

A scenario is used to model a square root situation. The lesson focuses on determing square roots, principal square roots or positive square roots, negative square roots, and extracting the square root from both sides of the equation. In second activity, the terms radical expression and radicand are given. An example of solving for the approximate value of a square root containing a radicand that is not a perfect square is provided. Students will solve similar problems within the context of the scenario. In the last activity, students solve for an exact answer by factoring out all numbers in the radicand that are not prime numbers, and expressing the answer in radical form. Extracting the square root is also used to rewrite expressions containing a variable and a constant term that are squared.

Warm Up

1. $(8)^2 = ?$
   64

2. $(-8^2) = ?$
   64

3. List different ways to factor 64.
   $1 \cdot 64$
   $2 \cdot 32$
   $4 \cdot 16$
   $8 \cdot 8$

4. What is the prime factorization of 64?
   $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

5. Express 64 four different ways using a power.
   $(8)^2$
   $(-8)^2$
   $(2)^6$
   $(-2)^6$

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13.6

Could It Be Groovy to Be a Square?

Approximating and Rewriting Radicals

LEARNING GOALS

In this lesson, you will:

- Determine the square root of perfect squares.
- Determine the approximate square root of given values.
- Determine the exact value of a square root of given values.
- Rewrite radicals by extracting perfect squares.

KEY TERMS

- square root
- positive square root
- principal square root
- negative square root
- extract the square root
- radical expression
- radicand

One of the most brilliant of ancient Greek mathematicians was a man named Pythagoras. He believed that every number could be expressed as a ratio of two integers.

Yet, legend tells us that one day at sea, one of Pythagoras's students pointed out to him that the diagonal of a square which measures 1 unit by 1 unit would be $\sqrt{2}$, a number which could not possibly be represented as a ratio of two integers.

This student was allegedly thrown overboard and the rest of the group was sworn to secrecy!

Problem 1

A scenario is represented by an equation containing a number squared. Students will use the model equation to solve for unknowns. The definition of square root is given and the descriptions of principal square root or positive square root, and negative square root are provided. An example of extracting the square root from both sides of the equation is given.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- Which variable is associated with the substitution of 9.5? Why?
- The solution to this problem is associated with which variable?
- Which variable is associated with the substitution of 7.6? Why?
- Which variable is associated with the substitution of 8.5? Why?
- Which variable is associated with the substitution of 81? Why?
- What number multiplied by itself is equal to 36?
- What is the square root of 36?
- Is the square root of 36 equal to 6 or −6? Why?

PROBLEM 1 Good Vibrations

Vanessa plays guitar and knows that when she plucks a guitar string, the vibrations result in sound waves, which are the compression and expansion of air particles. Each compression and expansion is called a cycle. The number of cycles that occur in one second is 1 hertz. The number of cycles that occur in one second can be referred to as "wave speed," or "frequency."

Vanessa also knows that tuning a guitar requires changing the tension of the strings. The tension can be thought of as the amount of stretch (in pounds of pressure per inch) on a string between two fixed points. A string with the correct tension produces the correct number of cycles per second over time, which produces the correct tone.

1. Consider a guitar string that has a tension of 0.0026 pound per inch. An equation that relates hertz h and tension t in pounds per inch is $h^2 = t$.

   a. Determine the string tension if the frequency is 9.5 hertz.

   $9.5^2 = t$

   $90.25 = t$

   The tension for a frequency of 9.5 hertz is 90.25 pounds per inch.

   b. Determine the string tension if the frequency is 7.6 hertz.

   $7.6^2 = t$

   $57.76 = t$

   The tension for a frequency of 7.6 hertz is 57.76 pounds per inch.

   c. Determine the string tension if the frequency is 8.5 hertz.

   $8.5^2 = t$

   $72.25 = t$

   The tension for a wave speed of 8.5 hertz is 72.25 pounds per inch.

2. What appears to happen to the tension as the frequency increases?

   The tension increases as the frequency increases.

3. Write an equation to determine the frequency produced by a string with a tension of 81 pounds per inch. Use the same variables you were given in Question 1.

   $h^2 = 81$

   The frequency must be 9 hertz because the square of 9 is 81.

4. Write an equation to determine the frequency produced by a string with a tension of 36 pounds per inch. Use the same variables you were given in Question 1.

   $h^2 = 36$

   The frequency must be 6 hertz because the square of 6 is 36.

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Grouping

Ask a student to read the information and definition. Complete Questions 5 and 6 as a class.

Guiding Questions for Share Phase, Question 5

- What is the sign of the product when a positive number is multiplied by a positive number?
- What is 9 times 9?
- What is the sign of the product when a negative number is multiplied by a negative number?
- What is -9 times -9?

Guiding Questions for Share Phase, Question 6

- What does the number of hertz in this problem situation represent?
- Is it possible for the number of hertz to be a negative number? Why not?

Notice that your answers to Questions 3 and 4 are *square roots* of 81 and 36. A number b is a **square root** of a if $b^2 = a$. So, 9 is the square root of 81 because $9^2 = 81$, and 6 is a square root of 36 because $6^2 = 36$.

5. Jasmine claims that 81 could have two square roots: 9 and -9. Maria says that there can be only one square root for 81, which is 9. Determine who is correct and explain why that student is correct.

Jasmine is correct because the product of a negative integer multiplied by itself will result in a positive product. Therefore, any positive number has two square roots, a positive square root and a negative square root.

In earlier grades, you may have seen problems in which you only determined one square root. However, there are in fact two square roots for every whole number, a **positive square root** (which is also called the **principal square root**) and a **negative square root**. This occurs because of the rule you learned about multiplying two negative numbers: when two negative numbers are multiplied together, the product is a positive number.

To solve the equation

$$h^2 = 81,$$

you can **extract the square root** from both sides of the equation.

$$\sqrt{h^2} = \pm\sqrt{81}$$
$$h = \pm 9$$

However, you must still be mindful of the solutions in the context of the problem.

6. Lincoln determined the frequency, in hertz, for a string with a tension of 121 pounds per inch. His work is shown.

Lincoln

$h^2 = 121$

$h = \pm\sqrt{121}$

$h^2 = \pm 11$

The frequency must be 11 hertz because the square of 11 is 121.

Explain why Lincoln is correct in the context of the problem.

Lincoln is correct because even though 121 has two square roots, 11 and -11, the solution for this problem will only be 11. There can only be positive number of hertz, there cannot be any negative number of hertz.

