Introduction to Quadratic Functions

12

Chapter 12 Overview

This chapter examines the graphical behavior of quadratic functions, including domain, range, increasing and decreasing, absolute maximum and absolute minimum, symmetry, and zeros. The relationship between the form of a quadratic function and the graph of a quadratic function is discussed, especially the key graphical characteristics identified from the form of the quadratic function. Transformations and dilations of quadratic functions are explored.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
12.1	Exploring Quadratic Functions	A.CED.1 A.CED.2 F.IF.4	1	This lesson provides two real world situations for students to model quadratic functions and explore the graphical behavior. A graphing calculator is then used to determine the absolute maximum or absolute minimum.	X	X		X	X
12.2	Comparing Linear and Quadratic Functions	A.SSE.1 A.CED.1 A.CED.2 F.IF.4 F.IF.6 F.LE.1.a	2	This lesson explores the first and second differences of linear and quadratic functions. Students will analyze tables and graphs of different functions to identify the function type. Questions focus students to examine the leading term of both functions to understand the affect the sign has on the graph of the function.				X	
12.3	Domain, Range, Zeros, and Intercepts	A.SSE.1 A.CED.1 A.CED.2 F.IF.4 F.IF.5 F.IF.7a	1	This lesson provides a real world situation that models vertical motion. Questions ask students to identify the domain, range, zeros, and intervals of increase and decrease.		X			X
12.4	Factored Form of a Quadratic Function	A.SSE.1.a A.SSE.3.a A.CED.1 A.CED.2 F.IF.4 F.IF.7a	2	This lesson provides opportunities for students to understand the significance of a quadratic function written in factored form. Questions focus students to graph a quadratic function to determine the zeros. They then compare the behaviors of the graph of the quadratic equation to the function written in factored form.		X	X		X

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
12.5	Investigating the Vertex of a Quadratic Function	A.SSE.1.a F.IF.4 F.IF.7a	1	This lesson provides opportunities for students to understand the significance of the line of symmetry with respect to quadratic functions. Questions focus students to graph a quadratic function to determine the vertex and axis of symmetry. They then use the axis of symmetry to determine additional points on the parabola.				X	X
12.6	Vertex Form of a Quadratic Function	A.SSE.1.a F.IF.4 F.IF.7.a	2	This lesson provides opportunities for students to identify and compare the key characteristics of a quadratic function written in standard form, factored form, and vertex form. Students are then given two functions written in standard form and will complete graphic organizers by writing the functions in factored form and vertex form, and then identifying the key features of each form.			X	X	X
12.7	Transformations of Quadratic Functions	F.BF.3 F.IF.7a	2	This lesson presents the basic quadratic function and investigates transformations and dilations of that function. Questions focus students to think about the transformations being performed on the function versus transformations performed on the argument of the function.					X

Skills Practice Correlation for Chapter 12

Lesson		Problem Set	Description
12.1	Exploring Quadratic Functions		Vocabulary
		1 – 6	Write quadratic functions in standard form
		7 – 12	Write quadratic functions in standard form that represent area as a function of width
		13 – 18	Determine and describe absolute maximums of functions
12.2	Comparing Linear and Quadratic Functions		Vocabulary
		1 – 6	Graph tables of values then describe the function represented by the graph
		7 – 12	Calculate first and second difference for tables of values then describe the function represented by the table
12.3	Domain, Range, Zeros, and Intercepts		Vocabulary
		1 – 6	Graph functions represented by problem situations then identify absolute maximums, zeros, domains, and ranges
		7 – 12	Use interval notation to represent intervals described
		13 – 18	Identify intervals of increase and decrease for given graphs of functions
12.4	Factored Form of a Quadratic Function		Vocabulary
		1 – 6	Factor expressions
		7 – 12	Determine x-intercepts of quadratic functions in factored form
		13 – 18	Write quadratic functions in factored form with given characteristics
		19 – 24	Determine x-intercepts for functions using a graphing calculator then write the function in factored form
		25 – 30	Determine x-intercepts for functions then write the function in factored form
12.5	Investigating the Vertex of a Quadratic Function		Vocabulary
		1 – 6	Write functions representing vertical motion problem situations
		7 – 12	Identify vertexes and axes of symmetry for vertical motion problem situations
		13 – 18	Determine axes of symmetry of parabolas
		19 – 24	Determine vertexes of parabolas
		25 – 30	Determine other points on a parabola given information about the parabola

12

Lesson		Problem Set	Description
12.6	Vertex Form of a Quadratic Function		Vocabulary
		1 – 6	Determine vertexes of quadratic functions given in vertex form
		7 – 12	Determine vertexes of quadratic function given in standard form
		13 – 18	Determine x-intercepts of quadratic functions given in standard form
		19 – 24	Identify the form of quadratic functions as standard, factored, or vertex then describe key characteristics based on the equation
		25 – 30	Write equations for quadratic functions with given characteristics
12.7	Transformations of Quadratic Functions		Vocabulary
		1 – 6	Describe transformations performed on functions to create new functions
		7 – 12	Describe transformations performed on functions to create new functions
		13 – 18	Describe transformations performed on functions to create new functions
		19 – 24	Represent functions as vertical dilations of other functions using coordinate notation
		25 – 30	Write equations in vertex form given characteristics then sketch the function
		31 – 36	Determine transformations necessary to translate the graph of a function into the graph of another function

12

12.1

Up and Down or Down and Up

Exploring Quadratic Functions

LEARNING GOALS

In this lesson, you will:

- Model real-world problems using quadratic functions.
- Analyze tables, graphs, and equations for quadratic functions.
- Use the Distributive Property to write a quadratic equation in standard form.
- Compare graphs of quadratic functions.
- Use a graphing calculator to determine the absolute minimum or absolute maximum of a quadratic function.

KEY TERMS

- standard form (general form) of a quadratic function
- parabola

ESSENTIAL IDEAS

- Real world problems are modeled using quadratic functions.
- The graph of a quadratic function is a parabola.
- Quadratic functions are modeled using multiple representations such as equations, tables, and graphs.
- The Distributive Property is used to write a quadratic equation in standard form.
- The graphing calculator is used to determine the absolute minimum or absolute maximum of a quadratic function.
- When the coefficient of the x^2 term in a quadratic function is a positive number, the graph of the function, the parabola, opens upward.
- When the coefficient of the x^2 term in a quadratic function is a negative number, the graph of the function, the parabola, opens downward.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-CED Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F-IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

Overview

Scenarios are used that are represented by a quadratic function. The first scenario is an area problem and the second scenario is the classic handshake problem. Students will write the expressions associated with the scenarios and use the expressions to write a quadratic function for each situation. Using a graphing calculator they then graph the parabola and compute the absolute maximum or the absolute minimum relevant to the problem situation. In the last activity, students conclude that when the coefficient of the x^2 term is a positive number, the parabola opens upward, and when the coefficient of the x^2 term is a negative number, the parabola opens downward.

Warm Up

Rewrite each expression using the distributive property and then simplify.

1. $4(3 + 6)$

$4(3) + 4(6)$

$12 + 24$

36

2. $-5(x + 7)$

$-5(x) - 5(7)$

$-5x - 35$

3. $x(-2 - 9)$

$x(-2) - x(9)$

$-2x - 9x$

$-11x$

4. $x(8 + x)$

$x(8) + x(x)$

$8x + x^2$

12

12

Up and Down or Down and Up

12.1

Exploring Quadratic Functions

LEARNING GOALS

In this lesson, you will:

- Model real-world problems using quadratic functions.
- Analyze tables, graphs, and equations for quadratic functions.
- Use the Distributive Property to write a quadratic equation in standard form.
- Compare graphs of quadratic functions.
- Use a graphing calculator to determine the absolute minimum or absolute maximum of a quadratic function.

KEY TERMS

- standard form (general form) of a quadratic function
- parabola

Take a necklace and hold it at both ends with your fingers. What function would you say describes the shape of the necklace? Quadratic? Wrong. This shape is formed by a catenary function.

From bridges to spider webs to architectural arches, catenaries are all around us—and they look so much like parabolas, it's easy to be fooled.

In future math classes—distant future—you'll learn about the differences between catenary functions and quadratic functions. For now, let's just focus on quadratics.

12

Problem 1

A scenario is used to explore a quadratic function. Students will write expressions to represent the length, width, and area of a rectangular enclosure. The Distributive property is used to rewrite the expression for the area of the enclosure then the area function is rewritten in standard form. A series of diagrams show how the area of the enclosure changes as the width of the enclosure diminishes. Using a graphing calculator students then graph the quadratic function which is a parabola. Instructions are given to calculate the absolute maximum using the Maximum function on the graphing calculator. Finally, students will determine the dimensions of the enclosure that will provide the maximum area.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Question 1 with a partner. Then share the responses as a class.

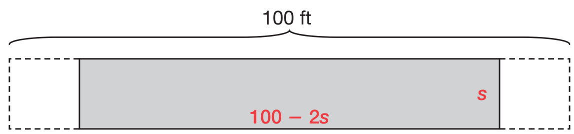
12

Guiding Questions for Share Phase, Question 1

- What is the relationship between the length and width of a square?

PROBLEM 1 Who Put the Dogs In?

A dog trainer is fencing in an enclosure, represented by the shaded region in the diagram. The trainer will also have two square-shaped storage units on either side of the enclosure to store equipment and other materials. She can make the enclosure and storage units as wide as she wants, but she can't exceed 100 feet in total length.



1. Let s represent a side length, in feet, of one of the square storage units.

   a. Write an expression to represent the width of the enclosure. Label the width in the diagram. Explain your reasoning.

   The storage unit is a square, so all 4 sides are the same length. This means that the width of the enclosure can be represented by s.

   b. Write an expression to represent the length of the enclosure. Label the length in the diagram. Explain your reasoning.

   Since the entire length of the enclosure plus the two storage units is 100 feet, the length of the enclosure is $100 - s - s$, or $100 - 2s$.

   c. Write an expression to represent the area of the enclosure. Explain your reasoning.

   The area of the enclosure is the product of its width and length. So, the area is $s(100 - 2s)$.

- If the original length of the rectangular enclosure was 100 feet and a length of $2s$ is removed, what operation is used to represent this new length?
- What do you need to know to determine the area of a rectangle?
- How is the area of a rectangle determined?
- What is the length and width of the rectangular enclosure?
- What is s times 100?
- What is s times $-2s$?

Grouping

- Ask a student to read the example and definitions and complete Question 2 as a class.
- Have students complete Questions 3 and 4 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- If there is no constant in the function written in standard form, what is the value of c?
- Does the length of the rectangle depend on the area, or does the area of the rectangle depend on the length of the side?

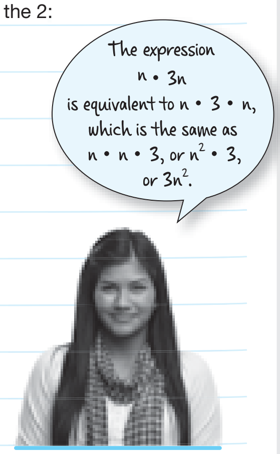
You know that you can use the Distributive Property to rewrite a mathematical expression.

For example, to rewrite $2(5 - 4)$, you can distribute the 2:

$$2(5 - 4) = 2(5) - 2(4)$$

You can also distribute variables:

$$n(4 - 3n) = n(4) - n(3n)$$
$$= 4n - 3n^2$$



2. Use the Distributive Property to rewrite the expression representing the area of the enclosure. Show your work.

$s(100 - 2s) = s(100) - s(2s)$
$= 100s - 2s^2$

The expression you wrote is a quadratic expression. Recall that a quadratic function written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$, is in **standard form,** or **general form**. In this form, a and b represent numerical coefficients and c represents a constant.

3. Write the area of the enclosure as a function, $A(s)$, in standard form. Then, identify a, b, and c for the function.

$A(s) = -2s^2 + 100s + 0$
$a = -2$, $b = 100$, and $c = 0$

4. Identify the independent and dependent quantities and their units in this problem situation.

The independent quantity is the side length, in feet, of each storage unit (or the width of the enclosure). The dependent quantity is the area of the enclosure in square feet.

Grouping

Have students complete Questions 5 through 7 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 5 through 7

- Looking at the first and second diagram, does the area of the rectangular enclosure appear to increase or decrease?
- Looking at the second and third diagram, does the area of the rectangular enclosure appear to increase or decrease?
- Looking at the third and fourth diagram, does the area of the rectangular enclosure appear to increase or decrease?
- Looking at the fourth and fifth diagram, does the area of the rectangular enclosure appear to increase or decrease?
- Looking at the fifth and sixth diagram, does the area of the rectangular enclosure appear to increase or decrease?
- Looking at the sixth and seventh diagram, does the area of the rectangular enclosure appear to increase or decrease?
- Looking at the seventh and eighth diagram, does the area of the rectangular enclosure appear to increase or decrease?
- How will the increase in area followed by a decrease in area affect the graph of the area function?

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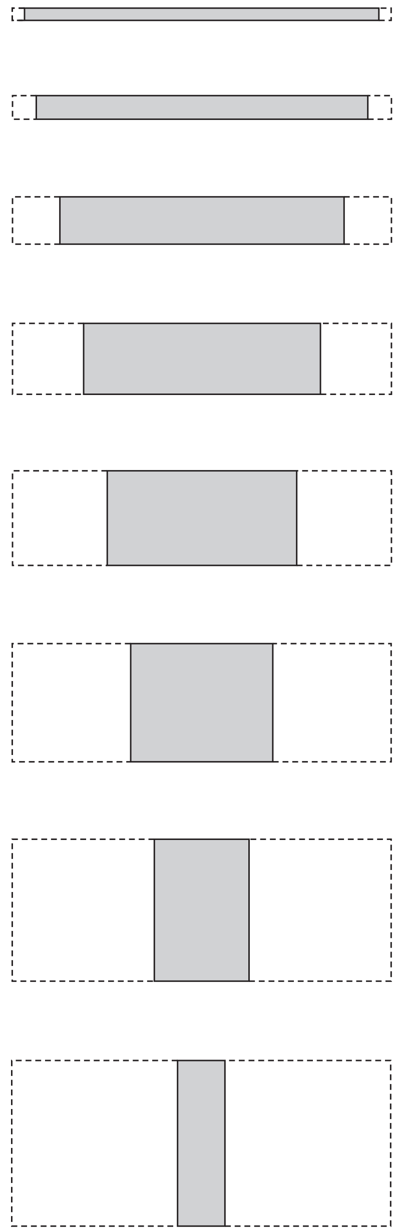
The progression of diagrams at the right shows how the area of the enclosure, $A(s)$, changes as the side length s of each square storage unit increases.



5. Describe how the area of the enclosure changes.

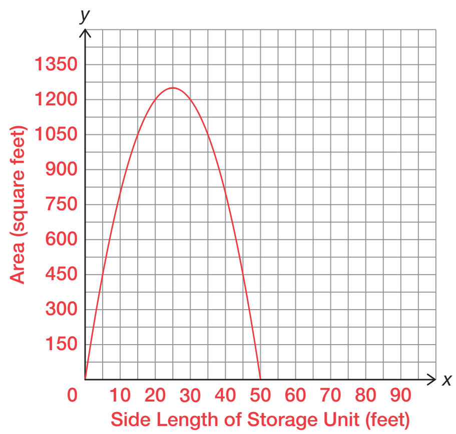
As s increases, the area of the enclosure first increases and then decreases.

6. Predict what the graph of the function will look like. Explain your reasoning.

Answers will vary.

The graph will be a smooth curve. It will be in the shape of an upside-down U because as the side length s increases, the area of the enclosure increases and then decreases.

7. Use a graphing calculator to graph the function you wrote in Question 3. Then sketch the graph and label the axes. How can you tell the graph is a quadratic function?



It is a quadratic function because the graph is a smooth curve that increases and then decreases in an upside-down U shape.

- Why do you think the graph of the parabola is upside down?
- Where on the graph do you think the area of the rectangular enclosure is the greatest?
- Where on the graph do you think the area of the rectangular enclosure is the least?

Grouping

- Ask a student to read the definition and complete Question 8 as a class.
- Have students complete Questions 9 and 10 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 9 and 10

- What is the difference between a maximum and an absolute maximum?
- Can the area of the rectangular enclosure ever be greater than 125 square feet? Why or why not?
- If the width of the rectangular enclosure is 25 feet, how do you determine the length of the rectangular enclosure?
- Is the length always longer than the width of a rectangular enclosure?

The shape that a quadratic function forms when graphed is called a **parabola**.

8. Think about the possible areas of the enclosure. Is there a maximum area that the enclosure can contain? Explain your reasoning in terms of the graph and in terms of the problem situation.

Yes. There is a point on the graph of the function that represents the greatest area that the enclosure can contain. Because the trainer can't exceed 100 feet in length, as the side length of the storage units increases, the area of the enclosure first increases but then must decrease to make room for the extra length of the storage units.

You can use a graphing calculator to identify the absolute maximum of a quadratic function.

Step 1: Enter the function and press **GRAPH**.

Step 2: Press **2ND** and then **CALC**. Select **4: maximum**.

Step 3: Move your cursor to any point on the curve where the graph is increasing and press **ENTER**.

Step 4: Move your cursor to a point directly opposite the point in Step 3 on the curve where the graph is decreasing and press **ENTER**. Then press **ENTER** one more time.

A graph is increasing when it is going up from left to right. It is decreasing when it is going down from left to right.

9. Use a graphing calculator to determine the absolute maximum of $A(s)$. Then describe what the x- and y-coordinates of the absolute maximum represent in this problem situation.

The absolute maximum of the function is at (25, 1250).
The x-coordinate of 25 represents the side length in feet of each storage unit that produces the maximum area.
The y-coordinate of 1250 represents the maximum area in square feet of the enclosure.

10. Determine the dimensions of the enclosure that will provide the maximum area. Show your work and explain your reasoning.

I can substitute 25 for s in the expression for the length of the enclosure, $100 - 2s$.
$100 - 2(25)$
$100 - 50 = 50$
The dimensions of 25 feet $\times$ 50 feet provide the maximum area of the enclosure.

Problem 2

The scenario is represented by a quadratic function. Students will determine values of the function then organize the values in a table and write an expression for *n* handshakes. Using the expression from the table, students then write a quadratic function representing the situation. A graphing calculator is used to graph the function and students determine the absolute minimum of the function.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 6 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

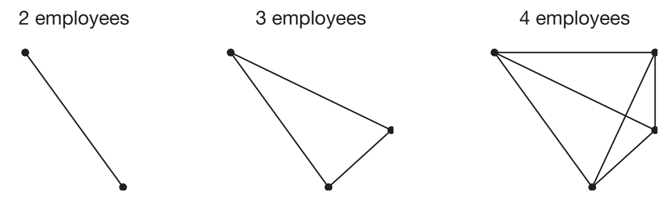
12

- When the first person and second person shake hands, is this counted as one or two handshakes?
- How many points are used to represent 5 people shaking hands with each other?

PROBLEM 2 Handshake Problem

Suppose that there is a monthly meeting at CIA headquarters for all employees. How many handshakes will it take for every employee at the meeting to shake the hand of every other employee at the meeting once?

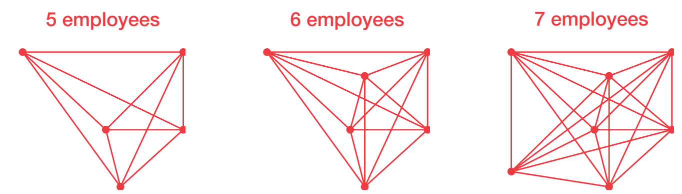
1. Use the figures shown to determine the number of handshakes that will occur between 2 employees, 3 employees, and 4 employees.



2 employees: 1 handshake; 3 employees: 3 handshakes; 4 employees: 6 handshakes

2. Draw figures to represent the number of handshakes that occur between 5 employees, 6 employees, and 7 employees and determine the number of handshakes that will occur in each situation.

Sample figures shown.



5 employees: 10 handshakes; 6 employees: 15 handshakes; 7 employees: 21 handshakes

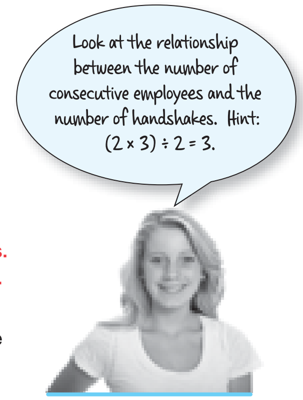
Guiding Questions for Share Phase, Questions 3 through 7

- What is 2 times 3? What can you do to the product to determine the number of handshakes for 3 employees?
- What is 3 times 4? What can you do to the product to determine the number of handshakes for 4 employees?
- What is 4 times 5? What can you do to the product to determine the number of handshakes for 5 employees?
- What is 5 times 6? What can you do to the product to determine the number of handshakes for 6 employees?
- What is 6 times 7? What can you do to the product to determine the number of handshakes for 7 employees?
- If *n* represents a number of employees, how would you express one employee less than *n*?
- What is $n-1$ times *n*? What can you do to the product to determine the number of handshakes for *n* employees?
- Is the graph of this quadratic function right side up or upside down? Why?

3. Complete the table to record your results.

Number of Employees	2	3	4	5	6	7	n
Number of Handshakes	1	3	6	10	15	21	$\frac{n(n-1)}{2}$

4. Write an expression in the table to represent the number of handshakes given any number of employees.

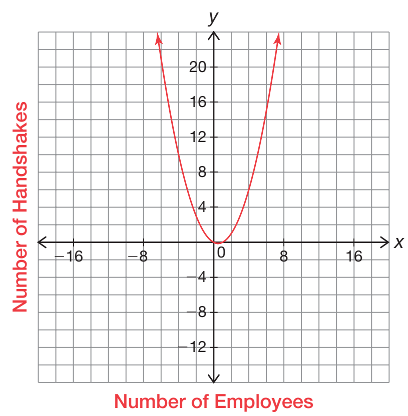


5. Rewrite the expression representing the handshake pattern as the quadratic function $H(n)$ in standard form. Identify the independent and dependent quantities.

$H(n) = \frac{1}{2}n^2 - \frac{1}{2}n$

The independent quantity is the number of employees. The dependent quantity is the number of handshakes.

6. Graph this function on a graphing calculator. Sketch the graph on the coordinate plane and label the axes.



7. Determine the absolute minimum of $H(n)$. Then, describe what the *x*- and *y*-coordinates of this point represent in this problem situation.

The absolute minimum is at $\left(\frac{1}{2}, \frac{-1}{8}\right)$. This doesn't make sense, because you can't have $\frac{1}{2}$ of an employee and $-\frac{1}{8}$ handshake.

The fewest number of handshakes occurs when there are 0 employees. So, in terms of this problem situation, the absolute minimum is at (0, 0) or (1, 0).

- How is a minimum different than an absolute minimum?
- When does the graph of a quadratic function have an absolute maximum?
- When does the graph of a quadratic function have an absolute minimum?

12

Talk the Talk

Students will compare the quadratic functions in the previous problems. They conclude that when the coefficient of the x^2 term is a positive number, the parabola opens upward, and when the coefficient of the x^2 term is a negative number, the parabola opens downward.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- If the storage area side length is 0 feet, does the storage area exist? Why not?
- If the storage area side length is 50 feet, does the storage area exist? Why not?
- How is the coefficient of the x^2 term different in the two problems in this lesson?
- Which way does the parabola open if the coefficient of the x^2 term is a positive number?
- Which way does the parabola open if the coefficient of the x^2 term is a negative number?

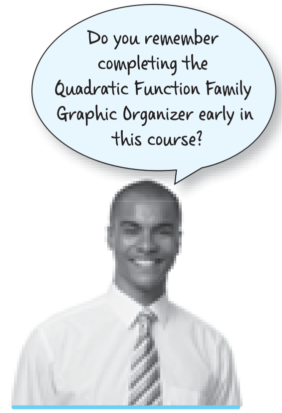
12

Talk the Talk

Let's compare the two functions you graphed in this lesson. Graph both functions together on a graphing calculator.

Area of dog enclosure: $A(s) = -2s^2 + 100s$

Handshake Pattern: $H(n) = \frac{1}{2}n^2 - \frac{1}{2}n$



1. Describe the domain in terms of the function and each problem situation.

   Area of the dog enclosure: In terms of the function, the domain is the set of all real numbers. In terms of the problem situation, the domain is $s > 0$.

