Three-Dimensional Figures

11

Chapter 11 Overview

This chapter focuses on three-dimensional figures. The first two lessons address rotating and stacking two-dimensional figures to created three-dimensional solids. Cavalieri's principle is presented and is used to derive the formulas for a volume of a cone, pyramid, and sphere. The chapter culminates with the topics of cross sections and diagonals in three dimensions.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
11.1	Rotating Two-Dimensional Figures through Space	G.GMD.4	1	This lesson explores rotations of two-dimensional figures through space. Questions ask students to identify the solids formed by rotating two-dimensional figures.	X				
11.2	Translating and Stacking Two-Dimensional Figures	G.GMD.4 G.MG.3	2	This lesson explores translating and stacking of two-dimensional figures. Questions ask students to identify the solids formed by translating and stacking two-dimensional figures.	X			X	
11.3	Application of Cavalieri's Principles	G.GMD.1 G.GMD.2 G.GMD.4	1	This lesson presents Cavalieri's principle for two-dimensional figures and three-dimensional solids. Questions ask students to estimate the area of two-dimensional figures and the volume of three-dimensional solids figures using Cavalieri's principles.	X			X	
11.4	Volume of Cones and Pyramids	G.MG.1 G.GMD.4	2	This lesson presents an informal argument for the derivation of the formulas for the volume of a cone and the volume of a pyramid. Questions ask students to derive the volume formulas based on stacking and rotating two-dimensional figures.	X	X			
11.5	Volume of a Sphere	G.GMD.4	1	This lesson presents an informal argument for the derivation of the formula for the volume of a sphere. Questions walk students through the steps of the argument using properties of cones, cylinders, and hemispheres.	X		X		

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Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
11.6	Using Volume Formulas	G.GMD.3 G.MG.1	1	This lesson provides student with the opportunity to solve problems using the volume formulas for a pyramid, a cylinder, a cone, and a sphere. Questions call for students to determine which formula to use in order to determine volume or other dimensions in a variety of scenarios.	X		X		
11.7	Cross Sections	G.GMD.4 G.MG.1	2	This lesson provides student with the opportunity to explore cross sections of solids. Questions ask students to identify the shape of cross sections between planes and three-dimensional solids.	X		X		
11.8	Diagonals in Three Dimensions	G.MG.1 G.MG.3	1	This lesson focuses on determining the length of diagonals in rectangular prisms. Questions ask students to calculate the length of the diagonal of rectangular prisms by using the Pythagorean and by deriving a formula.	X		X		

Skills Practice Correlation for Chapter 11

Lesson		Problem Set	Objectives
11.1	Rotating Two-Dimensional Figures through Space		Vocabulary
		1 – 6	Identify solid figures formed from rotating given plane figures
		7 – 12	Relate the dimensions of solid figures and plane figures rotated to create the solid figures
11.2	Translating and Stacking Two-Dimensional Figures		Vocabulary
		1 – 6	Identify solid figures formed from the translation of a plane figure
		7 – 12	Identify solid figures formed from the stacking of congruent plane figures
		13 – 18	Identify solid figures formed from the stacking of similar plane figures
		19 – 24	Relate the dimensions of solid figures and plane figures
11.3	Application of Cavalieri's Principles		Vocabulary
		1 – 3	Use Cavalieri's principles to estimate the approximate area or volume of irregular or oblique figures
11.4	Volume of Cones and Pyramids	1 – 10	Calculate the volume of cones
		11 – 20	Calculate the volume of square pyramids
11.5	Volume of a Sphere		Vocabulary
		1 – 10	Calculate the volume of spheres
11.6	Using Volume Formulas	1 – 6	Calculate the volume of pyramids
		7 – 14	Calculate the volume of cylinders
		15 – 22	Calculate the volume of cones
		23 – 26	Calculate the volume of spheres
11.7	Cross Sections	1 – 10	Describe the shape of cross sections
		11 – 16	Sketch and describe cross sections given descriptions
		17 – 24	Determine the shape of cross sections parallel and perpendicular to the base of solid figures
		25 – 30	Draw solid figures given the shape of a cross section

Lesson		Problem Set	Objectives
11.8	Diagonals in Three Dimensions	1 – 6	Draw three-dimensional diagonals
		7 – 12	Determine the length of three-dimensional diagonals
		13 – 18	Sketch triangles using the two-dimensional diagonals and dimensions of solid figures
		19 – 24	Use the two-dimensional diagonals of solid figures to determine the length of the three-dimensional diagonal
		25 – 30	Use a three-dimensional diagonal formula to solve problems

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11.1

Whirlygigs for Sale!

Rotating Two-Dimensional Figures through Space

LEARNING GOALS

In this lesson, you will:

- Apply rotations to two-dimensional plane figures to create three-dimensional solids.
- Describe three-dimensional solids formed by rotations of plane figures through space.

KEY TERM

- disc

ESSENTIAL IDEAS

- Rotations are applied to two-dimensional plane figures.
- Three-dimensional solids are formed by rotations of plane figures through space.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-GMD Geometric Measurement and Dimension

Visualize relationships between two-dimensional and three dimensional objects

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Overview

Models of two-dimensional figures are rotated through space. Students analyze the three-dimensional solid images associated with the rotation. Technically, the rotation of a single point or collection of points changes the location of the point or collection of points. Applied to the rotating pencil activities in this lesson, it is not an actual solid that results from rotating the pencil, rather an image to the eye that is associated with this motion.

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Warm Up

1. List objects you have seen that spin but do not require batteries.

   Answers will vary.

   A top, a gyroscope, a jack, a ball, a coin, a yo-yo

2. Describe how these toys are able to spin.

   These objects are powered by energy sources such as flicking a wrist, twirling a few fingers, or pulling a string.

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11

Whirlygigs for Sale!

Rotating Two-Dimensional Figures through Space

11.1

LEARNING GOALS

In this lesson, you will:

- Apply rotations to two-dimensional plane figures to create three-dimensional solids.
- Describe three-dimensional solids formed by rotations of plane figures through space.

KEY TERM

- disc

Throughout this chapter, you will analyze three-dimensional objects and solids that are "created" through transformations of two-dimensional plane figures.

But, of course, solids are not really "created" out of two-dimensional objects. How could they be? Two-dimensional objects have no thickness. If you stacked a million of them on top of each other, their combined thickness would still be zero. And translating two-dimensional figures does not really create solids. Translations simply move a geometric object from one location to another.

However, *thinking* about solid figures and three-dimensional objects as being "created" through transformations of two-dimensional objects is useful when you want to see how volume formulas were "created."

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Problem 1

A scenario is used which prompts students to tape a rectangle to a pencil and rotate the pencil.

They identify the three-dimensional solid image associated with this rotation as a cylinder and relate the dimension *s* of the rectangle to the dimensions of the image of the solid. This activity is repeated with a circle, and a triangle.

Grouping

- Ask students to read the information. Discuss as a class.
- Have students complete Questions 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- If the rectangle was turned lengthwise and then taped, how would that affect the image of the solid associated with the rotation?
- If the rectangle wasn't taped in the middle, but taped along a side, how would that affect the image of the solid associated with the rotation?

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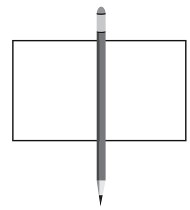
PROBLEM 1 Rectangular Spinners

You and a classmate are starting a summer business, making spinning toys for small children that do not require batteries and use various geometric shapes.

Previously, you learned about rotations on a coordinate plane. You can also perform rotations in three-dimensional space.

1. You and your classmate begin by exploring rectangles.
   a. Draw a rectangle on an index card.
   b. Cut out the rectangle and tape it along the center to a pencil below the eraser as shown.
   c. Hold on to the eraser with your thumb and index finger such that the pencil is resting on its tip. Rotate the rectangle by holding on to the eraser and spinning the pencil. You can get the same effect by putting the lower portion of the pencil between both palms of your hands and rolling the pencil by moving your hands back and forth.



   d. As the rectangle rotates about the pencil, the image of a three-dimensional solid is formed. Which of these solids most closely resembles the image formed by the rotating rectangle?

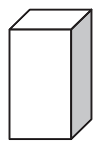


Figure 1

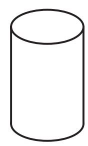


Figure 2

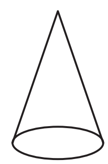


Figure 3

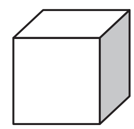


Figure 4

   The image formed by the rotating rectangle closely resembles Figure 2.

   e. Name the solid formed by rotating the rectangle about the pencil.
   The solid formed by rotating the rectangle about the pencil is a cylinder.

   f. Relate the dimensions of the rectangle to the dimensions of this solid.
   The width of the rectangle is the radius of the cylinder. The length of the rectangle is the height of the cylinder.

- If the rectangle was taped in a diagonal fashion to the pencil, how would that affect the image of the solid associated with the rotation?
- Will the image associated with this rotation always be a cylinder?

Grouping

Have students complete Question 2 with a partner. Then have students share their responses as a class.

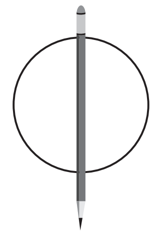
Guiding Questions for Share Phase, Question 2

- If the circle was turned 90 degrees and then taped, how would that affect the image of the solid associated with the rotation?
- If the circle wasn't taped in the middle, but taped slightly to the right or to the left of the middle, how would that affect the image of the solid associated with the rotation?
- If the circle was taped along its circumference to the pencil, how would that affect the image of the solid associated with the rotation?
- Will the image associated with this rotation always be a sphere?

2. You and your classmate explore circles next.

a. Draw a circle on an index card.

b. Cut out the circle and tape it along the center to a pencil below the eraser as shown.



c. Hold on to the eraser with your thumb and index finger such that the pencil is resting on its tip. Rotate the circle by holding on to the eraser and spinning the pencil. You can get the same effect by putting the lower portion of the pencil between both palms of your hands and rolling the pencil by moving your hands back and forth.

Remember, a circle is the set of all points that are equal distance from the center. A **disc** is the set of all points on the circle and in the interior of the circle.

d. As the disc rotates about the pencil, the image of a three-dimensional solid is formed. Which of these solids most closely resembles the image formed by the rotating disc?

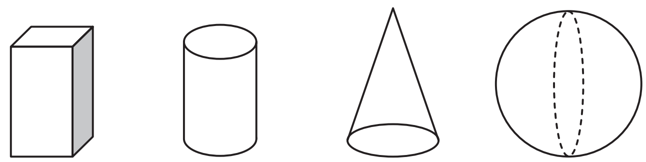


Figure 1 Figure 2 Figure 3 Figure 4

The image formed by the rotating disc closely resembles Figure 4.

e. Name the solid formed by rotating the circle about the pencil.

The solid formed by rotating the circle about the pencil is a sphere.

f. Relate the dimensions of the disc to the dimensions of this solid.

The radius of the disc is also the radius of the sphere.

Grouping

Have students complete Question 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

- If the triangle was turned sideways and then taped, how would that affect the image of the solid associated with the rotation?
- If the triangle wasn't taped in the middle, but taped along a side, how would that affect the image of the solid associated with the rotation?
- If the triangle was taped upside down to the pencil, how would that affect the image of the solid associated with the rotation?
- Will the image associated with this rotation always be a cone?

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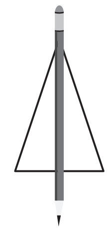
3. You and your classmate finish by exploring triangles.



a. Draw a triangle on an index card.

b. Cut out the triangle and tape it lengthwise along the center to a pencil below the eraser as shown.

c. Hold on to the eraser with your thumb and index finger such that the pencil is resting on its tip. Rotate the triangle by holding on to the eraser and spinning the pencil. You can get the same effect by putting the lower portion of the pencil between both palms of your hands and rolling the pencil by moving your hands back and forth.

d. As the triangle rotates about the pencil, the image of a three-dimensional solid is formed. Which of these solids most closely resembles the image formed by the rotating triangle?

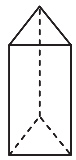


Figure 1

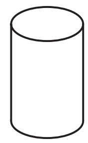


Figure 2

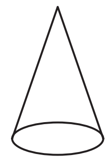


Figure 3

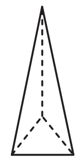


Figure 4

The image formed by the rotating triangle closely resembles Figure 3.

e. Name the solid formed by rotating the triangle about the pencil.

The solid formed by rotating the triangle about the pencil is a cone.

f. Relate the dimensions of the triangle to the dimensions of this solid.

The base of the triangle is the diameter of the cone. The height of the triangle is the height of the cone.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Associate a word in the first column to a word in the second column. Explain your reasoning.

Triangle	Sphere
Rectangle	Cone
Circle	Cylinder

Triangle—Cone

The image of a cone can be visualized by rotating a triangle on a pencil.

Rectangle—Cylinder

The image of a cylinder can be visualized by rotating a rectangle on a pencil.

Circle—Sphere

The image of a sphere can be visualized by rotating a circle on a pencil.

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11.2

Cakes and Pancakes

Translating and Stacking Two-Dimensional Figures

LEARNING GOALS

In this lesson, you will:

- Apply translations to two-dimensional plane figures to create three-dimensional solids.
- Describe three-dimensional solids formed by translations of plane figures through space.
- Build three-dimensional solids by stacking congruent or similar two-dimensional plane figures.

KEY TERMS

- isometric paper
- right triangular prism
- oblique triangular prism
- right rectangular prism
- oblique rectangular prism
- right cylinder
- oblique cylinder

ESSENTIAL IDEAS

- Rigid motion is used in the process of redrawing two-dimensional plane figures as three-dimensional solids.
- Models of three-dimensional solids are formed using translations of plane figures through space.
- Models of two-dimensional plane figures are stacked to create models of three-dimensional solids.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-GMD Geometric Measurement and Dimension

Visualize relationships between two-dimensional and three dimensional objects

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

3. Apply geometric methods to solve design problems.

Overview

Models of two-dimensional figures are translated in space using isometric dot paper. Students analyze the three-dimensional solid images associated with the translations. Technically, the translation of a single point or collection of points changes the location of the point or collection of points. Applied to the activities in this lesson, it is not an actual solid that results from translating the plane figure, rather an image to the eye that is associated with this movement. The activity that includes stacking the two-dimensional models should provide opportunities for these types of discussions as well. Both right and oblique prisms and cylinders are used in this lesson.

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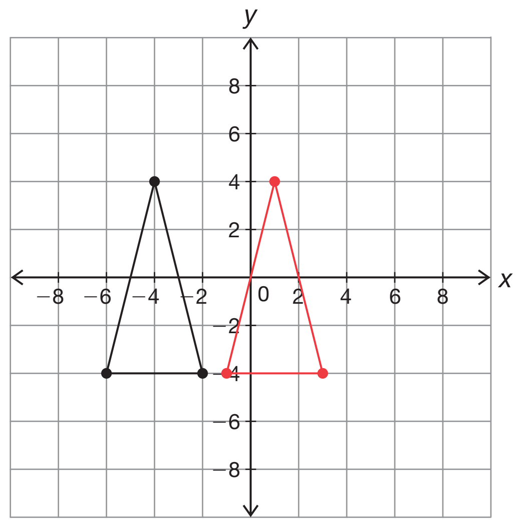
Warm Up

1. Translate the triangle on the coordinate plane in a horizontal direction.



2. Describe the translation of each vertex.

   Answers will vary.

   Each vertex was moved to the right 5 units.

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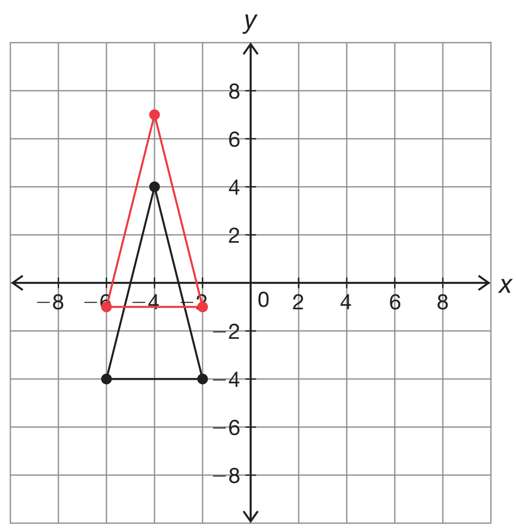
3. Translate the triangle on the coordinate plane in a vertical direction.

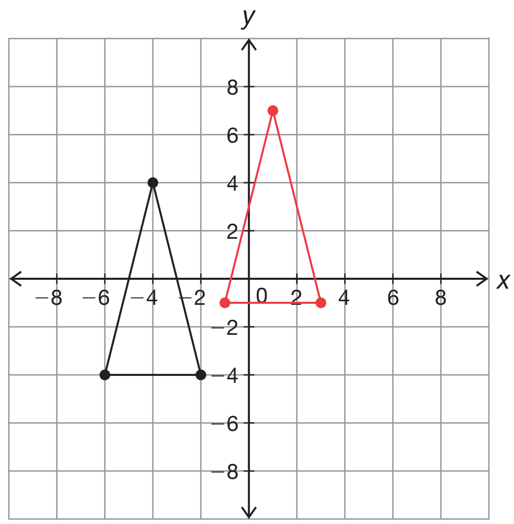


4. Describe the translation of each vertex.

Answers will vary.

Each vertex was moved to the up 3 units.

5. Translate the triangle on the coordinate plane in a diagonal direction.



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6. Describe the translation of each vertex.

Answers will vary.

Each vertex was moved to the right 5 units and up 3 units.

7. What do you suppose is the difference between translating a triangle on a coordinate plane, as you have done in the previous questions, and translating a triangle through space?

Answers will vary.

When a figure is translated on a coordinate plane, the vertices are moved in two directions, left or right, and up or down. When a figure is translated through space, space is three-dimensional so the vertices move in three directions, left or right, up or down, and backward or forward.

11.2

Cakes and Pancakes

Translating and Stacking Two-Dimensional Figures

LEARNING GOALS

In this lesson, you will:

- Apply translations to two-dimensional plane figures to create three-dimensional solids.
- Describe three-dimensional solids formed by translations of plane figures through space.
- Build three-dimensional solids by stacking congruent or similar two-dimensional plane figures.

KEY TERMS

- isometric paper
- right triangular prism
- oblique triangular prism
- right rectangular prism
- oblique rectangular prism
- right cylinder
- oblique cylinder

You may never have heard of isometric projection before, but you have probably seen something like it many times when playing video games.

Isometric projection is used to give the environment in a video game a three-dimensional effect by rotating the visuals and by drawing items on the screen using angles of perspective.

One of the first uses of isometric graphics was in the video game Q*bert, released in 1982. The game involved an isometric pyramid of cubes. The main character, Q*bert, starts the game at the top of the pyramid and moves diagonally from cube to cube, causing them to change color. Each level is cleared when all of the cubes change color. Of course, Q*bert is chased by several enemies.

While it may seem simple now, it was extremely popular at the time. Q*bert had his own line of toys, and even his own animated television show!