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Problem 2

The scenario from Problem 1 is used with larger numbers. The definitions of radical expressions and radicand are given. An example of determining the approximate square root when the radicand is not a perfect square is provided. Students will use the example to solve for unknowns related to the scenario.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Question 1 with a partner. Then share the responses as a class.
- Ask a student to read the information and definition. Discuss as a class.
- Have students complete Questions 2 through 4 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 1

- Which variable is associated with the substitution of 440? Why?
- The solution to this problem is associated with which variable?
- How is this problem different than the previous problems?

PROBLEM 2 Ah, That's Close Enough . . .

Generally, experts agree that humans can hear frequencies that are between 15 hertz and 20,000 hertz.

Recall the equation $h^2 = t$. The A-string on a guitar has a frequency of 440 hertz (cycles per second).

1. Determine the string tension if the frequency of the A-string is 440 hertz.

$$h^2 = t$$
$$440^2 = t$$
$$193{,}600 = t$$

The string tension for a frequency of 440 hertz is 193,600 pounds per inch.

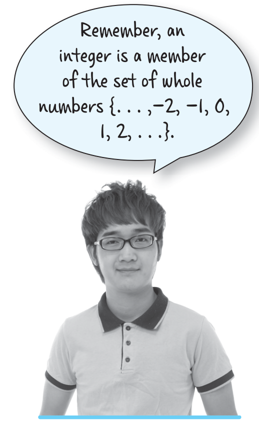
If $440^2 = 193{,}600$, then $\sqrt{193{,}600} = 440$. This second expression is a **radical expression** because it involves a radical symbol ($\sqrt{\ }$). The **radicand** is the expression enclosed within the radical symbol. In the expression $\sqrt{193{,}600}$, 193,600 is the radicand.

2. Write an equation to determine the frequency of a string with a tension of 75 pounds per inch.

$h^2 = 75$

3. Write your equation as a radical expression.

$h = \sqrt{75}$



4. Can you predict whether the frequency will be a positive integer? Explain why or why not.

No. I do not think that the frequency will be a positive integer because the radicand is not a perfect square.

Guiding Questions for Share Phase, Questions 2 through 4

- Will any whole number squared result in a product equal to 75?
- Is the number 75 a perfect square number?
- What is the closest perfect square number that is less than 75?
- What is the closest perfect square number that is greater than 75?
- Which perfect square number is closest to 75?

Grouping

- Ask a student to read the information and example. Discuss as a class.
- Have students complete Questions 5 through 7 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 5 through 7

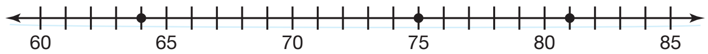
- What is the closest perfect square number that is less than 42?
- What is the closest perfect square number that is greater than 42?
- Which perfect square number is closest to 42?
- Is 6.5 an approximate answer or an exact answer? Why?
- What is the closest perfect square number that is less than 50?
- What is the closest perfect square number that is greater than 50?
- Which perfect square number is closest to 50?
- Is 7.1 an approximate answer or an exact answer?
- What is the closest perfect square number that is less than 136?

You can also estimate the square roots of numbers that are not perfect squares.

You can determine the approximate value of $\sqrt{75}$.

Determine the perfect square that is closest to but less than 75.
Then determine the perfect square that is closest to but greater than 75.

$64 \le 75 \le 81$



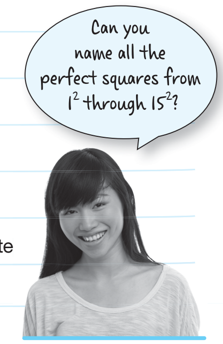
Determine the square roots of the perfect squares.

$\sqrt{64} = 8$ $\quad \sqrt{75} = ?$ $\quad \sqrt{81} = 9$

Now that you know that $\sqrt{75}$ is between 8 and 9, you can test the squares of numbers between 8 and 9.

$8.6^2 = 73.96$ $\quad 8.7^2 = 75.69$

Since 75.69 is closer to 75 than 73.96, 8.7 is the approximate square root of $\sqrt{75}$.



5. Determine the approximate frequency for each string tension given. First, write a quadratic equation with the information given, and then approximate your solution to the nearest tenth.

   a. 42 pounds per inch

   $h^2 = 42$

   $h = \sqrt{42}$

   $6.4^2 = 40.96$ $\quad 6.5^2 = 42.25$

   A string with a tension of 42 pounds per inch produces a frequency of approximately 6.5 hertz.

   b. 50 pounds per inch

   $h^2 = 50$

   $h = \sqrt{50}$

   $7^2 = 49$ $\quad 7.1^2 = 50.41$

   A string with a tension of 50 pounds per inch produces a frequency of approximately 7.1 hertz.

Guiding Questions for Share Phase, Questions 5 and 6

- What is the closest perfect square number that is greater than 136?
- Which perfect square number is closest to 136?
- Is 11.7 an approximate answer or an exact answer?
- Can the frequency be greater than the pressure given? Why not?
- What is the closest perfect square number that is less than 756?
- What is the closest perfect square number that is greater than 756?
- Which perfect square number is closest to 756?

Problem 3

Students will calculate ‘exact’ answers in similar situations by factoring out all numbers that are not prime numbers in the radicand and writing the answer as a radical expression. An example is provided. Extracting the square root is also used to simplify expressions containing a variable and a constant term that are squared.

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Grouping

- Ask a student to read the information and example. Complete Question 1 as a class.
- Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

c. 136 pounds per inch

$h^2 = 136$

$h = \sqrt{136}$

$11.6^2 = 134.56 \quad 11.7^2 = 136.89$

A string with a tension of 136 pounds per inch produces a frequency of approximately 11.7 hertz.

6. The lowest key on a piano has a string tension of 756 pounds of pressure per inch. What is the frequency of the lowest key on the piano? Write an equation and approximate your answer to the nearest tenth.

$h^2 = 756$

$h = \sqrt{756}$

$27^2 = 729 \quad 28^2 = 784$

$27.4^2 = 750.76 \quad 27.5^2 = 756.25$

The lowest key on the piano produces a frequency that is approximately 27.5 hertz.

PROBLEM 3 No! It Must Be Exact

There are times when an exact solution is necessary. For example, acoustical engineers may need to calculate the exact solution when designing sound stages or studios.

