Arcs and Sectors of Circles

10

Chapter 10 Overview

This chapter explores inscribed and circumscribed polygons as well as circles. Students determine relationships between central angles, arcs, arc lengths, areas of parts of circles, as well as linear velocity and angular velocity.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
10.1	Inscribed and Circumscribed Triangles and Quadrilaterals	G.C.3	2	This lesson provides opportunities for students to use construction tools to construct inscribed and circumscribed polygons. Questions ask students to prove the Inscribed Right Triangle-Diameter Theorem and its Converse, as well as the Inscribed Quadrilateral-Opposite Angles Theorem.	X				
10.2	Arc Length	G.C.5 G.MG.1	2	This lesson introduces arc length and radian measure. Questions ask students to calculate minor and major arc lengths in degree and radian measure for problem situations.	X	X	X	X	
10.3	Sectors and Segments of a Circle	G.C.5 G.MG.1	1	This lesson explores concentric circles and introduces the area of a sector of a circle as well as the area of a segment of a circle. Students will derive a formula to determine the area of a sector, and calculate areas in problem situations.	X	X	X		
10.4	Circle Problems	G.MG.1 G.MG.3	2	This lesson explores circular velocity. Questions ask students to determine linear velocity and angular velocity, as well as apply concepts explored in the chapter to problem situations.	X				

Skills Practice Correlation for Chapter 10

Lesson		Problem Set	Objectives
10.1	Inscribed and Circumscribed Triangles and Quadrilaterals		Vocabulary
		1 – 8	Draw inscribed triangles through given points and determine if the triangles are right triangles
		9 – 14	Draw inscribed triangles through given points and determine indicated angle measures
		15 – 20	Draw inscribed quadrilaterals through given points and determine indicated angle measures
		21 – 26	Construct circles inscribed in polygons
		27 – 32	Use inscribed and circumscribed polygons to write proofs
10.2	Arc Length		Vocabulary
		1 – 6	Calculate the ratio of arc lengths to circle circumferences
		7 – 12	Write expressions to calculate the lengths of arcs
		13 – 20	Calculate arc lengths
		21 – 28	Calculate arc lengths
		29 – 34	Use information to answer questions about circles
		35 – 40	Use central angle measures, arc lengths, and radii to solve for measures
10.3	Sectors and Segments of a Circle		Vocabulary
		1 – 8	Calculate the area of sectors
		9 – 14	Calculate the area of segments
		15 – 20	Use information to determine the radii of circles
10.4	Circle Problems		Vocabulary
		1 – 6	Use given arc measures to determine the measures of indicated angles
		7 – 12	Calculate the area of sectors
		13 – 16	Calculate the area of segments
		17 – 22	Determine linear velocity and angular velocity

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Replacement for a Carpenter's Square

10.1

Inscribed and Circumscribed Triangles and Quadrilaterals

LEARNING GOALS

In this lesson, you will:

- Inscribe a triangle in a circle.
- Explore properties of a triangle inscribed in a circle.
- Circumscribe a triangle about a circle.
- Inscribe a quadrilateral in a circle.
- Explore properties of a quadrilateral inscribed in a circle.
- Circumscribe a quadrilateral about a circle.
- Prove the Inscribed Right Triangle–Diameter Theorem.
- Prove the Inscribed Right Triangle–Diameter Converse Theorem.
- Prove the Inscribed Quadrilateral–Opposite Angles Theorem.

KEY TERMS

- inscribed polygon
- Inscribed Right Triangle–Diameter Theorem
- Inscribed Right Triangle–Diameter Converse Theorem
- circumscribed polygon
- Inscribed Quadrilateral–Opposite Angles Theorem

ESSENTIAL IDEAS

- An inscribed polygon is a polygon drawn inside a circle such that each vertex of the polygon touches the circle.
- The Inscribed Right Triangle-Diameter Theorem states: "When a triangle inscribed in a circle such that one side of the triangle is a diameter, the triangle is a right triangle."
- The Inscribed Right Triangle-Diameter Converse Theorem states: "When a right triangle is inscribed in a circle, the hypotenuse is the diameter of the circle."
- A circumscribed polygon is a polygon drawn outside a circle such that each side of the polygon is tangent to a circle.
- The Inscribed Quadrilateral-Opposite Angles Theorem states: "When a quadrilateral is inscribed in a circle, the opposite angles are supplementary."

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-C Circles

Understand and apply theorems about circles

3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

Overview

Students write conjectures and prove theorems related to inscribed and circumscribed polygons using two-column formats. Students apply the theorems to solve problem situations. Construction tools are used in this lesson.