   Handshake Pattern: In terms of the function, the domain is the set of all real numbers. In terms of the problem situation, the domain is $s \geqslant = 0$.

2. How can you determine whether the graph of a quadratic function opens up or down based on the equation?

   When the sign of the coefficient of the x^2 term is negative, the graph opens down. When the sign of the coefficient is positive, the graph opens up.

3. How can you determine whether the graph of a quadratic function has an absolute minimum or an absolute maximum?

   When the a value is greater than 0, the graph will have an absolute minimum.

   When the a value is less than 0, the graph will have an absolute maximum.

Be prepared to share your solutions and methods.

Check for Students' Understanding

The width of a rectangular painting is 5 inches shorter than its length.

1. Write algebraic expressions to represent the width and the length of the painting.

   w = width

   $w + 5$ = length

2. Write an equation to model the area of the painting.

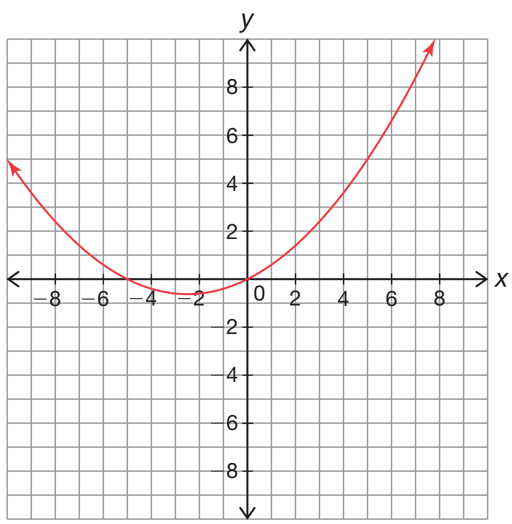
   $a = w(w + 5)$

   $a = w^2 + 5w$

3. Create a table of values to include possible widths, lengths, and areas of the painting.

Width (inches)	Length (inches)	Area (square inches)
w	$w + 5$	$w(w + 5)$
4	9	36
5	10	50
6	16	96
10	25	250

4. Sketch a graph to model the area of the painting.



5. Identify the x- and y-intercepts.

   The y-intercept is (0, 0) and the x-intercepts are (−5, 0) and (0, 0).

12

6. What do the *x*- and *y*-intercepts mean with respect to this problem situation?

The *x*- and *y*-intercept at (0, 0) means that if the width of the painting is 0, then the area of the painting is 0. The *x*-intercept at (−5, 0) does not make sense in the problem situation, the width of the painting cannot be a negative measurement.

7. Does this problem have an absolute maximum or an absolute minimum? Is it relevant to this problem situation?

This problem has an absolute minimum. It is negative so it is not relevant to the problem situation.

12

12.2

Just U and I

Comparing Linear and Quadratic Functions

LEARNING GOALS

In this lesson, you will:

- Identify linear and quadratic functions from multiple representations.
- Compare graphs, tables, and equations for linear and quadratic functions.
- Analyze graphs of linear and quadratic functions.
- Determine if a function is linear or quadratic by analyzing the first and second differences.

KEY TERMS

- leading coefficient
- second differences

ESSENTIAL IDEAS

- Real world problems are modeled using linear and quadratic functions.
- The first differences are the differences between successive output values when successive input values have a difference of 1.
- The first differences of a linear function are a constant whereas the first differences of a quadratic function changes.
- The second differences are the differences between consecutive values of the first differences.
- The second differences of a linear function are 0 whereas the second differences of a quadratic function are a constant.
- The leading coefficient of a function is the numerical coefficient of the term with the greatest power.
- The sign of the leading coefficient of a quadratic function determines if the parabola opens upward or downward.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context.

A-CED Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F-IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

F-LE Linear, Quadratic, and Exponential Models

Construct and compare linear, quadratic, and exponential models and solve problems

1. Distinguish between situations that can be modeled with linear functions and with exponential functions.

a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

Overview

The first scenario used is represented by a linear function, the length as a function of the width and a quadratic function, the area as a function of the width. Students will complete a table of values and graph the functions. They compare the *x*- and *y*-intercepts of each type of graph. In the second activity tables of values for two functions are given and students determine the first differences to identify which table describes a liner function and which table describes a quadratic function. The rates of change of each type of function are compared. They are given the equations and graphs of both functions and will determine the *x*-intercept of the linear function algebraically and the *x*-intercepts of the quadratic function using a graphing calculator. Questions focus students on the leading term of both functions and affect the sign of the leading term has on the graph of the function. In the third activity, students then compute the first and second differences of four different functions and generalize about the first and second differences associated with linear and quadratic functions. They also sketch the graph of each function and compare the sign of the first and second differences to the graphic behavior of the function.

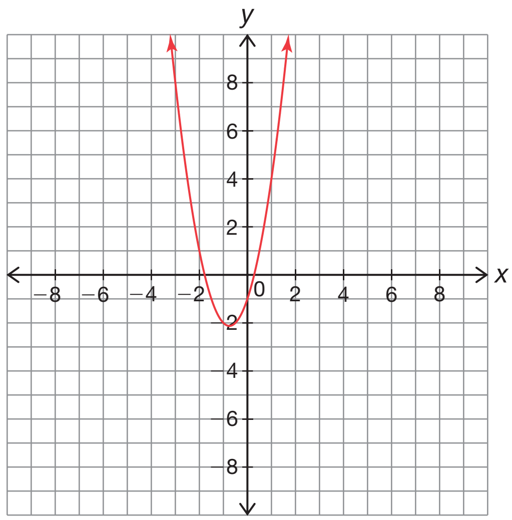
12

Warm Up

Use a graphing calculator to sketch each graph and determine whether each quadratic function has an absolute minimum or an absolute maximum.

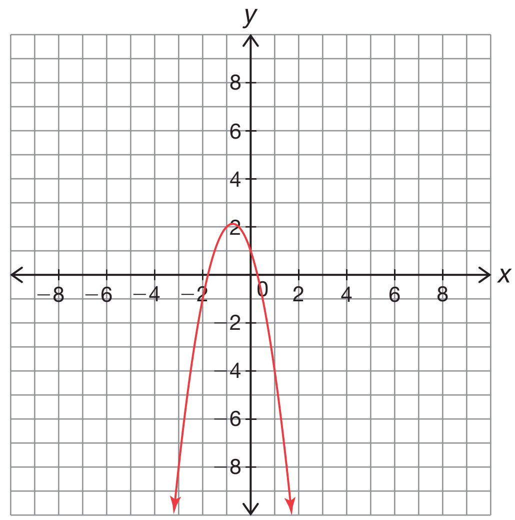
1. $y = 2x^2 + 3x - 1$

This quadratic function has an absolute maximum.



2. $y = -2x^2 - 3x + 1$

This quadratic function has an absolute minimum.



3. Predict if this function will have an absolute minimum or an absolute maximum. Explain your reasoning.

$y = -\frac{1}{2}x^2 - 3x + 1$

This quadratic function has an absolute maximum because the x^2 term has a negative coefficient.

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12.2

Just U and I

Comparing Linear and Quadratic Functions

LEARNING GOALS

In this lesson, you will:

- Identify linear and quadratic functions from multiple representations.
- Compare graphs, tables, and equations for linear and quadratic functions.
- Analyze graphs of linear and quadratic functions.
- Determine if a function is linear or quadratic by analyzing the first and second differences.

KEY TERMS

- leading coefficient
- second differences

Where do we get the word "parabola"? Dictionaries tell us that the word comes from Greek, and is related to the word "parable."

In Greek, "para" means "beside" or "alongside." And "bole" means "throwing." The actual meaning of the word *parabola* comes from slicing a cone exactly parallel to its side. A parabolic shape is often used in satellite receivers to focus incoming signals to a central point. This type of shape is also used to send a beam of energy outward from a central point—like in a car's headlights.

Where else have you seen parabolas?

Problem 1

A scenario is used to explore a linear and quadratic function. Students will complete a table listing different widths, lengths, and areas using 16 yards of fencing. They then create a linear graph that shows the length as a function of the width. Next, they create a quadratic graph that shows the area as a function of the width. Students then compare the *x*- and *y*-intercepts of each type of graph.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 10 with a partner. Then share the responses as a class.

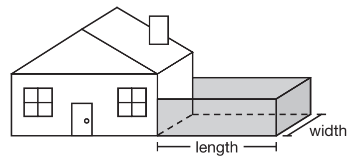
Guiding Questions for Share Phase, Questions 1 through 3

12

- If each length is 8 yards, is there enough fencing to build the width? Why not?
- If there is not enough fencing to build the width, is there a rectangular dog run?
- If the width of the dog run is 2 yards, how much fencing is left to build each length?
- What do you need to know to compute the area of the rectangular dog run?
- How do you determine the area of the rectangular dog run?

PROBLEM 1 Deciding on the Dimensions

Two dog owners have 16 yards of fencing to build a dog run beside their house. The dog owners want the run to be in the shape of a rectangle, and they want to use the side of their house as one side of the dog run. A rough sketch of what they have in mind is shown.



1. Complete the table to show different widths, lengths, and areas that can occur with sixteen yards of fencing.

Width	Length	Area
yards	yards	square yards
0	8	0
2	7	14
4	6	24
6	5	30
8	4	32
10	3	30
12	2	24
14	1	14
16	0	0

2. Describe what happens to the length as the width of the dog run increases. Why do you think this happens?
   The length decreases as the width increases because the total amount of fencing available is constant.

3. Describe what happens to the area as the width of the dog run increases.
   The area increases and then decreases as the width increases.

- If each length of the dog run is 6 yards, how much fencing is left to build the width?
- If the width of the dog run is 16 yards, is there enough fencing to build the lengths? Why not?
- If there is not enough fencing to build the lengths, is there a rectangular dog run?

Guiding Questions for Share Phase, Questions 4 through 7

- How did you determine the lower and upper bounds to view a complete graph?
- If the graph displays the length as a function of the width, what label is associated with the *x*-axis? What label is associated with the *y*-axis?
- Is length and width considered linear? Why or why not?
- What unit of measure is associated with the length and width of the dog run?
- How do you know the function is a linear function?
- Is the slope of the function positive or negative? How do you know?
- What is the slope of the function? What is the *y*-intercept?
- Are negative values for *x* or *y* relevant to this problem situation? Why or why not?

4. Describe what happens to the length and area as the width of the dog run decreases.

The length increases as the width decreases.
The area increases and then decreases as the width decreases.

5. Describe what happens to the width and area as the length of the dog run increases. Describe what happens to the width and area as the length of the dog run decreases.

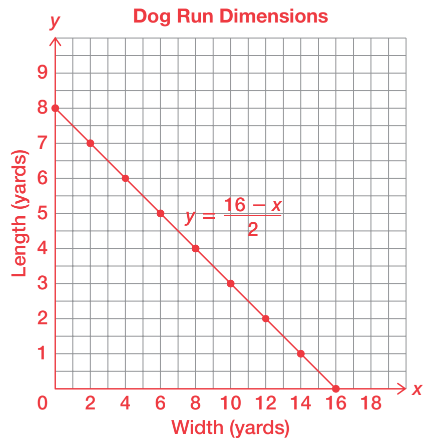
The width decreases as the length increases. The area increases and then decreases as the length increases.
The width increases as the length decreases. The area increases and then decreases as the length decreases.

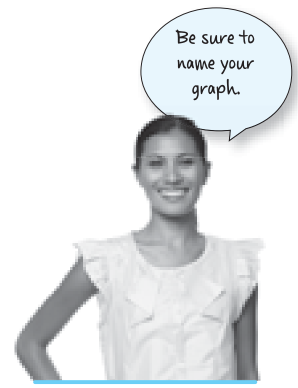
6. Compare how the area changes as the width changes to how the area changes as the length changes.

The changes are the same in that the area increases and then decreases when either measurement increases or decreases. Given a particular length and width, an increase of the width by two yards has the same effect on the area as an increase in the corresponding length by one yard.

7. Let $L(w)$ represent the length of the dog run as a function of the width. Create a graph to show this relationship. First, choose your bounds and intervals. Be sure to label your graph clearly.

Variable Quantity	Lower Bound	Upper Bound	Interval
Width	0	20	2
Length	0	10	1





Guiding Questions for Share Phase, Questions 8 and 9

- Is area a linear function? Why or why not?
- What unit of measure is associated with the area of the dog run?
- How do you know the area function is quadratic?
- Does the graph of the area in terms of the width have an absolute minimum or an absolute maximum?
- What is the absolute maximum of the quadratic function? What does it mean with respect to the problem situation?

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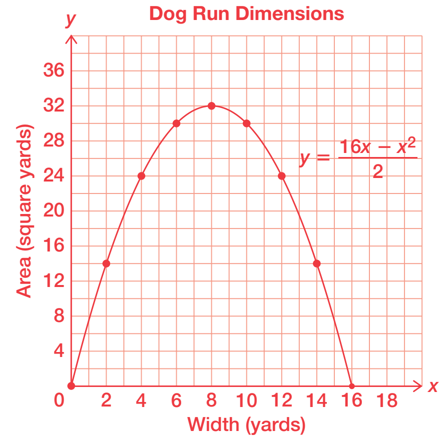
8. Let $A(w)$ represent the area of the dog run as a function of the width. Create a graph to show this relationship. First, choose your bounds and intervals. Be sure to label your axes and name your graph.

Variable Quantity	Lower Bound	Upper Bound	Interval
Width	0	20	2
Area	0	40	4



9. Let's compare and contrast the graphs of the two functions.

   $L(w)$: The length of the dog run as a function of the width.

   $A(w)$: The area of the dog run as a function of the width.

   a. Describe the type of function represented by each graph. Explain your reasoning.

   The graph of $L(w)$ is a linear function because the graph is a straight line.

   The graph of $A(w)$ is a quadratic function because the graph is a parabola.

   b. State the domain in terms of each function and the problem situation.

   The domain of $L(w)$ is the set of all real numbers. In terms of the problem situation, the domain is $0 < w < 16$.

   The domain of $A(w)$ is the set of all real numbers. In terms of the problem situation, the domain is $0 < w < 16$.

Guiding Questions for Share Phase, Questions 9 and 10

- Is it possible for a linear function to have no *y*-intercepts? Why not?
- Is it possible for a linear function to have an infinite number of *y*-intercepts? How?
- Is it possible for a quadratic function to have no *y*-intercepts? How?
- Is it possible for a quadratic function to have an infinite number of *y*-intercepts? Why not?
- Is it possible for a quadratic function to have exactly one *y*-intercept? How?

c. Determine the *y*-intercepts of each graph and interpret the meaning of each in terms of the problem situation.

The *y*-intercept of the graph of the linear function is 8. It indicates the length when the width is 0, which is not appropriate for the dog run.

The *y*-intercept of the graph of the quadratic function is 0. The *y*-intercept indicates the area when the width is 0, which is not appropriate for the dog run.

d. Describe the rates of change for each graph.

The linear function has a constant negative rate of change and the quadratic function has a varying rate of change.

10. Determine the dimensions that provide the greatest area. Use the graphical representations to explain your reasoning.

The greatest possible area is 32 square yards. The width would be 8 yards, and the length would be 4 yards.

The coordinates of the absolute maximum of the quadratic function provide the width and the greatest area. I then used the linear graph to determine the corresponding length.

12

Problem 2

Two tables of values are given. Students will determine which table describes a linear function and which table describes a quadratic function by computing the first differences. The equations and graphs of both functions are given and students identify the *y*-intercepts of each graph and describe the rate of change for each graph. Students then explore the sign of a leading term in a function and what affect the sign of the lead term has on the graph of the function. They also calculate the second differences.

Grouping

- Ask a student to read the information and complete Question 1 as a class.
- Have students complete Questions 2 through 7 with a partner. Then share the responses as a class.

12

Guiding Questions for Discuss Phase, Questions 1 and 2

- Is there anything in either table of values that would suggest one the tables describe a quadratic equation?
- If one of the tables describes a quadratic function, why do all of the *y*-values appear to be associated to different *x*-values?
- To compute the first differences, the *x*-values need to be consecutive, are they consecutive in each table of values?

PROBLEM 2 **U and I**

Tables A and B represent two different functions. One is a linear function, and one is a quadratic function.

Table A

x	*A(x)*
−2	7.5
−1	7.25
0	7
1	6.75
2	6.5

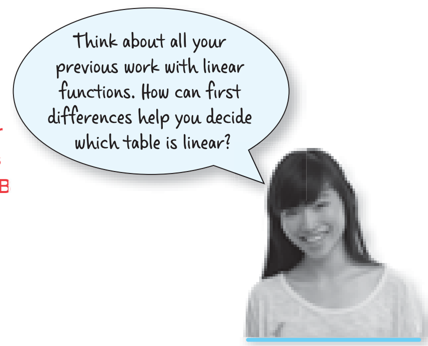
Table B

x	*B(x)*
−2	−15
−1	−7.25
0	0
1	6.75
2	13

1. Which table do you think represents each type of function? Explain your reasoning.

   Answers will vary.

   I think that Table A represents a linear function because it always decreases by the same amount. I think that Table B represents a quadratic function because it decreases by less and less.



2. Calculate the first differences for each function. What patterns do you notice?

   The first differences in Table A are all the same. For Table B, the first differences are changing.

- What is the difference between 7 and 6.75?
- What is the difference between 6.75 and 6.5?
- What is the difference between −15 and −7.25?
- What is the difference between −7.25 and 0?
- Do these first differences support your prediction in Question 1? Why or why not?

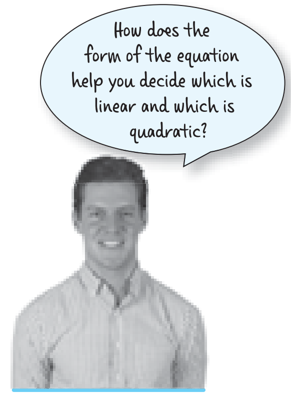
Guiding Questions for Share Phase, Questions 3 through 5

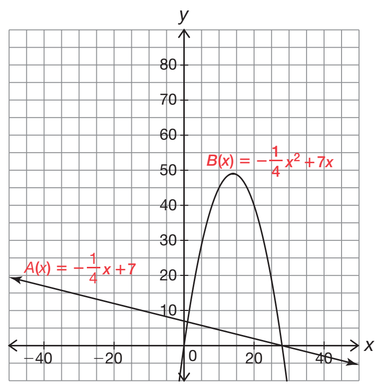
- Looking at the graph of the quadratic function, where on the graph are the values in the table? Does this explain why each y-value in the table was associated with only one x-value?
- Is the linear function an increasing function or a decreasing function? Why?
- Can the quadratic function be described as an increasing function or a decreasing function? Why or why not?
- What do the first differences tell you about the function?
- How do you determine the y-intercept of the linear function algebraically?
- How do you determine the y-intercepts of the quadratic function using a graphing calculator?

The graphs of the two functions are shown. The two equations that represent the linear and quadratic graphs are:

$$y = -\frac{1}{4}x + 7$$

$$y = -\frac{1}{4}x^2 + 7x$$





3. Identify the graph that represents Table A and the graph that represents Table B. Then rewrite each equation as the function $A(x)$ or $B(x)$ and label the graph appropriately. Was your prediction in Question 1 correct?

   See graph.

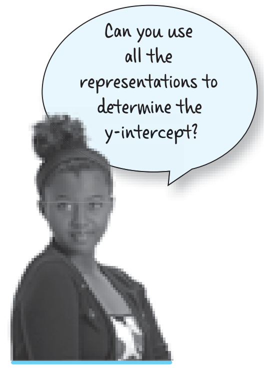
4. Describe the rate of change for each graph. Explain your reasoning.

   The linear function has a constant negative rate of change.
   The quadratic function has a changing rate of change.

5. Determine the y-intercept of each function. Explain how you know.

   The y-intercept of the quadratic function is at (0, 0), and the y-intercept of the linear function is at (0, 7).

   For each function, the y-intercept is where the graph crosses the y-axis. I determined the y-intercept from the tables. It is the y-coordinate of the point where the x-value is 0.



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Guiding Questions Share Phase, Questions 6 and 7

- What is the leading coefficient in the linear function?
- What is the leading coefficient in the quadratic function?
- What does a negative leading coefficient in a linear function tell you about the graph of the function?
- What does a negative leading coefficient in a quadratic function tell you about the graph of the function?
- Why calculate the second difference? What additional information does it give you about the function?

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The **leading coefficient** of a function is the numerical coefficient of the term with the greatest power. Recall that a power has two elements: the base and the exponent.

6. Identify the leading coefficient of each function. Then, describe how the sign of the leading coefficient affects the behavior of each graph.

The leading coefficient of the linear function is $-\frac{1}{4}$. Because this coefficient is negative, the linear function decreases.

The leading coefficient of the quadratic function is $-\frac{1}{4}$. Because this leading coefficient is negative, the quadratic function opens downward.

Let's explore the table of values one step further and analyze the *second differences*. **Second differences** are the differences between consecutive values of the first differences.

7. Calculate the second differences for each function. What do you notice?

The second differences for the linear function are all 0. The second differences for the quadratic function are all the same number. They are constant.

Problem 3

Students are given 4 tables of values consisting of 2 linear functions and 2 quadratic functions. They will identify which tables are associated with linear functions and which tables are associated with quadratic functions. Next, they compute and compare the first and second differences in each table of values. They then conclude that linear functions have a constant first difference and a second difference of 0, whereas quadratic functions have a changing first difference and a constant second difference. Finally, they sketch each function and associate the graphic behaviors of the functions to the signs of the first and second differences.

Grouping

Have students complete Questions 1 through 4 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 1

- How did you compute the first difference?
- How did you compute the second difference?
- What do the first differences in parts (a) and (c) have in common?
- What do the second differences in parts (a) and (c) have in common?

PROBLEM 3 Second Differences

1. Analyze the form of each equation and determine if it is linear or quadratic. Then complete each table to calculate the first and second differences.

a. $y = 2x$ linear

x	y	First Differences	Second Differences
−3	−6		
		2	
−2	−4		0
		2	
−1	−2		0
		2	
0	0		0
		2	
1	2		0
		2	
2	4		0
		2	
3	6		

b. $y = 2x^2$ quadratic

x	y	First Differences	Second Differences
−3	18		
		−10	
−2	8		4
		−6	
−1	2		4
		−2	
0	0		4
		2	
1	2		4
		6	
2	8		4
		10	
3	18		

c. $y = -x + 4$ linear

x	y	First Differences	Second Differences
−3	7		
		−1	
−2	6		0
		−1	
−1	5		0
		−1	
0	4		0
		−1	
1	3		0
		−1	
2	2		0
		−1	
3	1		

d. $y = -x^2 + 4$ quadratic

x	y	First Differences	Second Differences
−3	−5		
		5	
−2	0		−2
		3	
−1	3		−2
		1	
0	4		−2
		−1	
1	3		−2
		−3	
2	0		−2
		−5	
3	−5		

- What is the leading term in parts (a) and (c)? What is the degree of this function?
- What do the first differences in parts (b) and (d) have in common?
- What do the second differences in parts (b) and (d) have in common?
- What is the leading term in parts (b) and (d)? What is the degree of this function?

12

Guiding Questions for Share Phase, Questions 2 and 3

- Did you use the points in the tables of values to graph each function? If not, how did you graph each function?
- Do the graphs in parts (a) and (c) support your previous conclusions about the functions?
- Do the graphs in parts (b) and (d) support your previous conclusions about the functions?

2. What do you notice about the first and second differences of the:

a. linear functions.

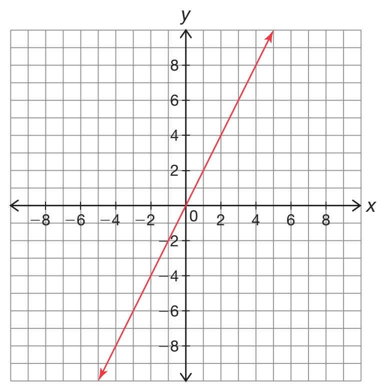
Linear functions have constant first differences and second differences of 0.

b. quadratic functions.