Laura makes and designs acoustical tiles for recording studios. These tiles are used to reflect different instrument tones of different frequencies in the studio. One of her clients has requested that the area of each square acoustical tile needs to be 75 square inches. Because these tiles can be lined up in rows or columns and they affect other factors in a recording studio, Laura needs to determine the exact side measure of each acoustical tile.

Consider a square acoustical tile with an area of 75 square inches.

You can set up and solve an equation to determine exact side length of the square.

$$s^2 = 75$$

$$s = \sqrt{75}$$

You can also rewrite $\sqrt{75}$ in a different form to help show the value. First, rewrite the product of 75 to include any perfect square factors, and then extract the square roots of those perfect squares.

$$\begin{aligned}\sqrt{75} &= \sqrt{25 \cdot 3}\\ &= \sqrt{25} \cdot \sqrt{3}\\ &= \sqrt{5 \cdot 5} \cdot \sqrt{3}\\ &= 5\sqrt{3}\end{aligned}$$

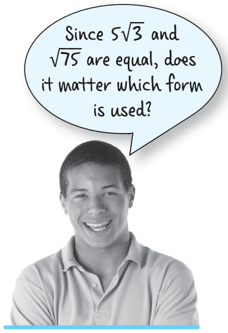
The exact measure of each side of the acoustical tile is $\sqrt{75}$, or $5\sqrt{3}$ inches.

1. Estimate the value of $5\sqrt{3}$. Explain your reasoning.

I know that $\sqrt{3}$ is about 1.75, so (1.75)(5) is 8.75. So, $5\sqrt{3}$ is approximately 8.75.

2. Compare your approximation of $5\sqrt{3}$ to the approximation of $\sqrt{75}$ from the worked example in Problem 2. What do you notice?

Answers will vary.



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Grouping

Have students complete Questions 3 and 4 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- What is a prime number?
- How many ways can you factor 20?
- What is the prime factorization of 20?
- What perfect squares are within the prime factorization of 20?
- Is $2\sqrt{5}$ an approximate answer or an exact answer?
- How many ways can you factor 26?
- What is the prime factorization of 26?
- What perfect squares are within the prime factorization of 26?
- Is $\sqrt{26}$ an approximate answer or an exact answer?
- How many ways can you factor 64?
- What is the prime factorization of 64?
- What perfect squares are within the prime factorization of 64?
- Is 8 an approximate answer or an exact answer?
- How many ways can you factor 18?
- What is the prime factorization of 18?

3. Rewrite each radical by extracting all perfect squares, if possible.

a. $\sqrt{20}$

$$\begin{aligned}\sqrt{20} &= \sqrt{4 \cdot 5}\\ &= \sqrt{4} \cdot \sqrt{5}\\ &= 2\sqrt{5}\end{aligned}$$

b. $\sqrt{26}$

The number 26 has factors of 2 and 13. Therefore the radical expression $\sqrt{26}$ cannot be rewritten, because neither factor is a perfect square.

c. $\sqrt{64}$

$\sqrt{64} = 8$

4. For each given area, write an equation to determine the side measurements of the square acoustical tiles. Then, determine the exact side measurement of each square acoustical tile.

a. 18 square inches

$$\begin{aligned}18 &= s^2\\ \sqrt{18} &= \sqrt{9 \cdot 2}\\ &= \sqrt{9} \cdot \sqrt{2}\\ &= 3\sqrt{2}\end{aligned}$$

The exact side measurement of a square acoustical tile with an area of 18 square inches is $3\sqrt{2}$ inches.

b. 116 square inches

$$\begin{aligned}116 &= s^2\\ \sqrt{116} &= \sqrt{4 \cdot 29}\\ &= \sqrt{4} \cdot \sqrt{29}\\ &= 2\sqrt{29}\end{aligned}$$

The exact side measurement of a square acoustical tile with an area of 116 square inches is $2\sqrt{29}$ inches.

- What perfect squares are within the prime factorization of 18?
- Is $3\sqrt{2}$ an approximate answer or an exact answer?
- How many ways can you factor 116?
- What is the prime factorization of 116?
- What perfect squares are within the prime factorization of 116?

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Guiding Questions for Share Phase, Question 5

- When a radical is squared what happens to the radicand?
- How do you 'undo' an expression that is squared?
- What is the approximate value of $\sqrt{83}$?
- What operation was used to isolate the variable?
- Why does this problem have two possible answers?
- What is the approximate value of $\sqrt{55}$?
- What operation was used to isolate the variable?
- How do you know the solution is the exact solution and not an approximation of the solution?
- What differentiates an approximate solution and an exact solution?
- Does an approximate solution have two possible answers? Why or why not?
- Does an exact solution have two possible answers? Why or why not?

Can extracting the square root also be used for expressions containing a variable and a constant term that are squared?

Consider the equation $(x - 1)^2 = 17$. How can you determine the value of x?

First take the square root of both sides of the equation.

$$(x - 1)^2 = 17$$

$$\sqrt{(x - 1)^2} = \pm\sqrt{17}$$

$$x - 1 = \pm\sqrt{17}$$

Then, isolate the variable to determine the value of the variable.

$$x - 1 + 1 = \pm\sqrt{17} + 1$$

$$x = 1 \pm \sqrt{17}$$

$$x \approx 1 \pm 4.12$$

$$x \approx -3.12 \quad \text{or} \quad 5.12$$

5. Determine the approximate solutions for each of the given equations.

a. $(r + 8)^2 = 83$

$\sqrt{(r + 8)^2} = \pm\sqrt{83}$

$r + 8 = \pm\sqrt{83}$

$r = -8 \pm \sqrt{83}$

$r \approx 1.11$ or -17.11

b. $(17 - d)^2 = 55$

$\sqrt{(17 - d)^2} = \sqrt{55}$

$17 - d = \pm\sqrt{55}$

$-d = -17 \pm \sqrt{55}$

$d = 17 \pm \sqrt{55}$

$d \approx 17 \pm 7.42$

$d \approx 9.58$ or 24.42

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Guiding Questions for Share Phase, Question 6

- How many ways can you factor 18?
- What is the prime factorization of 18?
- How many ways can you factor 116?
- What is the prime factorization of 116?
- How many ways can you factor 24?
- What is the prime factorization of 24?
- How many ways can you factor 99?
- What is the prime factorization of 99?