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Problem 1

A scenario in which students use isometric paper to create the images of three-dimensional solids on two-dimensional paper is the focus of this problem. The three-dimensional solids highlighted in this problem are both right and oblique; triangular prisms, rectangular prisms, and cylinders. Students conclude that the lateral faces of prisms are parallelograms. Rigid motion helps students visualize how models of solids can be formed from models of plane figures.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

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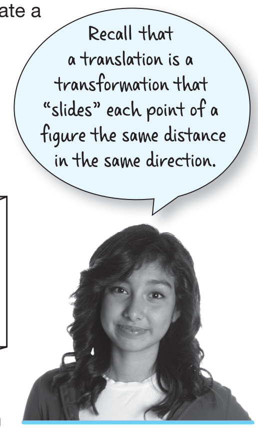
Guiding Questions for Share Phase, Questions 1 through 4

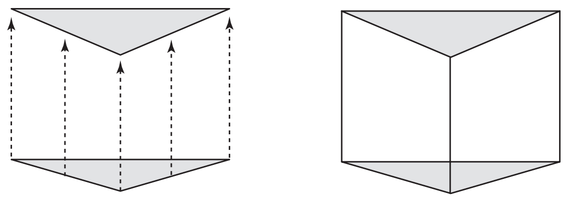
- Are the sides or lateral faces formed by parallel lines? How do you know?
- What is a prism?
- How are prisms named?
- What is a triangular prism?
- What are some properties of a triangular prism?
- How many units and in what direction did you translate the triangle on the isometric paper?

PROBLEM 1 These Figures Take the Cake

You can translate a two-dimensional figure through space to create a model of a three-dimensional figure.



1. Suppose you and your classmate want to design a cake with triangular bases. You can imagine that the bottom triangular base is translated straight up to create the top triangular base.



   a. What is the shape of each lateral face of this polyhedron formed by this translation?

   Each lateral face is a rectangle.

   b. What is the name of the solid formed by this translation?

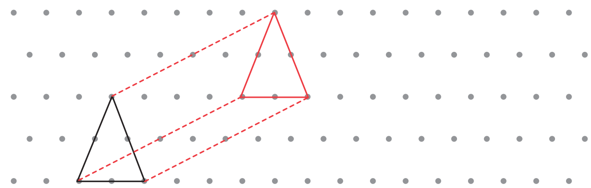
   The solid formed by this translation is a triangular prism.

A two-dimensional representation of a triangular prism can be obtained by translating a triangle in two dimensions and connecting corresponding vertices. You can use **isometric paper**, or dot paper, to create a two-dimensional representation of a three-dimensional figure. Engineers often use isometric drawings to show three-dimensional diagrams on "two-dimensional" paper.

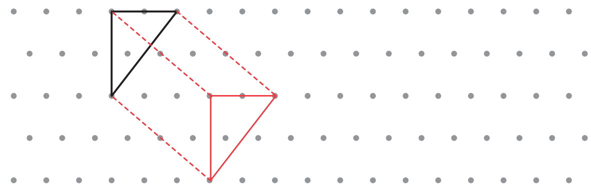
- When you connected the corresponding vertices, do the sides appear to be parallel to each other?
- Do the lateral faces appear to be parallelograms? How do you know?
- Are you translating the triangle through space in a direction that is perpendicular to the plane containing the triangle? How do you know?
- What is the difference between an oblique triangular prism and a right triangular prism?

2. Translate each triangle to create a second triangle. Use dashed line segments to connect the corresponding vertices.

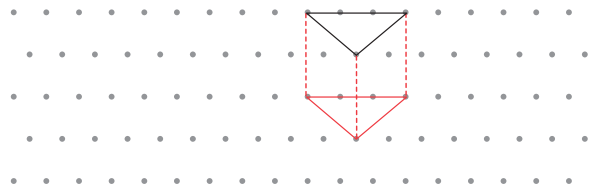
a. Translate this triangle in a diagonal direction.



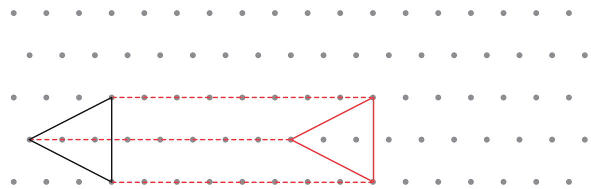
b. Translate this right triangle in a diagonal direction.



c. Translate this triangle vertically.



d. Translate this triangle horizontally.



3. What do you notice about the relationship among the line segments connecting the vertices in each of your drawings?

The line segments appear to be parallel to each other, and they are congruent.

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Grouping

Have students complete Questions 5 through 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 9

- Are the sides or lateral faces formed by parallel lines? How do you know?
- What is the difference between a triangular prism and a rectangular prism?
- What do a triangular prism and rectangular prism have in common?
- What are some properties of a rectangular prism?
- How is isometric dot paper different than Cartesian graph paper?
- How does isometric dot paper help visualize three dimensions?
- How many units and in what direction did you translate the rectangle on the isometric paper?
- When you connected the corresponding vertices, do the sides appear to be parallel to each other?
- Do the lateral faces appear to be parallelograms? How do you know?

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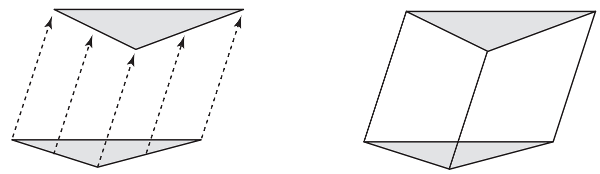
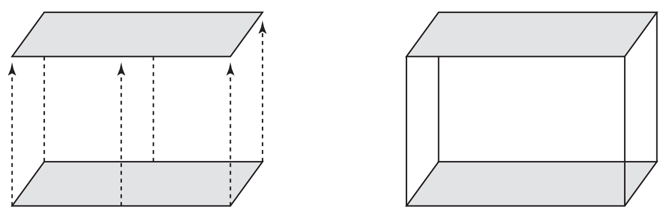
When you translate a triangle through space in a direction that is perpendicular to the plane containing the triangle, the solid formed is a **right triangular prism**. The triangular prism cake that you and your classmate created in Question 1 is an example of a right triangular prism. When you translate a triangle through space in a direction that is *not* perpendicular to the plane containing the triangle, the solid formed is an **oblique triangular prism**. An example of an oblique triangular prism is shown.



4. What is the shape of each lateral face of an oblique triangular prism?
Each lateral face is a parallelogram.

5. Suppose you and your classmate want to design a cake with rectangular bases. You can imagine that the bottom rectangular base is translated straight up to create the top rectangular base.



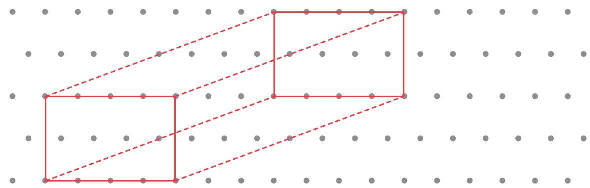
a. What is the shape of each lateral face of the solid figure formed by this translation?
Each lateral face would be a rectangle.

b. What is the name of the solid formed by this translation?
The solid formed by this translation is a rectangular prism.

- Are you translating the rectangle through space in a direction that is perpendicular to the plane containing the rectangle? How do you know?
- What is the difference between an oblique rectangular prism and a right rectangular prism?

A two-dimensional representation of a rectangular prism can be obtained by translating a rectangle in two dimensions and connecting corresponding vertices.

6. Draw a rectangle and translate it in a diagonal direction to create a second rectangle. Use dashed line segments to connect the corresponding vertices.



7. Analyze your drawing.

   a. What do you notice about the relationship among the line segments connecting the vertices in the drawing?

   The line segments appear to be parallel to each other, and they are congruent.

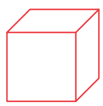
   b. What is the name of a rectangular prism that has all congruent sides?

   A rectangular prism with all congruent sides is a cube.

   c. What two-dimensional figure would you translate to create a rectangular prism with all congruent sides?

   To create a rectangular prism with all congruent sides, I would translate a square.

   d. Sketch an example of a rectangular prism with all congruent sides.



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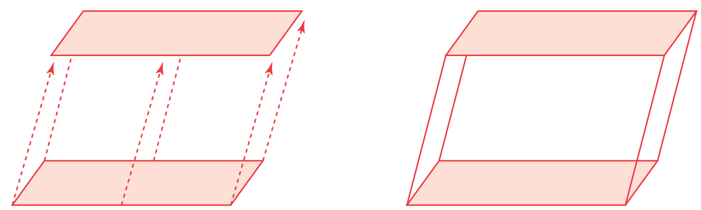
When you translate a rectangle through space in a direction that is perpendicular to the plane containing the rectangle, the solid formed is a **right rectangular prism**. The rectangular prism cake that you and your classmate created in Question 8 is an example of a right rectangular prism. When you translate a rectangle through space in a direction that is *not* perpendicular to the plane containing the rectangle, the solid formed is an **oblique rectangular prism**.

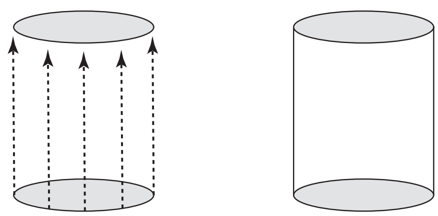
8. What shape would each lateral face of an oblique rectangular prism be?

Each lateral face would be a parallelogram.

9. Sketch an oblique rectangular prism.



10. Suppose you and your classmate want to design a cake with circular bases. You can imagine that the bottom circular base, a disc, is translated straight up to create the top circular base.



a. What shape would the lateral face of the solid figure formed by this translation be?

The lateral face would be a rectangle.

b. What is the name of the solid formed by this translation?

The solid formed by this translation is a cylinder.

Grouping

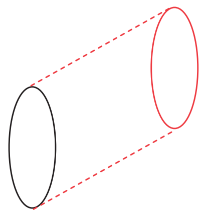
Have students complete Questions 10 through 14 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 10 through 14

- Are the sides or lateral faces formed by parallel lines? How do you know?
- What is the difference between a prism and a cylinder?
- What do a prism and cylinder have in common?
- What are some properties of a cylinder?
- When you connected the corresponding tops and bottoms of the cylinder, do the sides appear to be parallel to each other?
- Do the lateral faces appear to be parallelograms? How do you know?
- Are you translating a circle through space in a direction that is perpendicular to the plane containing the circle? How do you know?
- What is the difference between an oblique cylinder and a right cylinder?

A two-dimensional representation of a cylinder can be obtained by translating an oval in two dimensions and connecting the tops and bottoms of the ovals.

11. Translate the oval in a diagonal direction to create a second oval. Use dashed line segments to connect the tops and bottoms of the ovals.



12. What do you notice about the relationship among the line segments in the drawing?

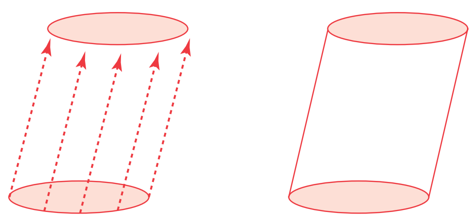
The line segments appear to be parallel to each other, and they are congruent.

When you translate a disc through space in a direction that is perpendicular to the plane containing the disc, the solid formed is a **right cylinder**. The cylinder cake that you and your classmate created in Question 13 is an example of a right cylinder. When you translate a disc through space in a direction that is *not* perpendicular to the plane containing the disc, the solid formed is an **oblique cylinder**.

13. What shape would the lateral face of an oblique cylinder be?

The lateral face would be a parallelogram.

14. Sketch an oblique cylinder.



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Problem 2

Within the context of a situation, students stack a variety of shapes to determine the solid formed. At first they stack congruent circles, congruent triangles, and congruent rectangles. Then they stack circles of different sizes, squares of different sizes, and other similar polygons. The solids determined in this problem include cylinders, prisms, cones, and pyramids.

Grouping

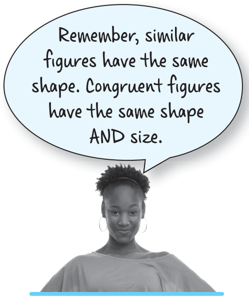
Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

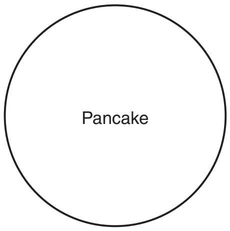
- What information do you need to know about one pancake to determine the height of the stack of pancakes?
- What unit of measure is easiest to use when measuring the height of a pancake?
- What do you suppose is the approximate height of one pancake?
- How many pancakes would be a reasonable portion? Why?
- Is the radius of the disc the same or different than the radius of the cylinder?

PROBLEM 2 Congruent and Similar

The math club at school is planning a pancake breakfast as a fund-raiser. Because this is a fund-raiser for the math club, the pancakes will use various geometric shapes!



1. Imagine you stack congruent circular pancakes on top of each other.



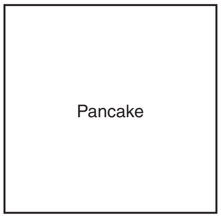
   a. What is the name of the solid formed by this stack of pancakes?

   The solid formed by stacking congruent circular pancakes is a cylinder.

   b. Relate the dimensions of a single circular pancake to the dimensions of this solid.

   The radius of the circular pancakes is the radius of the cylinder. The height of the stack of circular pancakes is the height of the cylinder.

2. Imagine you stack congruent square pancakes on top of each other.



   a. What is the name of the solid formed by this stack of pancakes?

   The solid formed by stacking congruent square pancakes is a right rectangular prism.

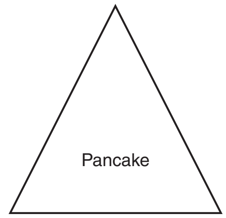
   b. Relate the dimensions of a single square to the dimensions of this solid.

   The length of the side of a square pancake is also the length and width of the right rectangular prism. The height of the stack of pancakes is the height of the right rectangular prism.

- Is the height of the stack of discs the same or different than the height of the cylinder?
- Is the length of a side of the square pancake a consideration when determining the height of the stack of pancakes? Why or why not?
- Do you suppose these square pancakes are larger or smaller than the circular pancakes? Why?
- How many pancakes would be a reasonable portion? Why?

- Is the length of the side of a square pancake the same or different than the width of the right rectangular prism?
- Is the height of the stack of square pancakes the same or different than the height of the right rectangular prism?
- Is the length of a side of the triangular pancake a consideration when determining the height of the stack of pancakes? Why or why not?
- Do you suppose these triangular pancakes are larger or smaller than the circular or square pancakes? Why?
- How many pancakes would be a reasonable portion? Why?
- Is the length of the base of a triangular pancake the same or different than the length of the base of the right triangular prism?
- Is the height of the stack of triangular pancakes the same or different than the height of the right triangular prism?

3. Imagine you stack congruent triangular pancakes on top of each other.



a. What is the name of the solid formed by this stack of pancakes?

The solid formed by stacking congruent triangular pancakes is a triangular prism.

b. Relate the dimensions of the triangle to the dimensions of this solid.

The dimensions of the triangular pancake (length of the base and the height) are also the dimensions of the base of the triangular prism. The height of the stack of pancakes is the height of the triangular prism.

4. What do you notice about the three-dimensional solids created by stacking congruent figures?

Stacking congruent circles creates a right cylinder.

Stacking congruent triangles or squares creates a right prism.

5. What type of solid would be formed by stacking congruent rectangles? pentagons? hexagons?

Stacking congruent polygons creates a right prism. The shape of the figure being stacked is the shape of the base of the prism.

11

Grouping

Have students complete Questions 6 through 12 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 6 through 12

- How is the stack of similar square pancakes different than the stack of congruent square pancakes?
- How is the stack of similar equilateral triangular pancakes different than the stack of congruent triangular pancakes?
- What is the name of a pyramid with a pentagonal base?
- How is the base of the pyramid related to the dimensions of the pyramid?
- What is the name of a pyramid with a hexagonal base?
- How many hexagons do you suppose are needed to create a hexagonal pyramid?
- What do you suppose would be the radius of the top most disc?
- How is this stack of similar discs different than the stack of congruent discs?

11

6. Imagine you stack similar circular pancakes on top of each other so that each layer of the stack is composed of a slightly smaller pancake than the previous layer.

a. What is the name of the solid formed by this stack of pancakes?

The solid formed by stacking similar circular pancakes is a cone.

b. Relate the dimensions of a single pancake to the dimensions of the solid.

The radius of the pancake at the bottom layer is the radius of the cone. The height of the stack of pancakes is the height of the cone.

7. Imagine you stack similar square pancakes on top of each other so that each layer of the stack is composed of a slightly smaller pancake than the previous layer.

a. What is the name of the solid formed by this stack of pancakes?

The solid formed by stacking similar square pancakes is a rectangular pyramid.

b. Relate the dimensions of a single pancake to the dimensions of the solid.

The length and width of the pancake at the bottom layer is the length and width of the base of the pyramid. The height of the stack of pancakes is the height of the pyramid.

8. Imagine you stack similar triangular pancakes on top of each other so that each layer of the stack is composed of a slightly smaller pancake than the previous layer.

a. What is the name of the solid formed by this stack of pancakes?

The solid formed by stacking similar triangular pancakes is a triangular pyramid.

b. Relate the dimensions of a single pancake to the dimensions of the solid.

The dimensions of the pancake at the bottom layer are the dimensions of the base of the pyramid. The height of the stack of pancakes is the height of the pyramid.

9. What do you notice about the three-dimensional solids created by stacking similar figures?

Stacking similar circles creates a cone.

Stacking similar triangles or squares creates a pyramid.

10. What type of solid would be formed by stacking similar rectangles? pentagons? hexagons?

Stacking congruent polygons creates a pyramid. The shape of the figure being stacked is the shape of the base of the pyramid.

11. Use what you have learned in this lesson to make an informal argument that explains the volume formulas for prisms and cylinders.

The base of a cylinder is a circle with an area of πr^2. When this area is translated a distance of h, the resulting formula is $\pi r^2 h$.

The base of a prism is a polygon. The area of the base depends on the type of polygon, but I can denote this area with B. The prism is created by translating (or stacking) the area of the base a distance (or height) of h. So, the resulting volume formula is Bh.

11

12. Complete the graphic organizer to record the volume formulas and the transformations you have used to create the solid figures.

Prisms

V = (area of base) $\times$ height

$V = Bh$

Transformations:

- translating polygons
- stacking polygons

Pyramids

$V = \frac{1}{3}Bh$

Transformations:

- stacking similar polygons

Volume Formulas and Transformations

Cylinders

$V = \pi r^2 h$

Transformations:

- translating a circle
- rotating a rectangle
- stacking circles

Cones

$V = \frac{1}{3}\pi r^2 h$

Transformations:

- rotating a right triangle
- stacking similar circles

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Talk the Talk

Students are given several different actions. Some of the actions could result in forming the same solid. They cut out separate cards containing each action and sort them into groups such that each group forms the same solid. They also label each group using the name of the solid and draw an example.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Talk the Talk

1. Which of the following actions could result in forming the same solid? Cut out the cards shown and sort them into groups that could each form the same solid figure. Then, draw an example of each solid figure and label each group. Explain how you sorted the actions. Be sure to name the solid that best represents the object.

translating an isosceles triangle	translating a right triangle	translating a square	translating a rectangle
translating a circle	rotating a rectangle	rotating a triangle	rotating a circle
stacking congruent circles	stacking similar circles	stacking congruent rectangles	stacking similar rectangles
stacking congruent squares	stacking similar squares	stacking congruent triangles	stacking similar triangles

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Group 1: Sphere

rotating a circle	stacking similar circles

Group 2: Cone

rotating a triangle	stacking similar circles

Group 3: Cylinder

rotating a rectangle	translating a circle	stacking congruent circles

Group 4: Pyramid

stacking similar triangles	stacking similar squares	stacking similar rectangles

Group 5: Prism

translating a square	stacking congruent squares	translating a rectangle	stacking congruent rectangles
translating an isosceles triangle	translating a right triangle	stacking congruent triangles	

Be prepared to share your solutions and methods.