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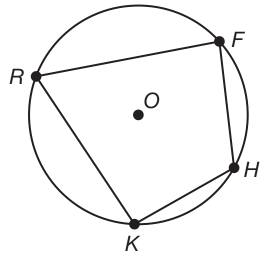
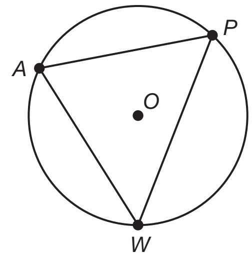
Warm Up



1. Nicki argued that $\angle R$ and $\angle H$ in the diagram above must be supplementary angles. Explain Nicki's reasoning.

   Nicki knew a circle has 360° and the measure of an inscribed angle is equal to half the measure of its intercepted arc. Using this information, she realized when combined, the sum of the two intercepted arcs for $\angle R$ and $\angle H$ covered the entire circle, and the sum of the measures of $\angle R$ and $\angle H$ must be half the sum of their combined intercepted arcs.



2. Yesterday, Ms. Angle taught her students how to determine the measure of an inscribed angle. Mitchell told his geometry lab partner that he finally understood why a circle is always 360°. Use triangle *PAW* drawn in the diagram to explain what Mitchell was thinking.

   Mitchell learned that the measure of an inscribed angle is equal to half the measure of its intercepted arc. In the diagram, the intercepted arcs of $\angle P$, $\angle A$, and $\angle W$ combined form the entire circle. He knew the sum of the measures of the three angles is equal to 180° using the Triangle Sum Theorem, so he concluded the sum of the measures of the three intercepted arcs must be twice the sum of the measures of the angles.

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10.1

Replacement for a Carpenter's Square

Inscribed and Circumscribed Triangles and Quadrilaterals

LEARNING GOALS

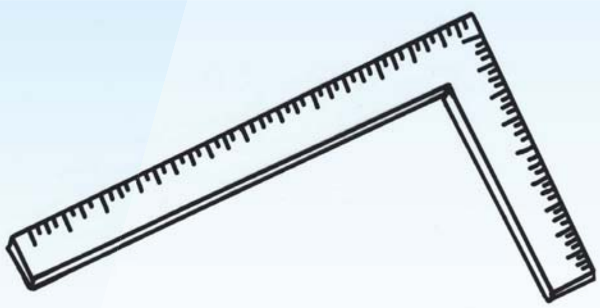
In this lesson, you will:

- Inscribe a triangle in a circle.
- Explore properties of a triangle inscribed in a circle.
- Circumscribe a triangle about a circle.
- Inscribe a quadrilateral in a circle.
- Explore properties of a quadrilateral inscribed in a circle.
- Circumscribe a quadrilateral about a circle.
- Prove the Inscribed Right Triangle–Diameter Theorem.
- Prove the Inscribed Right Triangle–Diameter Converse Theorem.
- Prove the Inscribed Quadrilateral–Opposite Angles Theorem.

KEY TERMS

- inscribed polygon
- Inscribed Right Triangle–Diameter Theorem
- Inscribed Right Triangle–Diameter Converse Theorem
- circumscribed polygon
- Inscribed Quadrilateral–Opposite Angles Theorem

A carpenter's square is a tool that is used to create right angles. These "squares" are usually made of a strong material such as metal so that the right angle is not easily bent or broken. It is a useful tool for stair and roof framing, especially to ensure that all building codes are being maintained.



Problem 1

The scenario is about a carpenter's square. The carpenter's square is damaged and students use a compass, straightedge to help the carpenter recreate a right angle. The definition of inscribed polygon is given. Students write the conjecture; a triangle inscribed in a circle such that one side of the triangle is a diameter is a right triangle, and its converse. Then, they prove both the conjecture and the converse conjecture. The definition of a circumscribed polygon is given and students are asked to circumscribe a triangle using a compass and straightedge. The incenter of the triangle will help students determine the radius of the circle.

Grouping

- Discuss and complete Question 1 as a class.
- Have students complete Questions 2 through 4 with a partner. Then have students share their responses as a class.

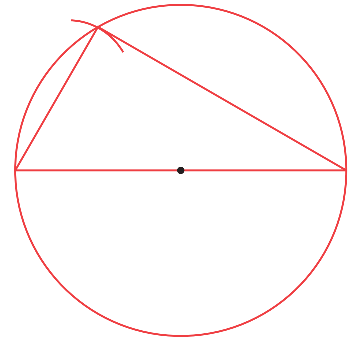
Guiding Questions for Share Phase, Questions 2 through 4

- How is a diameter of a circle related to a semicircle?
- How is an inscribed right angle related to a semicircle?
- How is an inscribed right angle related to a diameter?