6. Rewrite each radical by extracting all perfect squares, if possible.

a. $\sqrt{18}$

$$\begin{aligned}\sqrt{18} &= \sqrt{9 \cdot 2} \\ &= \sqrt{9} \cdot \sqrt{2} \\ &= 3\sqrt{2}\end{aligned}$$

b. $\sqrt{116}$

$$\begin{aligned}\sqrt{116} &= \sqrt{4 \cdot 29} \\ &= \sqrt{4} \cdot \sqrt{29} \\ &= 2\sqrt{29}\end{aligned}$$

c. $5\sqrt{24}$

$$\begin{aligned}5\sqrt{24} &= 5 \cdot \sqrt{4 \cdot 6} \\ &= 5\sqrt{4} \cdot \sqrt{6} \\ &= 5 \cdot 2\sqrt{6} \\ &= 10\sqrt{6}\end{aligned}$$

d. $7\sqrt{99}$

$$\begin{aligned}7\sqrt{99} &= 7 \cdot \sqrt{9 \cdot 11} \\ &= 7\sqrt{9} \cdot \sqrt{11} \\ &= 7 \cdot 3\sqrt{11} \\ &= 21\sqrt{11}\end{aligned}$$

Be prepared to share your solutions and methods.

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Check for Students' Understanding

Solve the equation shown for an approximate solution and an exact solution.

$(p + 10)^2 = 20$

$(p + 10)^2 = 20$
$\sqrt{(p + 10)^2} = \pm\sqrt{20}$
$p + 10 = \pm\sqrt{20}$
$p = -10 \pm \sqrt{20}$
$p \approx -5.53$ *or* -14.48

$(p + 10)^2 = 20$
$\sqrt{(p + 10)^2} = \pm\sqrt{20}$
$p + 10 = \pm\sqrt{20}$
$p = -10 \pm \sqrt{20}$
$p = -10 \pm 2\sqrt{5}$
$p = -10 + 2\sqrt{5}$ *or* $-10 - 2\sqrt{5}$

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13.7

Another Method

Completing the Square

LEARNING GOALS

In this lesson, you will:

- Determine the roots of a quadratic equation by completing the square.
- Complete the square geometrically and algebraically.

KEY TERMS

- completing the square

ESSENTIAL IDEAS

- Completing the square is a process for writing a quadratic expression in vertex form which then allows you to solve for the zeros.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Write expressions in equivalent forms to solve problems.

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

A-REI Reasoning with Equations and Inequalities

Solve equations and inequalities in one variable

4. Solve quadratic equations in one variable.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Overview

Students will use their knowledge of perfect square trinomials from the previous lesson to solve for the zeros of quadratic functions that cannot be easily factored. Students begin by geometrically completing the square and adjusting the equation representing the geometric model accordingly. The method 'Completing the Square' is described and students then practice this solution method to determine the zeros of quadratic functions. An example of completing the square and solving for the zeros of the quadratic function is provided and students will use this to model other solutions.

Warm Up

Solve the equation shown for the exact solution.

$(k + 5)(k + 5) = 90$

$(k + 5)(k + 5) = 90$

$(k + 5)^2 = 90$

$\sqrt{(k + 5)^2} = \pm\sqrt{90}$

$k + 5 = \pm\sqrt{90}$

$k + 5 = \sqrt{90}$ $\qquad$ $k + 5 = -\sqrt{90}$

$k = -5 + 3\sqrt{10}$ $\qquad$ $k = -5 - 3\sqrt{10}$

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13.7

Another Method

Completing the Square

LEARNING GOALS

In this lesson, you will:

- Determine the roots of a quadratic equation by completing the square.
- Complete the square geometrically and algebraically.

KEY TERMS

- completing the square

Can you construct a square that has the exact same area as a given circle using only a compass and a straightedge?

Well, no. And this was proven to be impossible in 1882, making pi a “transcendental” irrational number.

Unfortunately, it seems that no one in Indiana got the message at the time. In 1897, the Indiana state legislature, via amateur mathematician Edwin Goodwin, tried to pass a law declaring that there was a solution to this famous problem—known as completing the square.

Problem 1

Students will factor and calculate the zeros of quadratic functions then conclude that all quadratic functions cannot be factored. They then use a graphing calculator to identify the zeros of a function that cannot be factored.

Grouping

Have students complete Questions 1 through 5 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- What are the possible ways to factor the constant term of the quadratic function?
- Will any set of factors of the constant term satisfy the middle term of the quadratic function?
- Which functions could not be factored?

PROBLEM 1 Where Are the Zeros?

1. Factor each quadratic function and determine the zeros, if possible.

   a. $f(x) = x^2 + 5x + 4$

   $(x + 1)(x + 4) = 0$

   $x + 1 = 0$ or $x + 4 = 0$

   $x = -1$ or $x = -4$

   b. $f(x) = x^2 - 4x + 2$

   No real zeros

   c. $f(x) = x^2 + 5x + 2$

   No real zeros

   d. $f(x) = x^2 - 4x - 5$

   $(x - 5)(x + 1) = 0$

   $x - 5 = 0$ or $x + 1 = 0$

   $x = 5$ or $x = -1$

2. Were you able to determine the zeros of each function in Question 1 by factoring? Explain your reasoning.

   No. The functions in parts (b) and (c) cannot be factored because no factors of the constant can provide the necessary middle terms.

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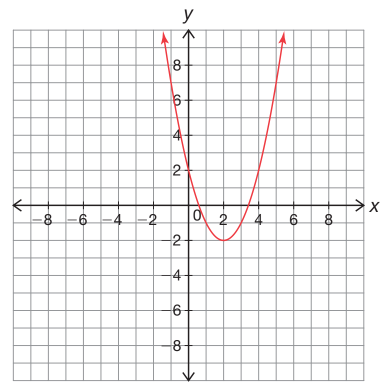
Guiding Questions for Share Phase, Question 3

- Do all quadratic functions have two zeros?
- Where are the zeros located on the graph of the quadratic function?

Problem 2

Students will use their knowledge of perfect square trinomials to construct a process to solve any quadratic equation. They begin by geometrically constructing an area model that contains the given two terms and rebuild the rectangle into a square by splitting the area rectangle in half and rearranging the pieces. After writing the area of each piece on the model, students will rewrite the original expression to include all pieces of the square model which is a perfect square trinomial. An example of completing the square to determine the zeros of a quadratic function is given. Students then use the example to solve for the zeros of other quadratic functions.