Check for Students' Understanding

1. Name the geometric solid associated with each situation.
   a. A stack of 100 copies of the same book
      rectangular prism
   b. A stack of 100 tires. The tires range in size from very large tires to very small tires
      cone
   c. A stack of 100 stop signs
      octagonal prism

2. Create a stacking situation that best resembles each geometric solid.
   a. A cylinder
      Answers will vary.
      Example Response: one hundred music CDs
   b. A square pyramid
      Answers will vary.
      Example Response: fifty similar progressively smaller squares
   c. A triangular prism
      Answers will vary.
      Example Response: forty yield signs

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11.3

Cavalieri's Principles

Application of Cavalieri's Principles

LEARNING GOALS

In this lesson, you will:

- Explore Cavalieri's principle for two-dimensional geometric figures (area).
- Explore Cavalieri's principle for three-dimensional objects (volume).

KEY TERM

- Cavalieri's principle

ESSENTIAL IDEAS

- If two regions in a plane are located between two parallel lines in the plane and all the lines parallel to these two lines intersects both regions in line segments of equal length, then the area of the two regions are equal.
- If two solids of equal altitude, and the sections formed by planes parallel to and at the same distance from their bases are equal, then the volumes of the two solids are equal.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-GMD Geometric Measurement and Dimension

Explain volume formulas and use them to solve problems

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone.
2. Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

Visualize relationships between two-dimensional and three dimensional objects

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Overview

Students approximate the area of an irregularly shaped figure by dividing the figure into familiar polygons and finish with showing Cavalieri's principle for two-dimensional figures; If the lengths of one-dimensional slices-just a line segment-of the two figures are the same, then the figures have the same area. Next, students first approximate the volume of a right rectangular prism and an oblique rectangular prism and end with showing Cavalieri's principle for three-dimensional figures; If, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal.

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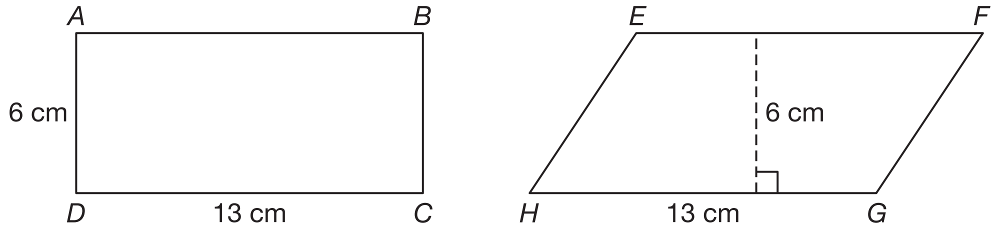
Warm Up

Consider rectangle *ABCD* and parallelogram *EFGH*.



1. What is the length of every line segment drawn in the interior of rectangle *ABCD* and parallel to line segments *AB* and *CD*?

   The every line segment drawn in the interior of rectangle *ABCD* and parallel to line segments *AB* and *CD* is equal to 13 cm.

2. What is the length of every line segment drawn in the interior of parallelogram *EFGH* and parallel to line segments *EF* and *GH*?

   The every line segment drawn in the interior of parallelogram *EFGH* and parallel to line segments *EF* and *GH* is equal to 13 cm.

3. Compare the area of rectangle *ABCD* to the area of parallelogram *EFGH*.

   The area of rectangle *ABCD* and the area of parallelogram *EFGH* are both equal to 78 square centimeters.

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Cavalieri's Principles

Application of Cavalieri's Principles

11.3

LEARNING GOALS

In this lesson, you will:

- Explore Cavalieri's principle for two-dimensional geometric figures (area).
- Explore Cavalieri's principle for three-dimensional objects (volume).

KEY TERM

- Cavalieri's principle

Bonaventura Cavalieri was an Italian mathematician who lived from 1598 to 1647. Cavalieri is well known for his work in geometry as well as optics and motion.

His first book dealt with the theory of mirrors shaped into parabolas, hyperbolas, and ellipses. What is most amazing about this work is that the technology to create the mirrors that he was writing about didn't even exist yet!

Cavalieri is perhaps best known for his work with areas and volumes. He is so well known that he even has a principle named after him—Cavalieri's principle.

Problem 1

A strategy for approximating the area of an irregularly shaped figure is given. Students answer questions related to the situation which lead to Cavalieri's Principle for two-dimensional figures.

Grouping

- Ask a student to read the information aloud. Discuss as a class.
- Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

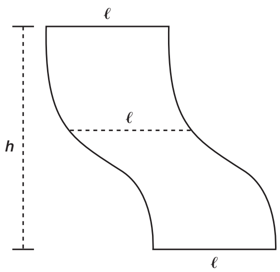
Guiding Questions for Share Phase, Questions 1 through 5

- What information do you know about the rectangles dividing the curved figure?
- What is the height of each rectangle if there are 10 rectangles and the total height is represented by h?
- What do the variables represent in the diagram?
- How is the area of one rectangle determined?
- How is the area of ten rectangles determined?
- If the ten rectangles were arranged differently, would the total area remain the same or change?
- Are the areas of the two figures the same or is one figure larger than the other?

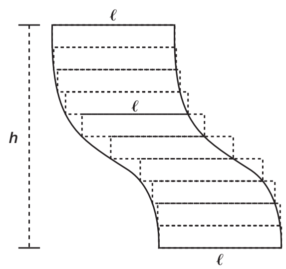
11

PROBLEM 1 Approximating the Area of a Two-Dimensional Figure

One strategy for approximating the area of an irregularly shaped figure is to divide the figure into familiar shapes and determine the total area of all of the shapes. Consider the irregular shape shown. The distance across any part of the figure is the same.



1. You can approximate the area by dividing the irregular shape into congruent rectangles. To start, let's divide this shape into 10 congruent rectangles.



   a. What is the length, the height, and the area of each congruent rectangle?
   The length of each congruent rectangle is ℓ, and the height of each congruent rectangle is $\frac{h}{10}$. The area of each congruent rectangle is $\frac{\ell h}{10}$.

   b. What is the approximate area of the irregularly shaped figure?
   The approximate area of the irregularly shaped figure is $\frac{\ell h}{10} \cdot 10 = \ell h$.

2. If this irregularly shaped figure were divided into 1000 congruent rectangles, what would be the area of each congruent rectangle? What would be the approximate area of the figure?
   The area of each congruent rectangle is $\frac{\ell h}{1000}$.
   The approximate area of the irregularly shaped figure would be $\frac{\ell h}{1000} \cdot 1000 = \ell h$.

- Do the two figures contain the same number of rectangles?
- Are all of the rectangles in both figures the same size?
- Can you think of a different example of using Cavalieri's principle?

3. If this irregularly shaped figure were divided into n congruent rectangles, what would be the area of each congruent rectangle? What would be the approximate area of the figure?

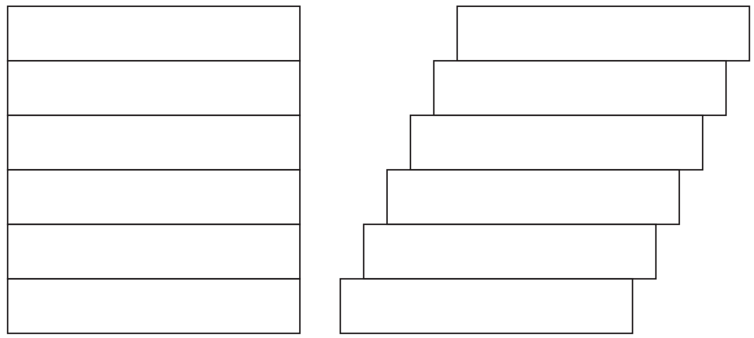
The area of each congruent rectangle is $\frac{\ell h}{n}$.

The approximate area of the irregularly shaped figure would be $\frac{\ell h}{n} \cdot n = \ell h$.

4. If the irregularly shaped figure were divided into only one rectangle, what would be the approximate area of the figure?

The approximate area of the irregularly shaped figure would be $\frac{\ell h}{1} \cdot 1 = \ell h$.

5. Compare the area of the two figures shown. Each rectangle has a height of h and a base equal to length ℓ.



The approximate area of both figures is $\frac{\ell h}{6} \cdot 6 = \ell h$.

You have just explored **Cavalieri's principle** for two-dimensional figures, sometimes called the method of indivisibles. If the lengths of one-dimensional slices—just a line segment—of the two figures are the same, then the figures have the same area. This is best illustrated by making several slices to one figure and pushing them to the side to form a second figure.

11

Problem 2

A right rectangular prism and an oblique rectangular prism are used to show Cavalieri's Principle for three-dimensional figures.

Grouping

- Instruct students to read the information. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- Is the cross section of the right rectangular prism different than the cross section of the oblique rectangular prism?
- Do perpendicular lines intersect at corners of the oblique rectangular prism?
- What do the variables represent in the diagram?
- How is the volume of one slice determined?
- How is the volume of ten slices determined?
- Are the volumes of the two figures the same or is one figure larger than the other?
- Do the two figures contain the same number of slices?

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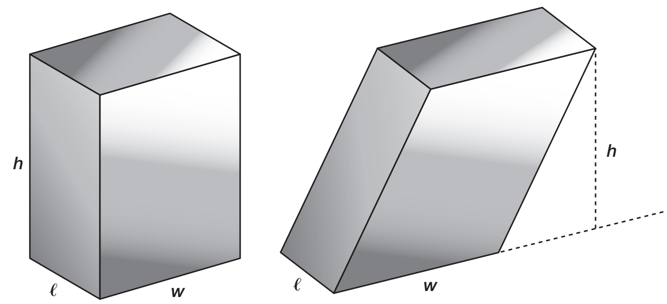
PROBLEM 2 Cavalieri's Principle for Volume

Consider the right rectangular prism and the oblique rectangular prism shown.

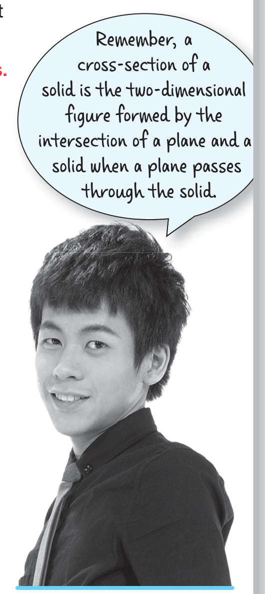


Right rectangular prism   Oblique rectangular prism

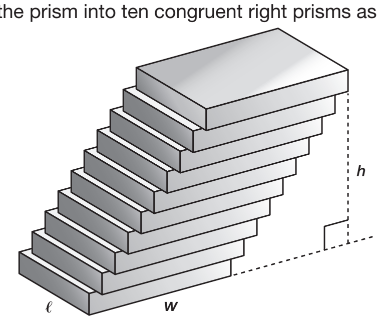
1. What geometric figure represents a cross section of each that is perpendicular to the base?
   A rectangle best represents the cross sections of the prisms.

2. What are the dimensions of one cross section?
   Each rectangular cross section has dimensions $\ell \times w$.

3. What is the volume of the right rectangular prism?
   The volume of the right rectangular prism is $V = \ell wh$.



4. Approximate the volume of the oblique rectangular prism by dividing the prism into ten congruent right prisms as shown.



The volume of one rectangular prism is $V = \ell \cdot w \cdot \frac{h}{10}$.

The volume of the oblique rectangular prism is $V = \left(\ell \cdot w \cdot \frac{h}{10}\right)10 = \ell wh$.

- Are all of the slices the same size?
- Can you think of a different example of using Cavalieri's principle?

Grouping

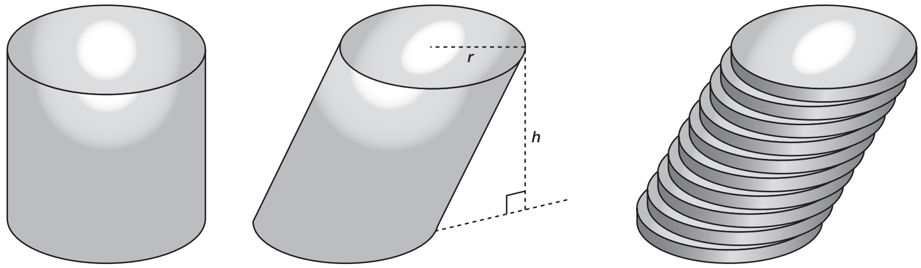
Have students complete Questions 5 through 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 10

- Is the cross section of the right cylinder different than the cross section of the cylinder?
- What do the variables represent in the diagram?
- How is the volume of one slice determined?
- How is the volume of ten slices determined?
- Are the areas of the two figures the same or is one figure larger than the other?
- Do the two figures contain the same number of slices?
- Are all of the slices the same size?
- How is the example using rectangular prisms similar to the example using cylinders?
- Can you think of a different example of using Cavalieri's principle?

You have just explored Cavalieri's principle for three-dimensional figures. Given two solids included between parallel planes, if every plane cross section parallel to the given planes has the same area in both solids, then the volumes of the solids are equal. In other words, if, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal.

For a second example of this principle, consider a right cylinder and an oblique cylinder having the same height and radii of equal length.



Right cylinder   Oblique cylinder

5. What geometric figure best represents the ten cross sections of the cylinders?
   A smaller cylinder best represents the cross sections of the cylinders.

6. What are the dimensions of one cross section?
   The dimensions of the smaller cylinder are the radius (r), and the height $\left(\frac{h}{10}\right)$.

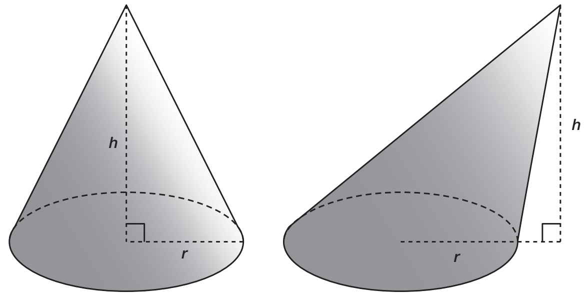
7. What is the volume of one cross section?
   The volume of one cross section is $V = \pi r^2\left(\frac{h}{10}\right) = \frac{\pi r^2 h}{10}$.

8. What is the volume of the oblique cylinder?
   The volume of the oblique cylinder is $V = \pi r^2\left(\frac{h}{10}\right) \cdot 10 = \pi r^2 h$.

9. What is the volume of the right cylinder?
   The volume of the right cylinder is $V = \pi r^2 h$.

You have just shown the volume of a right cylinder and the volume of an oblique cylinder are equal, provided both cylinders have the same height and radii of equal length.

10. Using Cavalieri's principle, what can you conclude about the volume of these two cones, assuming the heights are equal and the radii in each cone are congruent?



Using Cavalieri's principle, I can conclude the volumes of both cones are equal.

It is important to mention that the Cavalieri principles do not compute exact volumes or areas. These principles only show that volumes or areas are equal without computing the actual values. Cavalieri used this method to relate the area or volume of one unknown object to one or more objects for which the area or volume could be determined.

11

Talk the Talk

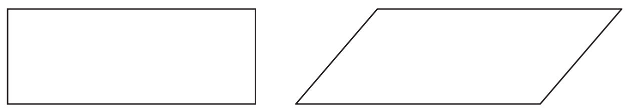
Students explain how Cavalieri's principle is used to determine the area formula for a parallelogram using the area formula of a rectangle.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Talk the Talk

1. Consider the rectangle and the parallelogram shown to be of equal height with bases of the same length.



Knowing the area formula for a rectangle, how is Cavalieri's principle used to determine the area formula for the parallelogram?

The area of the rectangle is $A = bh$. Because the height of both figures is any fixed number, the widths of the parallel segments in the parallelogram have the same length as the corresponding segments in the rectangle. Therefore, the area of the parallelogram must be the same as the area of the rectangle, or $A = bh$.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Use the appropriate words from the list to fill in the blanks.

area	two-dimentional
congruent	three-dimensional
similar	volume

1. Based on Cavalieri's principle, the area of a ___two-dimensional___ figure can be estimated by adding the area of congruent slices of the figure.

2. Based on Cavalieri's principle, the volume of an oblique cylinder can be estimated by adding the ___volume___ of congruent discs of the solid.

11

11

11.4

Spin to Win

Volume of Cones and Pyramids

LEARNING GOALS

In this lesson, you will:

- Rotate two-dimensional plane figures to generate three-dimensional figures.
- Give an informal argument for the volume of cones and pyramids.

ESSENTIAL IDEAS

- Rotations are applied to two-dimensional plane figures.
- Three-dimensional solids are formed by rotations of plane figures through space.
- An informal argument links Cavalieri's principle to the formula for the volume of a cylinder.
- An informal argument links Cavalieri's principle to the formula for the volume of a cone.
- An informal argument links Cavalieri's principle to the formula for the volume of a pyramid.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

G-GMD Geometric Measurement and Dimension

Visualize relationships between two-dimensional and three dimensional objects

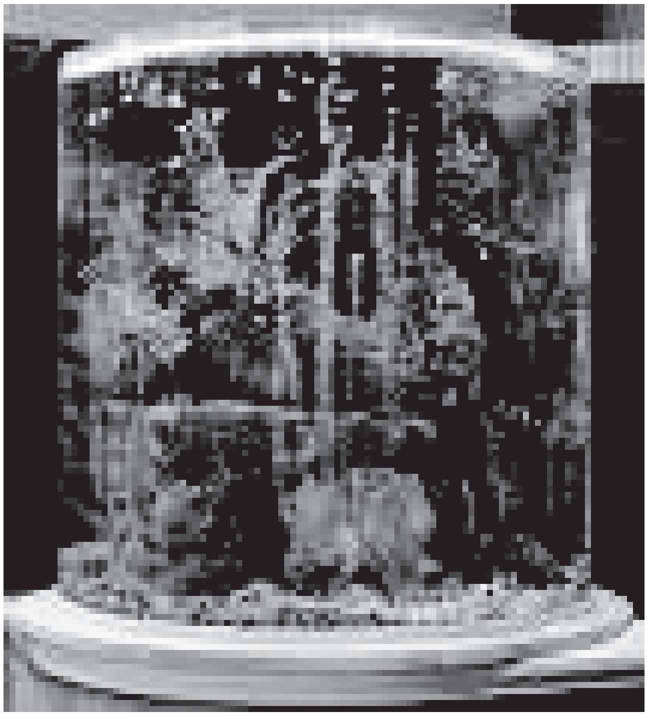
4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Overview

Models of two-dimensional figures such as rectangles and triangles are rotated through space on an axis. Students analyze the three-dimensional solid images associated with the rotation. Similar to the activities in the first lesson of this chapter, the rotation of a single point or collection of points changes the location of the point or collection of points. The rotation on the axis does not result in an actual solid, rather an image to the eye that is associated with this rotation. Through rotation or stacking strategies, the volumes of a cylinder, cone and pyramid are explored.