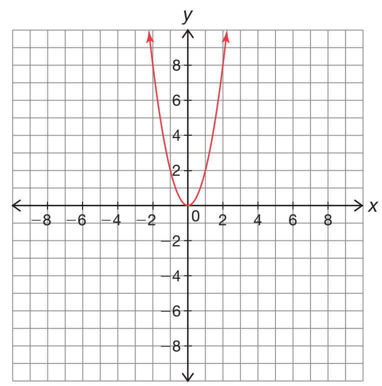
Quadratic functions have changing first differences and constant second differences.

3. Sketch the graphs represented by the equations in Question 1.

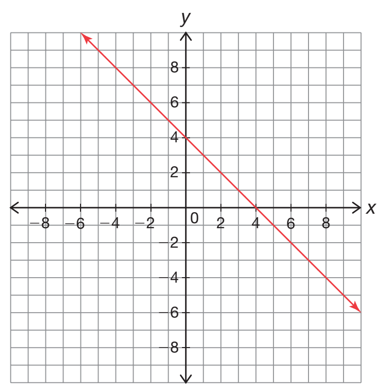
a. $y = 2x$



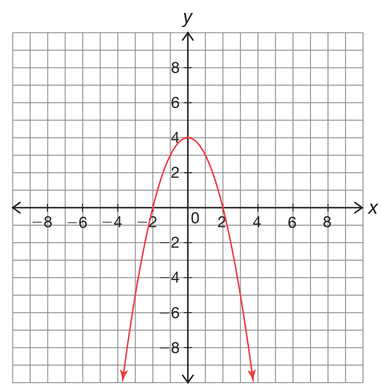
b. $y = 2x^2$



c. $y = -x + 4$



d. $y = -x^2 + 4$



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Guiding Questions for Share Phase, Question 4

- Do the signs of the leading terms in each function support the graphic behavior of the function? Why or why not?
- Could a quadratic function be considered both increasing and decreasing? Why?

4. Compare the signs of the first and second differences for each function and its graph.

a. How do the signs of the first differences for a linear function relate to the graph either increasing or decreasing?

If the signs of the first differences are positive, the linear function is increasing. If the signs of the first differences are negative, the linear function is decreasing.

b. How do the signs of the first differences and the signs of the second differences for quadratic functions relate to the graph of the quadratic either increasing or decreasing or opening upward or downward?

When the first differences are negative, the quadratic is decreasing. When the first differences are positive, the quadratic is increasing. If the second differences are positive, the parabola opens upward. If the second differences are negative, then the parabola opens downward.

Talk the Talk

Students will compare the quadratic functions in the previous problems. Using a graphing calculator, they then determine the *x*- and *y*-intercepts and their relevance to the problem situation. Students conclude that when the coefficient of the x^2 term is a positive number, the parabola opens upward, and when the coefficient of the x^2 term is a negative number, the parabola opens downward.

Grouping

Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

Talk the Talk

1. Describe how to determine when an equation represents a:

a. linear function.

A linear function is always in the form $f(x) = mx + b$.

b. quadratic function.

A quadratic function is in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$.

2. Describe how to determine when a table of values represents a:

a. linear function.

Linear functions have constant first differences and second differences of 0.

b. quadratic function.

Quadratic functions have either increasing or decreasing first differences and constant second differences.

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Guiding Questions for Share Phase, Questions 1 and 2

- What is the standard form a linear equation? Can a linear equation be written a different way? How?
- What is the standard form of a quadratic equation? Can a quadratic equation be written a different way? How?
- Why would anyone want to compute a first difference? What is its importance?
- Why would anyone want to compute a second difference? What is its importance?

3. Describe how the first and second differences describe the rate of change of a:

a. linear function.

The first differences of a linear function show that the function is increasing or decreasing at a constant rate. The second differences of 0 mean that the rate of change doesn't change over the domain.

b. quadratic function.

The first differences of a quadratic function show that the rate of change changes across the domain.
The second differences show that the rate of change changes at a constant rate over the domain.

4. Describe how to determine when an equation represents a:

a. linear function that increases.

The rate of change is positive.

b. linear function that decreases.

The rate of change is negative.

c. quadratic function that opens upward.

The *a* value is positive.

d. quadratic function that opens downward.

The *a* value is negative.

5. Describe how to determine the *y*-intercept given any function.

I can determine the *y*-intercept of any function by substituting $x = 0$ into the function. When the function is written using function notation, the *y*-intercept is the constant.

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Be prepared to share your solutions and methods.

Check for Students' Understanding

The path of a diver in the water can be modeled by the function $y = 0.16x^2 - 3.2x$, where x is the horizontal distance in feet that the diver travels and y is the depth in feet of the diver.

1. Complete the table of values to represent the depth as a function of the horizontal distance.

Horizontal Distance feet	Depth feet
x	$0.16x^2 - 3.2x$
0	0
1	−3.04
5	−12
10	−16
15	−12

2. Does this problem have an absolute maximum or an absolute minimum? Is it relevant to this problem situation?

This problem has an absolute minimum. It has a negative *y*-value which represents the depth so it is relevant to the problem situation.

3. Does the graph of the function open upward or downward? Explain your reasoning.

This graph of the function opens upward because the lead term has a positive coefficient.

4. Can the table of values be used to compute the first and second differences between successive output values? Why or why not?

The table of values should not be used to compute the first or second differences between the successive output values because the successive input values do not have a difference of 1.

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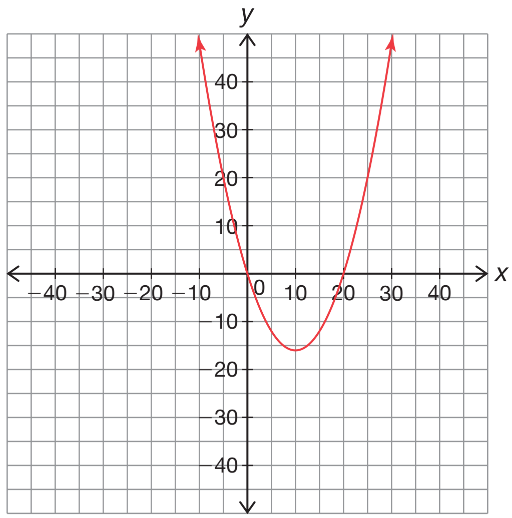
5. Sketch a graph to model the path of the diver.



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12.3

Walking the . . . Curve?

Domain, Range, Zeros, and Intercepts

LEARNING GOALS

In this lesson, you will:

- Describe the domain and range of quadratic functions.
- Determine the x-intercept(s) of a graph of a quadratic function.
- Understand the relationship of the zeros of a quadratic function and the x-intercepts of its graph.
- Analyze quadratic functions to determine intervals of increase and decrease.
- Solve a quadratic function graphically.

KEY TERMS

- vertical motion model
- zeros
- interval
- open interval
- closed interval
- half-closed interval
- half-open interval

ESSENTIAL IDEAS

- A vertical motion model is a quadratic equation that models the height of an object at a given time.
- The vertical motion equation is $g(t) = -16t^2 + v_0t + h_0$, where $g(t)$ represents the height of the object in feet, t represent s the time in seconds that the object has been moving, v_0 represents the initial velocity of the object in feet per second, and the h_0 represents the initial height of the object in feet.
- The zeros of a quadratic function are the x-intercepts.
- The roots of a quadratic function are the solutions to the quadratic equation.
- An interval is the set of real numbers between two given numbers.
- Interval notation is used to describe the interval of the domain in which the function is increasing and the interval of the domain in which the function is decreasing.
- An open interval (a, b) describes the set of all numbers between a and b, but not including a or b.
- A closed interval $[a, b]$ describes the set of all numbers between a and b, including a and b.
- A half-closed or half-open interval $(a, b]$ describes the set of all numbers between a and b, including b but not including a. Or $[a, b)$ describes the set of all numbers between a and b, including a but not including b.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context.

A-CED Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F-IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

Overview

Students are given an equation for vertical motion. The scenario uses this quadratic function to describe a falling object. Students will write the function that describes the problem situation and use a graphing calculator to graph the function and answer related questions. The terms zeros, roots, are defined and students describe the domain and range of the quadratic function representing the problem situation. Next, intervals, and interval notation are introduced. Students use interval notation to describe the intervals at which the domain of the function increases and decreases. In the last activity, students are given various graphs of quadratic functions, and then identify the domain, range, zeros, and intervals of increase and decrease.

Warm Up

Determine whether each quadratic function contains an absolute minimum or an absolute maximum.

1. $f(x) = -\frac{1}{2}x^2 + 3x - 1$

   This quadratic function has an absolute maximum.

2. $f(x) = x^2 - 3x + 1$

   This quadratic function has an absolute minimum.

3. $f(x) = -5x(2 - x)$

   This quadratic function has an absolute maximum.

4. $f(x) = 2x(1 - x)$

   This quadratic function has an absolute minimum.

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12

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12.3

Walking the . . . Curve?

Domain, Range, Zeros, and Intercepts

LEARNING GOALS

In this lesson, you will:

- Describe the domain and range of quadratic functions.
- Determine the x-intercept(s) of a graph of a quadratic function.
- Understand the relationship of the zeros of a quadratic function and the x-intercepts of its graph.
- Analyze quadratic functions to determine intervals of increase and decrease.
- Solve a quadratic function graphically.

KEY TERMS

- vertical motion model
- zeros
- interval
- open interval
- closed interval
- half-closed interval
- half-open interval

The first liquid-fueled rocket in history was created by Robert Goddard, who was a university professor and inventor, claiming more than 200 patents.

Goddard led a team that launched more than two dozen rockets in a span of 15 years. He was ridiculed in the press for his research, and his efforts received little support.

Yet, his invention paved the way for spaceflight, over 40 years later!

Problem 1

A scenario is used to explore a vertical motion quadratic function. Students are given the vertical motion equation and use it to write a function that represents the scenario. They will use a graphing calculator to graph the function and answer questions related to the scenario. The definitions of zeros and roots of a quadratic function are given. Students then describe the domain and range of the function representing the scenario. Intervals and their symbolic notations are introduced. An open interval, a closed interval, and a half-closed or half-open interval are described. Students will use interval notation to describe the domain of the problem situation.

Grouping

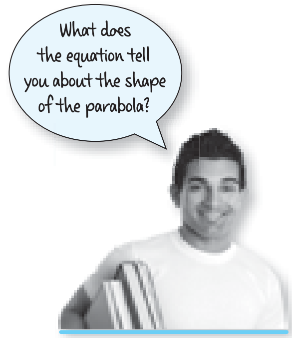
- Ask a student to read the information and complete Question 1 as a class.
- Have students complete Questions 2 and 3 with a partner. Then share the responses as a class.

PROBLEM 1 Model Rocket

Suppose you launch a model rocket from the ground. You can model the motion of the rocket using a *vertical motion model*. A **vertical motion model** is a quadratic equation that models the height of an object at a given time. The equation is of the form

$$g(t) = -16t^2 + v_0t + h_0,$$

where $g(t)$ represents the height of the object in feet, t represents the time in seconds that the object has been moving, v_0 represents the initial vertical velocity (speed) of the object in feet per second, and h_0 represents the initial height of the object in feet.



1. Why do you think it makes sense that this situation is modeled by a quadratic function?

   When a rocket is launched into the air, at first its height increases over time. Then, its height will decrease over time until it reaches the ground. That is, it goes up and then comes back down.
   So, it makes sense that this situation is modeled by a parabola.

Suppose the model rocket has an initial velocity of 160 feet per second.

2. Write a function, $g(t)$, to describe the height of the model rocket in terms of time t.

   $v_0 = 160$ feet per second; $h_0 = 0$ seconds

   $g(t) = -16t^2 + 160t$

3. Describe the independent and dependent quantities.

   The independent quantity is time in seconds, and the dependent quantity is height in feet.

Guiding Questions for Discuss Phase, Question 1

- Does a parabola resemble the path of a rocket? Why or why not?
- If you launched a model rocket, and graphed its height in terms of time, would the actual path of the rocket be parabolic in shape or would it look more linear?
- What is the difference between the actual path of the rocket and the graph representing the vertical motion of the rocket?

Guiding Questions for Share Phase, Questions 2 and 3

- What does velocity mean in this situation?
- What did you substitute into the vertical motion equation for v_0?
- Does the height of the rocket depend on the time, or does the time depend on the height of the rocket?

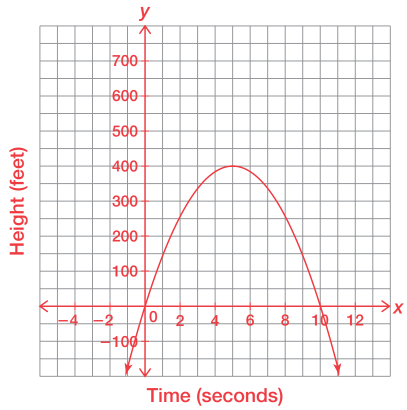
Grouping

Have students complete Questions 4 through 6 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 4 through 6

- What equation did you enter into the graphing calculator?
- If the graph of the function is a complete graph, what aspects of the graph should be viewable?
- What window allowed you to view a complete graph of the quadratic function?
- What in the equation helped you determine a reasonable viewing window?
- How did you determine the height of the rocket in 6 seconds?
- How did you determine when the height of the rocket was 200 feet?
- Where on the graph is the maximum height of the rocket?
- How did you determine the absolute maximum?

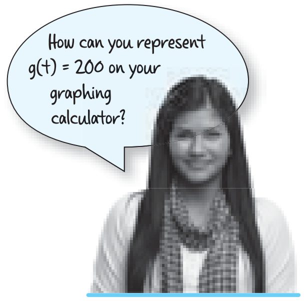
4. Use a graphing calculator to graph the function. Sketch the graph and label the axes.



5. Use a graphing calculator to answer each question.

a. What is the height of the model rocket at 6 seconds?

At 6 seconds, the model rocket is at a height of 384 feet.



b. After approximately how many seconds is the model rocket at a height of 200 feet?

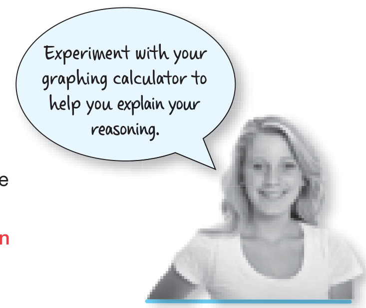
The model rocket is at a height of 200 feet after approximately 1.5 seconds and again after approximately 8.5 seconds.

c. What is the maximum height of the model rocket? When is the rocket at its maximum height?

The absolute maximum is at (5, 400).

This means that after 5 seconds, the rocket is at its maximum height of 400 feet.

6. You can use a graphing calculator and intersection points to determine the *x*-intercepts of a quadratic function.



a. What linear function can you graph along with the quadratic function to determine the *x*-intercepts? Explain your reasoning.

The *x*-intercepts of the quadratic function will be located where the graph crosses the *y*-axis. I can graph the line $y = 0$ and determine where this line and the graph of the quadratic function intersect.

12

Grouping

Ask a student to read the definitions. Discuss as a class. Complete Questions 7 through 9 as a class.

Guiding Questions for Discuss Phase, Questions 7 through 9

- Are the roots of a quadratic function and the zeros of a quadratic function the same thing? Why not?
- Can you use the word 'root' and 'zero' of a quadratic function interchangeably? Why not?
- Which quadrants of the coordinate plane are used to graph this quadratic function?
- Which, if any *y*-values are not associated with this quadratic function?
- Which, if any *x*-values are not associated with this quadratic function?
- How is the domain of this quadratic function different than the domain of the problem situation?
- How is the range of this quadratic function different than the range of the problem situation?
- Do negative values in the domain have meaning in this problem situation?
- Do negative values in the range have meaning in this problem situation?

12

b. Determine the *x*-intercepts of $g(t)$. Then, interpret the meaning in terms of this problem situation.

The *x*-intercepts are at (0, 0) and (10, 0).

The *x*-intercepts indicate the time when the rocket is launched and the time it lands.

The *x*-intercepts of a graph of a quadratic function are also called the **zeros** of the quadratic function.

You can use a graphing calculator to determine the zeros of a quadratic function.

Step 1: Press **2ND** and then **CALC**. Select **2: zero**.

Step 2: Determine the left and right bounds for each point that appears to be a zero. Then press **ENTER**.

7. Identify and describe the domain of the function in terms of the:

a. mathematical function you graphed.

The domain is all real numbers.

b. contextual situation.

In terms of the contextual situation, the domain is all real numbers greater than or equal to 0 and less than or equal to 10. The model rocket is launched at time 0, so anything before that doesn't make sense. It lands after 10 seconds, so times after that don't make sense.

8. Identify and describe the range of the function in terms of the:

a. mathematical function you graphed.

The range is all real numbers less than or equal to 400.

b. contextual situation.

The range in terms of the contextual situation is all real numbers less than or equal to 400 and greater than or equal to 0. Negative *y*-values don't make sense because the rocket cannot have negative height.

9. How is the range of a quadratic function related to its absolute maximum or minimum?

The absolute maximum or minimum of a quadratic function limits its range. There are no *y*-values of the function above the maximum or below the minimum.

Grouping

- Ask a student to read the definitions. Discuss as a class.
- Have students complete Questions 10 and 11 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 10 through 12

- How did you determine which interval notation to use when describing the interval of increase?
- How did you determine which interval notation to use when describing the interval of decrease?
- Does the absolute minimum or absolute maximum limit the x-values or limit the y-values of a quadratic function?
- Does the absolute minimum or absolute maximum limit the domain or limit the range of a quadratic function?

An **interval** is defined as the set of real numbers between two given numbers. To describe an interval, use this notation:

- An **open interval** (a, b) describes the set of all numbers between a and b, but <u>not</u> including a or b.
- A **closed interval** $[a, b]$ describes the set of all numbers between a and b, including a and b.
- A **half-closed** or **half-open interval** $(a, b]$ describes the set of all numbers between a and b, including b but not including a. Or, $[a, b)$ describes the set of all numbers between a and b, including a but not including b.

Intervals that are unbounded are written using the symbol for infinity, ∞.

The interval $[a, \infty)$ means all numbers greater than or equal to a.
The interval (a, ∞) means all numbers greater than a.

10. Use interval notation to describe the interval in which:

a. all numbers are less than a.
$(-\infty, a)$

b. all numbers are less than or equal to a.
$(-\infty, a]$

c. a is any real number.
$(-\infty, \infty)$

11. Use interval notation to describe the interval of the domain in which the model rocket function is:

a. increasing.
The function is increasing from negative infinity to 5.
$(-\infty, 5)$

b. decreasing.
The function is decreasing from 5 to positive infinity.
$(5, \infty)$

12. How does the absolute maximum or absolute minimum help you determine each interval?
The absolute maximum or absolute minimum is the turning point of a parabola. If the quadratic function has an absolute maximum, the x-value increases to the maximum and then decreases. If it has an absolute minimum, the x-value decreases to the minimum and then increases.

12

Problem 2

The graphs of 4 quadratic functions are given. Students will identify the domain, range, zeros, and the intervals of increase and decrease for each function.

Grouping

Have students complete Questions 1 through 4 with a partner. Then share the responses as a class.

Guiding Questions for Discuss Phase, Questions 1 through 4

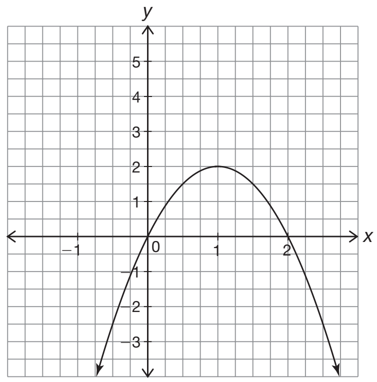
- What is the coefficient of the leading term in this quadratic equation?
- How did the sign of the leading term affect the graph of the quadratic equation?
- Is the domain or range of this quadratic function limited? How so?
- Do you think all quadratic functions have two zeros? Why or why not?
- What would a quadratic function that appears to have no real zeros look like?
- What would a quadratic function that appears to have one real zero look like?
- Is ∞ always used when describing the domain of a quadratic function? Why or why not?
- Is −∞ always used when describing the domain of a quadratic function? Why or why not?

12

PROBLEM 2 Intervals of Increase and Decrease

For each function shown, identify the domain, range, zeros, and the intervals of increase and decrease.

1. The graph shown represents the function $f(x) = -2x^2 + 4x$.



Domain: All real numbers

Range: All real numbers less than or equal to 2

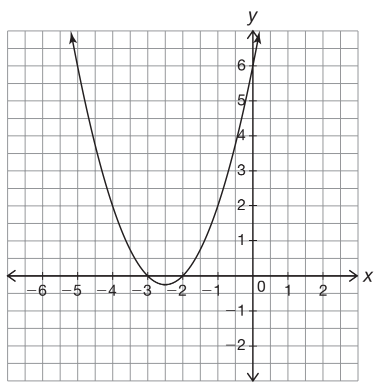
y-intercept: (0, 0)

Zeros: (0, 0), (2, 0)

Interval of increase: $(-\infty, 1)$

Interval of decrease: $(1, \infty)$

2. The graph shown represents the function $f(x) = x^2 + 5x + 6$.



Domain: All real numbers

Range: All real numbers greater than or equal to $-\frac{1}{4}$

y-intercept: (0, 6)

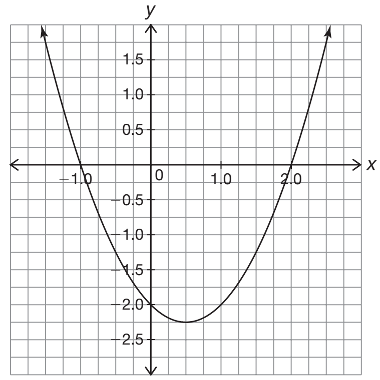
Zeros: (−2, 0), (−3, 0)

Interval of decrease: $(-\infty, -2.5)$

Interval of increase: $(-2.5, \infty)$

- Is −∞ always used to describe the range of a quadratic function? Why or why not?
- Is ∞ always used to describe the range of a quadratic function? Why or why not?

3. The graph shown represents the function $f(x) = x^2 - x - 2$.



Domain: All real numbers

Range: All real numbers greater than or equal to −2.25

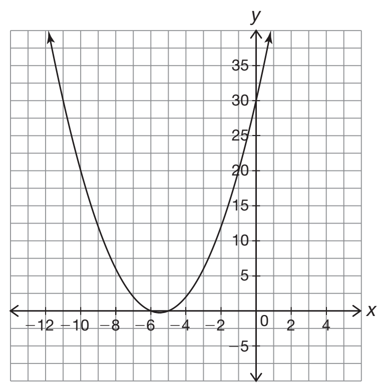
y-intercept: (0, −2)

Zeros: (−1, 0), (2, 0)

Interval of decrease: $(-\infty, 0.5)$

Interval of increase: $(0.5, \infty)$

4. The graph shown represents the function $f(x) = x^2 + 11x + 30$.



Domain: All real numbers

Range: All real numbers greater than or equal to $-\frac{1}{4}$

y-intercept: (0, 30)

Zeros: (−6, 0), (−5, 0)

Interval of decrease: $\left(-\infty, -\frac{11}{2}\right)$

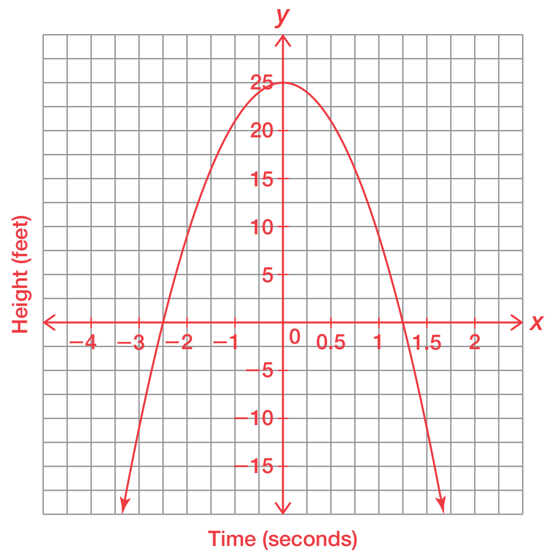
Interval of increase: $\left(\frac{11}{2}, \infty\right)$

Be prepared to share your solutions and methods.

Check for Students' Understanding

A coconut falls from a palm tree branch that is 25 feet above the ground. It can be modeled by the function $f(t) = -16t^2 + 25$ where $f(t)$ is the height of the coconut in feet and t is the time in seconds.