3. If you cannot factor a quadratic function, does that mean it does not have zeros?
   a. Graph the quadratic function from Question 1, part (b) on your calculator. Sketch the graph on the coordinate plane.



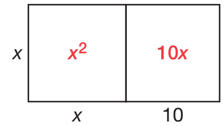
   b. Does this function have zeros? Explain your reasoning.
   Yes. This function does have zeros even though I could not determine them by factoring. I know it has zeros because it intersects the x-axis twice.

The quadratic function you graphed has zeros but cannot be factored, so we must find another method for calculating its zeros. You can use your understanding of the relationship among the coefficients of a perfect square trinomial to construct a procedure to solve any quadratic equation.

PROBLEM 2 Seeing the Square

Previously, you factored trinomials of the form $a^2 + 2ab + b^2$ as the perfect square $(a + b)^2$. This knowledge can help you when constructing a procedure for solving any quadratic equation.

1. The expression $x^2 + 10x$ can be represented geometrically as shown. Write the area of each piece in the center of the piece.



Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

13

Grouping

- Ask a student to read the information and definition. Discuss as a class.
- Have students complete Questions 3 through 5 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

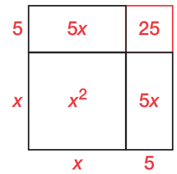
- What is the product of x times x?
- What is the product of x times 10?
- If the rectangle representing $10x$ is divided in half, what is the area of each half?
- Which piece now appears in the area model that was not included in the initial model?
- To keep the expression representative of the geometric model, what value must also be added to the expression?
- Is the resulting expression a perfect square trinomial?

Guiding Questions for Share Phase, Questions 3 through 5

- If the rectangle representing $8x$ is divided in half, what is the area of each half?
- Which piece now appears in the area model that was not included in the initial expression?
- To keep the expression representative of the geometric model, what value must also be added to the expression?

2. This figure can now be modified into the shape of a square by splitting the second rectangle in half and rearranging the pieces.

a. Label the side length of each piece and write the area of each piece in the center of the piece.



b. Do the two figures represent the same expression? Explain your reasoning.

Yes. The two figures represent the same expression because the area of each figure is the same.

$x^2 + 5x + 5x = x^2 + 10x$

c. Complete the figure so it is a square. Label the area of the piece you added.

See figure.

d. Add this area to your original expression. What is the new expression?

$x^2 + 10x + 25$

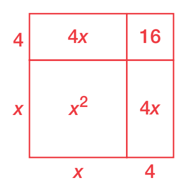
e. Factor this expression.

$x^2 + 10x + 25 = (x + 5)(x + 5)$

The process you worked through above is a method known as *completing the square*. **Completing the square** is a process for writing a quadratic expression in vertex form which then allows you to solve for the zeros.

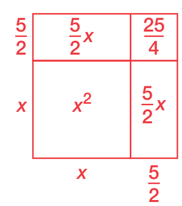
3. Use a geometric figure to complete the square for each expression. Then factor the resulting trinomial.

a. $x^2 + 8x$



$x^2 + 8x + 16 = (x + 4)^2$

b. $x^2 + 5x$



$x^2 + 5x + \frac{25}{4} = \left(x^2 + \frac{5}{2}\right)^2$

- Is the resulting expression a perfect square trinomial?
- If the rectangle representing $5x$ is divided in half, what is the area of each half?
- Which piece now appears in the area model that was not included in the initial expression?
- To keep the expression representative of the geometric model, what value must also be added to the expression?
- Is the resulting expression a perfect square trinomial?

13

4. Analyze your work in Question 3.

a. Explain how to complete the square on an expression $x^2 + bx$ where b is an integer.

To complete the square, divide b in half and then square the result. Add this to the original expression.

b. Describe how the coefficient of the middle term, b, is related to the constant term, c in each trinomial you wrote in Question 3.

The middle term's coefficient is twice the square root of the constant term.

5. Use the descriptions you provided in Question 4 to determine the unknown value of b or c that would make each expression a perfect square trinomial. Then write the expression as a binomial squared.

a. $x^2 - 8x + \underline{16} = \underline{(x - 4)^2}$

b. $x^2 + 5x + \underline{\frac{25}{4}} = \underline{\left(x + \frac{5}{2}\right)^2}$

c. $x^2 - \underline{20x} + 100 = \underline{(x - 10)^2}$

d. $x^2 + \underline{24x} + 144 = \underline{(x + 12)^2}$

Grouping

- Ask a student to read the information and example. Discuss as a class.
- Have students complete Questions 6 through 8 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 6 and 7

- How do you check to make sure these values are in fact roots of the quadratic equation?
- How will you know if these values satisfy the quadratic equation?
- When algebraically determining the roots of the equation, what is the first step?
- What constant was added to both sides of the equation? How did you determine this constant?

So how does completing the square help when trying to determine the roots of a quadratic equation that cannot be factored? Let's take a look.

Determine the roots of the equation $x^2 - 4x + 2 = 0$.

Isolate $x^2 - 4x$. You can make this into a perfect square trinomial.	$x^2 - 4x + 2 - 2 = 0 - 2$ $x^2 - 4x = -2$
Determine the constant term that would complete the square. Add this term to both sides of the equation.	$x^2 - 4x + \_\_\_\_ = -2 + \_\_\_\_$ $x^2 - 4x + 4 = -2 + 4$ $x^2 - 4x + 4 = 2$
Factor the left side of the equation.	$(x - 2)^2 = 2$
Determine the square root of each side of the equation.	$\sqrt{(x - 2)^2} = \pm\sqrt{2}$ $x - 2 = \pm\sqrt{2}$
Set the factor of the perfect square trinomial equal to each of the square roots of the constant. Solve for x.	$x - 2 = \pm\sqrt{2}$ $x - 2 = \sqrt{2}$ or $x - 2 = -\sqrt{2}$ $x = 2 + \sqrt{2}$ or $x = 2 - \sqrt{2}$ $x \approx 3.414$ or $x \approx 0.5858$ The roots are approximately 3.41 and 0.59.

6. Check the solutions in the worked example by substituting each into the original equation.

Check:

$x = 3.414$	$x = 0.5858$
$3.414^2 - 4(3.414) + 2 = 0$	$0.5858^2 - 4(0.5858) + 2 = 0$
$11.655 - 13.656 + 2 \approx 0$	$0.3432 - 2.343 + 2 \approx 0$

7. Do your solutions match the zeros you sketched on your graph in Problem 1, Question 3? Explain how you determined your answer.