Warm Up

A Cylindrical Fish Tank provides a 360° view!



- The height of the cylindrical fish tank is 30″.
- The length of the diameter of the base is 27.5″.
- One US gallon is equal to approximately 231 cubic inches.

Calculate the amount of water the tank will hold.

Volume of the cylinder:

$V = \pi r^2 h$

$V = (\pi)(13.75)^2\ (30) \approx 17809.6875 \text{ in}^3$

$\frac{17809.6875}{231} \approx 77.1$ gallons

11

11

Spin to Win

Volume of Cones and Pyramids

11.4

LEARNING GOALS

In this lesson, you will:

- Rotate two-dimensional plane figures to generate three-dimensional figures.
- Give an informal argument for the volume of cones and pyramids.

Imagine that you are, right now, facing a clock and reading the time on that clock—let's imagine that it's 2:28.

Now imagine that you are blasted away from that clock at the speed of light, yet you are still able to read the time on it (of course you wouldn't be able to, really, but this is imagination!).

What time would you see on the clock as you traveled away from it at the speed of light? The light bouncing off the clock travels at the speed of light, so, as you travel farther and farther away from the clock, all you could possibly see was the time on the clock as it was right when you left.

Einstein considered this "thought experiment" among many others to help him arrive at groundbreaking theories in physics.

Problem 1

Rotating a rectangle about an axis and stacking congruent discs are two methods used for creating the image of a cylinder. Students determine the area of an average disc which leads to the volume formula for any cylinder.

Grouping

Ask students to read introduction and worked example. Discuss Questions 1 and 2 as a class. Discuss as a class.

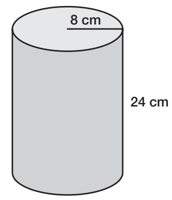
Guiding Questions for Discuss Phase

- How are the formulas for the area of a circle and volume of a cylinder related?
- What is the height of each disc if there are 10 discs and the total height is represented by *h*?
- What is the area of each disc in the cylinder?
- Are all discs in the cylinder congruent?
- What do the variables in the diagram represent?

PROBLEM 1 Building Cylinders

You have learned that when two-dimensional shapes are translated, rotated, or stacked, they can form three-dimensional solids. You can also use transformations and stacking to build formulas for three-dimensional figures.

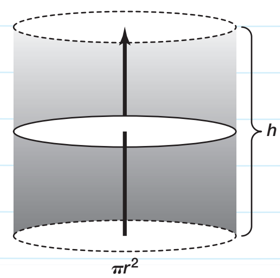
1. Determine the volume for the cylinder shown. Show your work.



To calculate the volume of the cylinder, I can use the formula $V = \pi r^2 h$.
The volume is $\pi(8^2)(24)$, or approximately 4825.49 cubic centimeters.

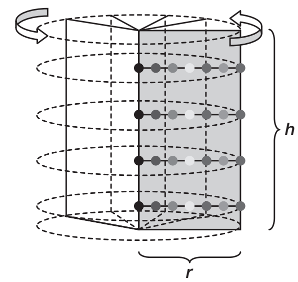
To derive the formula for the volume of a cylinder, you can think of the cylinder as an infinite stack of discs, each with an area of πr^2. These discs are stacked to a height of *h*, the height of the cylinder.

So, the volume of the cylinder is $\pi r^2 \times h$, or $\pi r^2 h$.

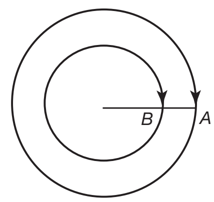


2. Can you choose any disc in the cylinder and multiply the area by the height to calculate the volume of the cylinder? Explain your reasoning.
Yes. Every disc that makes up the cylinder is congruent. So, you can choose any disc and multiply its area by the height to calculate the volume of the cylinder.

Another way to think about a cylinder is the rotation of a rectangle about one of its sides as shown. The rotation of the set of points that make up the rectangle forms the cylinder.



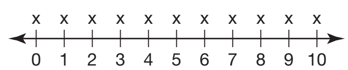
To determine the volume of the cylinder, you can multiply the area of the rectangle by the distance that the points of the rectangle rotate. However, the points of the rectangle don't all rotate the same distance. Consider a top view of the cylinder. The distance that point *A* rotates is greater than the distance that point *B* rotates.



You can't calculate the distance that each point rotates because there are an infinite number of points, each rotating a different distance. But you can use an average, or typical point, of the rectangle.

11

3. Consider the dot plot shown.



a. What is the median of the data?

The median of the data is the middle value when the data are ordered from least to greatest or greatest to least. The median of these data is 5.

b. Describe how you could determine the median of these data without doing any calculation.

There are 11 values that are evenly spaced from 0 to 10, so the median should be the exact center value of the data set. There are five values to the left of 5 and five values to the right of 5, so 5 is the median of these data.

Grouping

Have students complete Question 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

- What is the purpose of determining the median of the data?
- How is the median connected to the rotating rectangle that forms the cylinder and the volume of the cylinder?

Grouping

Have students complete Questions 4 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 8

- How is the rotating rectangle related to the area of the cylinder?
- Do you prefer the infinite stack of discs or the rotating for deriving the volume formula for a cylinder? Explain.

11

4. What is the location of the average, or typical, point of the rectangle in terms of the radius and the height? Explain your reasoning.

The average, or typical, point is the center point of the rectangle. The center of the width of the rectangle is $\frac{1}{2}r$. The center of the height of the rectangle is $\frac{1}{2}h$. So, the average, or typical, point of the rectangle is located at $\left(\frac{1}{2}r, \frac{1}{2}h\right)$.

5. What is the area of the rectangle that is rotated?

The area of the rectangle is rh.

6. Use the average point of the rectangle to calculate the average distance that the points of the rectangle rotate. Explain your reasoning.

Because the average point of the rectangle is located at $\frac{1}{2}r$, the average distance that all of the points of the rectangle are rotated is the circumference of a circle with a radius of $\frac{1}{2}r$.

$$\text{Circumference} = 2\pi\left(\frac{1}{2}r\right)$$
$$= \pi r$$

The average distance that all of the points of the rectangle are rotated is πr.

7. To determine the volume of the cylinder, multiply the area of the rectangle by the average distance that the points of the rectangle rotate. Calculate the volume of the cylinder.

To determine the volume formula, I multiply the area of the rectangle, rh, by the average distance that all of the points of the rectangle are rotated, πr.

$\pi r(rh) = \pi r^2 h$

8. Compare the volume that you calculated in Question 2 to the volume that you calculated in Question 7. What do you notice?

Both methods give the formula for the volume of a cylinder, $\pi r^2 h$.

Problem 2

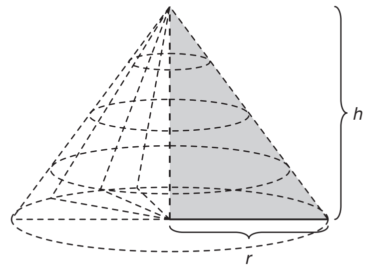
Rotating a right triangle about an axis and stacking an infinite number of similar discs are two methods for creating the image of a cone. The area of an average disc is calculated in which the length of the radius is determined algebraically. This leads to the volume formula for any cone.

Grouping

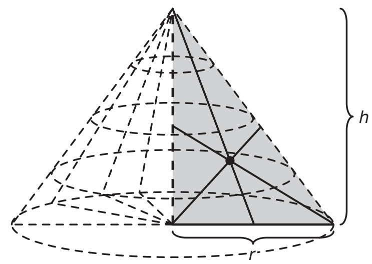
- Ask students to read introduction and worked examples. Then discuss as a class.
- Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

PROBLEM 2 Building Cones

You can also think about a cone as a rotation of a right triangle about one of its legs. You can use the same strategy of determining the average, or typical, point in a right triangle to derive the formula for the volume of a cone. The figure shown represents a cone formed by rotating a right triangle about one of its legs.

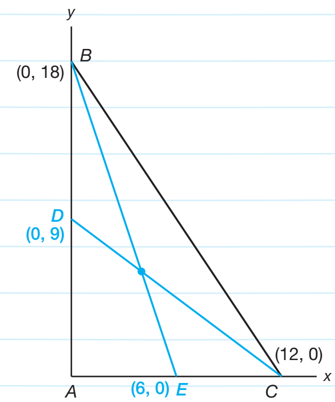


Previously, you explored points of concurrency of a triangle. Recall that the centroid is the point of concurrency of the three medians of a triangle. A median connects a vertex to the midpoint of the opposite side. You can use the centroid as the typical point of the triangle to derive the volume. The centroid of the right triangle is shown.



Let's review how to calculate the coordinates of the centroid of triangle *ABC*.

Triangle *ABC* has vertices at *A* (0, 0), *B* (0, 18), and *C* (12, 0).



First, determine the locations of the midpoints of side *AB* and side *AC*.

- Midpoint of side *AC*: $\left(\frac{0+12}{2}, \frac{0+0}{2}\right)$, or (6, 0)
- Midpoint of side *AB*: $\left(\frac{0+0}{2}, \frac{0+18}{2}\right)$, or (0, 9)

Next, determine an equation representing each median.

- The slope of $\overline{BE}$ is -3, and the *y*-intercept is 18.

  Slope of $\overline{BE} = \frac{0-18}{6-0} = -\frac{18}{6} = -3$

  So, the equation of $\overline{BE}$ is $y = -3x + 18$.

- The slope of $\overline{CD}$ is $-\frac{3}{4}$, and the *y*-intercept is 9.

  Slope of $\overline{CD} = \frac{0-9}{12-0} = -\frac{9}{12} = -\frac{3}{4}$

  So, the equation of $\overline{CD}$ is $y = -\frac{3}{4}x + 9$.

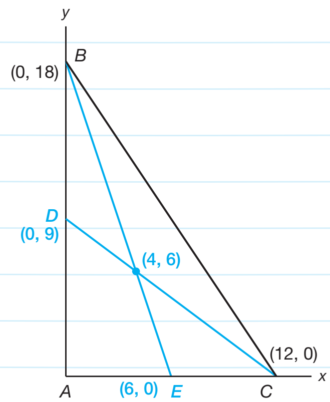
Solve the system of equations to determine the coordinates of the centroid.

$$\begin{cases} y = -3x + 18 \\ y = -\frac{3}{4}x + 9 \end{cases}$$

$$-\frac{3}{4}x + 9 = -3x + 18$$
$$-\frac{3}{4}x = -3x + 9$$
$$\frac{1}{4}x = x - 3$$
$$x = 4$$

$$y = -3x + 18$$
$$y = -3(4) + 18$$
$$y = 6$$

The centroid of the triangle is located at (4, 6).



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Guiding Questions for Share Phase, Questions 1 through 5

- When computing the average radius, how many radii are considered?
- When computing the average height, how many heights are considered?
- Why is the average radius located where the two lines intersect?
- What is the slope formula?
- What is the slope-intercept form for a line?
- What are the two linear equations that determine the average length of a radius?
- What is the length of the radius of the entire cone?
- What is the length of the average radius?
- Is 4 one-third of 12?
- What is the slope and y-intercept of the first line?
- What is the slope and y-intercept of the second line?
- What fraction is associated with the radius of a typical disc?

11

1. What is the area of triangle *ABC*?

The base of the triangle is 12 units, and the height is 18 units, so the area of the triangle is $\frac{1}{2}(12)(18)$, or 108 square units.

2. What is the average distance that all the points of the triangle are rotated? Show your work and explain your reasoning.

Because the centroid has an x-coordinate of 4, the average distance that all the points of the rectangle are rotated is the circumference of a circle with a radius of 4. So, the average distance all the points are rotated is $2\pi(4)$, or 8π, units.

3. Determine the volume of the cone. Show your work and explain your reasoning.

The volume of the cone is the product of the area of the triangle, 108 square units, and the average distance all the points of the triangle are rotated, 8π.

108 square units $\times$ 8π units $=$ 864π cubic units

4. Use the formula for the volume of a cone, $\frac{1}{3}\pi r^2 h$, to calculate the volume of the cone. Show your work.

$$\frac{1}{3}\pi(12)^2(18) = \frac{1}{3}\pi(2592)$$
$$= 864\pi \text{ cubic units}$$

5. Compare the volume that you calculated by rotating the right triangle to the volume that you calculated using the formula for the volume of a cone. What do you notice?

The volume calculations result in the same volume.

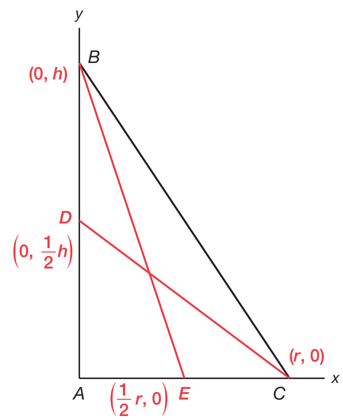
Grouping

Have students complete Question 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- The centroid is the point of concurrency for the three medians of a triangle. But, the solution only shows two of the medians. Is there a mistake? Explain.
- If two different medians of the triangle were used, would it result in the same volume formula?
- Does it matter which two medians are used for deriving the formula of a cone?

6. Derive the formula for the volume of any cone with radius, r, and height, h, by rotating a right triangle with vertices at (0, 0), (0, h), and (r, 0).



First, I determine the locations of the midpoints of side $\overline{AB}$ and side $\overline{AC}$.

- Midpoint of side $\overline{AC}$: $\left(\frac{0+r}{2}, \frac{0+0}{2}\right)$, or $\left(\frac{1}{2}r, 0\right)$
- Midpoint of side $\overline{AB}$: $\left(\frac{0+0}{2}, \frac{0+h}{2}\right)$, or $\left(0, \frac{1}{2}h\right)$

Next, I determine an equation representing each median.

- The slope of median $\overline{CD}$ is $-\frac{h}{2r}$, and the y-intercept is $\frac{1}{2}h$.

  Slope of median $\overline{CD} = \frac{0-\frac{1}{2}h}{r-0} = -\frac{\frac{1}{2}h}{r} = -\frac{h}{2r}$

  So, the equation of median $\overline{CD}$ is $y = -\frac{h}{2r}(x) + \frac{1}{2}h$.

- The slope of median $\overline{BE}$ is $-\frac{2h}{r}$, and the y-intercept is h.

  Slope of median $\overline{BE} = \frac{0-h}{\frac{1}{2}r-0} = -\frac{h}{\frac{1}{2}r} = -\frac{2h}{r}$

  So, the equation of median $\overline{BE}$ is $y = -\frac{2h}{r}(x) + h$.

Then, I solve the system of equations to determine the coordinates of the centroid.

$$-\frac{h}{2r}(x) + \frac{1}{2}h = -\frac{2h}{r}(x) + h \qquad y = -\frac{2h}{r}\left(\frac{r}{3}\right) + h$$

$$\left(\frac{2h}{r} - \frac{h}{2r}\right)x = \frac{1}{2}h \qquad y = -\frac{2hr}{3r} + h$$

$$\frac{3h}{2r}(x) = \frac{1}{2}h \qquad y = -\frac{2h}{3} + \frac{3h}{3}$$

$$x = \frac{h}{2}\cdot\frac{2r}{3h} = \frac{r}{3} \qquad y = \frac{h}{3}$$

The centroid of any triangle is located at $\left(\frac{1}{3}r, \frac{1}{3}h\right)$.

The centroid of a triangle is located at $\left(\frac{1}{3}r, \frac{1}{3}h\right)$, so the average distance all the points of a triangle are rotated to create a cone is $2\pi\left(\frac{1}{3}\right)r$, or $\frac{2}{3}\pi r$.

The area of the triangle is $\frac{1}{2}(rh)$.

So, the volume of any cone is $\frac{2}{3}\pi r \times \frac{1}{2}(rh) = \frac{2}{6}\pi r^2 h$, or $\frac{1}{3}\pi r^2 h$.

11

Talk the Talk

Students write an informal argument to show how Cavalieri's principle can be applied to determining the volume of a pyramid.

Grouping

Have students complete Questions 1–6 with a partner. Then have students share their responses as a class.

11

PROBLEM 3 And Now, Pyramids

You can use what you know about the similarities and differences between cylinders and cones to make conjectures about the volume of pyramids.

1. Compare different ways to create cylinders and cones. What similarities and differences are there between creating cylinders and cones:

   a. by stacking?

   Both solid figures are created by stacking circles. A cylinder is created by stacking congruent circles. A cone is created by stacking similar circles that are not congruent.

   b. by rotating?

   For both cylinders and cones, I determine an average point of the polygon that is rotated to create the solid figure. I use this average point to calculate the average distance that all of the points of the polygon are rotated. I, then, multiply the area of the polygon by this average distance to determine the volume formula.

   To create a cylinder, I rotate a rectangle with an average point located at $\left(\frac{1}{2}r, \frac{1}{2}h\right)$.

   To create a cone, I rotate a triangle with an average point located at $\left(\frac{1}{3}r, \frac{1}{3}h\right)$.

2. Analyze the formulas for the volumes of the cylinder and cone.

$$\text{Volume of cylinder} = \pi r^2 h$$
$$\text{Volume of cone} = \frac{1}{3}\pi r^2 h$$

   a. Which part of each formula describes the area of the base of each solid figure, B? Explain why.

   The expression πr^2 describes the area of the base of each solid figure, because the base of each solid is a circle. The area of a circle is given as $A = \pi r^2$.

   b. Which part of each formula describes the height of each solid figure?

   For each solid figure, the variable h in the volume formula represents the height.

   c. Rewrite the volume formulas for each solid figure using the variables B and h.

   Cylinder: $V = Bh$

   Cone: $V = \frac{1}{3}Bh$

3. How are the formulas for the volumes of a cylinder and cone similar and different?

Both formulas involve multiplying the area of the base by the height. The volume formula for a cone is $\frac{1}{3}$ the volume of a cylinder with the same base area and height, so I have to multiply Bh by $\frac{1}{3}$ to obtain the volume of a cone.

4. Now compare different ways to create prisms and pyramids. What similarities and differences are there between creating prisms and pyramids:

   a. by stacking?

   Both solid figures are created by stacking polygons. A prism is created by stacking congruent polygons. A pyramid is created by stacking similar polygons that are not congruent.

   b. by rotating?

   Prisms and cylinder cannot be created using rotations.

5. Analyze the formula for the volume of a prism.

   Volume of prism: $V = Bh$

   a. Which part of the formula represents the area of the base?

   The variable B represents the area of the base.

   b. Which part of the formula represents the height?

   The variable h represents the height.

6. Based on your answers to Questions 1 through 5, what conjecture can you make about the formula for the volume of any pyramid? Explain your reasoning.