1. Graph the quadratic function.



2. After how many seconds does the coconut hit the ground?

The coconut hits the ground in 1.25 seconds

3. What are the zeros of the quadratic function?

The zeros or the x-intercepts are (−1.25, 0) and (1.25, 0).

4. Use interval notation to describe the domain and range of the function.

Domain: $(-\infty, \infty)$

Range: $(-\infty, 25]$

5. Use interval notation to describe the domain and range of this problem situation.

Domain: $[0, 1.25]$

Range: $[25, 0]$

12

Are You Afraid of Ghosts?

Factored Form of a Quadratic Function

12.4

LEARNING GOALS

In this lesson, you will:

- Factor the greatest common factor from an expression.
- Write a quadratic function in factored form.
- Determine the *x*-intercepts from a quadratic function written in factored form.
- Determine an equation of a quadratic function given its *x*-intercepts.

KEY TERMS

- factor an expression
- factored form

ESSENTIAL IDEAS

- To factor an expression is to use the Distributive Property in reverse to rewrite the expression as a product of factors.
- Factoring algebraic expressions requires factoring out the greatest common factor from all of the terms.
- When factoring an expression that contains a negative leading coefficient, it is convention to factor out the negative sign.
- A quadratic function written in factored form is in the form $f(x) = a(x - b)(x - c)$, where $a \neq 0$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context.

a. Interpret parts of an expression, such as terms, factors, and coefficients.

Write expressions in equivalent forms to solve problems.

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

a. Factor a quadratic expression to reveal the zeros of the function it defines.

A-CED Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F-IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

   a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

Overview

A scenario represented by a quadratic function is used to guide students through the process of writing a quadratic equation in factored form. An example provides a model for using the Distributive Property in reverse and students apply it to the problem situation. Using a graphing calculator, students will graph the quadratic function and determine key characteristics of the function. Next, they explore the importance of the intercepts and the absolute maximum with respect to the problem situation. Students then practice writing quadratic functions in factored form given key characteristics. They are also given quadratic functions written in factored form and will describe the key characteristics.

Warm Up

Use the Distributive Property to simplify each expression.

1. $f(x) = -x(x + 10)$

   $f(x) = -x^2 - 10x$

2. $f(x) = -8x(11 - x)$

   $f(x) = 8x^2 - 88x$

3. $f(x) = -6(x - 4)$

   $f(x) = -6x + 24$

4. $f(x) = 7(x + 1)$

   $f(x) = 7x + 7$

12

12

12.4

Are You Afraid of Ghosts?

Factored Form of a Quadratic Function

LEARNING GOALS

In this lesson, you will:

- Factor the greatest common factor from an expression.
- Write a quadratic function in factored form.
- Determine the x-intercepts from a quadratic function written in factored form.
- Determine an equation of a quadratic function given its x-intercepts.

KEY TERMS

- factor an expression
- factored form

Touring allegedly "haunted" houses in the United States is a big business. Go to Tombstone, Arizona, where you're told that you can hear ghosts laughing and yelling. Or go to Hotel Jerome in Aspen, Colorado. This hotel is said to be haunted by a young boy. How about Bannack, Montana—a ghost town where you might meet a ghost named Dorothy and hear babies crying?

12

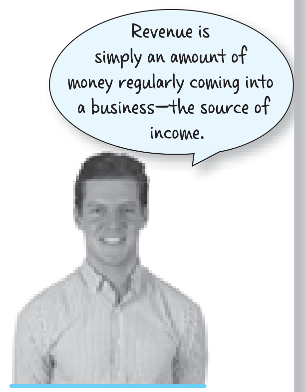
Problem 1

A scenario is used to explore a quadratic function written in factored form. Students will write expressions to represent aspects of the problem situation arranging the terms such that the variable term comes first. An example of factoring out the greatest common factor is given and students then use the example to model the same behavior with their expressions. The general equation of a quadratic function written in factored form is provided and students combine their expressions to rewrite the equation in this form. They will use a graphing calculator to graph the quadratic function and determine key characteristics such as the x-intercepts, the y-intercept, and the absolute maximum. They then compare the behaviors of the graph to the quadratic equation written in factored form.

PROBLEM 1 Making the Most of the Ghosts

The Jacobson brothers own and operate their own ghost tour business. They take tour groups around town on a bus to visit the most notorious "haunted" spots throughout the city. They charge \$50 per tour. Each summer, they book 100 tours at that price. The brothers are considering a decrease in the price per tour because they think it will help them book more tours. They estimate that they will gain 10 tours for every \$1 decrease in the price per tour.

Let's consider the revenue for the ghost tour business. In this situation, the revenue is the number of tours multiplied by the price per tour.



1. Let x represent the change in the price per tour. Write an expression to represent the number of tours booked if the decrease in price is x dollars per tour. Write the expression so that the variable term comes first.

   $10x + 100$

2. Write an expression to represent the price per tour if the brothers decrease the price x dollars per tour. Write the expression so that the variable term comes first.

   $50 - x = -x + 50$

To **factor an expression** means to use the Distributive Property in reverse to rewrite the expression as a product of factors.

When factoring algebraic expressions, you can factor out the greatest common factor from all the terms.

> Consider the expression $12x + 42$.
>
> The greatest common factor of $12x$ and 42 is 6. Therefore, you can use the Distributive Property in reverse to rewrite the expression.
>
> $$12x + 42 = 6(2x) + 6(7)$$
> $$= 6(2x + 7)$$
>
> So, the factored expression is $6(2x + 7)$.

3. Rewrite the expression from Question 1 by factoring out the greatest common factor.

   $10x + 100 = 10(x + 10)$

Grouping

Ask a student to read the information and example. Complete Questions 1 through 5 as a class.

Guiding Questions for Discuss Phase, Questions 1 and 2

- If the Jacobson brothers charge \$50 per tour, how many tours will they book?
- If the Jacobson brothers charge \$49 per tour, how many tours do they estimate they will book?
- If the Jacobson brothers charge \$48 per tour, how many tours do they estimate they will book?
- If the Jacobson brothers charge \$47 per tour, how many tours do they estimate they will book?
- If the Jacobson brothers charge \$46 per tour, how many tours do they estimate they will book?
- What is an example of using the Distributive Property?

- What is an example of using the Distributive Property in reverse?
- How do you determine the greatest common factor?
- How can you tell if the expression has been completely factored?

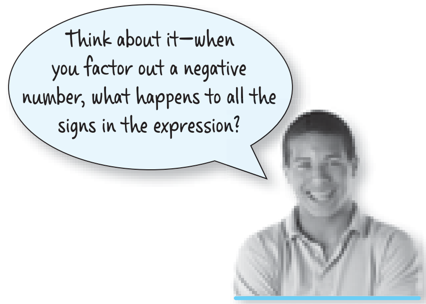
Guiding Questions for Share Phase, Questions 3 and 4

- What is the greatest common factor the expression $10x + 100$?
- What is the prime factorization of 10?
- What is the prime factorization of 100?
- Does the x-term have a negative coefficient?
- How do you factor out a negative coefficient?
- What would happen if the value of a was equal to zero? Would the equation be a quadratic equation? Why not?
- If the value of b or c were equal to zero, would the equation be a quadratic equation? Why?

When factoring an expression that contains a negative leading coefficient, it is a convention to factor out the negative sign. Remember that a coefficient is a number that is multiplied by a variable in an algebraic expression.

4. Rewrite the expression from Question 2 by factoring out -1.

$-x + 50 = -1(x - 50)$

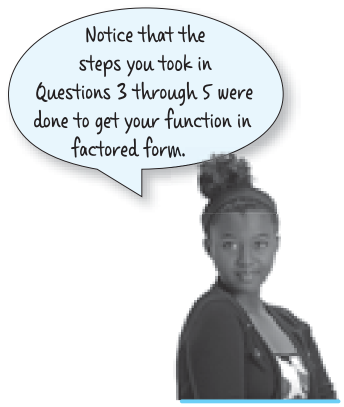


You can write a quadratic function in factored form to represent the Jacobson brothers' revenue, $r(x)$, in terms of the decrease in price per tour, x.

A quadratic function written in **factored form** is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$.

5. Use your expressions from Questions 3 and 4 to first represent the revenue, $r(x)$, as the number of tours times the price per tour. Then write $r(x)$ in factored form.

Revenue	=	**Number of Tours**	•	**Price per Tour**
↓		↓		↓
$r(x)$	=	$10(x + 10)$	•	$-1(x - 50)$
$r(x)$	=	$-10(x + 10)(x - 50)$		



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Grouping

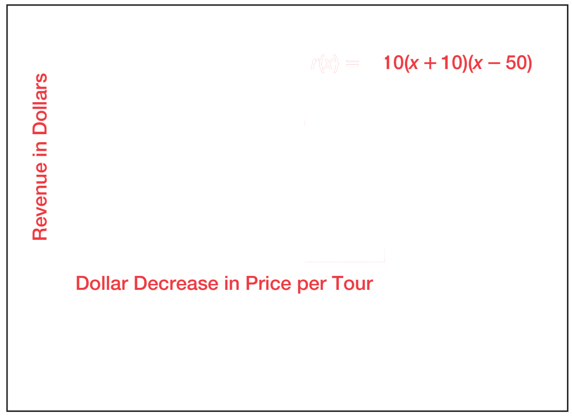
Have students complete Questions 6 through 8 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 5 through 8

- Why do you think you need to use a number as large as 10,000 for an upper bound?
- Do the suggested bounds provide a view of a complete graph? Why or why not?
- Are negative *x* or *y* values relevant in this problem situation? Why not?
- If the price of each tour was reduced by $50, would the Jacobson brothers have any revenue?
- If the Jacobson brothers do not reduce the price of the tour, what will be their revenue?
- Does the absolute maximum in the graph describe the maximum reduction in the price of the tour or the maximum revenue the Jacobson brothers can earn?
- Is it possible to determine the *x*-intercepts from the factored form of a quadratic function? If so, how?
- Is it possible to determine the *y*-intercept from the factored form of a quadratic function? If so, how?
- Is it possible to determine the zeros from the factored form of a quadratic function? If so, how?

12

6. Graph $r(x)$ on a graphing calculator using the bounds $[-10, 50] \times [0, 10{,}000]$. Sketch and label your graph.



7. Use a graphing calculator to determine each key characteristic. Then, interpret the meaning of each in terms of this problem situation.

a. *x*-intercepts

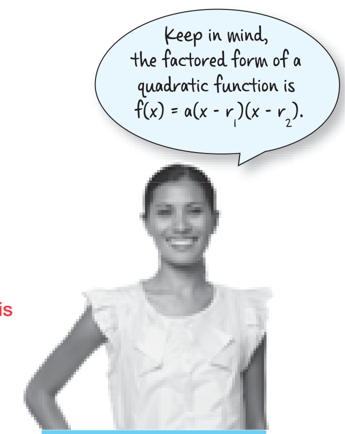
The *x*-intercepts are $(-10, 0)$ and $(50, 0)$.
Each represents the decrease in the price per tour for which the revenue is 0 dollars.

b. *y*-intercept

The *y*-intercept is $(0, 5000)$.
It represents the revenue when there is no decrease in the price per tour.

c. absolute minimum or absolute maximum

The function has a maximum at $(20, 9000)$. This means that the brothers will receive a maximum revenue of $9000 if they decrease the price per tour by $20.



8. Compare your answers in Question 7 to the factored form of the function you wrote in Question 5. Which, if any, key characteristics do you think you can determine directly from the equation of the function when written in factored form?

You can determine the *x*-intercepts from the factored form of the function.
You can also determine whether the parabola opens upward or downward.

Problem 2

The graphs of 4 quadratic functions are given. Students will identify the domain, range, zeros, and the intervals of increase and decrease for each function.

Grouping

Ask a student to read the information and complete Question 1 as a class.

Guiding Questions for Discuss Phase, Question 1

- If the parabola opens downward, what should be the sign of the coefficient of the leading term?
- Is the coefficient of the leading term in each equation positive or negative?
- Should *a* have a value that is greater than zero or a value that is less than zero? Why?

PROBLEM 2 Exploring Factored Form

1. A group of students are working together on the problem shown.

   Write a quadratic function in factored form to represent a parabola that opens downward and has zeros at (4, 0) and (−1, 0).

Maureen
My function is
$k(x) = -(x - 4)(x + 1)$.

Micheal
My function is
$d(x) = \frac{1}{2}(x - 4)(x + 1)$.

Tim
My function is
$m(x) = 2(x - 4)(x + 1)$.

Tom
My function is
$k(x) = -2(x - 4)(x + 1)$.

Dianne
My function is
$t(x) = -0.5(x - 4)(x + 1)$.

Judy
My function is
$f(x) = -(x + 4)(x - 1)$.

a. Use your graphing calculator to graph each student's function. What are the similarities among all the graphs? What are the differences among the graphs.

   All graphs are parabolas. Tom, Dianne, Judy, and Maureen all have parabolas that open downward, while Tim's and Micheal's parabolas open upward. Tom's, Micheal's, and Tim's parabolas are the same "width," while the others are wider, with Dianne's being the widest.

b. How is it possible to have more than one correct function?

   In this case, the given characteristics only described the direction the parabola opened and the specific zeros of the function. The width of the parabola is not given.

12

Grouping

- Have students complete Question 2 on their own. Then share the responses as a class.
- Have students complete Question 3 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 2

- If the parabola opens upward, what is the sign of the coefficient of the leading term in the equation?
- If the parabola opens downward, what is the sign of the coefficient of the leading term in the equation?
- If the *x*-intercept is (2, 0), should the factor be written as $(x - 2)$ or $(x + 2)$? Why?
- If the *x*-intercept is (−3, 0), should the factor be written as $(x - 3)$ or $(x + 3)$? Why?
- Should *a* have a value that is greater than zero or *a* value that is less than zero? Why?
- If the values of either *b* or *c* have a value of zero, how does that affect the general form of the quadratic equation?
- If the values of both *b* and *c* have a value of zero, how does that affect the general form of the quadratic equation?

Guiding Questions for Share Phase, Question 3

- If 4 is a zero of the quadratic function, how is this zero written as an *x*-intercept?
- If 4 is a zero of the quadratic function, how is this zero written as a factor?
- If −3 is a zero of the quadratic function, how is this zero written as an *x*-intercept?
- If −3 is a zero of the quadratic function, how is this zero written as a factor?
- If 0 is a zero of the quadratic function, how is this zero written as an *x*-intercept?
- If 0 is a zero of the quadratic function, how is this zero written as a factor?

12

c. What would you tell Micheal, Tim, and Judy to correct their functions.

Micheal had the correct zeros but her parabola is opening upward. So, she should choose a negative *a* value.

Tim had the correct zeros but his parabola is opening upward. So, he should choose a negative *a* value.

Judy's parabola opens downward, but her function shows zeros at (−4, 0) and (1, 0). So, she should rewrite the function as $f(x) = -(x - 4)(x + 1)$.

d. How many possible functions can you write to represent the given characteristics? Explain your reasoning.

There are an infinite number of equations for the function because the coefficient *a* does not affect the *x*-intercepts when written in factored form. All possible equations are of the form $f(x) = a(x - 4)(x + 1)$, where $a < 0$.

2. For a quadratic function written in factored form $f(x) = a(x - r_1)(x - r_2)$:

a. what does the sign of the variable *a* tell you about the graph?

The sign of *a* tells you whether the parabola opens upward (positive *a*) or downward (negative *a*).

b. what do the variables r_1 and r_2 tell you about the graph?

The variables r_1 and r_2 represent the *x*-coordinates of the *x*-intercepts. The *x*-intercepts are $(r_1, 0)$ and $(r_2, 0)$.

3. Use the given information to write a quadratic function in factored form, $f(x) = a(x - r_1)(x - r_2)$.

Values for *a* will vary.

a. The parabola opens upward and the zeros are (2, 0) and (4, 0).

$f(x) = a(x - 2)(x - 4)$ for $a > 0$

b. The parabola opens downward and the zeros are (−3, 0) and (1, 0).

$f(x) = a(x + 3)(x - 1)$ for $a < 0$

c. The parabola opens downward and the zeros are (0, 0) and (5, 0).

$f(x) = ax(x - 5)$ for $a < 0$

d. The parabola opens upward and the zeros are (−2.5, 0) and (4.3, 0).

$f(x) = a(x + 2.5)(x - 4.3)$ for $a > 0$

Grouping

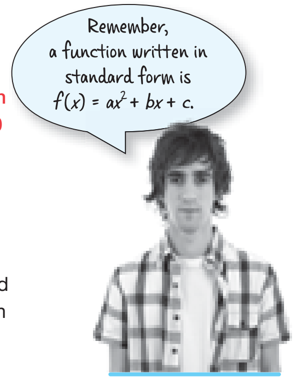
Have students complete Question 5 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 5

- Is this function written in factored form? How do you know?
- Are any of the coefficients of the x-terms in the factors of the quadratic equation negative? How do you change the coefficient of the x-term to positive?
- If a factor in the quadratic equation is written as $(x - 2)$, how is it written as an x-intercept?
- If there is only one factor written in the quadratic equation, what does that tell you about one of the zeros of the function that does not appear?

4. Compare your quadratic functions with your classmates' functions. How does the a value affect the shape of the graph?

As compared to $a = 1$, when the a value is less than 1, the graph is wider. As the a value gets closer to 0 the graph keeps getting wider. When the a value is greater than 1, the graph gets more narrow.



5. Use a graphing calculator to determine the zeros of each function. Sketch each graph using the zeros and y-intercept. Then, write the equation of the function in factored form.

a. $h(x) = x^2 - 8x + 12$

zeros: $(6, 0)$ and $(2, 0)$

factored form: $h(x) = (x - 6)(x - 2)$

b. $r(x) = -2x^2 + 6x + 20$

zeros: $(-2, 0)$ and $(5, 0)$

factored form: $r(x) = -2(x + 2)(x - 5)$

c. $w(x) = -x^2 - 4x$

zeros: $(-4, 0)$ and $(0, 0)$

factored form: $w(x) = -(x - 0)(x + 4)$

or $w(x) = -x(x + 4)$

d. $c(x) = 3x^2 - 3$

zeros: $(-1, 0)$ and $(1, 0)$

factored form: $v(x) = 3(x + 1)(x - 1)$

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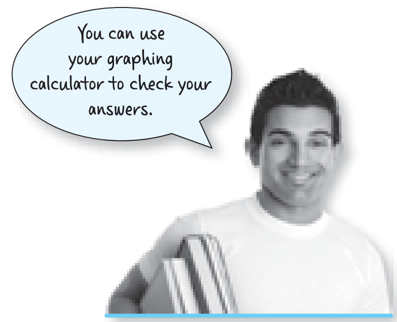
6. Determine the zeros of the function. Write the function in factored form if it is not already in factored form.



a. $f(x) = (x - 2)(x - 7)$

The function is in factored form.
The zeros are (2, 0) and (7, 0).

b. $v(x) = x(2x + 6)$

The function in factored form is $v(x) = 2(x - 0)(x + 3) = 2x(x + 3)$.
The zeros are (0, 0) and (−3, 0).

c. $g(x) = (x + 1)(5 - x)$

The function in factored form is $g(x) = -1(x + 1)(x - 5) = -(x + 1)(x - 5)$.

The zeros are (−1, 0) and (5, 0).

d. $p(x) = (-9 - 3x)(x + 4)$

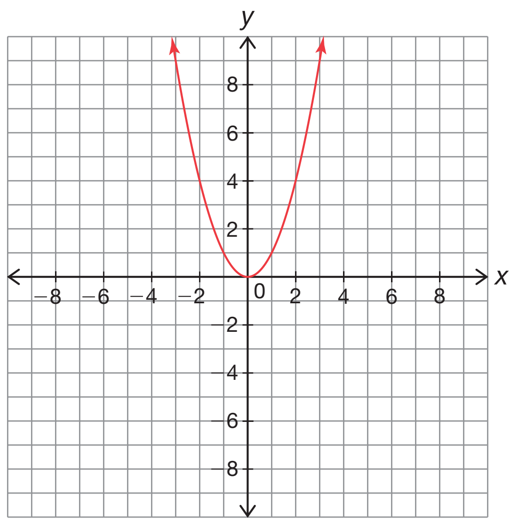
The function in factored form is $p(x) = -3(3 + x)(x + 4) = -3(x + 3)(x + 4)$.
The zeros are (−3, 0) and (−4, 0).

Be prepared to share your solutions and methods.

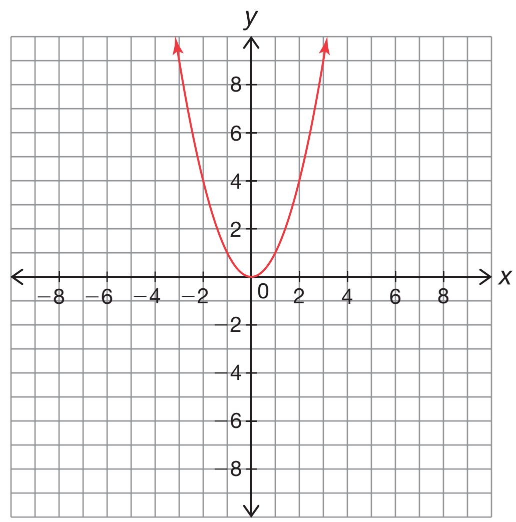
Check for Students' Understanding

1. Sketch a graph of each function.

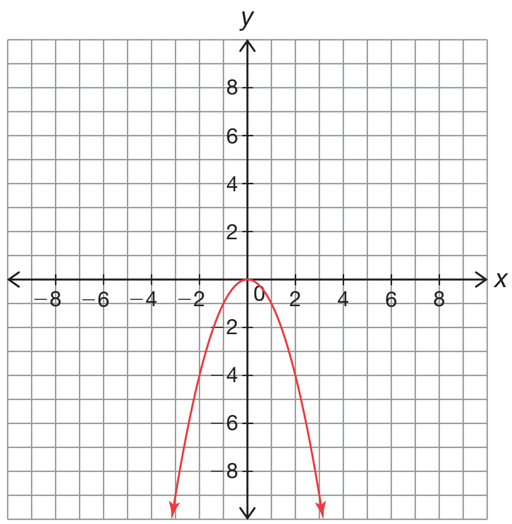
   a. A quadratic function.



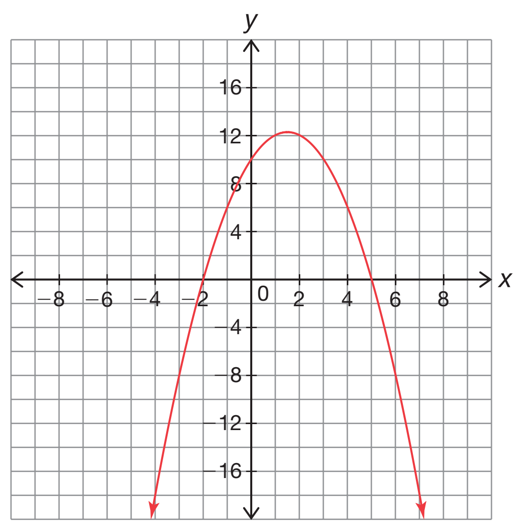
   b. A quadratic function that passes through the origin.



   c. A quadratic function that opens downward.



   d. A quadratic function that opens downward and has a *x*-intercepts at (−2, 0) and (5, 0).



2. What is a possible equation that describes the function you graphed in Question 5? Use interval notation to describe the domain and range of the function.

   One possible equation for the graph of the function is $f(x) = -(x + 2)(x - 5)$.

   Domain: $(-\infty, \infty)$

   Range: $(-\infty, 12.25]$

12

Just Watch that Pumpkin Fly!

12.5

Investigating the Vertex of a Quadratic Function

LEARNING GOALS

In this lesson, you will:

- Interpret parts of a quadratic function in terms of a problem situation.
- Use a calculator to determine the x-intercept(s), y-intercept, and absolute maximum or minumum of a quadratic function.
- Solve a quadratic function graphically.
- Determine the vertex of a quadratic function.
- Use symmetric points to determine the location of the vertex of a parabola.
- Use the vertex to determine symmetric points on a parabola.

KEY TERMS

- vertex
- axis of symmetry

ESSENTIAL IDEAS

- The vertex of a parabola is the lowest or highest point on the curve.
- The axis of symmetry of a parabola is the vertical line that passes through the vertex and divides the parabola into two mirror images.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context.

a. Interpret parts of an expression, such as terms, factors, and coefficients.