PROBLEM 1 In Need of a New Tool

A carpenter is working on building a children's playhouse. She accidentally drops her carpenter's square, and the right angle gets bent. She still needs to cut out a piece of plywood that is in the shape of a right triangle. So, the carpenter gets out her compass and her straightedge to get the job done.

1. Use the steps to re-create how the carpenter created the right triangle.
   a. The hypotenuse of the triangle needs to be 6 centimeters. Use your ruler and open your compass to 3 centimeters. In the space provided, draw a circle with a diameter of 6 centimeters. Use the given point as the center.
   b. Use your straightedge to draw a diameter on the circle.
   c. One of the legs of the triangle is to be 4 centimeters long. Open your compass to 4 centimeters. Place the point of your compass on one of the endpoints of the diameter and draw an arc that passes through the circle.
   d. Use your straightedge to draw segments from the endpoints of the diameter to the intersection of the circle and the arc.
   e. Use your protractor to verify that this triangle is a right triangle.

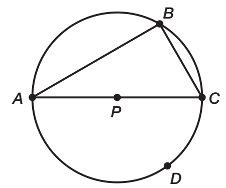


An **inscribed polygon** is a polygon drawn inside a circle such that each vertex of the polygon touches the circle.

- How many degrees are in a semicircle?
- How many semicircles are in every circle?
- When a triangle is inscribed in a circle, are the three interior angles always inscribed angles?
- What is the hypothesis of the Inscribed Right Triangle-Diameter Conjecture?
- What is the conclusion of the Inscribed Right Triangle-Diameter Conjecture?

- What is the hypothesis of the Inscribed Right Triangle-Diameter Converse Conjecture?
- What is the conclusion of the Inscribed Right Triangle-Diameter Converse Conjecture?

2. Consider $\triangle ABC$ that is inscribed in circle P.



a. What do you know about $\overline{AC}$?

Line segment AC is a diameter of circle P.

b. What do you know about $m\overset{\frown}{ADC}$? Explain your reasoning.

The measure of $\overset{\frown}{ADC}$ is 180° because it is a semicircle.

c. What does this tell you about $m\angle ABC$? Explain your reasoning.

The measure of $\angle ABC$ is 90° because its measure is half the measure of its intercepted arc.

d. What kind of triangle is $\triangle ABC$? How do you know?

Triangle ABC is a right triangle because it contains a right angle.

3. Write an Inscribed Right Triangle–Diameter Conjecture about the kind of triangle inscribed in a circle when one side of the triangle is a diameter.

If a triangle is inscribed in a circle such that the hypotenuse of the triangle is a diameter of the circle, then the triangle is a right triangle.

4. Write the converse of the conjecture you wrote in Question 3. Do you think this statement is also true?

If a right triangle is inscribed in a circle, then the hypotenuse of the triangle is a diameter of the circle.

Yes. I think the converse of the conjecture is true.

Grouping

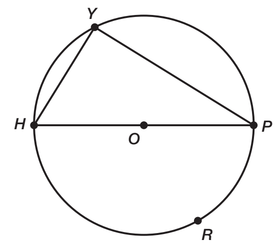
Have students complete Questions 5 and 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 6

- What do you know about the measure of arc *HRP*?
- Is arc *HRP* a semicircle? Explain.
- Which inscribed angle is associated with arc *HRP*?
- What is the measure of the inscribed angle associated with arc *HRP*? Explain.
- How is proving the converse of this theorem similar to proving the theorem?
- If the theorem and its converse theorem are both true? Can they be rewritten as a biconditional statement?
- What biconditional statement describes this theorem and its converse?

It would appear that $\triangle ABC$ is a right triangle and that $\angle B$ is a right angle. This observation can be stated as a theorem and then proved.

5. Create a proof of the Inscribed Right Triangle–Diameter Conjecture.



Given: $\triangle HYP$ is inscribed in circle O such that $\overline{HP}$ is the diameter of the circle.

Prove: $\triangle HYP$ is a right triangle.