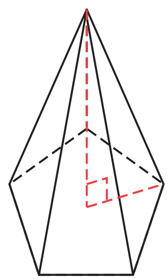
I think the volume formula for a pyramid is $\frac{1}{3}$ the volume of a prism with the same base area and height. A prism is created by stacking congruent polygons, which is similar to creating a cylinder by stacking congruent circles. A pyramid is created by stacking similar polygons that are not congruent, which is similar to creating a cone by stacking similar circles.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Jody was asked to approximate the volume of a pyramid. She first had to decide how to determine the height of the pyramid. She needs your help.

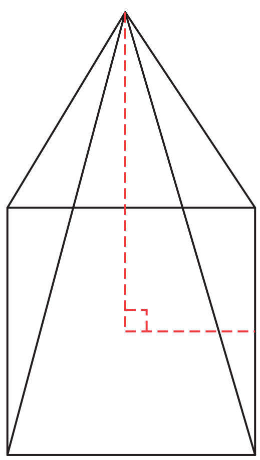
1. Draw a line segment that represents the height of the pentagonal pyramid.



2. Describe a triangle that can be formed to calculate the height of the pyramid.

   A triangle that can be formed to calculate the height of the pyramid connects the vertex to the center point of the base to a corner of the base.

3. Draw a line segment that represents the height of the square pyramid.



4. Describe a triangle that can be formed to calculate the height of the pyramid.

   A triangle that can be formed to calculate the height of the pyramid connects the vertex to the center point of the base to a midpoint of a side of the base.

11

5. Jody wanted to determine the height of this polyhedron. She knew that it was the vertical distance from the center point of the base (point *P*) to the vertex (point *T*), but she wasn't sure if the triangle she wanted to form should connect to point *S*, a corner of the base, as shown in figure 1 or point *A*, the midpoint of the side of the base, as shown in figure 2. She plans to solve for the distance from point *P* to point *T* using the Pythagorean Theorem.

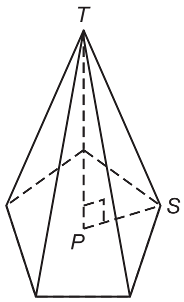


Figure 1

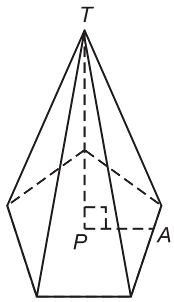


Figure 2

What advice would you give Jody?

I would tell Jody that either choice would result in the same answer. She should think about what other distances are known in the situation, because she will need to know the length of two sides to use the Pythagorean Theorem. If she is given the length of line segments *PS* and *TS*, she should use figure 1. If she is given the length of line segments *PA* and *TA*, she should use figure 2.

11

11

11.5

Spheres à la Archimedes

Volume of a Sphere

LEARNING GOALS

In this lesson, you will:

- Derive the formula for the volume of a sphere.

KEY TERMS

- sphere
- radius of a sphere
- diameter of a sphere
- great circle of a sphere
- hemisphere
- annulus

ESSENTIAL IDEAS

- A sphere is the set of all points in three dimensions that are equidistant from a given point called the center.
- The radius of a sphere is a line segment drawn from the center of the sphere to a point on the sphere.
- The diameter of a sphere is a line segment drawn between two points on the sphere passing through the center.
- A cross section of a solid is the two dimensional figure formed by the intersection of a plane and a solid when a plane passes through the solid.
- A great circle of a sphere is a cross section of a sphere when a plane passes through the center of the sphere.
- A hemisphere is half of a sphere bounded by a great circle.
- The volume formula for a sphere is: $V = \frac{4}{3}\pi r^3$, where V is the volume, and r is the radius of the sphere.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-GMD Geometric Measurement and Dimension

Visualize relationships between two-dimensional and three dimensional objects

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Overview

This lesson leverages rotation and stacking strategies to derive the formula for the volume of a sphere using a cylinder, cone, and hemisphere with equal heights and radii.

11

Warm Up

1. What is larger, the volume of Earth or the surface area of Earth?

   The unit of measure for volume is cubic and the unit of measure for area is quadratic. They cannot be compared.

2. Is it possible to compare the amount of sand it would take to fill a model of the Earth and the amount of sand it would take to cover the entire surface of the model? What information would you need? Explain.

   Yes, they could be compared because both tasks involve calculating the volume of sand. You would need to know the depth of sand to cover the surface and you would need to know the radius of Earth

11

11

Spheres à la Archimedes

Volume of a Sphere

11.5

LEARNING GOAL

In this lesson, you will:

- Derive the formula for the volume of a sphere.

KEY TERMS

- sphere
- radius of a sphere
- diameter of a sphere
- great circle of a sphere
- hemisphere
- annulus

Archimedes of Syracuse, Sicily, who lived from 287 BC to 212 BC, was an ancient Greek mathematician, physicist, and engineer. Archimedes discovered formulas for computing volumes of spheres, cylinders, and cones.

Archimedes has been honored in many ways for his contributions. He has appeared on postage stamps in East Germany, Greece, Italy, Nicaragua, San Marino, and Spain. His portrait appears on the Fields Medal for outstanding achievement in mathematics. You can even say that his honors are out of this world. There is a crater on the moon named Archimedes, a mountain range on the moon named the Montes Archimedes, and an asteroid named 3600 Archimedes!

Problem 1

The scenario is about Archimedes of Syracuse, Sicily (c. 287–212 BC) who discovered the formula for the volume of a sphere using a cone, cylinder and hemisphere.

The problem begins with definitions of sphere, radius of a sphere, diameter of a sphere, cross section, great circle of a sphere and hemisphere. Students answer questions to determine the area of a cross section of the annulus formed by a cone placed inside a cylinder.

Grouping

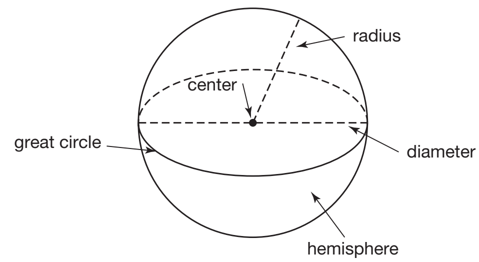
- Ask students to read introduction and definitions. Discuss as a class.
- Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- What polygon is the base of the small cone?
- Which formula is used to determine the area of the base of the small cone?
- What polygon is the base of the large cone?
- Which formula is used to determine the area of the base of the large cone?
- What is the difference between the two area formulas?
- If the area of the base of the smaller cone is removed from the cross section of the cylinder, how would you describe the shape of this region?
- What operation is used to remove the base of the smaller cone from the cross section of the cylinder?
- When rewriting the expression, what is common to both terms?
- What do the variables *b* and *r* represent?

PROBLEM 1 Starting with Circles . . . and Cones

Recall that a circle is the set of all points in two dimensions that are equidistant from the center of the circle. A sphere can be thought of as a three-dimensional circle.



A **sphere** is the set of all points in three dimensions that are equidistant from a given point called the center.

The **radius of a sphere** is a line segment drawn from the center of the sphere to a point on the sphere.

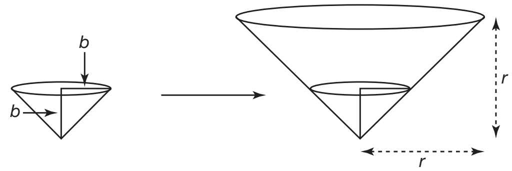
The **diameter of a sphere** is a line segment drawn between two points on the sphere passing through the center.

A **great circle of a sphere** is a cross section of a sphere when a plane passes through the center of the sphere.

A **hemisphere** is half of a sphere bounded by a great circle.

The cone shown on the left has a height and a radius equal to b. The height and the radius form two legs of a right triangle inside the cone. The hypotenuse lies along the side of the cone.

The cone shown on the right is an enlargement of the first cone. It also has a height that is equal to its radius, r. The smaller cone is shown inside the larger cone.



1. Write an expression to describe the area of:

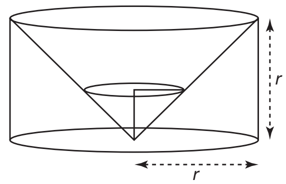
a. the base of the smaller cone.

πb^2

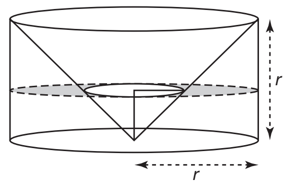
b. the base of the larger cone.

πr^2

2. Place both cones inside a cylinder with the same radius and height as the larger cone.



Then, make a horizontal cross section through the cylinder just at the base of the smaller cone.



What is the area of this cross section? Explain your reasoning.

The cross section is a circle with a radius of r. So, the area of the cross section is πr^2.

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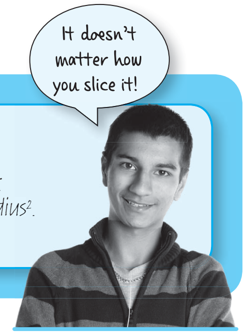
3. Amy makes the following statement about the horizontal cross section of a cylinder.



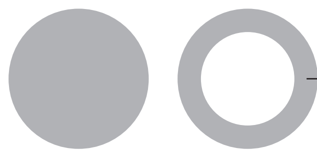
Amy

It doesn't matter where you make the cross section in a cylinder. The cross section will always be a circle with an area of $\pi \times \text{radius}^2$.

Explain why Amy is correct.

You can think of a cylinder as a translation of a circle. That circle's radius doesn't change over the translation, because a translation preserves congruency. So, no matter where the cross section is on a cylinder, it will have the same radius as every other cross section. That means it will also have the same area as every other cross section.

The image on the left shows the cross section of the cylinder, including the base of the cone. The image on the right shows the cross section of the cylinder with the base of the cone removed. The area bound between the two concentric circles shown on the right is called the **annulus**.



4. Calculate the area of the annulus of the cylinder. Explain your reasoning.

The area of the circular cross section is πr^2. The area of the base of the smaller cone is πb^2. So, the area of the annulus is $\pi r^2 - \pi b^2$.

5. Show how you can use the Distributive Property to rewrite your expression from Question 4.

$\pi(r^2 - b^2)$

Now that you have a formula that describes the area of the annulus of the cylinder, let's compare this with a formula that describes the area of a cross section of a hemisphere with the same height and radius as the cylinder.

11

Problem 2

Using a hemisphere and cylinder with a cone with the same radius and equal heights, students write an algebraic expression representing the area of the circular cross section of the hemisphere. They conclude that this area is equal to the area of the cross section in the previous problem.

Grouping

- Ask students to read introduction. Discuss as a class.
- Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

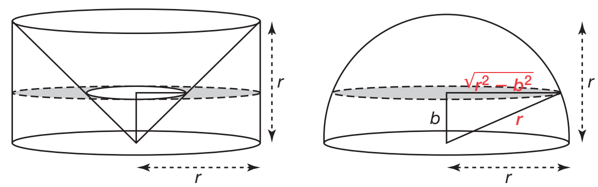
Guiding Questions for Share Phase, Questions 1 through 7

- How does the shape of the cross section in the hemisphere compare to the shape of the cross section in the cylinder?
- Where is the center point of the hemisphere located?
- What do you know about the distance from the center point of the hemisphere to any point on the hemisphere?
- What do the variables b and r represent?
- Is it possible for two sides of the right triangle to be equal in length? Why or why not?

PROBLEM 2 And Now Hemispheres

The diagram shows a hemisphere with the same radius and height of the cylinder from Problem 1.

The diagram also shows that there is a cross section in the hemisphere at the same height, b, as that in the cylinder.



1. Describe the shape of the cross section shown in the hemisphere.

   The cross section is a circle.

2. Analyze the hemisphere. Write expressions for the side lengths of the right triangle in the diagram. Label the diagram with the measurements.

   The length of the shortest leg is b, and the length of the hypotenuse is r, because a sphere or hemisphere is defined by all the points that are equidistant from a center point. Using the Pythagorean Theorem, I can determine that the horizontal side length is $\sqrt{r^2 - b^2}$.

3. Lacy says that the length of the horizontal side has a measure of r. Is Lacy correct? Explain your reasoning.

   Lacy is not correct. The hypotenuse of a right triangle is the longest side of that triangle. The horizontal length could be close to the hypotenuse, but it can't be as long as the hypotenuse, which has a length of r.

4. Write an expression to describe the area of the cross section in the hemisphere. Explain your reasoning.

   The cross section has a radius of $\sqrt{r^2 - b^2}$. So, the area of the cross section is $\pi \times \text{radius}^2$, or $\pi(r^2 - b^2)$.

- How is the Pythagorean Theorem used to write an expression representing the horizontal side length of the right triangle in the diagram of the hemisphere?
- Does the expression describing the area of the cross section in the hemisphere look familiar?
- Does every cross section of the hemisphere and the cylinder, with the cone removed, have the same area?
- What is the volume formula of a cone?

11

- What is the volume formula of a cylinder?
- Is the volume of the hemisphere equal to the volume of the cylinder with equal height and radius minus the volume of the cone with equal height and radius?
- Considering the hemisphere's height is equal to its radius, how can the formula be rewritten?
- How is the volume formula for a hemisphere used to write the volume formula for a sphere?

5. Compare the area of the cross section of the hemisphere to the area of the annulus of the cylinder. What do you notice?

The areas of both cross sections at each height are equal to each other. They are both equal to $\pi(r^2 - b^2)$.

6. Stacy says that the volume of the cylinder and the volume of the hemisphere are not the same. But, if you remove the volume of the cone from the volume of the cylinder, then the resulting volume would be the same as the volume of the hemisphere. Is Stacy correct? Explain why or why not.

Stacy is correct. The area of any annulus of the cylinder is equal to the area of the circular cross section minus the area of the base of the cone. This difference is equal to the area of any horizontal cross section of the hemisphere. So, the volume of the cylinder, with the volume of the cone removed, is equal to the volume of the hemisphere with the same radius and height.

7. Write the formula for the volume of a sphere. Show your work and explain your reasoning.

The volume of the hemisphere is equal to the volume of the cylinder minus the volume of the cone:

Volume of hemisphere: $\pi r^2 h - \left(\frac{1}{3}\right)\pi r^2 h = \left(\frac{3}{3}\right)\pi r^2 h - \left(\frac{1}{3}\right)\pi r^2 h$

$= \left(\frac{2}{3}\right)\pi r^2 h$

Because a hemisphere's height is equal to its radius, its volume can be written as $\left(\frac{2}{3}\right)\pi r^2 \times r$, or $\left(\frac{2}{3}\right)\pi r^3$.

The volume of the sphere is twice the volume of the hemisphere, so the volume of the sphere is

$2 \times \left(\frac{2}{3}\right)\pi r^3$, or $\left(\frac{4}{3}\right)\pi r^3$.

Be prepared to share your solutions and methods.

11

Check for Students' Understanding

1. For over seven years, John Bain has spent his life creating the Worlds Largest Rubber Band Ball. The ball is completely made of rubber bands. Each rubber band was individually stretched around the ball creating a giant rubber band ball. The weight of the ball is over 3,120 pounds, the circumference is 15.1 feet, the cost of the materials was approximately $25,000 and the number of rubber bands was 850,000.

   Calculate the volume of the giant rubber band ball. Use 3.14 for pi.

   $C = 2\pi r$

   $15.1 = 2\pi r$

   $r \approx 2.4$

   $V = \frac{4}{3}\pi(2.4)^3$

   $V = \frac{4}{3}\pi(13.824)$

   $V \approx 57.88 \text{ ft}^3$

   The volume of the rubber band ball is approximately 57.88 ft^3.

2. The world's largest twine ball is in Darwin, Minnesota. It weighs 17,400 pounds and was created by Francis A. Johnson. He began this pursuit in March of 1950. He spent four hours a day, every day wrapping the ball. At some point, the ball had to be lifted with a crane to continue proper wrapping. It took Francis 39 years to complete. Upon completion, it was moved to a circular open air shed on his front lawn for all to view.

   If the volume of the world's largest twine ball is 7234.56 cubic feet, determine the radius. Use 3.14 for pi.

   $7234.56 = \frac{4}{3}\pi r^3$

   $5425.92 = \pi r^3$

   $1728 = r^3$

   $\sqrt[3]{1728} = r$

   $12 = r$

   The radius of the world's largest ball of twine is 12 feet.

11

11

11.6

Turn Up the . . .

Using Volume Formulas

LEARNING GOALS

In this lesson, you will:

- Apply the volume formulas for a pyramid, a cylinder, a cone, and a sphere to solve problems.

ESSENTIAL IDEAS

- The volume formulas for a pyramid, cylinder, cone, and a sphere are used to solve application problems.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-GMD Geometric Measurement and Dimension

Explain volume formulas and use them to solve problems

3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

Students apply the volume formulas of a cylinder, cone, pyramid, and sphere in different problem situations.

11

Warm Up

Ice Cream Cone Piñata

Carly asked her parents to make a piñata for her birthday party. A piñata is a brightly-colored papier-mâché, cardboard, or clay container, originating in Mexico, and filled with any combination of candy or small toys suspended from a height for blindfolded children to break with sticks. Her parents decided to make the piñata in the shape of her favorite dessert, an ice cream cone. They stuffed only the cone portion of the piñata.

- The height of the cone is 34″.
- The length of the diameter of the base is 24″.

Calculate the amount of space (cubic feet) in the cone that will be filled with goodies.

(144 square inches equal 1 square foot)

(1728 cubic inches equals 1 cubic foot)

Volume of the cone:

$V = \frac{1}{3}Bh$

$V = \frac{1}{3}(144\pi)(34) = 1632\pi \approx 5124.5 \text{ in}^3 \approx 3 \text{ ft}^3$

11

11

Turn Up the . . .

Using Volume Formulas

11.6

LEARNING GOAL

In this lesson, you will:

- Apply the volume formulas for a pyramid, a cylinder, a cone, and a sphere to solve problems.

A mnemonic is a device used to help you remember something. For example, the mnemomic "My Very Energetic Mother Just Served Us Noodles" can be used to remember the order of the planets in the Solar System.

What about volume formulas? Can you come up with mnemonics to remember these so you don't have to look them up?

Maybe you can use this one to remember the formula for the volume of a cylinder:

Cylinders are: "Perfectly Ready 2 Hold"

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ \pi & \cdot & r^2 & \cdot & h \end{array}$$

Try to come up with mnemonics for the other volume formulas that you have learned!

11

Problem 1

A standard sheet of paper is used to create two different sized cylinders. Students first predict which cylinder contains the greater volume and then use mathematics to verify their answer. Next, students write a strategy to determine the approximate volume of an odd shaped vase and then use mathematics to support their estimate.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

11

Guiding Questions for Share Phase, Questions 1 through 6

- If the two cylinders are made from the same sized sheet of paper, how can they be different sizes?
- What formula is used to determine the length of the radius of the base of the cylinder?
- What formula is used to determine the height of the cylinder?
- What formula is used to determine the volume of the cylinder?
- Is the volume exact or approximate? Why?
- Which variable was squared and used as a multiplier in the volume formula?