Yes. By using the zero function on my calculator I can determine that the first zero is equal to 0.5858 and the second zero is equal to 0.3414.

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Guiding Questions for Share Phase, Question 8

- If the equation is quadratic, will there always be two distinct zeros?
- Could there be only a single value for the zero of a quadratic equation?
- If there was exactly one value for root, what could you conclude about the graph of the quadratic equation?
- Are the roots an exact solution or an approximate solution? How do you know?

8. Determine the roots of each equation by completing the square.

a. $x^2 - 6x + 4 = 0$

$x^2 - 6x = -4$

$x^2 - 6x + 9 = -4 + 9$

$(x - 3)^2 = 5$

$\sqrt{(x - 3)^2} = \pm\sqrt{5}$

$x - 3 = \pm\sqrt{5}$

$x - 3 = \sqrt{5}$ or $x - 3 = -\sqrt{5}$

$x = 3 + \sqrt{5}$ or $x = 3 - \sqrt{5}$

The roots are $3 \pm \sqrt{5}$.

b. $x^2 - 12x + 6 = 0$

$x^2 - 12x = -6$

$x^2 - 12x + 36 = -6 + 36$

$x^2 - 12x + 36 = 30$

$(x - 6)^2 = 30$

$\sqrt{(x - 6)^2} = \pm\sqrt{30}$

$x - 6 = \pm\sqrt{30}$

$x = 6 \pm\sqrt{30}$

$x \approx 11.477$ or $x \approx 0.523$

The roots are $6 \pm \sqrt{30}$.

Check:

$x = 11.477$

$11.477^2 - 12(11.477) + 6 = 0$

$131.72 - 137.72 + 6 \approx 0$

OR

$x = 0.523$

$0.523^2 - 12(0.523) + 6 = 0$

$0.274 - 6.276 + 6 \approx 0$

You can identify the axis of symmetry and the vertex of any quadratic function written in standard form by completing the square.

$$y = ax^2 + bx + c$$

Step 1: $y - c = ax^2 + bx$

Step 2: $y - c = a\left(x^2 + \frac{b}{a}x\right)$

Step 3: $y - c + a\left(\frac{b}{2a}\right)^2 = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right)$

Step 4: $y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$

Step 5: $y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$

Step 6: $y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$

Notice that the a-value was factored out before completing the square!

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9. Explain why $a\left(\frac{b}{2a}\right)^2$ was added to the left side of the equation in Step 3.

You have to multiply the term $\left(\frac{b}{2a}\right)^2$ by a to maintain balance of the equation.

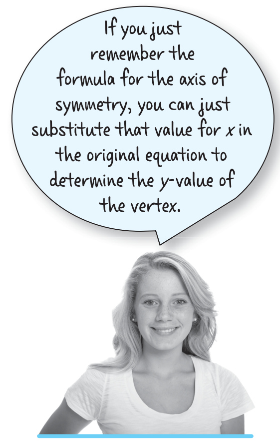
10. Given a quadratic function in the form $y = ax^2 + bx + c$,

   a. identify the axis of symmetry.

   The axis of symmetry is $x = -\frac{b}{2a}$.

   b. identify the location of the vertex.

   The vertex is $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.



11. Rewrite each quadratic equation in vertex form. Then identify the axis of symmetry and the location of the vertex in each.

   a. $y = x^2 + 8x - 9$

$$y + 9 = x^2 + 8x$$
$$y + 9 + 16 = x^2 + 8x + 16$$
$$y + 25 = (x + 4)^2$$
$$y = (x + 4)^2 - 25$$

   The axis of symmetry is $x = -4$ and the vertex is at $(-4, 25$

   b. $y = 3x^2 + 2x - 1$

$$y + 1 = 3x^2 + 2x$$
$$y + 1 = 3\left(x^2 + \frac{2}{3}x\right)$$
$$y + 1 + 3\left(\frac{1}{9}\right) = 3\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right)$$
$$y + 1 + \frac{1}{3} = 3\left(x + \frac{1}{3}\right)^2$$
$$y = \frac{4}{3} = 3\left(x + \frac{1}{3}\right)^2$$
$$y = 3\left(x + \frac{1}{3}\right)^2 - \frac{4}{3}$$

   The axis of symmetry is $x = -\frac{1}{3}$ and the vertex is at $\left(-\frac{1}{3}, -\frac{4}{3}\right)$.

12. Determine the roots and the location of the vertex of $y = x^2 + 20x + 36 = 0$.

$$y = x^2 + 20x + 36$$
$$y - 36 = x^2 + 20x$$
$$y - 36 + 100 = x^2 + 20x + 100$$
$$y + 64 = (x + 10)^2$$
$$y = (x + 10)^2 - 64$$

The vertex is at $(-10, -64)$.

$$64 = (x + 10)^2$$
$$\sqrt{64} = \sqrt{(x + 10)^2}$$
$$\pm 8 = x + 10$$
$$8 = x + 10 \quad \text{or} \quad -8 = x + 10$$
$$-2 = x \quad \text{or} \quad -18 = x$$

The roots are -2 and -18.

Be prepared to share your solutions and methods.

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Check for Students' Understanding

Determine the roots of the quadratic equation by completing the square. Solve for exact solutions and approximate solutions. Check your solutions.

$y = x^2 - 6x - 12$

$x^2 - 6x = 12$

$x^2 - 6x + 9 = 12 + 9$

$x^2 - 6x + 9 = 21$

$(x - 3)^2 = 21$

$\sqrt{(x - 3)^2} = \pm\sqrt{21}$

$x - 3 = \pm\sqrt{21}$

$x = 3 \pm \sqrt{21}$

$x \approx 7.58$ *or* $x \approx -1.58$

$x \approx 7.58$

$(7.58)^2 - 6(7.58) \approx 12$

$57.456 - 45.48 \approx 12$

$x \approx -1.58$

$(-1.58)^2 - 6(-1.58) \approx 12$

$2.496 + 9.48 \approx 12$

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Chapter 13 Summary

KEY TERMS

- polynomial (13.1)
- term (13.1)
- coefficient (13.1)
- monomial (13.1)
- binomial (13.1)
- trinomial (13.1)
- degree of a term (13.1)
- degree of a polynomial (13.1)
- Zero Product Property (13.4)
- Converse of Multiplication Property of Zero (13.4)
- roots (13.4)
- difference of two squares (13.5)
- perfect square trinomial (13.5)
- difference of two cubes (13.5)
- sum of two cubes (13.5)
- square root (13.6)
- positive square root (13.6)
- principal square root (13.6)
- negative square root (13.6)
- extract the square root (13.6)
- radical expression (13.6)
- radicand (13.6)
- completing the square (13.7)