Statements	Reasons
1. $\triangle HYP$ is inscribed in circle O such that $\overline{HP}$ is the diameter of the circle.	**1.** Given
2. $\overset{\frown}{HRP}$ is a semicircle.	**2.** The diameter of a circle divides the circle into two semicircles
3. $m\overset{\frown}{HRP} = 180°$	**3.** Definition of semicircle
4. $m\angle HYP = 90°$	**4.** Inscribed Angle Converse Theorem
5. $\angle HYP$ is a right angle.	**5.** Definition of a right angle
6. $\triangle HYP$ is a right triangle.	**6.** Definition of a right triangle

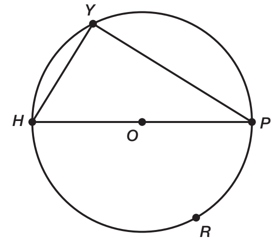
The **Inscribed Right Triangle–Diameter Theorem** states: "If a triangle is inscribed in a circle such that one side of the triangle is a diameter of the circle, then the triangle is a right triangle."

The converse of the Inscribed Right Triangle–Diameter Theorem can also be proved as a separate theorem. The converse is used in a situation when you are given an inscribed right triangle and want to conclude that one side of the inscribed triangle is a diameter of the circle.

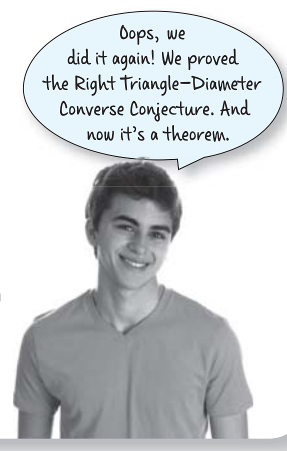
6. Create a proof of the Inscribed Right Triangle–Diameter Converse Conjecture.

   Given: Right $\triangle HYP$ is inscribed in circle O.

   Prove: $\overline{HP}$ is the diameter of circle O.



Statements	Reasons
1. Right $\triangle HYP$ is inscribed in circle O.	1. Given
2. $\angle HYP$ is a right angle.	2. Definition of a right triangle
3. $m\angle HYP = 90°$	3. Definition of a right angle
4. $m\overset{\frown}{HRP} = 180°$	4. Inscribed Angle Theorem
5. $\overset{\frown}{HYP}$ is a semicircle.	5. Definition of a semicircle
6. $\overline{HP}$ is the diameter of circle O.	6. Definition of diameter



The **Inscribed Right Triangle–Diameter Converse Theorem** states: "If a right triangle is inscribed in a circle, then the hypotenuse is a diameter of the circle."

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Grouping

Have students complete Question 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 7

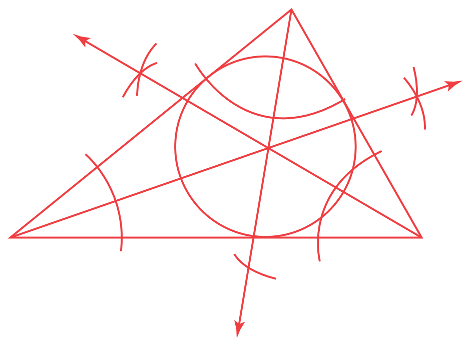
- What is the difference between an inscribed polygon and a circumscribed polygon?
- Were you able to draw a circle inside a triangle such that the circle was tangent to all three sides?
- What do you remember about the incenter of a triangle?
- How is locating the incenter of the triangle helpful in this situation?
- How is the distance from the incenter to any side of the triangle important in this situation?

A **circumscribed polygon** is a polygon drawn outside a circle such that each side of the polygon is tangent to a circle.

7. Mr. Scalene asks his geometry class to draw a triangle and to use a compass to draw a circle inside the triangle such that the circle was tangent to each side of the triangle. See if you can do this.

a. Use a straightedge to draw a triangle.

b. Use your compass to construct a circle inside the triangle such that each side of the triangle is tangent to the circle.



I know that the incenter of a triangle is equidistant to each side of the triangle. I located the incenter of the triangle by constructing the angle bisector of each angle, and then I used the distance from the incenter to any side of the triangle as the radius of the circle.

Problem 2

Students use a compass and straightedge to inscribe a quadrilateral. A protractor is used to determine the measure of the four angles of the quadrilateral. Students conclude the opposite angles of the quadrilateral are supplementary. Students write and prove the Inscribed Quadrilateral-Opposite Angles Conjecture: "If a quadrilateral is inscribed in a circle, then the opposite angles are supplementary."

Grouping

- Discuss and complete Question 1 as a class.
- Have students complete Question 2 with a partner. Then have students share their responses as a class.