PROBLEM 1 On a Roll

1. A standard sized sheet of paper measures 8.5 inches by 11 inches. Use two standard sized sheets of paper to create two cylinders. One cylinder should have a height of 11 inches and the other cylinder should have a height of 8.5 inches.

2. Carol predicts that the cylinder with a height of 11 inches has a greater volume.
Lois predicts that the cylinder with a height of 8.5 inches has a greater volume.
Stu predicts that the two cylinders have the same volume.

   Predict which cylinder has the greatest volume.

   Answers will vary.
   Student response may include:
   I predict both cylinders will have the same volume because the same sized paper was used to create both cylinders.

3. Determine the radius and the height of each cylinder without using a measuring tool.

   Radius of cylinder with height of 8.5 inches:

   $C = 2\pi r$

   $11 = 2\pi r$

   $\frac{11}{2\pi} = r$

   $r \approx 1.75$

   The cylinder with a height of 8.5 inches has a radius of approximately 1.75 inches.

   Radius of cylinder with height of 11 inches:

   $C = 2\pi r$

   $8.5 = 2\pi r$

   $\frac{8.5}{2\pi} = r$

   $r \approx 1.35$

   The cylinder with a height of 11 inches has a radius of approximately 1.35 inches.

- What equation was used to determine the height of a cylinder with equal volume?
- Is the height exact or approximate? Why?
- Why does the radius have a more significant impact on the volume of the cylinder than the height? Explain.

4. Calculate the volume of each cylinder to prove or disprove your prediction and determine who was correct.

Volume of the cylinder with height of 8.5 inches:

$V = \pi r^2 h$

$= \pi(1.75)^2(8.5)$

≈ 81.74

Volume of the cylinder with height of 11 inches:

$V = \pi r^2 h$

$= \pi(1.35)^2(11)$

≈ 62.95

Lois is correct. The cylinder with a height of 8.5 inches has a greater volume.

5. Does the radius or the height have a greater impact on the magnitude of the volume? Explain your reasoning.

The radius has a greater impact on the volume than the height because the radius is squared, whereas the height is not squared. The height would have to have been substantially longer to create a larger volume.

6. Consider the volume of the cylinder with a height of 8.5 inches. What radius would be required to create a cylinder with a height of 11 inches that has the same volume?

$V = \pi r^2 h$

$81.74 = \pi(1.35)^2(h)$

$h \approx 14.3$

The height would need to be approximately 14.3 inches to create a cylinder of equal volume.

11

Problem 2

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Is there more than one correct strategy to approximate the volume of the vase?
- How do you decide which strategy will produce a more accurate result?
- Which volume formula(s) are used to approximate the volume of the vase?

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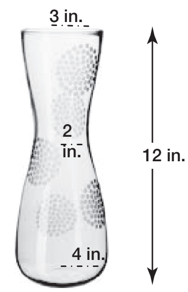
PROBLEM 2 **Let's Vase It**

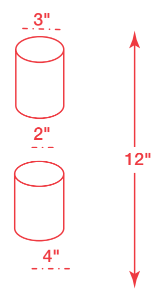
1. Describe a strategy for approximating the volume of the vase shown.



The volume of the upper and lower portions of the vase could be approximated as cylinders. I could average the length of the radii on both bases of each cylinder to get an approximate volume.

2. Determine the approximate volume of the vase.

One possible solution is using the volume formula for a cylinder as shown.



I used 2.5 as the length of the bases of the upper cylinder and 3 as the length of the bases of the lower cylinder.

$V = \pi r^2 h$

V (upper cylinder) $= (3.14)(2.5)^2(6) = 117.75$

V (lower cylinder) $= (3.14)(3)^2(6) = 169.56$

$117.75 + 169.56 = 287.31$

The approximate volume of the vase is 287.31 cubic units.

Problem 3

A hot air-balloon can be divided into a hemisphere and a cone to determine an approximate volume. Students determine the approximate cubic feet of hot air contained in a typical hot-air balloon. In the second scenario, students work with the volume of water in a great lake and a cup shaped like a cone. Next, students describe a cone that has the same radius and volume as a sphere. In the last activity, students consider a cylinder and square pyramid having the same height, but the radius of the cylinder and the side lengths of the square base of the pyramid are gradually increasing. They determine which solid's volume increases more rapidly.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- Is a hot-air balloon spherical? Why or why not?
- When approximating the volume, would it be better to think of the balloon as one solid or divide the hot-air balloon into two solids?
- A hot-air balloon can be divided into which familiar solids?

PROBLEM 3 Balloons, Lakes, and Graphs

1. A typical hot-air balloon is about 75 feet tall and about 55 feet in diameter at its widest point. About how many cubic feet of hot air does a typical hot-air balloon hold? Explain how you determined your answer.

The top of the balloon can be approximated by a hemisphere with a diameter of 55 feet and a radius of 27.5 feet. The formula for the volume of a hemisphere is $\frac{2}{3}\pi r^3$. So, the top would have a volume of $\frac{2}{3}\pi(27.5)^3$, or approximately 43,556.87 cubic feet.

The remaining part of the balloon can be approximated by a cone with the same diameter but with a height of 75 − 27.5, or 47.5, feet. The formula for the volume of a cone is $\frac{1}{3}\pi r^2 h$. So, the bottom of the balloon would have a volume of $\frac{1}{3}\pi(27.5)^2(47.5)$, or approximately 37,617.3 cubic feet.

The total volume of hot air the balloon could contain would be 81,174.17 cubic feet.

- What is the radius of the widest part of the balloon?
- What formulas were used to determine the volume of the balloon?
- Is the volume exact or approximate? Why?
- Is a unit conversion necessary to determine the time it takes to empty the lake?
- How do you convert cubic miles into cubic inches?
- What operation is used to determine the time it takes to empty the lake?
- What is the volume formula for a sphere?

11

- What is the volume formula for a cone?
- Which variables in the volume formulas are equal?
- How is the graph used to determine which solid increases more rapidly?
- How did you determine the graph of the cylinder?
- How did you determine the graph of the square pyramid?
- Which graph is steeper?

11

2. Lake Erie, the smallest of the Great Lakes by volume, still holds an impressive 116 cubic miles of water. Suppose you start today dumping out the entire volume of Lake Erie using a cone cup. A typical cone cup has a diameter of $2\frac{3}{4}$ inches and a height of 4 inches. About how long would it take you to empty the lake if you could dump out one cup per second?

Volume of cone cup: $\frac{1}{3}\pi\left(\frac{11}{8}\right)^2(4) = 7.92$ cubic inches

Volume of Lake Erie: 116 cubic miles is equal to 7,349,760 cubic inches

$7{,}349{,}760 \div 7.92 = 928{,}000$ cups, or 928,000 seconds

It would take me 15,467 minutes, or 257 hours, or 10.7 days working around the clock to empty the lake.

3. A cone and a sphere each have a radius of r units. The cone and the sphere also have equal volumes. Describe the height of the cone in terms of the radius. Show your work.

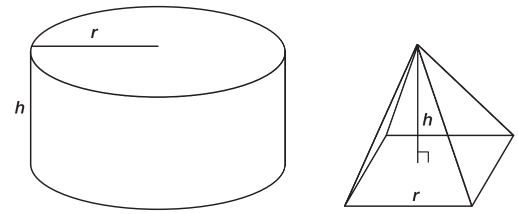
$$\frac{4}{3}\pi r^3 = \frac{1}{3}\pi r^2 h$$
$$4\pi r^3 = \pi r^2 h$$
$$4r^3 = r^2 h$$
$$4r = h$$

The height of any cone that has the same radius and volume of a sphere would be 4 times the radius of the sphere.

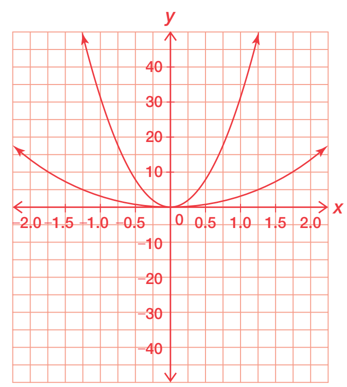
11

11

4. The diagram shows a cylinder and a square pyramid with the same height. The width of the pyramid is equal to the radius of the cylinder. Suppose that the radius of the cylinder is gradually increased and the side lengths of the square pyramid are also gradually increased. Which solid's volume would increase more rapidly? Explain your reasoning.



The formula for the volume of a cylinder is $\pi r^2 h$. The formula for the volume of a pyramid is $\frac{1}{3}s^2h$. In this problem, $s = r$ and the heights are equal. So, I want to compare $y = \pi r^2$ and $y = \frac{1}{3}r^2$. If I graph these two functions, I can see that the volume of the cylinder is growing more rapidly as the radius increases.



Be prepared to share your solutions and methods.

Check for Students' Understanding

The Great Pyramid of Giza also known as the Khufu's Pyramid, Pyramid of Khufu, and Pyramid of Cheops is located near Cairo, Egypt, and is one of the Seven Wonders of the Ancient World. The Great Pyramid was the tallest man-made structure in the world for over 3,800 years.

A side of the square base originally measured approximately 230.4 meters. The original height was approximately 146.7 meters and the slant height was approximately 186.5 meters.

Calculate the volume of the Great Pyramid.

Area of the square base: $A = (230.4)^2 = 53{,}084.2 \text{ m}^2$

Volume of the pyramid:

$V = \frac{1}{3}Bh$

$V = \frac{1}{3}(53084.2)(146.7) = 2{,}595{,}817.38 \text{ m}^3$

11

11

11.7

Tree Rings

Cross Sections

LEARNING GOALS

In this lesson, you will:

- Determine the shapes of cross sections.
- Determine the shapes of intersections of solids and planes.

ESSENTIAL IDEAS

- A cross section of a three-dimensional solid is a two-dimensional figure that is formed by the intersection of the solid and a plane.
- The cross section of a sphere results in a great circle, circle smaller than a great circle, or single point.
- The cross section of a cube results in a square, rectangle that is not a square, triangle, pentagon, or a hexagon.
- The cross section of a pyramid results in a square, isosceles trapezoid, quadrilateral, or single point.
- The cross section of a cone results in a circle, ellipse, parabola, triangle, or a single point.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-GMD Geometric Measurement and Dimension

Visualize relationships between two-dimensional and three dimensional objects

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

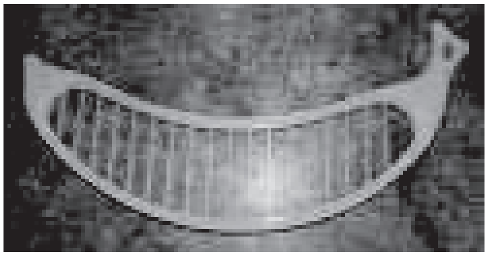
1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

Geometric solids are sliced with planes drawn parallel to the base, perpendicular to the base, and on an angle to the base which create a variety of cross sections. Students practice identifying solids given their cross sections, and cross sections given their solids.

11

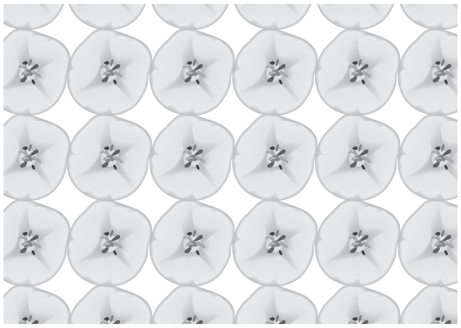
Warm Up



1. Describe the shapes of the banana pieces after using the banana slicer shown.

   The banana slicer slices the banana into circular shaped flat pieces exposing the interior of the banana. Most pieces are the same size but taper off as either end slice is cut.

2. Draw several slices of the banana.



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3. Describe the shapes of the pieces egg after using the egg slicer shown.

The egg slicer slices the egg into oval shaped flat pieces exposing a round yellow yolk with a white exterior. Each piece gradually gets smaller but is similar to all other pieces until the yolk disappears.

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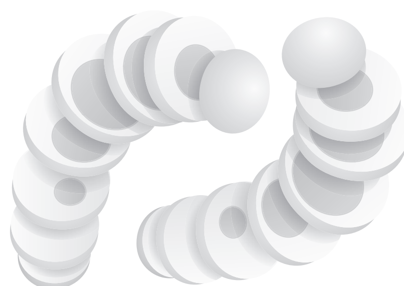
4. Draw several slices of the egg.



11.7

Tree Rings

Cross Sections

LEARNING GOALS

In this lesson, you will:

- Determine the shapes of cross sections.
- Determine the shapes of intersections of solids and planes.

Each year, a tree grows in diameter. The amount of growth of the diameter depends on the weather conditions and the amount of water that is available to the tree. If a tree is cut perpendicular to its trunk, you can see rings, one for each year of growth. The wider the ring, the greater the amount of growth in one year.

Dendrochronology, or tree-ring dating, is the method of dating trees based on an analysis of their rings. Often times, it is possible to date a tree to an exact year.

The oldest known tree in the world is a Great Basin bristlecone pine located in White Mountains, California. It is estimated to be over 5000 years old!

Problem 1

A cylinder is used to model a tree trunk. In questions 1 through 3, students examine the cross sections formed when a plane cuts through the tree trunk parallel to the base (circle), perpendicular to the base (rectangle) and on an angle to the base (ellipse). In questions 4 through 7, students sketch possible cross sections of a sphere, cube, square pyramid, and cone. In question 8 students are given a hexagonal prism and a pentagonal pyramid and sketch cross sections formed by the intersection of a plane parallel to and perpendicular to the base. In question 9 students are given cross sections and sketch the solids associated with the cross sections.

11

Grouping

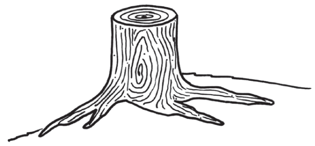
Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- When the plane cuts through the cylinder perpendicular to the altitude, what is the relationship between the plane and the base of the cylinder?

PROBLEM 1 Cutting a Tree Trunk

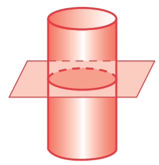
A section of a tree trunk is roughly in the shape of a cylinder as shown.



When a tree trunk is cut in order to see the tree rings, a cross section of the trunk is being studied.

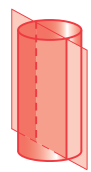
1. Suppose a plane intersects a cylinder parallel to its bases. What is the shape of the cross section? Sketch an example of this cross section.

   The cross section is a circle.



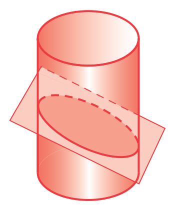
2. Suppose a plane intersects a cylinder perpendicular to its bases so that the plane passes through the centers of the bases. What is the shape of this cross section? Sketch an example of this cross section.

   The cross section is a rectangle.



3. Suppose a plane intersects a cylinder so that it is not parallel to its bases. What is the shape of this cross section? Sketch an example of this cross section.

   The cross section formed is an ellipse.

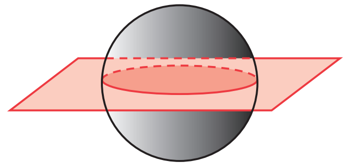


- When the plane cuts through the cylinder perpendicular to the base, what is the relationship between the plane and the altitude of the cylinder?
- What if the plane cuts through the cylinder at an angle along its height less than 90 degrees?
- What if the plane cuts through the cylinder at an angle along its height equal to 90 degrees?

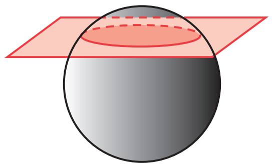
- What if the plane cuts through the cylinder at an angle along its height greater than to 90 degrees?
- Can additional cross sections be formed in Questions 1 through 5? If so, identify them and explain how they are formed.
- Is a circular cross section possible? If so, how?
- Is an elliptical cross section possible? If so, how?
- Is a rectangular cross section possible? If so, how?
- Is a square cross section possible? If so, how?
- Is a trapezoidal cross section possible? If so, how?
- Is a triangular cross section possible? If so, how?
- Is a single point cross section possible? If so, how?
- Is a pentagonal cross section possible? If so, how?
- Is a hexagonal cross section possible? If so, how?
- Is a parabolic cross section possible? If so, how?

4. Consider a sphere. Sketch and describe three different cross sections formed when a plane intersects a sphere.

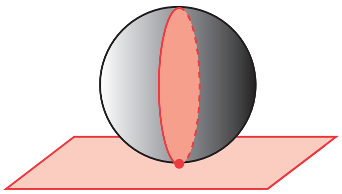
The cross section formed is a great circle.



The cross section formed is a circle smaller than a great circle.



The cross section formed is a single point.



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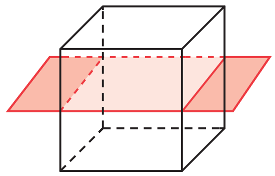
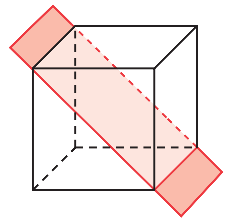
5. Consider a cube. Sketch and describe five different cross sections formed when a plane intersects a cube.

Answers will vary.

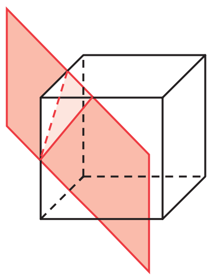
The cross section formed is a square.



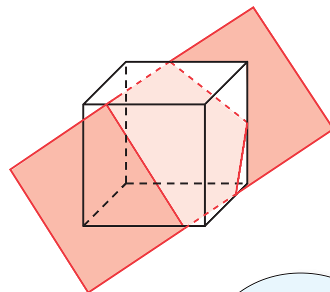
The cross section formed is a rectangle.



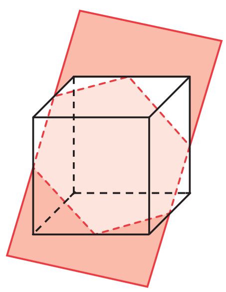
The cross section formed is a triangle.



The cross section formed is a pentagon.



The cross section formed is a hexagon.



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Sally says that it's not possible to form a circle or an octagon as a cross section of a cube. Is Sally correct? Explain your reasoning.

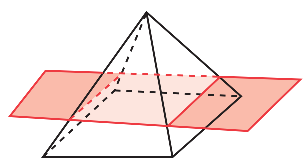
Sally is correct. When the cube is cut by a plane, each edge of the cross section relates to an intersection on one of the cube's faces with the plane. A cube has no curved faces, so it is not possible to intersect a plane and a cube to create a cross section with a curved segment on its perimeter.

A cube has six faces, so it is not possible to cut a cube with a plane and create an octagonal cross section.

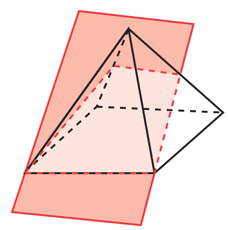
6. Consider a square pyramid. Sketch and describe four different cross sections formed when a plane intersects a square pyramid.

Answers will vary.

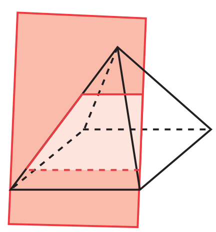
The cross section formed is a square.