F-IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

Overview

Students are given the equation for vertical motion. The scenario uses this quadratic function to describe a catapulted object. Students will write the function that describes this problem situation and use a graphing calculator to graph the function and answer questions related to the key characteristics of the graph, and then interpret them with respect to this problem situation. The terms vertex of a parabola and axis of symmetry of a parabola are introduced. Students will use these terms to describe key characteristics of the equation used to describe the scenario. A graphing calculator is used to explore the coordinates of points that are symmetrically located on the parabola. They then practice determining the vertex and the equation of the axis of symmetry for various parabolas, given key information.

Warm Up

1. Determine the distance between points (−6, 0) and (4, 0).
   The distance between the points is 10 units.

2. Determine the coordinates of the point that is located halfway between the points (−6, 0) and (4, 0).
   The coordinates of the point that is located halfway between points (−6, 0) and (4, 0) is (−1, 0).

3. Determine the distance between points (−4, 15) and (12, 15).
   The distance between the points is 16 units.

4. Determine the coordinates of the point that is located halfway between the points (−4, 15) and (12, 15).
   The coordinates of the point that is located halfway between points (−4, 15) and (12, 15) is (4, 15).

5. How did you determine the distance between the two points in Questions 1 and 3?
   To determine the distance between two points, subtract the *x*-coordinates of each point.

6. How did you determine the coordinates of the point located halfway between the two given points in Questions 2 and 4?
   To determine the coordinates of the point located halfway between the two given points, divide the distance between the two points by 2.

12

12

Just Watch that Pumpkin Fly!

12.5

Investigating the Vertex of a Quadratic Function

LEARNING GOALS

In this lesson, you will:

- Interpret parts of a quadratic function in terms of a problem situation.
- Use a calculator to determine the *x*-intercept(s), *y*-intercept, and absolute maximum or minimum of a quadratic function.
- Solve a quadratic function graphically.
- Determine the vertex of a quadratic function.
- Use symmetric points to determine the location of the vertex of a parabola.
- Use the vertex to determine symmetric points on a parabola.

KEY TERMS

- vertex
- axis of symmetry

Every year the county of Sussex, Delaware, holds a competition called the Punkin' Chunkin' World Championships, which is a pumpkin-throwing competition. Participants build machines that hurl pumpkins great distances. The winner is the person whose machine hurls the pumpkin the farthest.

There are different divisions based on the type of machine used.

- The Air Cannon Division includes machines that use compressed air to fire pumpkins.
- In the Catapult Division, catapults are composed of cords, springs, rubber, weights, or other mechanisms that create and store energy.
- The Centrifugal Division includes machines that have devices that spin at least one revolution before firing pumpkins.
- There is also the Trebuchet Division. Trebuchets are machines that have swinging or fixed counterweights that can fling pumpkins up and through the air.

What do you think the world-record distance is for hurling a pumpkin at the Punkin' Chunkin' World Championships? Which machine do you think was used?

12

Problem 1

A scenario is used to explore a vertical motion quadratic function. A model of the vertical motion equation is given. Students will write the function for the height in terms of the time relevant to this problem situation. They then use a graphing calculator to graph the function, determine the *x*- and *y*-intercepts, and identify the coordinates of the absolute maximum.

Grouping

- Ask a student to read the information and complete Question 1 as a class.
- Have students complete Questions 2 through 7 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 2 and 3

- What variable was used in the model equation when you substituted the value of 68?
- What variable was used in the model equation when you substituted the value of 128?
- What is the sign of the coefficient of the leading term?
- What did the sign of the coefficient of the leading term imply about the graph of the quadratic equation?
- If the sign of the coefficient of the leading term is negative, does the quadratic function have an absolute minimum or an absolute maximum?

12

PROBLEM 1 Punkin Chunkin by Catapult!

You can model the motion of a pumpkin released from a catapult using a vertical motion model. Remember, a vertical motion model is a quadratic equation that models the height of an object at a given time. The equation is of the form

$$y = -16t^2 + v_0t + h_0,$$

where y represents the height of the object in feet, t represents the time in seconds that the object has been moving, v_0 represents the initial vertical velocity (speed) of the object in feet per second, and h_0 represents the initial height of the object in feet.

1. Why do you think it makes sense that this situation is modeled by a quadratic function?

   If a pumpkin is catapulted in the air, at first its height increases over time. Then, its height will decrease over time until it reaches the ground. That is, it goes up and then comes back down. So, it makes sense that this situation is modeled by a parabola.

Suppose that a catapult hurls a pumpkin from a height of 68 feet at an initial vertical velocity of 128 feet per second.

2. Write a function for the height of the pumpkin $h(t)$ in terms of time t.

   $h(t) = -16t^2 + 128t + 68$

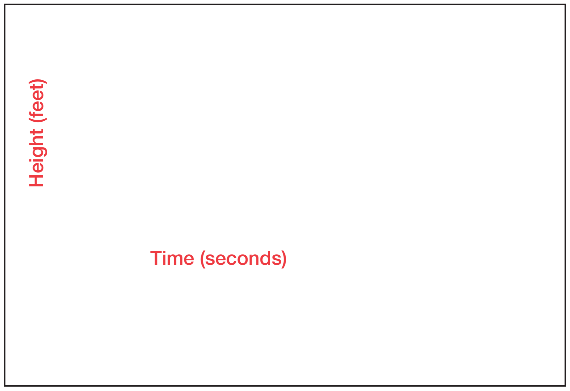
3. Does the function you wrote have an absolute minimum or an absolute maximum? How can you tell from the form of the function?

   The function has an absolute maximum.
   I can tell because the leading coefficient is negative.

Guiding Questions for Share Phase, Questions 4 through 7

- How did you use your graphing calculator to determine the *x*-intercepts?
- What is the height of the pumpkin at each *x*-intercept?
- What is the time in seconds at each *x*-intercept?
- Do negative *x*-values make sense in this problem situation? Why or why not?
- Do negative *y*-values make sense in this problem situation? Why or why not?
- How did you use your graphing calculator to determine the *y*-intercept?
- What is the height of the pumpkin at the *y*-intercept?
- What is the time in seconds at the *y*-intercept?

4. Graph the function on a graphing calculator using the bounds $[-1, 9] \times [0, 500]$. Sketch your graph and label the axes.



5. Use a graphing calculator to determine the zeros of the function. Then explain what each means in terms of the problem situation. Do each make sense in terms of this problem situation?

The zeros are (−0.5, 0) and (8.5, 0).

Each represents the time in seconds for which the height of the pumpkin is 0 feet.

The zero (8.5, 0) means that the pumpkin was at a height of 0 feet (on the ground) 8.5 seconds after it was catapulted, which makes sense.

The zero (−0.5, 0) means that the pumpkin was at a height of 0 feet 0.5 second before it was catapulted, and there is no way to know that, so this does not make sense.

6. Determine the *y*-intercept and interpret its meaning in terms of this problem situation.

The *y*-intercept is (0, 68). Students may use the equation of the function or a graphing calculator to determine the *y*-intercept.

The initial height of the pumpkin at 0 seconds was 68 feet.

7. Use a graphing calculator to determine the absolute minimum or maximum. Then explain what it means in terms of this problem situation.

The absolute maximum is (4, 324).
It means that the pumpkin reaches a maximum height of 324 feet after 4 seconds.

12

Problem 2

The terms vertex of a parabola and axis of symmetry of a parabola are given. Students will identify the vertex and the equation of the axis of symmetry of the graph that models the situation in Problem 1. They then use a graphing calculator to answer questions related to the scenario and explore coordinates of symmetric points on the parabola.

Grouping

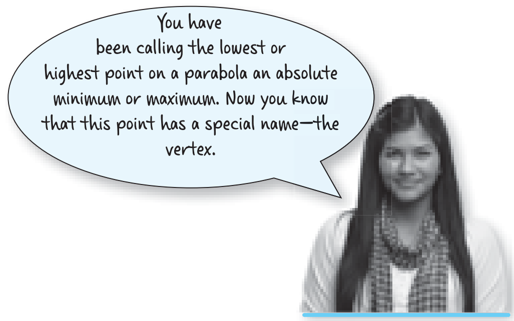
- Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.
- Have students complete Questions 3 through 7 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- If you folded the graph of a parabola along a horizontal line that goes through the middle of the parabola, would it produce two mirror images? Why not?
- If you folded the graph of a parabola along a vertical line that did not go through the vertex of the parabola, would it produce two mirror images? Why not?

PROBLEM 2 The Vertex of a Parabola and Symmetry

The **vertex** of a parabola is the lowest or highest point on the curve. The **axis of symmetry** of a parabola is the vertical line that passes through the vertex and divides the parabola into two mirror images.



Because the axis of symmetry always divides the parabola into two mirror images, you can say that a parabola is symmetric.

1. Because a parabola is symmetric, over which line would you fold it to show its symmetry? What is the equation of that line?
   You would fold the graph along the vertical line that goes through the minimum or maximum point, the vertex, to produce two mirror images.

   The equation of that line is x = the x-coordinate of the vertex.

Remember the Punkin Chunkin scenario in Problem 1, in which you wrote the function $h(t) = -16t^2 + 128t + 68$ for the height of the pumpkin in terms of time.

2. Identify the coordinates of the vertex of the graph and the equation for the axis of symmetry.
   The vertex is at (4, 324). The equation for the axis of symmetry is $x = 4$.

3. Use a graphing calculator to answer each question.
   a. When does the pumpkin reach a height of 128 feet?
      The pumpkin reaches a height of 128 feet after 0.5 second and after 7.5 seconds.
   b. When does the pumpkin reach a height of 180 feet?
      The pumpkin reaches a height of 180 feet after 1 second and after 7 seconds.
   c. When does the pumpkin reach a height of 308 feet?
      The pumpkin reaches a height of 308 feet after 3 seconds and after 5 seconds.

- Is a vertical line described by an equation that is written in the form $x = a$, or written in the form $y = a$, where a is a real number?
- What does the equation of the line of symmetry of a parabola have in common with the coordinates of the vertex of the parabola?
- If the vertex of the parabola has the coordinates (4, 324), is the equation for the line of symmetry $x = 4$, or $y = 324$?

Guiding Questions for Share Phase, Questions 3 through 6

- To determine when the pumpkin reaches a height of 128 feet, should the point on the graph have an *x*-value of 128, or a *y*-value of 128? Why?
- How did you use a graphing calculator to determine when the height of the pumpkin reaches 128 feet?
- What do the points (−0.5, 0), and (8.5, 0) have in common? How are they different?
- What do the points (0.5, 128), and (7.5, 128) have in common? How are they different?
- What do the points (1, 180), and (7, 180) have in common? How are they different?
- What do the points (3, 308), and (5, 308) have in common? How are they different?
- What is the distance between point (−0.5, 0) and the axis of symmetry?
- What is the distance between point (8.5, 0) and the axis of symmetry?
- What is the distance between point (0.5, 128) and the axis of symmetry?
- What is the distance between point (7.5, 128) and the axis of symmetry?
- What is the distance between point (1, 180) and the axis of symmetry?

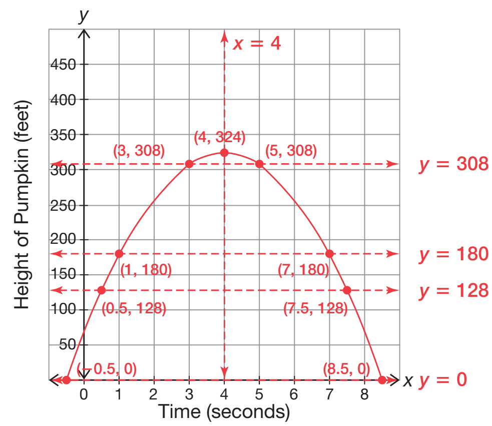
4. Use the information from Questions 2 and 3 to construct a graph.

a. Plot and label the vertex. Then draw and label the axis of symmetry.

b. Plot and label the points that correspond to the answers from Question 3.

c. Plot and label the points symmetric to the points from part (b).

d. Plot and label the zeros.



5. Analyze the symmetric points.

a. What do you notice about the *y*-coordinates?

The *y*-coordinates of the symmetric points are the same.

b. What do you notice about each point's horizontal distance from the axis of symmetry?

Both symmetric points are the same distance from the axis of symmetry.

6. How does the *x*-coordinate of each symmetric point compare to the *x*-coordinate of the vertex?

The *x*-coordinate of the vertex is halfway between the *x*-coordinates of the symmetric points. In other words, the *x*-coordinate of the vertex is the average, or midpoint, of the *x*-coordinates of symmetric points on the parabola with the same *y*-coordinates.

12

- What is the distance between point (7, 180) and the axis of symmetry?
- What is the distance between point (3, 308) and the axis of symmetry?
- What is the distance between point (5, 308) and the axis of symmetry?

Talk the Talk

Students will determine the axis of symmetry of a parabola given the *x*-intercepts, and given the coordinates of 2 symmetric points. They then determine the vertex of a parabola given the equation and the axis of symmetry, the equation and the *x*-intercepts, and the coordinates of 2 symmetric points.

Grouping

Have students complete Questions 1 through 3 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What is the distance between points (1, 0) and (5, 0)?
- What are the coordinates of the point located halfway between points (1, 0) and (5, 0)?
- What is the distance between points (−3.5, 0) and (4.1, 0)?
- What are the coordinates of the point located halfway between points (−3.5, 0) and (4.1, 0)?

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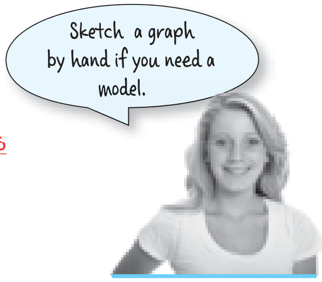
Talk the Talk

Use the given information to answer each question. Do not use a graphing calculator. Show your work.



1. Determine the axis of symmetry of each parabola.

a. The *x*-intercepts of the parabola are (1, 0) and (5, 0).

The axis of symmetry is $x = 3$ because $\frac{1 + 5}{2} = \frac{6}{2} = 3$.

b. The *x*-intercepts of the parabola are (−3.5, 0) and (4.1, 0).

The axis of symmetry is $x = 0.3$ because $\frac{-3.5 + 4.1}{2} = \frac{0.6}{2} = 0.3$.

c. Two symmetric points on the parabola are (−7, 2) and (0, 2).

The axis of symmetry is $x = -3.5$ because $\frac{-7 + 0}{2} = \frac{-7}{2} = -3.5$.

d. Describe how to determine the axis of symmetry given the *x*-intercepts of a parabola.

Determine the average of the *x*-coordinates of the two *x*-intercepts, *a*. The equation for the axis of symmetry is $x = a$.

2. Determine the location of the vertex of each parabola.

a. The function $f(x) = x^2 + 4x + 3$ has the axis of symmetry $x = -2$.

The axis of symmetry is $x = -2$, so the *x*-coordinate of the vertex is −2.

The *y*-coordinate when $x = -2$ is:

$$f(-2) = (-2)^2 + 4(-2) + 3$$
$$= 4 - 8 + 3$$
$$= -1$$

The vertex is at (−2, −1).

b. The equation of the parabola is $y = x^2 - 4$, and the x-intercepts are $(-2, 0)$ and $(2, 0)$.

The axis of symmetry is $x = 0$ because $\frac{-2 + 2}{2} = \frac{0}{2} = 0$. So, the x-coordinate of the vertex is 0.

The y-coordinate when $x = 0$ is:

$$\begin{aligned} f(0) &= 0^2 - 4 \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

The vertex is at $(0, -4)$.

c. The function $f(x) = x^2 + 6x - 5$ has two symmetric points $(-1, -10)$ and $(-5, -10)$.

The axis of symmetry is $x = -3$ because $\frac{-1 + (-5)}{2} = \frac{-6}{2} = -3$. So, the x-coordinate of the vertex is -3.

The y-coordinate when $x = -3$ is:

$$\begin{aligned} f(-3) &= (-3)^2 + 6(-3) - 5 \\ &= 9 + (-18) - 5 \\ &= -14 \end{aligned}$$

The vertex is at $(-3, -14)$.

d. Describe how to determine the vertex of a parabola given the equation and the axis of symmetry.

The axis of symmetry gives the x-coordinate of the vertex. Substitute this x-value into the equation for the parabola to determine the y-coordinate of the vertex.

3. Determine another point on each parabola.

a. The axis of symmetry is $x = 2$.

A point on the parabola is $(0, 5)$.

Another point on the parabola: $(4, 5)$

Answers may vary.

Another point on the parabola is a symmetric point that has the same y-coordinate as $(0, 5)$. The x-coordinate is:

$$\begin{aligned} \frac{0 + a}{2} &= 2 \\ 0 + a &= 4 \\ a &= 4 \end{aligned}$$

Another point on the parabola is $(4, 5)$.

12

b. The vertex is (0.5, 9).

An x-intercept is $(-2.5, 0)$.

Another point on the parabola: (3.5, 0)

Answers may vary.

The other x-intercept has a y-coordinate of 0. The x-coordinate is:

$$\frac{-2.5 + a}{2} = 0.5$$

$$-2.5 + a = 1$$

$$a = 3.5$$

Another point on the parabola is the other x-intercept, (3.5, 0).

c. The vertex is $(-2, -8)$.

A point on the parabola is $(-1, -7)$.

Another point on the parabola: $(-3, -7)$

Answers may vary.

Another point on the parabola is a symmetric point that has the same y-coordinate as $(-1, -7)$. The x-coordinate is:

$$\frac{-1 + a}{2} = -2$$

$$-1 + a = -4$$

$$a = -3$$

Another point on the parabola is $(-3, -7)$.

d. Describe how to determine another point on a parabola if you are given one point and the axis of symmetry.

Answers may vary.

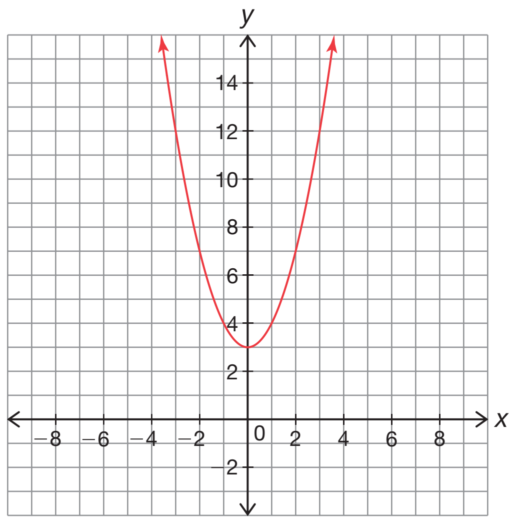
The axis of symmetry shows the average of the x-coordinates of two symmetric points on a parabola. Add the x-coordinate of one point to a variable, divide by 2, and set this equal to the axis of symmetry. Then, solve for the variable to determine the x-coordinate of the other point. The y-coordinate is the same as the y-coordinate for the given point.

12

Be prepared to share your solutions and methods.

Check for Students' Understanding

1. Sketch a quadratic function with a vertex at (0, 3).



2. How many possible parabolas have a vertex at (0, 3)?

   There are an infinite number of parabolas having a vertex at (0, 3).

3. Does your parabola open upward or downward?

   The parabola opens upward.

4. What is the equation of the axis of symmetry?

   The equation of the axis of symmetry is $x = 0$.

5. Do all of the possible parabolas with a vertex at (0, 3) have the same axis of symmetry? Explain your reasoning.

   Yes, all possible parabolas with a vertex at (0, 3) have an axis of symmetry that has an equation $x = 0$, because it is the x-coordinate of the vertex of the parabola that determines the equation of the axis of symmetry.

6. What are the x-intercepts?

   There are no x-intercepts.

7. Do all of the possible parabolas with a vertex at (0, 3) have the same x-intercepts? Explain your reasoning.

   No. All parabolas with a vertex at (0, 3) will not have the same x-intercepts. If the parabola opens downward, the x-intercepts will be a pair of symmetric points on the x-axis.

12

8. What is a possible equation that describes the function you graphed in Question 1?

One possible equation for the parabola is $y = x^2 + 3$.

9. Use interval notation to describe the domain and range of the function.

Domain: $(-\infty, \infty)$

Range: $[3, \infty)$

10. Do all of the possible parabolas with a vertex at (0, 3) have the same domain and range? Explain your reasoning.

No. All parabolas with a vertex at (0, 3) will have the same domain, but not the same range. If the parabola we drawn opening downward, the range would be $(-\infty, 3]$.

12

12.6

The Form Is "Key"

Vertex Form of a Quadratic Function

LEARNING GOALS

In this lesson, you will:

- Determine key characteristics of parabolas using a graphing calculator.
- Determine key characteristics of parabolas given their equations in standard form.
- Determine key characteristics of parabolas given their equations in factored form.
- Determine key characteristics of parabolas given their equations in vertex form.
- Write equations of parabolas given key characteristics of their graphs.

KEY TERM

- vertex form

ESSENTIAL IDEAS

- The standard form of a quadratic function is $f(x) = ax^2 + b\,x + c$, where $a \neq 0$.
- A quadratic function written in standard form shows which direction the parabola opens and the coordinates of the y-intercept.
- The factored form of a quadratic function is $f(x) = a\,(x - b)\,(x - c)$, where $a \neq 0$.
- A quadratic function written in factored form shows which direction the parabola opens and the coordinates of the x-intercepts.
- The vertex form of a quadratic function is $f(x) = a\,(x - h)^2 + k$, where $a \neq 0$, and (h, k) is the vertex.
- A quadratic function written in vertex form shows which direction the parabola opens and the coordinates of the vertex.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions.

1. Interpret expressions that represent a quantity in terms of its context.

   a. Interpret parts of an expression, such as terms, factors, and coefficients.

F-IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

Overview

The vertex form of a quadratic function is introduced. Students will explore which if any, key characteristics of a quadratic function can be determined if the function is written in standard form, factored form, and vertex form. They then practice using a graphing calculator to complete a table of values, sketch the graph, determine the coordinates of the vertex, and determine the coordinates of the *y*-intercept. They then rewrite quadratic functions in vertex form and factored form and identify which form a given quadratic function is written in. In the last activity, students complete a graphic organizer using the standard form of a quadratic function that opens upward, and then downward.

12

Warm Up

Use the graphing calculator to write each quadratic function in factored form.

1. $f(x) = x^2 + 11x + 30$

$f(x) = (x + 5)(x + 6)$

2. $f(x) = x^2 - 11x + 18$

$f(x) = (x - 9)(x - 2)$

Use the graphing calculator to write each quadratic function in standard form.

3. $f(x) = (x + 3)(x - 5)$

$f(x) = x^2 - 2x - 15$

4. $f(x) = (x - 1)(x - 8)$

$f(x) = x^2 - 9x + 8$

12

12

12.6

The Form Is "Key"

Vertex Form of a Quadratic Function

LEARNING GOALS

In this lesson, you will:

- Determine key characteristics of parabolas using a graphing calculator.
- Determine key characteristics of parabolas given their equations in standard form.
- Determine key characteristics of parabolas given their equations in factored form.
- Determine key characteristics of parabolas given their equations in vertex form.
- Write equations of parabolas given key characteristics of their graphs.

KEY TERM

- vertex form

Once upon a time, people believed that a god, named Atlas, held up all of the sky, or the "Celestial Sphere." And it was assumed that a deity made thunder and lightning and caused the Sun to shine and even crops to grow.

Of course, now we know that nothing whatsoever holds up the sky, except for gravity and the laws of physics. We know what makes lightning and thunder, and we know what causes crops to flourish or not. And we even know what causes the Sun to shine—and why it won't continue to shine forever.

Problem 1

Students will use a graphing calculator to complete a table of values, sketch a graph, tell whether the parabola opens up or down, determine the vertex, determine the x-intercepts, and determine the y-intercepts. They then determine which key characteristics can be determined directly from a function when it is written in standard form and when it is written in factored form.

Grouping

Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

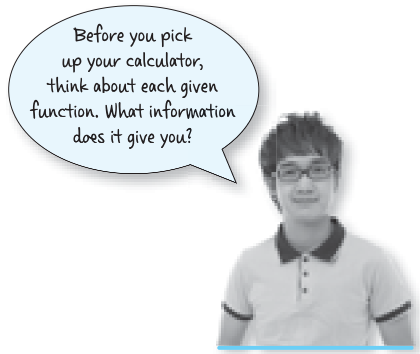
- How did you use a graphing calculator to complete the table of values?
- How did you use a graphing calculator to determine the coordinates of the vertex?
- Does the parabola have an absolute maximum or an absolute minimum?
- How did you use a graphing calculator to determine the zeros?
- How did you use a graphing calculator to determine the y-intercept?
- Does the coefficient of the x term in the equation of the function tell you anything about any key characteristics of the function?