13.1 Identifying Characteristics of Polynomial Expressions

A polynomial is an expression involving the sum of powers in one or more variables multiplied by coefficients. A polynomial in one variable is the sum of terms of the form ax^k where a, called the coefficient, is a real number and k is a non-negative integer. In general, a polynomial is of the form $a_1x^k + a_2x^{k-1} + \ldots a_nx^0$. Each of the products in a polynomial is called a term. Polynomials are named according to the number of terms: monomials have exactly 1 term, binomials have exactly 2 terms, and trinomials have exactly 3 terms. The exponent of a term is the degree of the term, and the greatest exponent in a polynomial is the degree of the polynomial. When a polynomial is written in standard form, the terms are written in descending order, with the term of the greatest degree first and ending with the term of the least degree.

Example

The characteristics of the polynomial $13x^3 + 5x + 9$ are shown.

	1st term	2nd term	3rd term
Term	$13x^3$	$5x$	9
Coefficient	13	5	9
Power	x^3	x^1	x^0
Exponent	3	1	0

There are 3 terms in this polynomial. Therefore, this polynomial is a trinomial. This trinomial has a degree of 3 because 3 is the greatest degree of the terms in the trinomial.

13.1 Adding and Subtracting Polynomial Expressions

Polynomials can be added or subtracted by identifying the like terms of the polynomial functions, using the Associative Property to group the like terms together, and combining the like terms to simplify the expression.

Example 1

Expression: $(7x^2 - 2x + 12) + (8x^3 + 2x^2 - 3x)$

The like terms are $7x^2$ and $2x^2$ and $-2x$ and $-3x$. The terms $8x^3$ and 12 are not like terms.

$(7x^2 - 2x + 12) + (8x^3 + 2x^2 - 3x)$

$8x^3 + (7x^2 + 2x^2) + (-2x - 3x) + 12$

$8x^3 + 9x^2 - 5x + 12$

Example 2

Expression: $(4x^4 + 7x^2 - 3) - (2x^2 - 5)$

The like terms are $7x^2$ and $2x^2$ and -3 and -5. The term $4x^4$ does not have a like term.

$(4x^4 + 7x^2 - 3) - (2x^2 - 5)$

$4x^4 + (7x^2 - 2x^2) + (-3 + 5)$

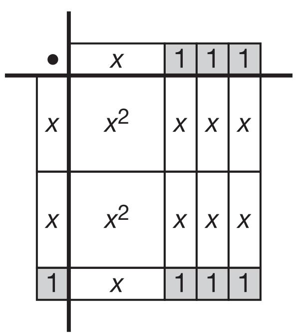
$4x^4 + 5x^2 + 2$

13.2 Modeling the Product of Polynomials

The product of 2 binomials can be determined by using an area model with algebra tiles. Another way to model the product of 2 binomials is a multiplication table which organizes the two terms of the binomials as factors of multiplication expressions.

Example 1

$(2x + 1)(x + 3)$



$(2x + 1)(x + 3) = 2x^2 + 7x + 3$

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Example 2

$(9x - 1)(5x + 7)$

$\bullet$	$9x$	-1
$5x$	$45x^2$	$-5x$
7	$63x$	-7

$(9x - 1)(5x + 7) = 45x^2 - 5x + 63x - 7$

$= 45x^2 + 58x - 7$

13.2 Using the Distributive Property to Multiply Polynomials

The Distributive Property can be used to multiply polynomials. Depending on the number of terms in the polynomials, the Distributive Property may need to be used multiple times.

Example

$(2x^2 + 5x - 10)(x + 7)$

$(2x^2 + 5x - 10)(x) + (2x^2 + 5x - 10)(7)$

$(2x^2)(x) + (5x)(x) - 10(x) + (2x^2)(7) + (5x)(7) - 10(7)$

$2x^3 + 5x^2 - 10x + 14x^2 + 35x - 70$

$2x^3 + 19x^2 + 25x - 70$

13.3 Factoring Polynomials by Determining the Greatest Common Factor

Factoring a polynomial means to rewrite the expression as a product of factors. The first step in factoring any polynomial expression is to determine whether or not the expression has a greatest common factor.

Example

Expression: $12x^3 + 4x^2 + 16x$

The greatest common factor is $4x$.

$12x^3 + 4x^2 + 16x = 4x(3x^2) + 4x(x) + 4x(4)$

$= 4x(3x^2 + x + 4)$

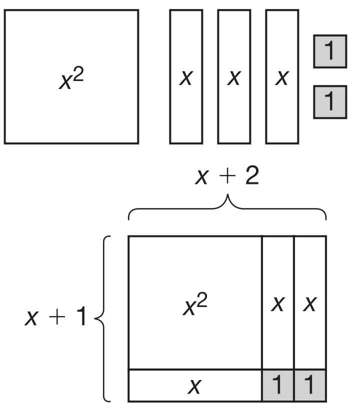
13.3 Factoring Trinomials

A quadratic expression can be written in factored form, $ax^2 + bx + c = a(x - r_1)(x - r_2)$, by using an area model with algebra tiles, multiplication tables, or trial and error. Factoring a quadratic expression means to rewrite it as a product of two linear expressions.

Example 1

Trinomial: $x^2 + 3x + 2$

Represent each part of the trinomial as a piece of the area model. Then use the parts to form a rectangle.



The factors of this trinomial are the length and width of the rectangle.
Therefore, $x^2 + 3x + 2 = (x + 1)(x + 2)$.

Example 2

Trinomial: $x^2 + 15x + 54$

•	x	9
x	x^2	$9x$
6	$6x$	54

So, $x^2 + 15x + 54 = (x + 6)(x + 9)$.