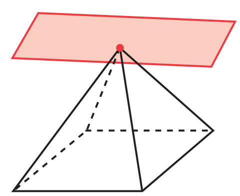
The cross section formed is an isosceles trapezoid.



The cross section formed is a quadrilateral.



The cross section formed is a single point.



Grouping

- Have students complete Questions 6 through 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 6 through 9

- Can additional cross sections be formed in Questions 6 and 7? If so, identify them and explain how they are formed.
- Identify the name of the given solids in Question 8.
- What other cross sections can be formed with the solids in Question 8?
- Which solids have a circle as a cross section but not a triangle?
- Which solids have a triangle as a cross section but not a circle?
- Does more than one solid have a triangle as a cross section?
- Which solids have a rectangle as a cross section?

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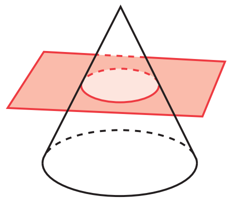
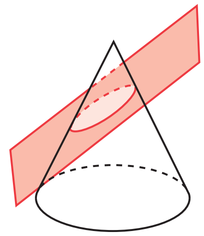
7. Consider a cone. Sketch and describe four different cross sections formed when a plane intersects a cone.

Answers will vary.

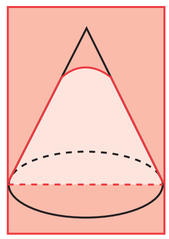
The cross section formed is a circle.



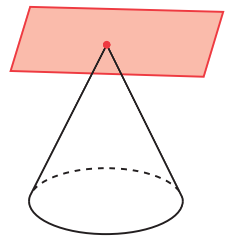
The cross section formed is an ellipse.



The cross section formed is a parabola.

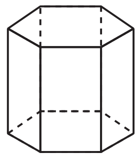


The cross section formed is a single point.

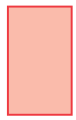


8. Use each solid to create two cross sections. Create one cross section parallel to a base and a second cross section perpendicular to a base. Then, identify each of the cross sections.

a.

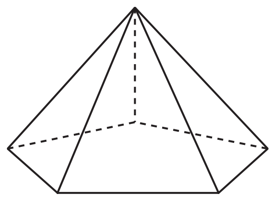


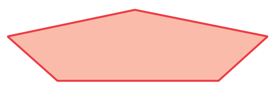
The cross section parallel to the base is a hexagon congruent to the hexagonal bases.



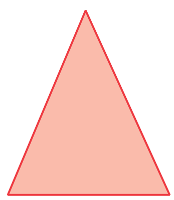
The cross section perpendicular to the base is a rectangle.

b.





The cross section parallel to the base is a pentagon similar to the pentagonal base.



The cross section perpendicular to the base is a triangle.

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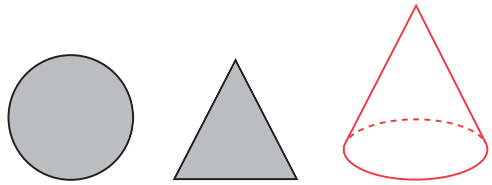
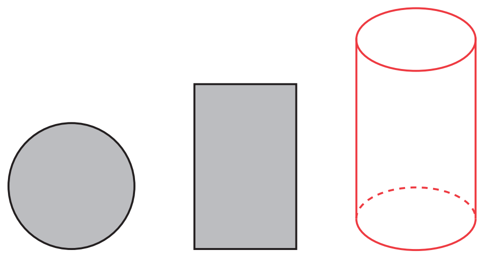
9. Draw a solid that could have the given cross sections.

a. A cross section parallel to a base and a cross section perpendicular to a base



The solid is a cone.

b. A cross section parallel to a base and a cross section perpendicular to a base



The solid is a cylinder.

c. Consider a prism placed on a table such that the base is horizontal. If a plane passes through the prism horizontally at any height, describe the cross sections.

All cross sections will be congruent to the bases of the prism.

d. Define a cylinder in terms of its cross sections.

A cylinder is a solid that has uniform circular cross sections.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Which of the following are not possible cross sections of a cube? If it is not possible, explain why it is not possible.

1. a hexagon
   possible

2. an octagon
   Not possible. When the cube is cut by a plane, each edge of the cross section relates to an intersection on one of the cube's faces with the plane. A cube has six faces, therefore it is not possible to cut a cube with one plane and create an octagonal cross section.

3. a rectangle
   possible

4. a equilateral triangle
   possible

5. a circle
   Not possible. A cube has no curved faces, so it is not possible to intersect a plane and a cube to create a cross section with a curved segment on its perimeter.

6. a triangle that is not equilateral
   possible

7. a pentagon
   possible

8. a parallelogram that is not a rectangle
   possible

9. a rectangle that is not a square
   possible

10. a square
    possible

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11.8

Two Dimensions Meet Three Dimensions

Diagonals in Three Dimensions

LEARNING GOALS

In this lesson, you will:

- Use the Pythagorean Theorem to determine the length of a diagonal of a solid.
- Use a formula to determine the length of a diagonal of a rectangular solid given the lengths of three perpendicular edges.
- Use a formula to determine the length of a diagonal of a rectangular solid given the diagonal measurements of three perpendicular sides.

ESSENTIAL IDEAS

- The Pythagorean Theorem is used to determine the length of a diagonal of a solid.
- The formula to determine the length of a diagonal of a rectangular solid is $d^2 = a^2 + b^2 + c^2$, where d represents the length of a diagonal, and a, b, and c represent the length, width and height of the rectangular solid.
- The formula to determine the length of a diagonal of a rectangular solid is $d^2 = \frac{1}{2}(d_1^2 + d_2^2 + d_3^2)$, where d represents the length of a three dimensional diagonal, and d_1, d_2, and d_3 represent the lengths of the diagonals on three perpendicular faces of the solid.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

3. Apply geometric methods to solve design problems.

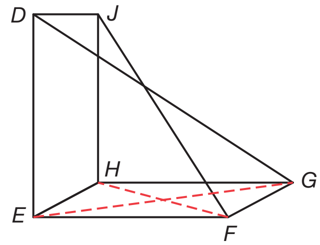
Overview

The length of a diagonal of a rectangular prism is calculated using different methods. Student use the Pythagorean Theorem and two other formulas to determine the length of a diagonal. One formula uses the length, width, and height of the rectangular prism and the other formula uses the diagonal lengths of three perpendicular faces.

11

Warm Up

Rectangle *DEHJ* is drawn perpendicular to rectangle *HEFG*.



Suppose the length of lines segments *DE*, *EF*, and *EH* are known. Explain a strategy for calculating the length of line segments *DG* and *JF*.

First connect points *E* and *G* to form line segment *EG*, and points *H* and *F* to form line segment *HF*. Since opposite sides of a rectangle are congruent, we know *DE* = *JH*, *DJ* = *EH* = *FG*, and *HG* = *EF*. Using the known values, the Pythagorean Theorem can be used to solve for the lengths of line segments *EG* and *HF*. Since $\overline{DG}$ is the hypotenuse of right triangle *DEG* and $\overline{JF}$ is the hypotenuse of right triangle *JHF*, we can use the Pythagorean Theorem again to solve for the length of line segments *DG* and *JF*.

11

11

11.8 Two Dimensions Meet Three Dimensions

Diagonals in Three Dimensions

LEARNING GOALS

In this lesson, you will:

- Use the Pythagorean Theorem to determine the length of a diagonal of a solid.
- Use a formula to determine the length of a diagonal of a rectangular solid given the lengths of three perpendicular edges.
- Use a formula to determine the length of a diagonal of a rectangular solid given the diagonal measurements of three perpendicular sides.

There are entire industries dedicated to helping people move from one location to another. Some people prefer to hire a moving company that will pack, move, and unpack all of their belongings. Other people choose to rent a van and do all the packing and unpacking themselves.

One of the most common questions heard during a move is, "Will it fit?" Moving companies pride themselves on being able to pack items as efficiently as possible. Sometimes it almost seems impossible to fit so much into so little a space.

What strategies would you use if you had to move and pack yourself?

11

Problem 1

A rectangular solid is used to model a box of roses. Students sketch three-dimensional diagonals and determine their lengths using the Pythagorean Theorem. They also practice using the formula $d^2 = a^2 + b^2 + c^2$ where d represents the length of a diagonal, and a, b, and c represent the length, width and height of the rectangular solid.

Grouping

- Ask students to read the introduction to Problem 1. Discuss as a class.
- Have students complete Questions 1 through 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 9

- Do the roses have to be positioned in the box parallel to the sides of the box?
- How would you describe the position of a rose of maximum length contained in the box?
- What is the definition of a diagonal?
- Does this definition apply to both two-dimensional and three-dimensional diagonals?
- How many three-dimensional diagonals can be drawn in the box?
- How many two-dimensional diagonals can be drawn on the bottom side of the box?
- What equation is used to determine the length of the second leg?
- What equation is used to determine the length of the three-dimensional diagonal?
- What equation is used to determine if Norton is correct?
- How do you suppose Norton made this discovery?

PROBLEM 1 A Box of Roses

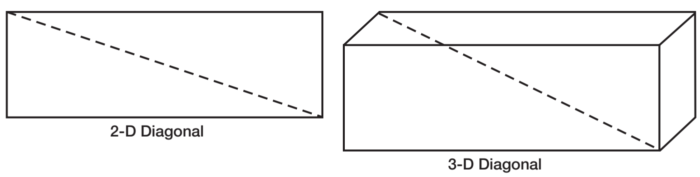
The dimensions of a rectangular box for long-stem roses are 18 inches in length, 6 inches in width, and 4 inches in height.

You need to determine the maximum length of a long-stem rose that will fit in the box without bended the rose's stem. You can use the Pythagorean Theorem to solve this problem.

1. What makes this problem different from all of the previous applications of the Pythagorean Theorem?

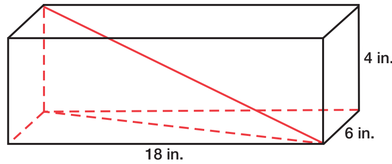
   I have always applied the Pythagorean Theorem in two dimensions. This problem will require me to use the Pythagorean Theorem in a three-dimensional situation.

2. Compare a two-dimensional diagonal to a three-dimensional diagonal. Describe the similarities and the differences.

   A two-dimensional diagonal and a three-dimensional diagonal are both line segments that connect any vertex to another vertex that does not share a face or an edge. A two-dimensional diagonal shares the same plane as the sides on which the figure is drawn. A three-dimensional diagonal does not share a plane with any side on which the figure is drawn.

3. Draw all of the sides in the rectangular solid you cannot see using dotted lines.



4. Draw a three-dimensional diagonal in the rectangular solid shown.

See image.

5. If the three-dimensional diagonal is the hypotenuse and an edge of the rectangular solid is a leg of the right triangle, where is the second leg?

The second leg is a diagonal drawn on the bottom of the box.

6. Draw the second leg using a dotted line.

See image.

7. Determine the length of the second leg.

$d^2 = 18^2 + 6^2$

$d^2 = 324 + 36$

$d^2 = 360$

$d = \sqrt{360} \approx 18.974$

The length of the second leg is approximately 18.974 inches.

8. Determine the length of the three-dimensional diagonal.

$d^2 = 18.974^2 + 4^2$

$d^2 = 360.013 + 16$

$d^2 = 376.013$

$d = \sqrt{376.013} \approx 19.39$

The length of the three-dimensional diagonal is approximately 19.39 inches.

11

9. Describe how you used the Pythagorean Theorem to calculate the length of the three-dimensional diagonal.

I formed a right triangle on the plane in which the three-dimensional diagonal was drawn such that it was the hypotenuse. However, I did not know the length of a leg of this triangle. I first applied the Pythagorean Theorem to determine the length of a leg using the plane on the bottom of the rectangular solid, and then I used the Pythagorean Theorem a second time to determine the length of the three-dimensional diagonal.

10. Norton tells his teacher he knows a shortcut for determining the length of a three-dimensional diagonal. He says, "All you have to do is calculate the sum of the squares of the rectangular solids' three perpendicular edges [the length, the width, and the height] and that sum would be equivalent to the square of the three-dimensional diagonal." Does this work? Use the rectangular solid in Question 3 and your answer in Question 8 to determine if Norton is correct. Explain your reasoning.

$SD^2 = 4^2 + 6^2 + 18^2$

$SD^2 = 16 + 36 + 324$

$SD^2 = 376$

$SD = \sqrt{376} \approx 19.39''$

Norton is correct.

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Problem 2

Rectangular solids are used in different contexts. Students determine the lengths of three-dimensional diagonals. They also practice using the formula $d^2 = \frac{1}{2}(d_1^2 + d_2^2 + d_3^2)$, where d represents the length of a three dimensional diagonal, and d_1, d_2, and d_3 represent the lengths of the diagonals on three perpendicular faces of the solid.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

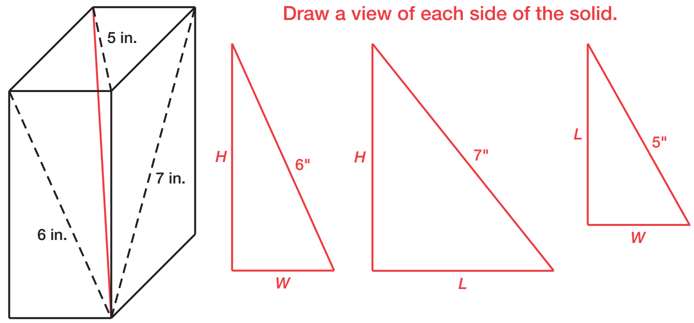
Guiding Questions for Share Phase, Questions 1 through 4

- Which panel of the rectangular solid is widest? How do you know?
- Why is it helpful to sketch each panel (front, side, top) separately?
- What variables were used to represent the unknown lengths on each panel?
- How many panels were associated with the height of the rectangular solid?
- How many panels were associated with the length of the rectangular solid?
- How many panels were associated with the width of the rectangular solid?
- Which variable did you solve for first?

PROBLEM 2 More Solids

1. A rectangular prism is shown. The diagonal across the front panel is 6 inches, the diagonal across the side panel is 7 inches, and a diagonal across the top panel is 5 inches. Determine the length of a three-dimensional diagonal.

Draw a view of each side of the solid.



Write equations representing each side of the solid.

$H^2 + W^2 = 6^2$ $\quad H^2 + L^2 = 7^2$ $\quad L^2 + W^2 = 5^2$

Solve for H^2 and L^2 and substitute into the third equation to calculate W.

$H^2 + W^2 = 36$	$36 - W^2 + L^2 = 49$	$W^2 + 13 + W^2 = 25$
$H^2 = 36 - W^2$	$L^2 - W^2 = 13$	$2W^2 = 12$
	$L^2 = W^2 + 13$	$W^2 = 6$
		$W = \sqrt{6}$

Substitute the value of W into the equations to calculate L and H.

$L^2 = W^2 + 13$	$H^2 = 36 - (\sqrt{6})^2$
$L^2 = (\sqrt{6})^2 + 13$	$H^2 = 36 - 6$
$L^2 = 6 + 13$	$H^2 = 30$
$L^2 = 19$	$H = \sqrt{30}$
$L = \sqrt{19}$	

The length is $\sqrt{19}$ inches, the width is $\sqrt{6}$ inches, and the height is $\sqrt{30}$ inches.

Calculate the three-dimensional diagonal.

$SD^2 = 5^2 + (\sqrt{30})^2$
$SD^2 = 25 + 30$
$SD^2 = 55$
$SD = \sqrt{55} \approx 7.42$

The length of a three-dimensional diagonal is $\sqrt{55}$ inches, or approximately 7.4 inches.

- Did you perform a substitution? Where?
- What variable did you solve for next?
- Was another substitution performed? Where?
- What equation is used to determine if Norton is correct?
- How do you suppose Norton made this discovery?

2. Norton tells his teacher that there is a much faster way to do Question 1. This time, he says, all you have to do is take one-half the sum of the squares of the diagonals on each dimension of the rectangular solid, and that would be equivalent to the square of the three-dimensional diagonal. Is Norton correct this time?

$SD^2 = \frac{1}{2}(5^2 + 6^2 + 7^2)$

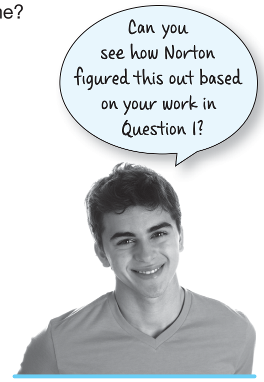
$SD^2 = \frac{1}{2}(25 + 36 + 49)$

$SD^2 = \frac{1}{2}(110)$

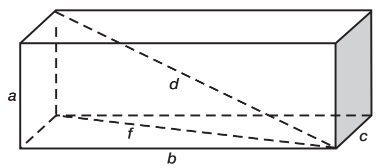
$SD^2 = 55$

$SD = \sqrt{55} \approx 7.4$

Norton is once again correct!



3. Write a formula to determine the three-dimensional diagonal in the rectangular prism in terms of the dimensions of the rectangular prism.

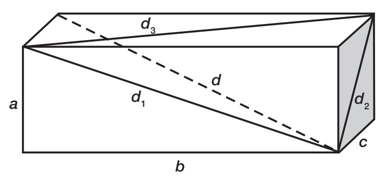


$f^2 = b^2 + c^2$

$d^2 = a^2 + f^2$

$d^2 = a^2 + b^2 + c^2$

4. Write a formula to determine the three-dimensional diagonal in the rectangular prism in terms of the lengths of the diagonals.



$d_1^2 = a^2 + b^2$

$d_2^2 = a^2 + c^2$

$d_3^2 = b^2 + c^2$

$2(a^2 + b^2 + c^2) = d_1^2 + d_2^2 + d_3^2$

$2d^2 = d_1^2 + d_2^2 + d_3^2$

$d^2 = \frac{1}{2}(d_1^2 + d_2^2 + d_3^2)$

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- How did you determine the formula for the three-dimensional diagonal?
- How will a three-dimensional diagonal help to solve this problem?
- How many inches are in eight feet?
- Are you able to use a formula to solve this problem? Which formula?
- Is the three-dimensional diagonal longer or shorter than the tree? How do you know?
- Under what constraints will the tree fit into the car?
- Are you able to use a formula to solve this problem? Which formula?
- Which dimensions are unknown?
- Is the striping considered two or three-dimensional diagonals?
- How did you determine the length of each diagonal?
- How many feet of striping are needed to design one pencil box?

PROBLEM 3 Box It Up

1. Your new part-time job with a landscaping business requires you to transport small trees in your own car. One particular tree measures 8 feet from the root ball to the top. The interior of your car is 62 inches in length, 40 inches in width, and 45 inches in height. Determine if the tree will fit inside your car. Explain your reasoning.

$d^2 = a^2 + b^2 + c^2$

$d^2 = 62^2 + 40^2 + 45^2$

$d^2 = 3844 + 1600 + 2025$

$d^2 = 7469$

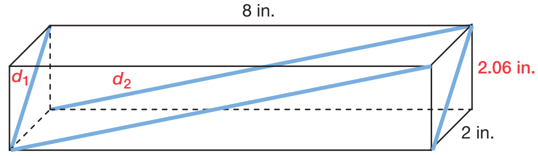
$d^2 = \sqrt{7469} \approx 86.42$

The height of the tree is 96 inches, and the three-dimensional diagonal is approximately 86.42 inches, so the tree will not fit inside my car.