12

PROBLEM 1 **Analyzing Forms**

1. For each function, use a graphing calculator to:

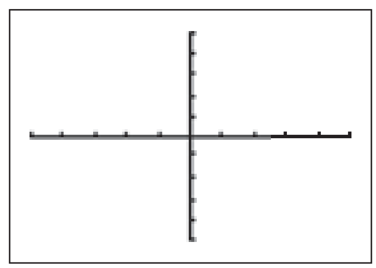
- complete the table of values.
- sketch the graph in the space shown using a window with the given bounds.
- tell whether the parabola opens up or down.
- determine the location of the vertex.
- determine the zero(s).
- determine the y-intercept.



a. $f(x) = x^2 + 2x - 3$

x	$f(x)$
−2	−3
−1	−4
0	−3
1	0
2	5

[−10, 10] × [−10, 10]



parabola opens: up

vertex: (−1, −4)

zero(s): (−3, 0) and (1, 0)

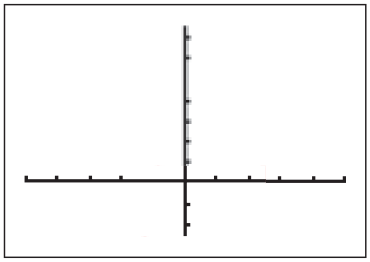
y-intercept: (0, −3)

- Does the constant in the equation tell you anything about any key characteristics of the function?
- Does the coefficient of the x^2 term in the equation of the function tell you anything about any key characteristics of the function?

b. $f(x) = -2x^2 + 6x + 20$

x	$f(x)$
−2	0
−1	12
0	20
1	24
2	24

[−10, 10] × [−10, 30]



parabola opens: down

vertex: (1.5, 24.5)

zero(s): (−2, 0) and (5, 0)

y-intercept: (0, 20)

The quadratic functions in Question 1 are written in standard form, $f(x) = ax^2 + bx + c$. You learned about the standard form of a quadratic function earlier in this chapter.

2. In Question 1, you determined whether the parabola opens up or down, the location of the vertex, the zeros, and the *y*-intercept using a graphing calculator.

Which, if any, of those key characteristics can you determine directly from a quadratic function when it is written in standard form?

The sign of the *a* value tells me whether the parabola opens up or down.
The *c* value tells me the *y*-intercept.

12

Grouping

Have students complete Questions 3 through 5 with a partner. Then share the responses as a class.

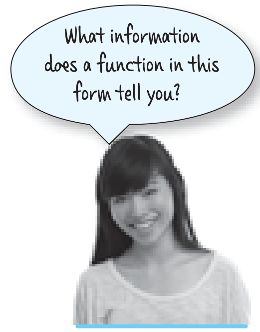
Guiding Questions for Share Phase, Questions 3 through 5

- Is this a quadratic function? How do you know?
- Where is the x^2 term in the equation of the function?
- How did you use a graphing calculator to complete the table of values?
- How did you use a graphing calculator to determine the coordinates of the vertex?
- Does the parabola have an absolute maximum or an absolute minimum?
- How did you use a graphing calculator to determine the x-intercepts?
- How did you use a graphing calculator to determine the y-intercept?
- Does the constant in the first set of parenthesis in the equation of the function tell you anything about any key characteristics of the function?
- Does the constant in the second set of parenthesis in the equation of the function tell you anything about any key characteristics of the function?

12

3. For each function, use a graphing calculator to:

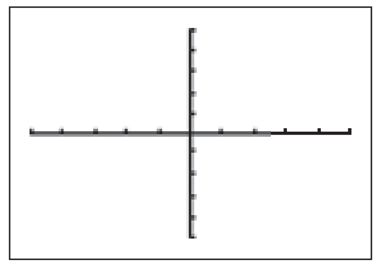
- complete the table of values.
- sketch the graph in the space shown using a window with the given bounds.
- tell whether the parabola opens up or down.
- determine the location of the vertex.
- determine the zero(s).
- determine the y-intercept.



a. $f(x) = (x - 1)(x + 3)$

x	$f(x)$
−2	−3
−1	−4
0	−3
1	0
2	5

$[-10, 10] \times [-10, 10]$



parabola opens: up

vertex: $(-1, -4)$

zero(s): $(-3, 0)$ and $(1, 0)$

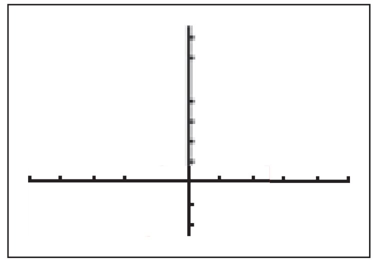
y-intercept: $(0, -3)$

- Does the number outside the sets of parentheses in the equation of the function tell you anything about any key characteristics of the function?
- If there is no number written outside the sets of parentheses, what does this tell you about the direction of the parabola?
- How is factored form different than standard form?

b. $f(x) = -2(x + 2)(x - 5)$

x	$f(x)$
−2	0
−1	12
0	20
1	24
2	24

[−10, 10] × [−10, 30]



parabola opens: down

vertex: (1.5, 24.5)

zero(s): (−2, 0) and (5, 0)

y-intercept: (0, 20)

4. Compare your answers in Question 1 with your answers in Question 3. What do you notice?

The functions are the same. They are just written in different forms.

The quadratic functions in Question 3 are written in factored form, $f(x) = a(x - r_1)(x - r_2)$. You learned about the factored form of a quadratic function earlier in this chapter.

5. What key characteristics can you determine directly from a quadratic function when it is written in factored form?

The sign of the *a* value tells you whether the parabola opens up or down. The r_1 and r_2 values tell you the *x*-intercepts.

12

Guiding Questions for Share Phase, Question 6

- How did you use a graphing calculator to complete the table of values?
- How did you use a graphing calculator to determine the coordinates of the vertex?
- Does the parabola have an absolute maximum or an absolute minimum?
- How did you use a graphing calculator to determine the zeros?
- How did you use a graphing calculator to determine the *y*-intercept?
- Does the constant in the in the equation of the function tell you anything about any key characteristics of the function?
- Does the constant in the set of parenthesis in the equation of the function tell you anything about any key characteristics of the function?

12

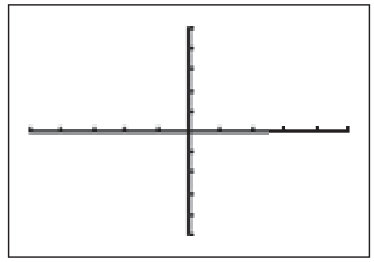
6. For each function, use a graphing calculator to:

- complete the table of values.
- sketch the graph in the space shown using a window with the given bounds.
- tell whether the parabola opens up or down.
- determine the location of the vertex.
- determine the zero(s).
- determine the *y*-intercept.

a. $f(x) = (x + 1)^2 - 4$

x	$f(x)$
−2	−3
−1	−4
0	−3
1	0
2	5

[−10, 10] × [−10, 10]



parabola opens: up

vertex: (−1, −4)

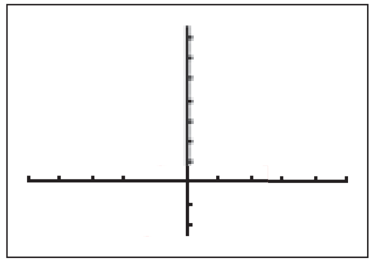
zero(s): (−3, 0) and (1, 0)

y-intercept: (0, −3)

b. $f(x) = -2(x - 1.5)^2 + 24.5$

x	$f(x)$
−2	0
−1	12
0	20
1	24
2	24

[−10, 10] × [−10, 30]



parabola opens: down

vertex: (1.5, 24.5)

zero(s): (−2, 0) and (5, 0)

y-intercept: (0, 20)

7. What do you notice when you compare the functions in Question 6 with the functions in Question 1?

The functions are the same. They are just written in different forms.

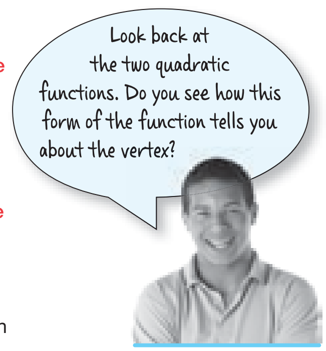
The quadratic functions in Question 6 are written in *vertex form*. A quadratic function written in **vertex form** is in the form $f(x) = a(x - h)^2 + k$, where $a \neq 0$.

Guiding Questions for Share Phase, Questions 8 through 10

- What key characteristic of the quadratic function does the point (h, k) describe?
- How is vertex form different from standard form?
- How is vertex form different from factored form?

8. What does the variable h represent in the vertex form of a quadratic function?

The variable h represents the x-coordinate of the vertex.



9. What does the variable k represent in the vertex form of a quadratic function?

The variable k represents the y-coordinate of the vertex.

10. What key characteristics can you determine directly from the quadratic function when it is written in vertex form?

You can determine whether the parabola opens up or down and the coordinates of the vertex from the vertex form of a quadratic function.

12

Talk the Talk

Students will graph five different quadratic equations using a graphing calculator to determine which of the equations contains the given direction and vertex. They also rewrite several quadratic functions in vertex form and then in factored form. Next, students identify the form of several quadratic equations. They then use given information to write possible equations for quadratic functions. Finally, students will complete a graphic organizer using the standard form of a quadratic function.

Grouping

Have students complete Question 1 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 1

12

- Are all of the students' equations written in vertex form?
- Does each equation describe a parabola that opens upward?
- Does each equation describe the correct y-coordinate of the vertex?
- Does each equation describe the correct x-coordinate of the vertex?

Talk the Talk

1. Simone, Teresa, Jesse, Leon, and David are working together on the problem shown.

Write a quadratic function to represent a parabola that opens upward and has a vertex at $(-6, -4)$.

Simone
My function is
$s(x) = 3(x + 6)^2 - 4$.

Teresa
My function is
$t(x) = \frac{1}{4}(x + 6)^2 - 4$.

Jesse
My function is
$j(x) = -3(x + 6)^2 - 4$.

David
My function is
$d(x) = (x + 6)^2 - 4$.

Leon
My function is
$f(x) = 2(x - 6)^2 + 4$.

a. Use a graphing calculator to graph each student's function. What are the similarities among all the graphs? What are the differences among the graphs?

All graphs are parabolas.
Simone, Teresa, Leon, and David all have parabolas that open upward, while Jesse's parabola opens downward.

Simone's and Jesse's parabolas are the same "width," while the others have wider graphs, with Teresa's being the widest.

b. How is it possible to have more than one correct function?

In this case, the given characteristics only described the direction the parabola opened and the vertex. The width of the parabola is not given.

c. What would you tell Jesse and Leon to correct their functions.

Jesse had the correct vertex but his parabola is opening downward. So, he should change the value of a to a positive number.

Leon's parabola is opening upward but he reversed the signs of the vertex. So, he should rewrite his function $f(x) = 2(x + 6)^2 - 4$.

Grouping

Have students complete Questions 2 through 4 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 2

- Are any key characteristics obvious when the function is written in the given form?
- Does the parabola open upward or downward? How do you know?
- Is the y-intercept helpful in writing the function in vertex form?
- Is the y-intercept helpful in writing the function in factored form?
- How did you determine the zeros?

d. How many possible functions can you write for the parabola described in this problem? Explain your reasoning.

There are an infinite number of functions because the coefficient a does not affect the coordinates of the vertex when an equation is written in factored form. All possible equations are of the form $f(x) = a(x + 6)^2 - 4$, where $a > 0$.

2. Use a graphing calculator to rewrite each quadratic function. First, determine the vertex of each and write the function in vertex form. Then, determine the zero(s) of each and write the function in factored form.

a. $h(x) = x^2 - 8x + 12$

vertex: $(4, -4)$

vertex form: $h(x) = (x - 4)^2 - 4$

zero(s): $(6, 0)$ and $(2, 0)$

factored form: $h(x) = (x - 6)(x - 2)$

b. $r(x) = -2x^2 + 6x + 20$

vertex: $(1.5, 24.5)$

vertex form: $r(x) = -2(x - 1.5)^2 + 24.5$

zero(s): $(5, 0)$ and $(-2, 0)$

factored form: $r(x) = -2(x - 5)(x + 2)$

c. $w(x) = -x^2 - 4x$

vertex: $(-2, 4)$

vertex form: $w(x) = -(x + 2)^2 + 4$

zero(s): $(0, 0)$ and $(4, 0)$

factored form: $w(x) = -(x - 0)(x - 4)$

d. $c(x) = 3x^2 - 3$

vertex: $(0, -3)$

vertex form: $c(x) = 3(x - 0)^2 - 3$ or $c(x) = 3x^2 - 3$

zero(s): $(-1, 0)$ and $(1, 0)$

factored form: $c(x) = 3(x - 1)(x + 1)$

12

Guiding Questions for Share Phase, Question 3

- What form of the function gives you the most information about the key characteristics of the function?
- What form of the function gives you the least information about the key characteristics of the function?
- How do you know if a function is written in standard form?
- How do you know if a function is written in factored form?
- How do you know if a function is written in vertex form?

3. Identify the form(s) of each quadratic function as either standard form, factored form, or vertex form. Then state all you know about each quadratic function's key characteristics, based only on the given equation of the function.

a. $f(x) = -(x - 1)^2 + 9$

vertex form

The parabola opens down and the vertex is (1, 9).

b. $f(x) = x^2 + 4x$

standard form

The parabola opens up and the y-intercept is (0, 0).

c. $f(x) = -\frac{1}{2}(x - 3)(x + 2)$

factored form

The parabola opens down and the x-intercepts are (3, 0) and (−2, 0).

d. $f(x) = x^2 - 5$

vertex form and standard form

The parabola opens up, the y-intercept is (0, −5), and the vertex is (0, −5).

Guiding Question for Share Phase, Question 4

How many possible equations will satisfy the key characteristics of this function?

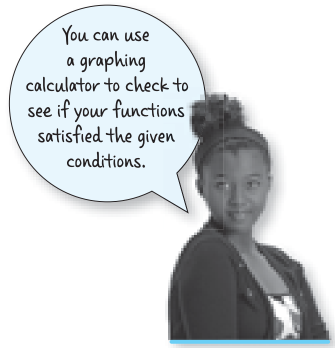
Grouping

Have students complete Question 5 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 5

- What is the sign of the coefficient of the x^2 term? What key characteristic of the function does this identify?
- Is the y-intercept obvious in the equation of the function?
- Are the zeros obvious in the equation of the function?
- Are the coordinates of the vertex obvious in the equation of the function?

4. Use the given information to write a possible equation for each quadratic function.
Values for a will vary.



a. The zeros are -4 and 6, and the parabola opens down.

$f(x) = a(x + 4)(x - 6)$ for $a < 0$

b. The vertex is (0, 3), and the parabola opens up.

$f(x) = a(x - 0)^2 + 3 = ax^2 + 3$ for $a > 0$

c. The vertex is $(-1, 1)$, and the parabola opens down.

$f(x) = a(x + 1)^2 + 1$ for $a < 0$

d. The zeros are 0 and 2, and the parabola opens up.

$f(x) = a(x + 0)(x - 2) = ax(x - 2) = ax^2 - 2ax$ for $a > 0$

5. Complete the graphic organizers on the next two pages using the standard form of the function given. Check the appropriate boxes for each function form.

Be prepared to share your solutions and methods.

12

Standard Form

Equation: $f(x) = x^2 + 2x - 3$

Identify the key features from standard form:

- ☒ parabola opens up/down
- ☐ location of vertex
- ☐ zeros
- ☒ *y*-intercept

Factored Form

Equation: $f(x) = (x - 1)(x + 3)$

Identify the key features from factored form:

- ☒ parabola opens up/down
- ☐ location of vertex
- ☒ zeros
- ☐ *y*-intercept

Graph of the Quadratic Function

Vertex Form

Equation: $f(x) = (x + 1)^2 - 4$

Identify the key features from vertex form:

- ☒ parabola opens up/down
- ☒ location of vertex
- ☐ zeros
- ☐ *y*-intercept

Standard Form

Equation: $f(x) = -2x^2 + 6x + 20$

Identify the key features from standard form:

- ☒ parabola opens up/down
- ☐ location of vertex
- ☐ zeros
- ☒ *y*-intercept

Factored Form

Equation: $f(x) = -2(x + 2)(x - 5)$

Identify the key features from factored form:

- ☒ parabola opens up/down
- ☐ location of vertex
- ☒ zeros
- ☐ *y*-intercept

Graph of the Quadratic Function

Vertex Form

Equation: $f(x) = -2(x - 1.5)^2 + 24.5$

Identify the key features from vertex form:

- ☒ parabola opens up/down
- ☒ location of vertex
- ☐ zeros
- ☐ *y*-intercept

12

Check for Students' Understanding

Identify any obvious key characteristics in each quadratic function.

1. $f(x) = x^2 - 11x + 18$

   The parabola opens upward, and the *y*-intercept is (0, 18).

2. $f(x) = -2(x + 11)(x - 3)$

   The parabola opens downward, and the *s*-intercept are (−11, 0) and (3, 0).

3. $f(x) = (x + 6)^2 - 2$

   The parabola opens upward, and the vertex is (−6, −2)

12

More Than Meets the Eye

Transformations of Quadratic Functions

12.7

LEARNING GOALS

In this lesson, you will:

- Translate quadratic functions.
- Reflect quadratic functions.
- Dilate quadratic functions.
- Write equations of quadratic functions given multiple transformations.
- Graph quadratic functions given multiple transformations.
- Identify multiple transformations given equations of quadratic functions.

KEY TERMS

- vertical dilation
- dilation factor

ESSENTIAL IDEAS

- A vertical translation of a function is a transformation in which a positive or negative constant is added to the basic function.
- A vertical translation shifts the graph of a function up or down.
- A horizontal translation of a function is a transformation in which a positive or negative constant is added to the argument of the basic function.
- A horizontal translation shifts the graph of the function to the left or to the right.
- A reflection of a function is a transformation in which the basic function is multiplied by -1.
- A reflection reflects the graph of the function over the x-axis.
- A vertical dilation of a function is a transformation in which the y-coordinate of every point on the graph of the function is multiplied by a common factor called the dilation factor.
- A vertical dilation stretches or shrinks the graph of a function.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-BF Building Functions

Build new functions from existing functions

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

F-IF Interpreting Functions

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

   a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

Overview

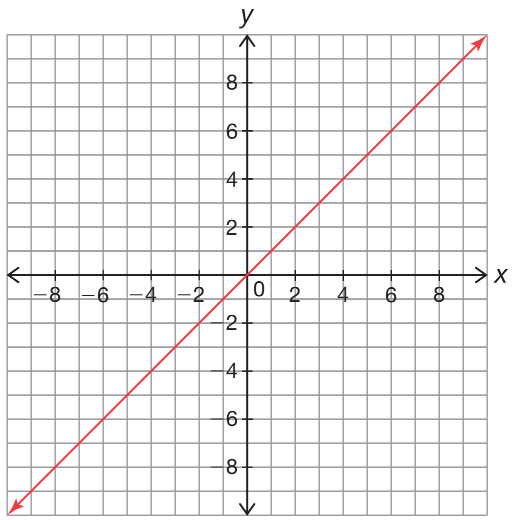
Basic quadratic functions and the argument of basic quadratic functions are transformed by operations such as addition, subtraction, and multiplication by -1. Students will use a graphing calculator to graph the basic function and the transformed function in each case and note the graphic behaviors of each type of transformation. Students then explore the effects of dilations by changing the coefficient of the leading term of a quadratic function. They will compare the graph of the basic function to the graph of the transformed function and note the shrinking or stretching that occurs and list ordered pairs in each situation. In the last activity, students use only equations of functions that have undergone transformations and describe each transformation without graphing the functions.

12

Warm Up

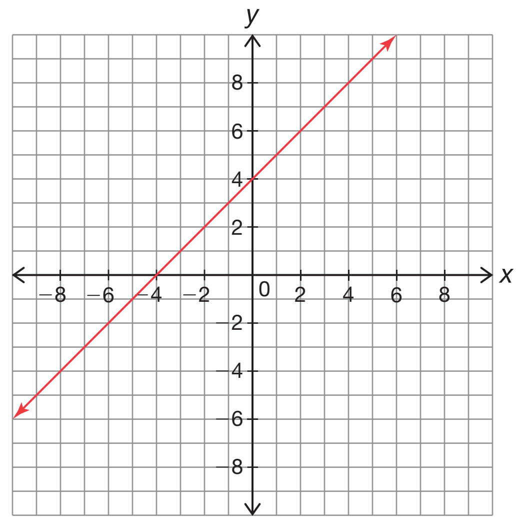
Use a graphing calculator to graph the linear equation $f(x) = x$.

1. Graph the linear function $f(x) = x$ and identify the y-intercept.



The y-intercept is (0, 0).

2. Graph the linear function $f(x) = x + 4$ and identify the y-intercept.



The y-intercept is (0, 4).

3. Molly is comparing the graph in Question 1 to the graph in Question 2. She thinks the graph shifted up 4 units. Lily is also comparing the graphs and she thinks the graph shifted to the left 4 units. Who is correct?

Both Molly and Lily are correct.

4. Without graphing the function, how do you think the graph of $f(x) = x$ will compare to the graph of $f(x) = x - 5$?

It will appear that the graph of $f(x) = x$ will have shifted down 5 units or to the right 5 units.

12

12

More Than Meets the Eye

Transformations of Quadratic Functions

12.7

LEARNING GOALS

In this lesson, you will:

- Translate quadratic functions.
- Reflect quadratic functions.
- Dilate quadratic functions.
- Write equations of quadratic functions given multiple transformations.
- Graph quadratic functions given multiple transformations.
- Identify multiple transformations given equations of quadratic functions.

KEY TERMS

- vertical dilation
- dilation factor

In 1854, Ignazio Porro did some transformational geometry and invented the "Porro Prism."

When an image is gathered by a traditional lens, it is upside down. Using right triangular prisms, Porro was able to turn the image upright.

There is a connection between Porro and binoculars too!

12

Problem 1

Students will use a graphing calculator to graph a basic quadratic function that has been vertically transformed, horizontally transformed, and reflected. They then compare the transformed graphs to the graph of the basic function and describe the graphic behavior in terms of shifts up or down, shifts to the left or right, and reflections over the *x*-axis. Students also rewrite the transformed equations in vertex form.

Grouping

Have students complete Question 1 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 1

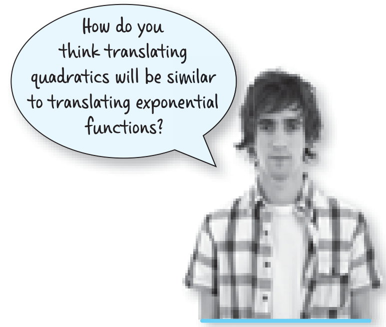
- Where is the graph of $c(x)$ in relation to the graph of the basic function?
- Where is the graph of $d(x)$ in relation to the graph of the basic function?
- What is the location of the vertex of the basic function?
- Does performing addition on the basic function raise or lower the graph of the function?
- What is location of the vertex of the transformed function?

12

PROBLEM 1 Translations and Reflections

1. Consider the three quadratic functions shown, where $g(x)$ is the basic function.

- $g(x) = x^2$
- $c(x) = x^2 + 3$
- $d(x) = x^2 - 3$



a. Write the functions $c(x)$ and $d(x)$ in terms of the basic function $g(x)$. For each, determine whether an operation is performed on the *function* $g(x)$ or on the *argument* of the function $g(x)$. Describe the operation.