13.4 Solving Quadratic Equations Using Factoring

The Zero Product Property states that if the product of two or more factors is equal to 0, at least one factor must be equal to 0. The property is also known as the Converse of the Multiplication Property of Zero. This property can be used to solve a quadratic equation. The solutions to a quadratic equation are called roots. To calculate the roots of a quadratic equation using factoring:

- Perform transformations so that one side of the equation is equal to 0.
- Factor the quadratic expression on the other side of the equation.
- Set each factor equal to 0.
- Solve the resulting equation for the roots. Check each solution in the original equation.

Example

Equation: $2x^2 + x = 6$

$$2x^2 + x = 6$$
$$2x^2 + x - 6 = 6 - 6$$
$$2x^2 + x - 6 = 0$$
$$(2x - 3)(x + 2) = 0$$
$$2x - 3 = 0 \quad \text{or} \quad x + 2 = 0$$
$$\frac{2x}{2} = \frac{3}{2}$$
$$x = 1.5 \qquad\qquad x = -2$$

13.4 Connecting the Zeros of a Function to the x-intercepts of a Graph

The x-intercepts of the graph of the quadratic function $f(x) = ax^2 + bx + c$ and the zeros of the function are the same as the roots of the equation $ax^2 + bx + c = 0$.

Example

Function: $f(x) = x^2 + 6x - 55$

$$x^2 + 6x - 55 = 0$$
$$(x + 11)(x - 5) = 0$$
$$x + 11 = 0 \quad \text{or} \quad x - 5 = 0$$
$$x = -11 \qquad\qquad x = 5$$

The zeros of the function $f(x) = x^2 + 6x - 55$ are $x = -11$ and $x = 5$.

13.5 Identifying Special Products of Degree 2

There are special products of degree 2 that have certain characteristics. A perfect square trinomial is a trinomial formed by multiplying a binomial by itself. A perfect square trinomial is in the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$. A binomial is a difference of two squares if it is in the form $a^2 - b^2$ and can be factored as $(a + b)(a - b)$.

Example 1

Expression: $4x^2 + 12x + 9$

$$4x^2 + 12x + 9$$
$$(2x + 3)(2x + 3)$$
$$(2x + 3)^2$$

This expression is a perfect square trinomial.

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Example 2

Expression: $x^2 - 49y^2$

$x^2 - 49y^2$

$x^2 - (7y)^2$

$(x + 7y)(x - 7y)$

This binomial is the difference of two squares.

13.5 Identifying Special Products of Degree 3

There are also special products of degree 3 that have certain characteristics. The difference of two cubes is an expression in the form $a^3 - b^3$ that can be factored as $(a - b)(a^2 + ab + b^2)$. The sum of two cubes is an expression in the form of $a^3 + b^3$ that can be factored as $(a + b)(a^2 - ab + b^2)$.

Example 1

Expression: $27x^3 - 125$

$27x^3 - 125$

$(3x)^3 - 5^3$

$(3x - 5)((3x)^2 + (3x)(5) + 5^2)$

$(3x - 5)(9x^2 + 15x + 25)$

Example 2

Expression: $64x^3 + 1$

$64x^3 + 1$

$(4x)^3 + 1^3$

$(4x + 1)((4x)^2 - (4x)(1) + 1^2)$

$(4x + 1)(16x^2 - 4x + 1)$

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13.6 Determining Approximate Square Roots of Given Values

A number b is a square root of a if $b^2 = a$. There are 2 square roots for every whole number: a positive square root, which is also called the principal square root, and a negative square root. To determine approximate square roots for given values, determine the perfect square that is closest to, but less than, the given value. Also, determine the perfect square that is closest to, but greater than, the given value. You can use these square roots to approximate the square root of the given number.

Example

Determine the approximate value of $\sqrt{40}$.

$36 \le 40 \le 49$

$\sqrt{36} = 6 \quad \sqrt{40} = ? \qquad \sqrt{49} = 7$

$6.3^2 = 39.69 \qquad 6.4^2 = 40.96$

The approximate value of $\sqrt{40}$ is 6.3.

13.6 Simplifying Square Roots

To simplify a square root, extract any perfect squares within the expression.

Example

Simplify $\sqrt{40}$.

$$\begin{aligned} \sqrt{40} &= \sqrt{4 \cdot 10} \\ &= \sqrt{4} \cdot \sqrt{10} \\ &= 2\sqrt{10} \end{aligned}$$

13.6 Extracting Square Roots to Solve Equations

The solution to an equation where one term contains a variable and a constant term that are squared can be determined by extracting the square root. To do so, take the square root of both sides of the equation, and then isolate the variable to determine the value of the variable.

Example

$$\begin{aligned} (x - 7)^2 &= 75 \\ \sqrt{(x - 7)^2} &= \sqrt{75} \\ x - 7 &= \pm\sqrt{75} \\ x &= 7 \pm \sqrt{75} \\ x &\approx 7 \pm 8.7 \\ x &\approx 15.7 \quad \text{or} \quad -1.7 \end{aligned}$$

13.7 Determining the Roots of a Quadratic Equation by Completing the Square

For a quadratic function that has zeros but cannot be factored, there exists another method for calculating the zeros of the function or solving the quadratic equation. Completing the square is a process for writing a quadratic expression in vertex form which then allows you to solve for the zeros.

When a function is written in standard form, $ax^2 + bx + c$, the axis of symmetry is $x = -\frac{b}{2a}$.

Example

Function: $f(x) = x^2 + 4x + 1$

$$x^2 + 4x + 1 = 0$$
$$x^2 + 4x = -1$$
$$x^2 + 4x + 4 = -1 + 4$$
$$x^2 + 4x + 4 = 3$$
$$(x + 2)^2 = 3$$
$$\sqrt{(x + 2)^2} = \pm\sqrt{3}$$
$$x + 2 = \pm\sqrt{3}$$
$$x = -2 \pm \sqrt{3}$$
$$x \approx -0.268, -3.732$$

The roots are $-2 \pm\sqrt{3}$.

The axis of symmetry is $x = -\frac{4}{2}$, or $x = -2$.

Check:

$$x = -0.268$$
$$(-0.268)^2 + 4(-0.268) + 1 \stackrel{?}{=} 0$$
$$0.0718 - 1.072 + 1 \stackrel{?}{=} 0$$
$$-0.0002 \approx 0$$

$$x = -3.732$$
$$(-3.732)^2 + 4(-3.732) + 1 \stackrel{?}{=} 0$$
$$13.9278 - 14.928 + 1 \stackrel{?}{=} 0$$
$$-0.0002 \approx 0$$

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