- How many feet of striping are needed to design two hundred pencil boxes?
- How was 20% profit used to determine the amount of the bid?

2. Andy's company is bidding on a project to create a decal design for pencil boxes. If the client likes Andy's design then they will order 200 boxes. The design plan with the decal stripes is shown. The length of the three-dimensional diagonal is approximately 8.5 inches.

If the striping will cost Andy \$0.59 per linear foot, how much should Andy bid to make sure he gets at least a 20% profit?



$8^2 + 2^2 + h^2 = 8.5^2$

$64 + 4 + h^2 = 72.25$

$68 + h^2 = 72.25$

$h^2 = 4.25$

$h = 2.06$ inches

$2.06^2 + 2^2 = d_1^2$

$8.25 = d_1^2$

$d_1 = \sqrt{8.25} \approx 2.87$ inches

$8^2 + 2.06^2 = d_2^2$

$68.2436 = d_2^2$

$d_2 = \sqrt{68.2436} \approx 8.26$ inches

8.26 inches $\times$ 2 = 16.52 inches

2.87 inches $\times$ 2 = 5.74 inches

16.52 inches + 5.74 inches = 22.26 inches or approximately 1.855 feet for each box.

$1.855 \cdot 200 = 371$ feet

$371 \times 0.59 = \$218.89$

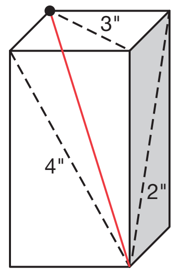
It will cost Andy \$218.89 for decal striping on all 200 boxes.

$218.89 \times 1.2 = \$262.67$

Andy will need to bid \$262.67 in order to make a 20% profit.

Be prepared to share your solutions and methods.

Check for Students' Understanding



The hole for a straw in a juice box is situated in the upper corner of the top of the box as shown. The diagonal measurements of three perpendicular sides of the box are given. If the straw is 3.8″, is it possible for the straw slip inside the box? Use a formula to answer the question.

$d^2 = \frac{1}{2}(d_1^{\,2} + d_2^{\,2} + d_3^{\,2})$

$d^2 = \frac{1}{2}(2^2 + 3^2 + 4^2)$

$d^2 = \frac{1}{2}(4 + 9 + 16)$

$d^2 = \frac{1}{2}(29)$

$d^2 = 14.5$

$d = \sqrt{14.5} \approx 3.808''$

The 3-D diagonal of the juice box is approximately 3.808″ and the straw is 3.8″ so it could possibly slip inside the juice box.

11

11

Chapter 11 Summary

KEY TERMS

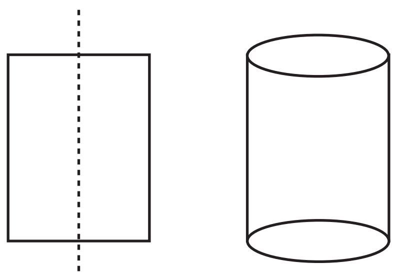
- disc (11.1)
- isometric paper (11.2)
- right triangular prism (11.2)
- oblique triangular prism (11.2)
- right rectangular prism (11.2)
- oblique rectangular prism (11.2)
- right cylinder (11.2)
- oblique cylinder (11.2)
- Cavalieri's principle (11.3)
- sphere (11.5)
- radius of a sphere (11.5)
- diameter of a sphere (11.5)
- great circle of a sphere (11.5)
- hemisphere (11.5)
- annulus (11.5)

11.1 Creating and Describing Three-Dimensional Solids Formed by Rotations of Plane Figures through Space

A three-dimensional solid is formed by rotating a two-dimensional figure around an axis. As the shape travels around in a somewhat circular motion through space, the image of a three-dimensional solid is formed. A rotated rectangle or square forms a cylinder. A rotated triangle forms a cone. A rotated disc forms a sphere. The radius of the resulting solid relates to the distance of the edge of the plane figure to the axis point.

Example

A cylinder is formed by rotating a rectangle around the axis. The width of the rectangle is equal to the radius of the cylinder's base.

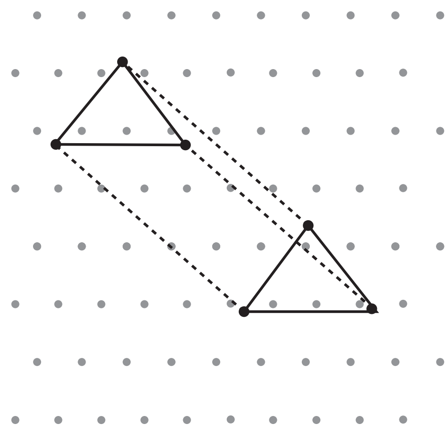


11.2 Creating and Describing Three-Dimensional Solids Formed by Translations of Plane Figures through Space

A two-dimensional drawing of a solid can be obtained by translating a plane figure through two dimensions and connecting the corresponding vertices. The solid formed by translating a rectangle is a rectangular prism. The solid formed by translating a square is a cube. The solid formed by translating a triangle is a triangular prism. The solid formed by translating a circle is a cylinder.

Example

A triangular prism is formed by translating a triangle through space.



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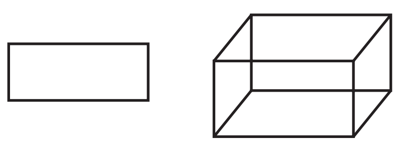
11.2 Building Three-Dimensional Objects by Stacking Congruent Plane Figures

Congruent shapes or figures are the same shape and the same size. A stack of congruent figures can form a solid shape. A stack of congruent circles forms a cylinder. A stack of congruent squares forms a rectangular prism or cube. A stack of congruent triangles forms a triangular prism. The dimensions of the base of the prism or cylinder are the same as the original plane figure.

Example

A rectangular prism can be formed by stacking rectangles.

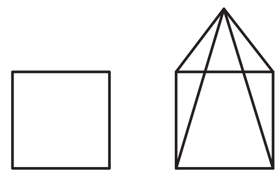


11.2 Building Three-Dimensional Objects by Stacking Similar Plane Figures

Similar shapes or figures have the same shape, but they do not have to be the same size. A stack of similar figures that is incrementally smaller with each layer form a solid shape. A stack of similar circles forms a cone. A stack of similar squares, rectangles, triangles, pentagons, hexagons, or other polygons forms a pyramid. The dimensions of the base of the cone or pyramid are the same as the original plane figure.

Example

A square pyramid is formed by stacking similar squares.



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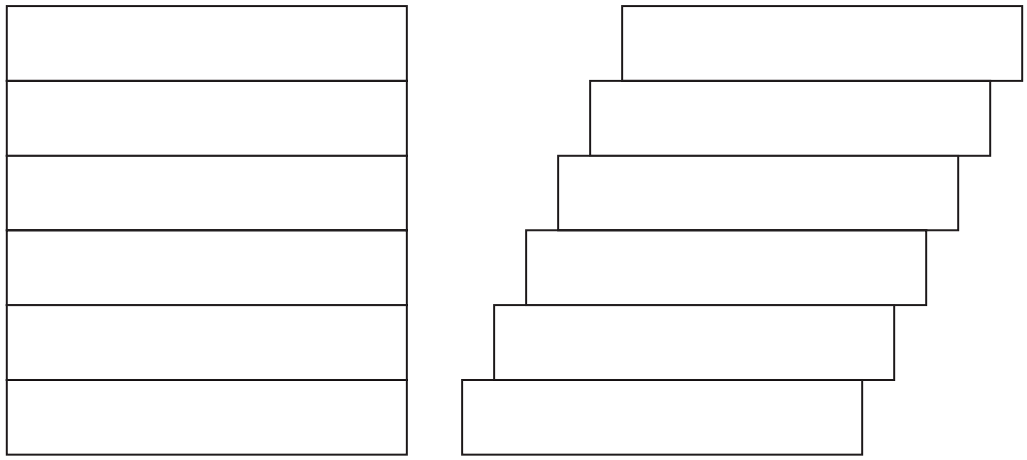
11.3 Applying Cavalieri's Principles

Cavalieri's principle for two-dimensional figures states that if the lengths of one-dimensional slices—just a line segment—of two figures are the same, then the figures have the same area.

Cavalieri's principle for three-dimensional figures states that if, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal.

Examples

Both figures have the same area:



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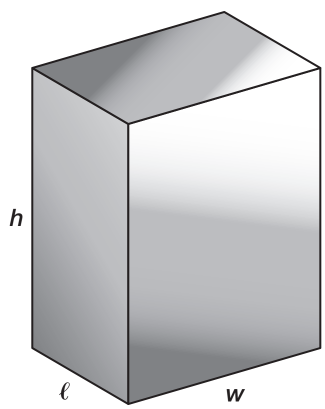
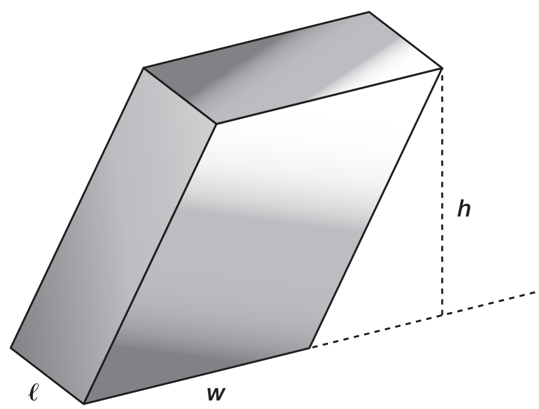
Both solids have the same volume:



Right rectangular prism



Oblique rectangular prism

11.4 Building the Cylinder Volume Formula

To build a volume formula for a cylinder, you can think of it as an infinite stack of discs, each with an area of πr^2. These discs are stacked to a height of h (the height of the cylinder).

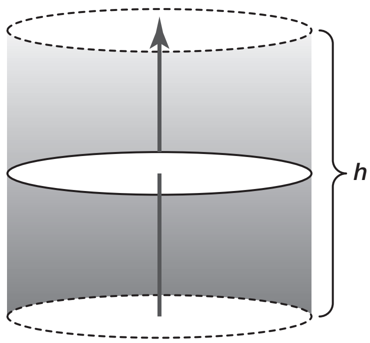
A cylinder can also be created by rotating a rectangle about one of its sides. To determine the volume formula for the cylinder, determine the average, or typical, point of the rectangle, which is located at $\left(\frac{1}{2}r, \frac{1}{2}h\right)$.

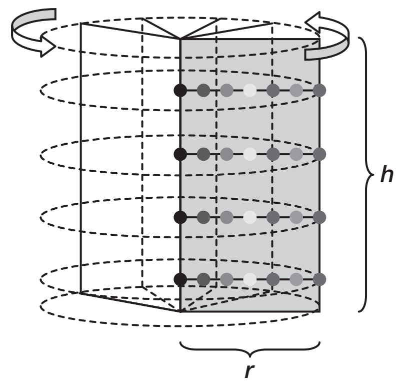
Then, use this point as the radius of a circle and calculate the circumference of that circle: $2\pi\left(\frac{1}{2}r\right) = \pi r$. This is the average distance that all the points of the rectangle are rotated to create the cylinder.

Finally, multiply this average distance by the area of the rectangle, rh, to determine the volume formula: $\pi r(rh) = \pi r^2 h$.

Examples

The volume of any cylinder is $\pi r^2 \times h$, or $\pi r^2 h$.





11.4 Building the Cone Volume Formula

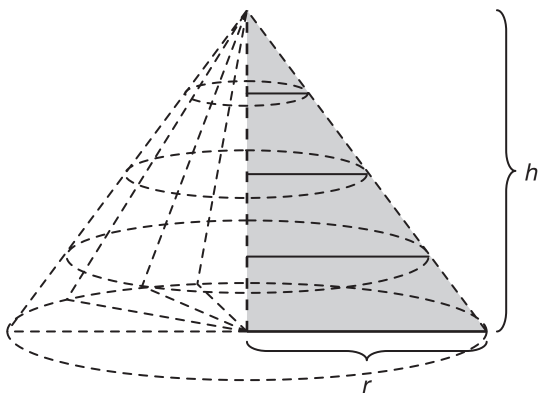
You can think of the volume of a cone as a stacking of an infinite number of similar discs at a height of *h*. You can also think of a cone as being created by rotating a right triangle.

To determine the volume formula for the cone, determine the average, or typical, point of the triangle, which is located at $\left(\frac{1}{3}r, \frac{1}{3}h\right)$. Then, use this point as the radius of a circle and calculate the circumference of that circle: $2\pi\left(\frac{1}{3}r\right) = \frac{2}{3}\pi r$. This is the average distance that all the points of the triangle are rotated to create the cone.

Finally, multiply this average distance by the area of the triangle, $\left(\frac{1}{2}rh\right)$, to determine the volume formula: $\frac{2}{3}\pi r \times \frac{1}{2}(rh) = \frac{1}{3}\pi r^2h$.

Example

The volume of any cone is $\frac{1}{3}\pi r^2h$.



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11.4 Determining the Pyramid Volume Formula

A pyramid is to a prism what a cone is to a cylinder. A prism is created by stacking congruent polygons, which is similar to creating a cylinder by stacking congruent circles. A pyramid is created by stacking similar polygons that are not congruent, which is similar to creating a cone by stacking similar circles. The volume of any pyramid is $\frac{1}{3}$ the volume of a prism with an equal base area and height.

Example

Volume of prism: $V = Bh$

Volume of pyramid: $V = \frac{1}{3}Bh$

11.5 Calculating Volume of Spheres

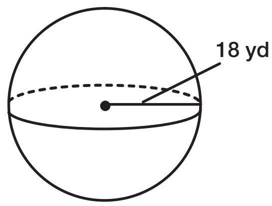
A sphere is the set of all points in three dimensions that are equidistant from a given point called the center.

The volume of a sphere is the amount of space contained inside the sphere. To calculate the volume of a sphere, use the formula $V = \frac{4}{3}\pi r^3$, where V is the volume of the sphere and r is the radius of the sphere.

The volume of a sphere is equal to twice the volume of a cylinder minus the volume of a cone with an equal base area and height.

Volume formulas can be used to solve problems.

Examples



$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(18^3) \\ &\approx \frac{4}{3}(3.14)(5832) \\ &\approx 24{,}416.6 \end{aligned}$$

The volume of the sphere is approximately 24,416.6 cubic yards.

Volume of sphere $=$ 2 $\times$ Volume of cylinder $-$ Volume of cone

$$2 \times \pi r^2 h - \frac{1}{3}\pi r^2 h = 2 \times \frac{2}{3}\pi r^2 h = \frac{4}{3}\pi r^2 h$$

In a sphere, h and r are equal, so the volume formula for a sphere can be written as $\frac{4}{3}\pi r^3$.

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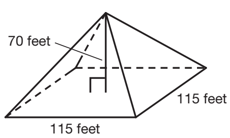
11.6 Solving Problems Using Volume Formulas

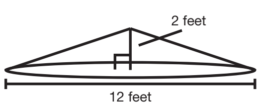
Formulas for the volumes of pyramids, cylinders, and cones can be used to solve problems.

Examples



$$V = \frac{1}{3}(115^2)(70) \approx 308{,}583$$

The volume of this pyramid is about 308,583 cubic feet.



$r = 6$, $h = 2$

$$V = \frac{1}{3}\pi(6^2)(2) = \frac{1}{3}(3.14)(36)(2) = 75.36$$

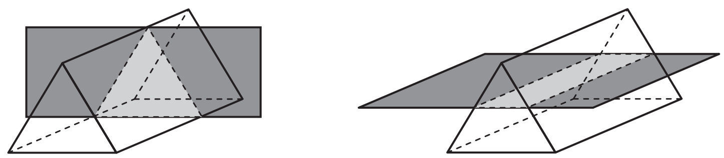
The volume of the cone is about 75.36 cubic feet.

11.7 Determining Shapes of Cross Sections

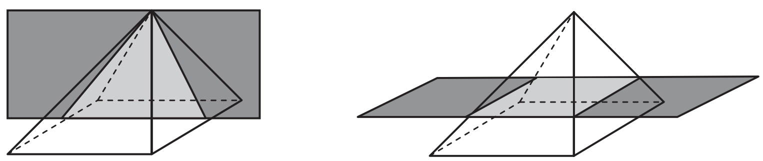
A cross section of a solid is the two-dimensional figure formed by the intersection of a plane and a solid when a plane passes through the solid.

Examples

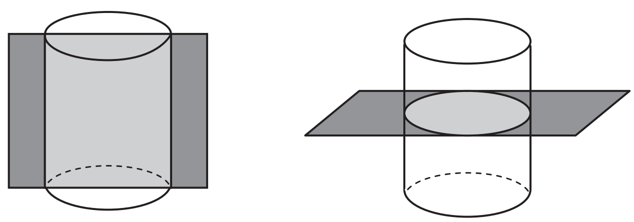
The cross sections of the triangular prism are triangles and rectangles.



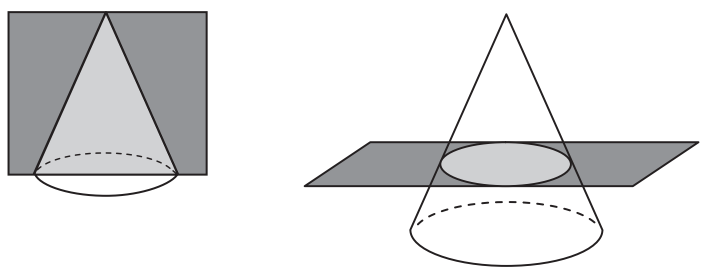
The cross sections of the square pyramid are triangles and squares.



The cross sections of the cylinder are circles and rectangles.



The cross sections of the cone are circles and triangles.



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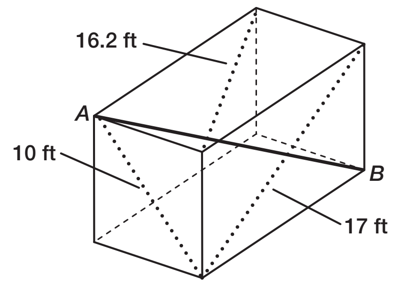
11.8 Determining Lengths of Spatial Diagonals

To determine the length of a spatial diagonal of a rectangular solid given the lengths of the diagonals of the faces of the solid, use the following formula, where d represents the length of a spatial diagonal, and d_1, d_2, and d_3 represent the lengths of the diagonals of the faces of the rectangular solid.

$$d^2 = \frac{1}{2}(d_1^2 + d_2^2 + d_3^2)$$

Examples



$$(AB)^2 = \frac{1}{2}(d_1^2 + d_2^2 + d_3^2)$$

$$(AB)^2 = \frac{1}{2}(10^2 + 17^2 + 16.2^2)$$

$$(AB)^2 = \frac{1}{2}(100 + 289 + 262.44)$$

$$(AB)^2 = \frac{1}{2}(651.44)$$

$$AB \approx 18$$

The length of the spatial diagonal AB is approximately 18 feet.

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