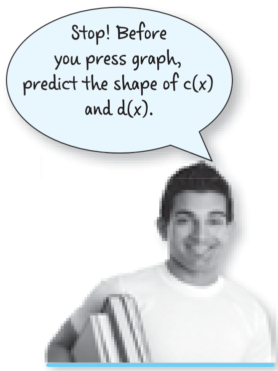
$c(x) = g(x) + 3$

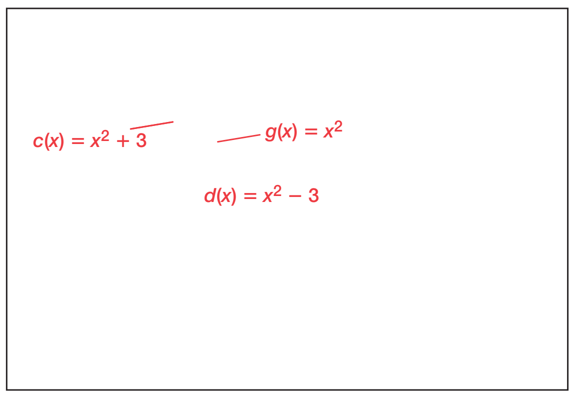
For $c(x)$, the operation of addition is performed on the function $g(x)$.

$d(x) = g(x) - 3$

For $d(x)$, the operation of subtraction is performed on the function $g(x)$.

b. Use a graphing calculator to graph each function with the bounds $[-10, 10] \times [-10, 10]$. Then, sketch the graph of each function on the coordinate plane provided. Label each graph.





- Does performing subtraction on the basic function raise or lower the graph of the function?
- What is location of the vertex of the transformed function?

Guiding Questions for Share Phase, Question 1

- If $h(x) = x^2 + n$, would the basic function $g(x) = x^2$ be raised or lowered? How many units?
- If $h(x) = x^2 + n$, where do you think the vertex would be located?
- If $j(x) = x^2 - n$, would the basic function $g(x) = x^2$ be raised or lowered? How many units?
- If $j(x) = x^2 - n$, where do you think the vertex would be located?

Grouping

Have students complete Question 2 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 2

- Where is the graph of $j(x)$ in relation to the graph of the basic function?
- Where is the graph of $k(x)$ in relation to the graph of the basic function?
- What is the location of the vertex of the basic function?
- Does performing addition on the argument of the basic function move the graph of the function to the left or move the graph to the right? Why do you suppose this happens?

c. Compare the graphs of $c(x)$ and $d(x)$ to the graph of the basic function. What do you notice?

The graph of $c(x)$ is 3 units up from the graph of the basic function.
The graph of $d(x)$ is 3 units down from the graph of the basic function.

d. Describe the type of transformation performed on $g(x)$ to result in:

- $c(x)$

  The graph of $g(x)$ is translated up 3 units.

- $d(x)$

  The graph of $g(x)$ is translated down 3 units.

Vertical translations are performed on $g(x)$ to result in $c(x)$ and $d(x)$.

e. Use coordinate notation to represent the vertical translation of each function.

- $c(x) = x^2 + 3$

  $(x, y) \rightarrow$ $(x, y + 3)$

- $d(x) = x^2 - 3$

  $(x, y) \rightarrow$ $(x, y - 3)$

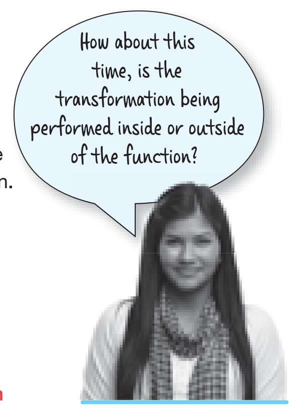
f. Write $c(x)$ and $d(x)$ in vertex form, $y - a(x - h)^2 + k$.

$c(x) =$ $(x - 0)^2 + 3$

$d(x) =$ $(x - 0)^2 - 3$

2. Consider the three quadratic functions shown, where $g(x)$ is the basic function.

- $g(x) = x^2$
- $j(x) = (x + 3)^2$
- $k(x) = (x - 3)^2$



a. Write the functions $j(x)$ and $k(x)$ in terms of the basic function $g(x)$. For each, determine whether an operation is performed on the *function* $g(x)$ or on the *argument* of the function $g(x)$. Describe the operation.

$j(x) =$ $g(x + 3)$

For $j(x)$, the operation of addition is performed on the argument of the function $g(x)$.

$k(x) =$ $g(x - 3)$

For $k(x)$, the operation of subtraction is performed on the argument of the function $g(x)$.

- What is the location of the vertex of the transformed function?
- Does performing subtraction on the argument of the basic function move the graph of the function to the left or move the graph to the right? Why do you suppose this happens?
- What is the location of the vertex of the transformed function?

12

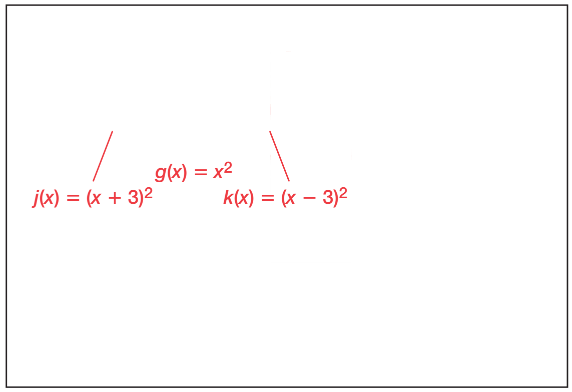
Guiding Questions for Share Phase, Question 2

- If $j(x) = (x + n)^2$, would the basic function $g(x) = x^2$ move to the left or move to the right? How many units?
- If $j(x) = (x + n)^2$, where do you think the vertex would be located?
- If $j(x) = (x - n)^2$, would the basic function $g(x) = x^2$ move to the left or move to the right? How many units?
- If $j(x) = (x - n)^2$, where do you think the vertex would be located?

Grouping

Have students complete Question 3 with a partner. Then share the responses as a class.

b. Use a graphing calculator to graph each function with the bounds $[-10, 10] \times [-10, 10]$. Then, sketch the graph of each function on the coordinate plane provided. Label each graph.



c. Compare the graphs of $j(x)$ and $k(x)$ to the graph of the basic function. What do you notice?

The graph of $j(x) = (x + 3)^2$ is 3 units to the left of the graph of the basic function.
The graph of $k(x) = (x - 3)^2$ is 3 units to the right of the graph of the basic function.

d. Describe the type of transformation performed on $g(x)$ to result in:

- $j(x)$

  The graph of $g(x)$ is translated left 3 units.

- $k(x)$

  The graph of $g(x)$ is translated right 3 units.

Horizontal translations are performed on $g(x)$ to result in both $j(x)$ and $k(x)$.

e. Use coordinate notation to represent the horizontal translation of each function.

- $j(x) = (x + 3)^2$

  $(x, y) \rightarrow$ $(x - 3, y)$

- $k(x) = (x - 3)^2$

  $(x, y) \rightarrow$ $(x + 3, y)$

f. Write $j(x)$ and $k(x)$ in vertex form.

$j(x) =$ $(x + 3)^2 + 0$

$k(x) =$ $(x - 3)^2 + 0$

3. Consider the three quadratic functions shown, where $g(x)$ is the basic function.

- $g(x) = x^2$
- $m(x) = -x^2$
- $n(x) = (-x)^2$

Guiding Questions for Share Phase, Question 3

- Where is the graph of $m(x)$ in relation to the graph of the basic function?
- Where is the graph of $n(x)$ in relation to the graph of the basic function?
- What is the location of the vertex of the basic function?
- Does performing multiplication by -1 on the basic function reflect the graph over the x-axis or reflect the graph over the y-axis?
- What is the location of the vertex of the transformed function?
- Does performing multiplication by -1 on the argument of the basic function reflect the graph over the x-axis or reflect the graph over the y-axis?
- If the location of the basic function remains unaltered after the multiplication by -1 to the argument, is it considered a transformed function?

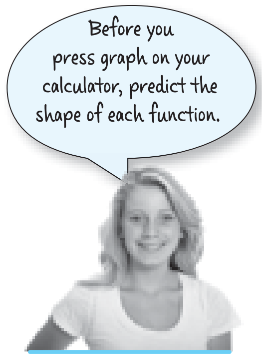
a. Write the functions $m(x)$ and $n(x)$ in terms of the basic function $g(x)$. For each, determine whether an operation is performed on the *function* $g(x)$ or on the *argument* of the function $g(x)$. Describe the operation.

$m(x) = -g(x)$

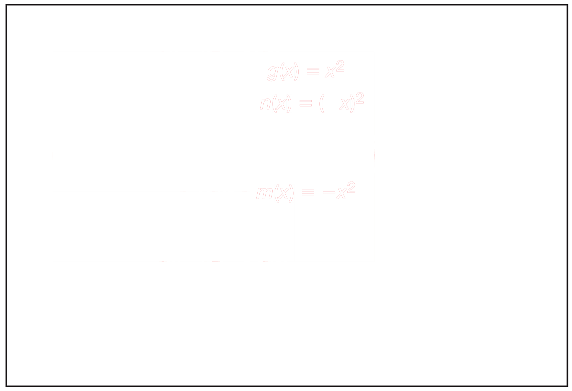
For $m(x)$, the operation of multiplication by -1 is performed on the function $g(x)$.

$n(x) = g(-x)$

For $n(x)$, the operation of multiplication by -1 is performed on the *argument* of the function $g(x)$.

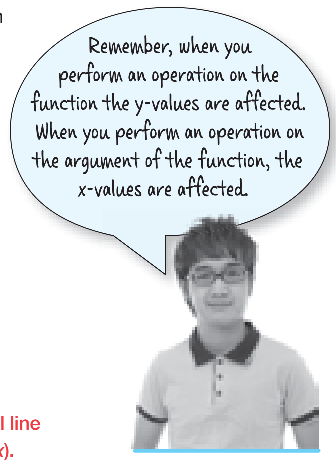


b. Use a graphing calculator to graph each function with the bounds $[-10, 10] \times [-10, 10]$. Then, sketch the graph of each function on the coordinate plane provided. Label each graph.



c. Compare the graphs of $m(x)$ and $n(x)$ to the graph of the basic function. What do you notice?

The graph of $m(x) = -x^2$ is a mirror image of the graph of the basic function, flipped vertically over the line $y = 0$. The graph of $n(x) = (-x)^2$ is the same as the graph of the basic function.



d. Describe the type of transformation performed on $g(x)$ to result in:

- $m(x)$

  The graph of $g(x)$ is reflected over the horizontal line $y = 0$.

- $n(x)$

  The graph of $g(x)$ is reflected over the vertical line $x = 0$, which produces the same graph as $g(x)$.

12

Grouping

Have students complete Question 4 with a partner. Then share the responses as a class.

Guiding Questions for Share Phase, Question 4

- What operation is used to transform the basic function?
- Was the operation performed on the basic function or the argument of the basic function?
- If b is a positive number, does the transformation raise or lower the basic function?
- If b is a negative number, does the transformation raise or lower the basic function?
- Was the operation performed on the basic function or the argument of the basic function?
- If b is a positive number, does the transformation move the basic function to the left or to the right?
- If b is a negative number, does the transformation move the basic function to the left or to the right?
- Was the operation performed on the basic function or the argument of the basic function?
- If the transformation is a result of multiplication by -1 of the basic function, is the graph reflected over the x-axis, or does the graph remain unaltered?

12

Reflections are performed on $g(x)$ to result in both $m(x)$ and $n(x)$.

e. Use coordinate notation to represent the reflections of each function.

- $m(x) = -x^2$

  $(x, y) \to$ $(x, -y)$

- $n(x) = (-x)^2$

  $(x, y) \to$ $(-x, y)$

f. What is special about a reflection of $g(x)$ over the vertical line $x = 0$? Explain your reasoning.

When $g(x)$ is reflected over the vertical line $x = 0$, the function does not change. Algebraically, this is because $g(x) = x^2$ produces the same values as $g(-x) = (-x)^2$, since squared values are always positive. Graphically, this is because the graph of $g(x) = x^2$ is symmetrical about the vertical line $x = 0$.

g. Write $m(x)$ and $n(x)$ in vertex form.

$m(x) =$ $(x + 0)^2 + 0$

$n(x) =$ $(-x - 0)^2 + 0$

4. Describe each graph in relation to its basic function.

a. Compare $f(x) = x^2 + b$ when $b > 0$ to the basic function $g(x) = x^2$.

The graph of $f(x)$ is translated b units up from the graph of $g(x)$.

b. Compare $f(x) = x^2 + b$ when $b < 0$ to the basic function $g(x) = x^2$.

The graph of $f(x)$ is translated b units down from the graph of $g(x)$.

c. Compare $f(x) = (x + b)^2$ when $b > 0$ to the basic function $g(x) = x^2$.

The graph of $f(x)$ is translated b units to the left of the graph of $g(x)$.

d. Compare $f(x) = (x + b)^2$ when $b < 0$ to the basic function $g(x) = x^2$.

The graph of $f(x)$ is translated b units to the right of the graph of $g(x)$.

e. Compare $f(x) = -x^2$ to the basic function $g(x) = x^2$.

The graph of $f(x)$ is a reflection of the graph of $g(x)$ about the horizontal line $y = 0$.

f. Compare $f(x) = (-x)^2$ to the basic function $g(x) = x^2$.

The graph of $f(x)$ is a reflection of the graph of $g(x)$ about the vertical line $x = 0$, which is the same as the graph of $g(x)$.

- If the transformation is a result of multiplication by -1 of the argument of the basic function, is the graph reflected over the x-axis, or does the graph remain unaltered?

Problem 2

Students will use a graphing calculator to graph basic quadratic functions that have been dilated. They then compare the transformed graphs to the graph of the basic function and describe the graphic behavior in terms of stretching and shrinking. The terms vertical dilation and dilation factor are introduced. Coordinate notation is used to indicate the vertical dilation.

Grouping

Have students complete Question 1 parts (a) through (f) with a partner. Then share the responses as a class.

Guiding Questions for Discuss Phase, Question 1 parts (a) through (f)

- Which function appears to be stretched wider than the basic function?
- Which function appears to be shrunken smaller than the basic function?
- Does a multiplier that is greater than 1 appear to stretch or shrink the basic function?
- Does a multiplier that is less than 1 appear to stretch or shrink the basic function?
- Why does it appear that only the *y*-coordinate of every point on the basic function is multiplied by the coefficient of the leading term?

PROBLEM 2 Dilation Station

1. Consider the three quadratic functions shown, where $g(x) = x^2$ is the basic function.

- $g(x) = x^2$
- $p(x) = 3x^2$
- $q(x) = \frac{1}{3}x^2$

a. Write the functions $p(x)$ and $q(x)$ in terms of the basic function $g(x)$. For each, determine whether an operation is performed on the *function* $g(x)$ or on the *argument* of the function $g(x)$. Describe the operation.

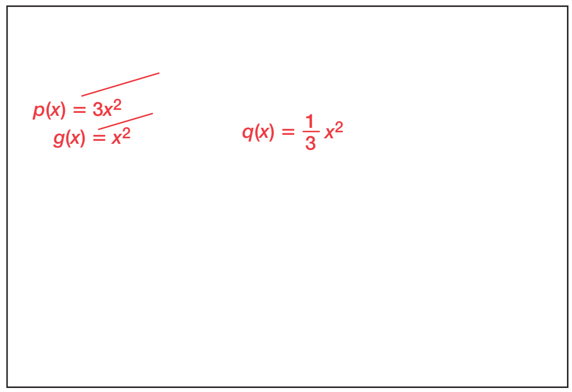
$p(x) =$ $3 \cdot g(x)$

The function $g(x)$ is multiplied by 3 to result in the equation for $p(x)$.

$q(x) =$ $\frac{1}{3} \cdot g(x)$

The function $g(x)$ is multiplied by $\frac{1}{3}$ to result in the equation for $q(x)$.

b. Use a graphing calculator to graph each function with the bounds $[-10, 10] \times [-10, 10]$. Then, sketch the graph of each function on the coordinate plane provided. Label each graph.



c. Compare the graphs of $p(x)$ and $q(x)$ to the graph of the basic function. What do you notice?

The graph of $p(x)$ has the same vertex as the graph of $g(x)$, but it appears that $g(x)$ has stretched up. The graph of $q(x)$ has the same vertex as the graph of $g(x)$, but it appears that $g(x)$ has shrunk down.

12

d. Complete the table of ordered pairs for the three given functions.

$g(x) = x^2$	$p(x) = 3x^2$	$q(x) = \frac{1}{3}x^2$
$(-2, 4)$	$(-2, 12)$	$\left(-2, \frac{4}{3}\right)$
$(-1, 1)$	$(-1, 3)$	$\left(-2, \frac{1}{3}\right)$
$(0, 0)$	$(0, 0)$	$(0, 0)$
$(1, 1)$	$(1, 3)$	$\left(-2, \frac{1}{3}\right)$
$(2, 4)$	$(2, 12)$	$\left(-2, \frac{4}{3}\right)$

e. Use your table to compare the ordered pairs of the graphs of $p(x)$ and $q(x)$ to the ordered pairs of the graph of the basic function $g(x)$. What do you notice?

For the same x-coordinate, the y-coordinate of $p(x)$ is 3 times the y-coordinate of $g(x)$. For the same x-coordinate, the y-coordinate of $q(x)$ is one-third the y-coordinate of $g(x)$.

A **vertical dilation** of a function is a transformation in which the y-coordinate of every point on the graph of the function is multiplied by a common factor called the **dilation factor**. A vertical dilation stretches or shrinks the graph of a function.

You can use the coordinate notation shown to indicate a vertical dilation.

$$(x, y) \rightarrow (x, ay), \text{ where } a \text{ is the dilation factor.}$$

Grouping

- Ask a student to read the definitions and example, then discuss as a class.
- Have students complete part (f) and Question 2 with a partner. Then share the responses as a class.

12

f. Use coordinate notation to represent the vertical dilation of each function.

- $p(x) = 3x^2$

  $(x, y) \rightarrow (x, 3y)$

- $q(x) = \frac{1}{3}x^2$

  $(x, y) \rightarrow \left(x, \frac{1}{3}y\right)$

Guiding Questions for Discuss Phase, Question 2

- Why do the fractional coefficients that are greater than 0 but less than 1 of the leading term stretch the basic function rather than shrink the function?
- Why do the coefficients of the leading term that are greater than 1 shrink the basic function rather than stretch the function?
- Which coordinate of every point on the basic function is multiplied by the coefficient of the leading term?

Problem 3

Students are given key characteristics of quadratic functions. Using these characteristics, they will write an equation to describe each function and sketch a graph of the function.

Grouping

Have students complete Questions 1 and 2 with a partner. Then share the responses as a class.

2. Describe each graph in relation to its basic function.

a. Compare $f(x) = ax^2$ when $a > 1$ to the basic function $g(x) = x^2$.

Each y-coordinate on the graph of $f(x)$ is a times each y-coordinate on the graph of $g(x)$, which stretches the graph vertically when $a > 1$.

b. Compare $f(x) = ax^2$ when $0 < a < 1$ to the basic function $g(x) = x^2$.

Each y-coordinate on the graph of $f(x)$ is a times each y-coordinate on the graph of $g(x)$, which shrinks the graph vertically when $0 < a < 1$.

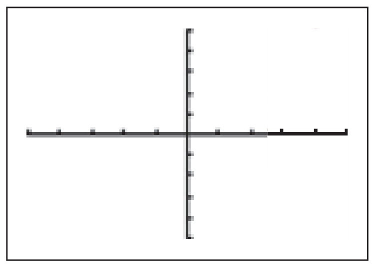
PROBLEM 3 Name That Parabola

1. Use the given characteristics to write a function and sketch a graph of $f(x)$.

a. Write a function in vertex form and sketch a graph that has these characteristics:

- The function is quadratic.
- The function is continuous.
- The parabola opens upward.
- The function is translated 5 units to the right of $f(x) = x^2$.

Equation: $f(x) =$ ______________



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Guiding Questions for Discuss Phase, Question 1

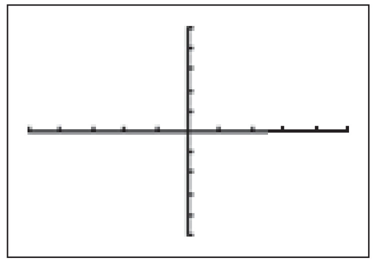
- How would you describe the shape of a quadratic function?
- What is the difference between a continuous function and a non-continuous function?
- If the parabola opens upward, what must be present in the equation of the function?
- If the parabola opens downward, what must be present in the equation of the function?
- If the parabola is translated to the right, what must be present in the equation of the transformed function?
- If the parabola is translated down, what must be present in the equation of the transformed function?
- If the parabola is vertically dilated by a factor of 2, what must be present in the equation of the transformed function?
- If the parabola is translated to the left, what must be present in the equation of the transformed function?
- If the parabola is translated up, what must be present in the equation of the transformed function?

12

b. Write a function in vertex form and sketch a graph that has these characteristics:

- The function is quadratic.
- The function is continuous.
- The parabola opens downward.
- The function is translated 1 unit down from $f(x) = -x^2$ and is vertically dilated with a dilation factor of 2.

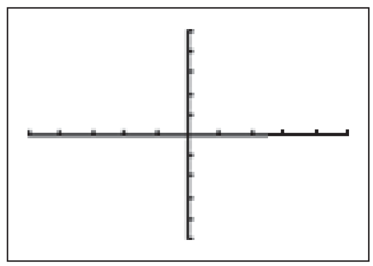
Equation: $f(x) = -2(x - 0)^2 - 1$



c. Write a function in vertex form and sketch a graph that has these characteristics:

- The function is quadratic.
- The function is continuous.
- The parabola opens upward.
- The function is translated 4 units down and 3 units to the left of $f(x) = x^2$.
- The function is vertically dilated with a dilation factor of $\frac{1}{4}$.

Equation: $f(x) = \frac{1}{4}(x + 3)^2 - 4$



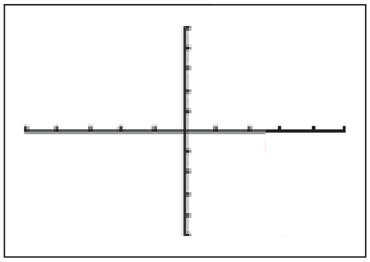
Guiding Questions for Share Phase, Question 2

- Is the basic function associated with a transformation or the argument of the basic function?
- Which directional shifts occur when the basic function is associated with the transformation?
- Which directional shifts occur when the argument of the basic function is associated with the transformation?
- Using only the equation of the function, how do you determine the number of transformations the basic function has undergone?

d. Write a function in vertex form and sketch a graph that has these characteristics:

- The function is quadratic.
- The function is continuous.
- The parabola opens downward.
- The function is translated 8 units up and 2 units to the right of $f(x) = x^2$.

Equation: $f(x) = -(x - 2)^2 + 8$



2. Based on the equation of each function, describe how the graph of each function compares to the graph of $g(x) = x^2$.

a. $w(x) = (x + 2)^2$

The graph of $w(x)$ is 2 units to the left of the graph of $g(x)$.

b. $t(x) = 3x^2 + 4$

Each y-coordinate of the graph of $t(x)$ is 3 times the y-coordinate of the graph of $g(x)$ (it is stretched vertically) and is 4 units up from the graph of $g(x)$.

c. $z(x) = -(x - 1)^2 - 10$

The vertex of $z(x)$ is 1 unit to the right and 10 units down from the vertex of $g(x)$, and $z(x)$ opens downward.

d. $r(x) = \frac{1}{2}(x + 6)^2 + 7$

Each y-coordinate of the graph of $r(x)$ is $\frac{1}{2}$ the y-coordinate of the graph of $g(x)$ (it is shrunk vertically) and is 6 units left and 7 units up from the graph of $g(x)$.

Be prepared to share your solutions and methods.

12

Check for Students' Understanding

Describe every transformation the function $f(x) = x^2$ has undergone in each.

1. $f(x) = (x + 10)^2 + 3$

   The parabola was translated 10 units to the left and 3 units up.

2. $f(x) = -(x - 6)^2$

   The parabola was reflected over the x-axis and translated 6 units to the right.

3. $f(x) = 5(x + 6)^2 - 2$

   The parabola vertically shrunk by a dilation factor of 5 and is horizontally translated 6 units to the left and vertically translated 2 units down.

12

Chapter 12 Summary

KEY TERMS

- standard form (general form) of a quadratic function (12.1)
- parabola (12.1)
- leading coefficient (12.2)
- second differences (12.2)
- vertical motion model (12.3)
- zeros (12.3)
- interval (12.3)
- open interval (12.3)
- closed interval (12.3)
- half-closed interval (12.3)
- half-open interval (12.3)
- factor an expression (12.4)
- factored form (12.4)
- vertex (12.5)
- axis of symmetry (12.5)
- vertex form (12.6)
- vertical dilation (12.7)
- dilation factor (12.7)

12.1 Writing Quadratic Functions in Standard Form

A quadratic function written in standard form, or general form, is in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. In this form, a and b represent numerical coefficients and c represents a constant.

Example

Cara wants to make an herb garden with a rectangular area in the center and equal square sections on both sides. The entire length of the garden can be no longer than 20 feet. Let g represent the width of each square section. The length would be $20 - 2g$.

The area of the rectangular part of the garden will be $g(20 - 2g)$.

Use the Distributive Property to rewrite the expression: $20g - 2g^2$.

Write the area as a function in standard form: $A(g) = -2g^2 + 20g + 0$.

$a = -2$; $b = 20$; $c = 0$

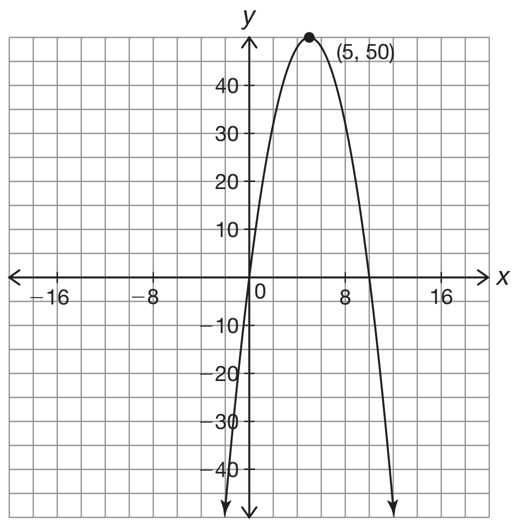
12

12.1 Identifying Maximums and Minimums of Quadratic Graphs

The shape that a quadratic function forms when graphed is called a parabola. A parabola is a smooth curve in a U-shape that is upside-down when *a* is negative and right-side-up when *a* is positive. You can use a graphing calculator to identify the absolute maximum or minimum of a quadratic function.

Example

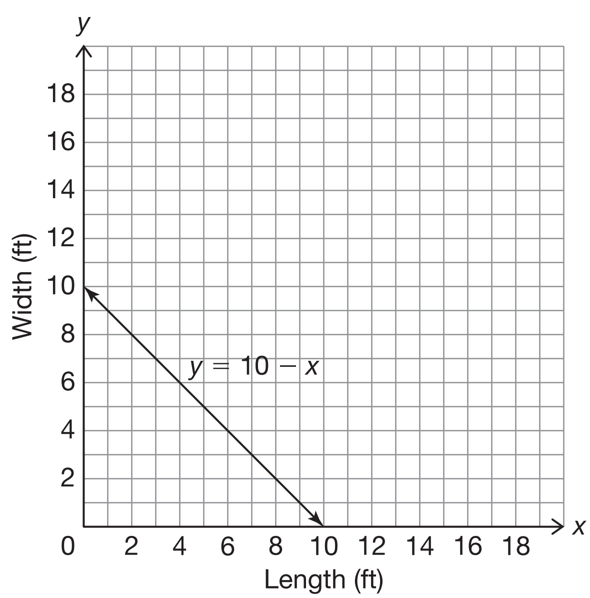
The function $A(g) = -2g^2 + 20g$ has an absolute maximum of (5, 50).



12.2 Creating and Analyzing Linear and Quadratic Graphs

You have 20 feet of fencing with which to enclose an area. The table of possible lengths and widths represents a linear function. The table with possible lengths and areas represents a quadratic function.

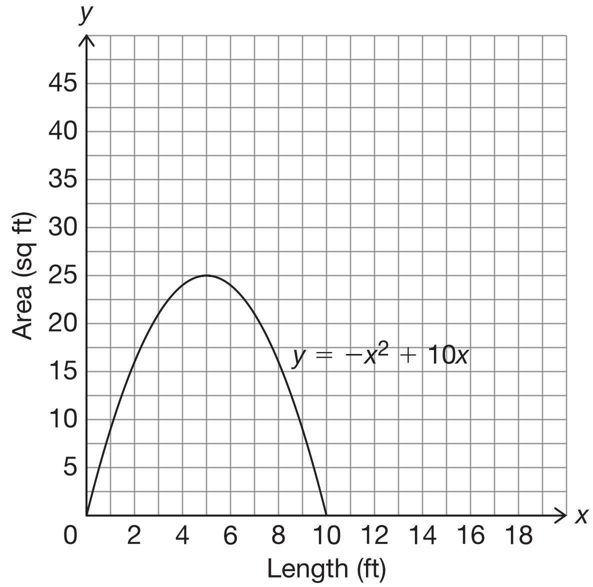
Length (ft)	Width (ft)
0	10
1	9
2	8
3	7
4	6
5	5
6	4
7	3
8	2
9	1
10	0



The *y*-intercept of 10 indicates the width when the length is 0 feet. The *x*-intercept of 10 indicates the length when the width is 0 feet. Neither point is appropriate for an enclosed area.

12

Length (ft)	Area (ft^2)
0	0
1	9
2	16
3	21
4	24
5	25
6	24
7	21
8	16
9	9
10	0



The *x*-intercepts are 0 and 10, indicating the length when the area is 0 feet. Neither point is appropriate for an enclosed area. The *y*-intercept is 0, indicating the area when the length is 0.

12.2 Identifying Linear and Quadratic Functions

First differences are the differences between successive output values when successive input values have a difference of 1. Second differences are the differences between consecutive values of first differences. Linear functions have constant first differences and second differences of 0. Quadratic functions have changing first differences and constant second differences.

Example

Linear function: $y = -x + 2$ Quadratic function: $y = 2x^2 - 3x$

x	y	First Differences	Second Differences
0	2		
		−1	
1	1		0
		−1	
2	0		0
		−1	
3	−1		0
		−1	
4	−2		

x	y	First Differences	Second Differences
0	0		
		−1	
1	−1		4
		3	
2	2		4
		7	
3	9		4
		11	
4	20		

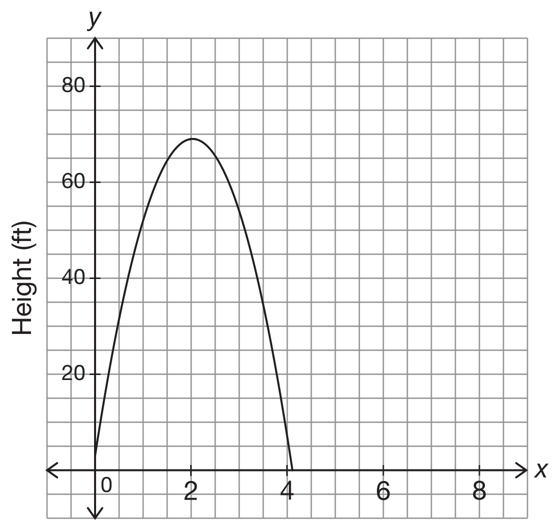
12.3

Identifying and Describing the Domain and Range of a Quadratic Function

The domain of a function is all the possible x-values. The range of a function is all the possible y-values. The domain and range can be different for the same function when considering the mathematical function alone compared to considering the contextual situation.

Example

A tennis ball is hit into the air from 3 feet above ground with a vertical velocity of 65 feet per second. The function that describes the height of the ball in terms of time is $g(t) = -16t^2 + 65t + 3$.



The domain is all real numbers from negative infinity to positive infinity. In terms of the problem situation, the domain is all real numbers greater than or equal to 0 and less than or equal to about 4.1. The range is all real numbers less than or equal to about 69. In terms of the problem situation, the range is all real numbers less than or equal to about 69 and greater than or equal to 0.

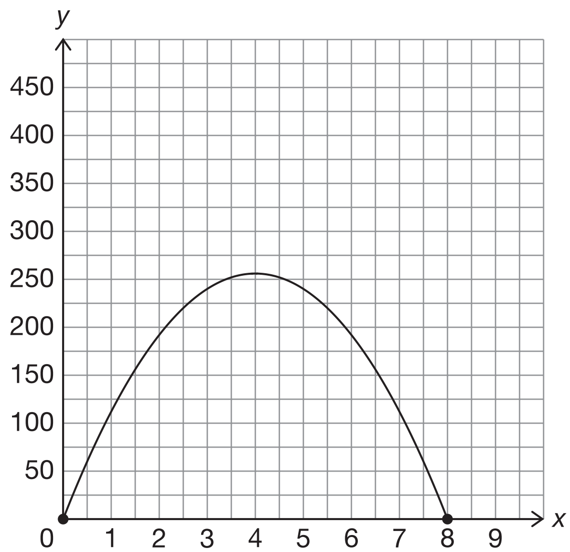
12

12.3 Identifying the Zeros of a Quadratic Function and the Roots of a Quadratic Equation

The x-intercepts of a graph of a quadratic function are also called the zeros of the quadratic function. The points where the graph crosses the x-axis are called the x-intercepts.

Example

A firework is launched into the air from the ground with a vertical velocity of 128 feet per second. The function that describes the height of the firework in terms of time is $g(t) = -16t^2 + 128t$. The x-intercepts, or zeros are (0, 0) and (8, 0).



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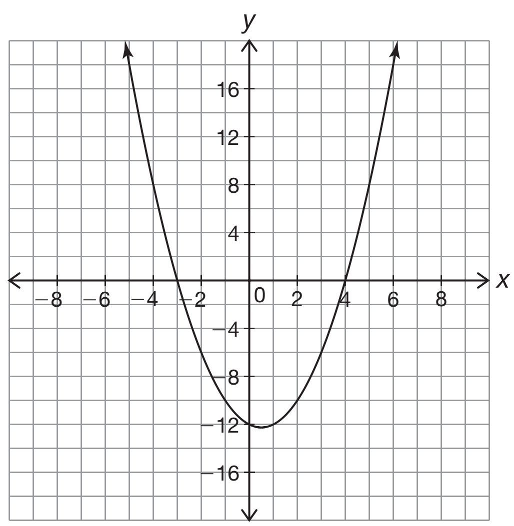
12.3

Determining Intervals of Increase and Decrease of a Quadratic Function

The absolute maximum or absolute minimum is the turning point of a parabola. If the quadratic function has an absolute maximum, the x-value increases to the maximum and then decreases. If it has an absolute minimum, the x-value decreases to the minimum and then increases. You can use interval notation to describe the interval of the domain in which the function is increasing and the interval of the domain in which the function is decreasing.

An interval is defined as the set of real numbers between two given numbers. An open interval (a, b) describes the set of all numbers between a and b, but not including a or b. A closed interval $[a, b]$ describes the set of all numbers between a and b, including a and b. A half-closed or half-open interval $(a, b]$ describes the set of all numbers between a and b, including b but not including a. Or, $[a, b)$ describes the set of all numbers between a and b, including a but not including b. Intervals that are unbounded are written using the symbol for infinity, ∞.

Example



12

The graph represents the function $f(x) = x^2 - x - 12$.

Domain: All real numbers

Range: All real numbers greater than or equal to -12.25

y-intercept: $(0, -12)$

Zeros: $(-3, 0)$, $(4, 0)$

Interval of decrease: $\left(-\infty, \frac{1}{2}\right)$

Interval of increase: $\left(\frac{1}{2}, \infty\right)$

12.4 Factoring the Greatest Common Factor from an Algebraic Expression

To factor an expression means to use the Distributive Property in reverse to rewrite the expression as a product of factors. When factoring algebraic expressions, you can factor out the greatest common factor from all the terms. When factoring an expression that contains a negative leading coefficient, it is convention to factor out the negative sign.

Example

$25x + 85 = 5(5x) + 5(17)$

$= 5(5x + 17)$

$-3x + 27 = -3(x) + (-3)(-9)$

$= -3(x - 9)$

12.4 Writing Quadratic Functions in Factored Form

A quadratic function written in factored form is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$.

Example

$f(x) = x^2 + 9x + 14$

$= 1(x + 2)(x + 7)$

12.4 Writing a Quadratic Function in Factored Form Given Its *x*-Intercepts

A quadratic function with *x*-intercepts $(r_1, 0)$ and $(r_2, 0)$, can be written in factored form as $f(x) = a(x - r_1)(x - r_2)$. The sign of a tells you whether the parabola opens upward (positive a) or downward (negative a).

Example

The quadratic function in factored form of a parabola that opens downward and has zeros at (3, 0) and (−2, 0) is:

$f(x) = a(x - 3)(x + 2)$ for $a < 0$

12.4 Determining *x*-Intercepts from Functions in Factored Form

For a quadratic function written in factored form $f(x) = a(x - r_1)(x - r_2)$, the variables r_1 and r_2 represent the *x*-coordinates of the *x*-intercepts. The *x*-intercepts are $(r_1, 0)$ and $(r_2, 0)$.

Example

$g(x) = (8 - 2x)(x + 3)$

The function in factored form is $g(x) = -2(x - 4)(x + 3)$.

The zeros are (4, 0) and (−3, 0).

12.5 Determining the Axis of Symmetry of Quadratic Functions

The axis of symmetry of a parabola is the vertical line that passes through the vertex and divides the parabola into two mirror images. The equation of that line is $x =$ the x-coordinate of the vertex. The value of the axis of symmetry is the average of the two x-intercepts.

Example

The x-intercepts of a parabola are $(-3, 0)$ and $(7, 0)$.

The axis of symmetry is $x = 2$ because $\frac{-3 + 7}{2} = \frac{4}{2} = 2$.

12.5 Determining the Vertex of Quadratic Functions

The vertex of a parabola is the lowest or highest point on the curve. The equation of the axis of symmetry is $x =$ the x-coordinate of the vertex. The y-coordinate of the vertex can be found by substituting the x-value into the equation and solving for y.

Example

The equation of a parabola is $f(x) = -x^2 - 6x + 7$.

The axis of symmetry is $x = -3$, so the x-coordinate of the vertex is -3.

The y-coordinate when $x = -3$ is:

$$f(-3) = -(-3)^2 - 6(-3) + 7$$
$$= -9 + 18 + 7$$
$$= 16$$

The vertex is at $(-3, 16)$.

12.5 Determining the Axis of Symmetry Using Symmetric Points

The x-coordinate of the vertex is halfway between the x-coordinates of symmetric points on a parabola. In other words, the x-coordinate of the vertex is the average, or midpoint, of the x-coordinates of points on the parabola with the same y-coordinates.

Example

The equation of a parabola is $f(x) = x^2 + 3x - 8$ and two symmetric points on the parabola are $(-7, 20)$ and $(4, 20)$.

The axis of symmetry is $x = -\frac{3}{2}$ because $\frac{-7 + 4}{2} = -\frac{3}{2}$.

12

12.5 Determining Symmetric Points on the Parabola Using a Vertex

The y-coordinates of symmetric points on a parabola are the same. Symmetric points are the same distance from the axis of symmetry. So, the x-coordinate of the vertex is the average, or midpoint, of the x-coordinates of points on the parabola with the same y-coordinates.

Example

The vertex of a parabola is (3, 5). A point on the parabola is (0, 3). Another point on the parabola is (6, 3) because:

$$\frac{0 + a}{2} = 3$$

$$0 + a = 6$$

$$a = 6$$

12.6 Identifying the Vertex of a Quadratic Function in Vertex Form

A quadratic function written in vertex form is in the form $f(x) = a(x - h)^2 + k$, where $a \neq 0$. The variable h represents the x-coordinate of the vertex. The variable k represents the y-coordinate of the vertex.

Example

$f(x) = x^2 - 8x + 15$

vertex: $(4, -1)$

vertex form: $f(x) = (x - 4)^2 - 1$

zero(s): (3, 0) and (5, 0)

factored form: $f(x) = (x - 3)(x - 5)$

12

12.6 Identifying Characteristics of a Parabola Given Its Equation in Different Forms

In a quadratic function written in standard form, the sign of the a value tells you whether the parabola opens up or down. The c value tells you the y-intercept. In a quadratic function written in factored form, the sign of the a value tells you whether the parabola opens up or down. The r_1 and r_2 values tell you the x-intercepts. In a quadratic function written in vertex form, the sign of the a value tells you whether the parabola opens up or down. The variable h represents the x-coordinate of the vertex and the variable k represents the y-coordinate of the vertex.

Example

a. $f(x) = (x + 3)^2 + 6$

The function is in vertex form.

The parabola opens up and the vertex is $(-3, 6)$.

b. $f(x) = -x^2 + 5x + 1$

The function is in standard form.

The parabola opens down and the y-intercept is $(0, 1)$.

c. $f(x) = (x + 2)(x - 5)$

The function is in factored form.

The parabola opens up and the zeros are $(-2, 0)$ and $(5, 0)$.

12.6 Writing an Equation of a Parabola Given Information about Its Graph

Given information about whether the parabola opens up or down, the y-intercept, x-intercepts, or vertex, you can write an equation in the appropriate form.

12

Example

a. The zeros are 3 and 7, and the parabola opens up.

$f(x) = a(x - 3)(x - 7)$ for $a > 0$

b. The vertex is (4, 2), and the parabola opens down.

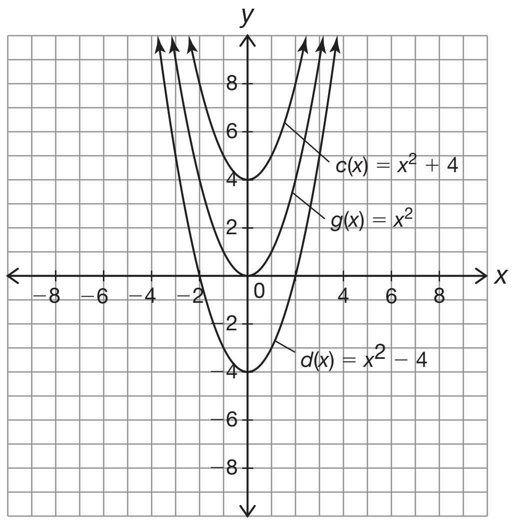
$f(x) = a(x - 4)^2 + 2$ for $a < 0$

12.7 Translating Quadratic Functions

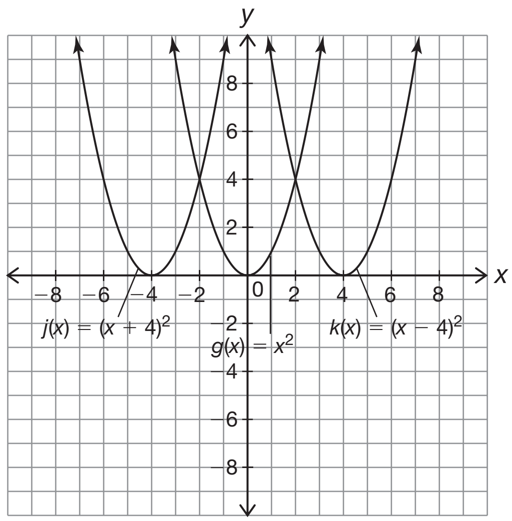
Vertical translations are performed on a basic quadratic function $g(x) = x^2$. Adding to the equation translates it up and subtracting translates it down. Horizontal translations are performed on the argument, x, of a basic quadratic function.

Example

$g(x) = x^2$	basic function
$c(x) = x^2 + 4$	$g(x)$ translated 4 units up, so $(x, y) \rightarrow (x, y + 4)$.
$d(x) = x^2 - 4$	$g(x)$ translated 4 units down, so $(x, y) \rightarrow (x, y - 4)$.



$g(x) = x^2$	basic function
$j(x) = (x + 4)^2$	$g(x)$ translated 4 units left, so $(x, y) \rightarrow (x - 4, y)$.
$k(x) = (x - 4)^2$	$g(x)$ translated 4 units right, so $(x, y) \rightarrow (x + 4, y)$.



12.7 Reflecting Quadratic Functions

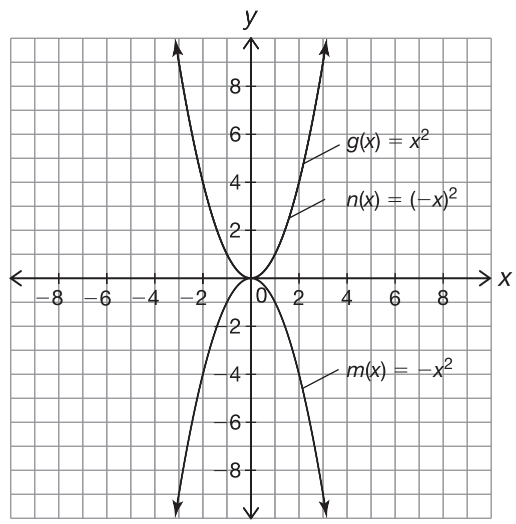
Multiplying the basic quadratic function by -1 results in a reflection over the line $y = 0$. Multiplying the argument of the basic quadratic function by -1 results in a reflection over the line $x = 0$, which ends up being the same as the original function because squared values are always positive.

Example

$g(x) = x^2$ basic function

$m(x) = -x^2$ $g(x)$ is reflected over $y = 0$, so $(x, y) \rightarrow (x, -y)$.

$n(x) = (-x)^2$ $g(x)$ is reflected over $x = 0$, so $(x, y) \rightarrow (-x, y)$.



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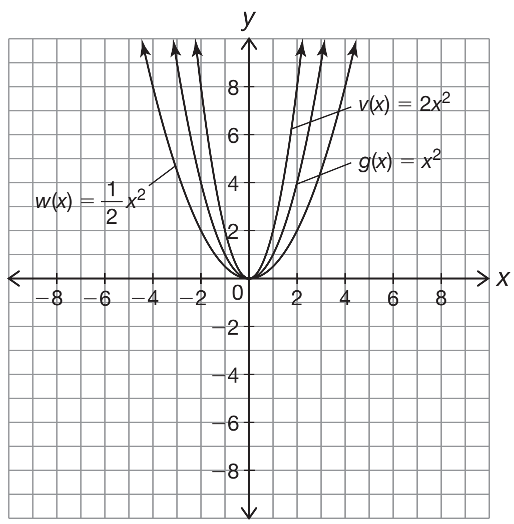
12.7 Dilating Quadratic Functions

A vertical dilation of a function is a transformation in which the y-coordinate of every point on the graph of the function is multiplied by a common factor called the dilation factor. A vertical dilation stretches or shrinks the graph of a function. When the dilation factor, a, is greater than 1, the graph of the function appears to be stretched vertically. When $0 < a < 1$, the function appears to shrink vertically. You can use the coordinate notation shown to indicate a vertical dilation.

$(x, y) \rightarrow (x, ay)$, where a is the dilation factor.

Example

$g(x) = x^2$	basic function
$v(x) = 2x^2$	$g(x)$ stretched by a dilation factor of 2, so $(x, y) \rightarrow (x, 2y)$.
$w(x) = \frac{1}{2}x^2$	$g(x)$ shrunk by a dilation factor of $\frac{1}{2}$, so $(x, y) \rightarrow (x, \frac{1}{2}y)$.



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12.7 Writing Equations Given Transformations

Use given characteristics including a transformation done to a basic equation to write a quadratic equation in vertex form and sketch its graph.

Example

The function is quadratic.

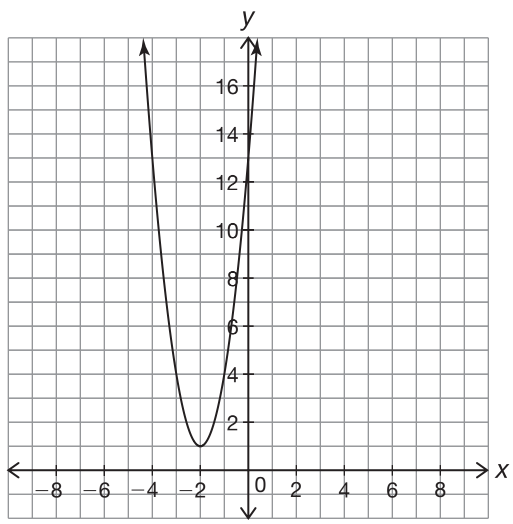
The function is continuous.

The parabola opens upward.

The function is translated 2 units to the left and 1 unit up from $f(x) = x^2$.

The function is vertically dilated with a dilation factor of 3.

Equation: $f(x) = 3(x + 2)^2 + 1$



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