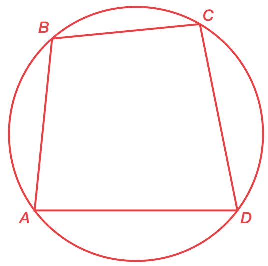
Guiding Questions for Share Phase, Question 2

- Which arc is associated with inscribed $\angle U$?
- Which arc is associated with inscribed $\angle D$?
- Which arc is associated with inscribed $\angle Q$?
- Which arc is associated with inscribed $\angle A$?
- What in $m\angle U + m\angle D$?
- Why does $m\angle U + m\angle D$ equal one half of the entire circle (360°)?
- What in $m\angle D + m\angle A$?
- Why does $m\angle D + m\angle A$ equal one half of the entire circle (360°)?

PROBLEM 2 Quadrilaterals and Circles

1. Consider the relationship between opposite angles of an inscribed quadrilateral.

   a. Use your compass to draw a circle.



   b. Use your straightedge to draw an inscribed quadrilateral that is not a parallelogram in your circle. Label the vertices of your quadrilateral.

   c. Use your protractor to determine the measures of the angles of the quadrilateral. What is the relationship between the measures of each pair of opposite angles?

   The opposite angles are supplementary.

   d. Write an Inscribed Quadrilateral–Opposite Angles Conjecture about the opposite angles of an inscribed quadrilateral.

   If a quadrilateral is inscribed in a circle, then opposite angles are supplementary.

It would appear that opposite angles of the inscribed quadrilateral are supplementary. This observation can be stated as a theorem and then proved.

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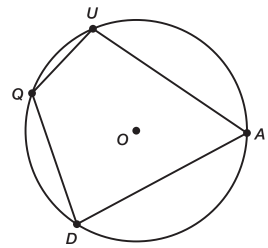
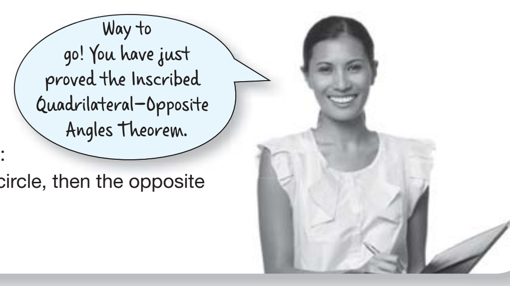
2. Create a proof of the Inscribed Quadrilateral–Opposite Angles Conjecture.



Given: Quadrilateral *QUAD* is inscribed in circle *O*.

Prove: $\angle Q$ and $\angle A$ are supplementary angles.
$\angle U$ and $\angle D$ are supplementary angles.

Statements	Reasons
1. Quadrilateral *QUAD* is inscribed in circle *O*.	1. Given
2. $m\overset{\frown}{QUA} + m\overset{\frown}{QDA} = 360°$ $m\overset{\frown}{UAD} + m\overset{\frown}{UQD} = 360°$	2. A circle is 360°
3. $m\angle U = \frac{1}{2}m\overset{\frown}{QDA}$ $m\angle D = \frac{1}{2}m\overset{\frown}{QUA}$ $m\angle Q = \frac{1}{2}m\overset{\frown}{UAD}$ $m\angle A = \frac{1}{2}m\overset{\frown}{UQD}$	3. Inscribed Angle Theorem
4. $m\angle U + m\angle D = \frac{1}{2}m\overset{\frown}{QUA} + \frac{1}{2}m\overset{\frown}{QDA}$ $m\angle Q + m\angle A = \frac{1}{2}m\overset{\frown}{UAD} + \frac{1}{2}m\overset{\frown}{UQD}$	4. Substitution Property
5. $m\angle U + m\angle D = \frac{1}{2}(m\overset{\frown}{QUA} + m\overset{\frown}{QDA})$ $m\angle Q + m\angle A = \frac{1}{2}(m\overset{\frown}{UAD} + m\overset{\frown}{UQD})$	5. Distributive Property
6. $m\angle U + m\angle D = \frac{1}{2}(360°)$ $m\angle Q + m\angle A = \frac{1}{2}(360°)$	6. Substitution Property steps 2 and 5
7. $m\angle U + m\angle D = 180°$ $m\angle Q + m\angle A = 180°$	7. Multiplication
8. $\angle U$ and $\angle D$ are supplementary angles. $\angle Q$ and $\angle A$ are supplementary angles.	8. Definition of supplementary angles



The **Inscribed Quadrilateral–Opposite Angles Theorem** states:
"If a quadrilateral is inscribed in a circle, then the opposite angles are supplementary."

Problem 3

Students explore when it is possible to inscribe a circle in a quadrilateral. They conclude that determining the inscribed circle in a quadrilateral is possible to do if and only if the four angle bisectors of the quad are concurrent.

Grouping

- Ask students to read introduction. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

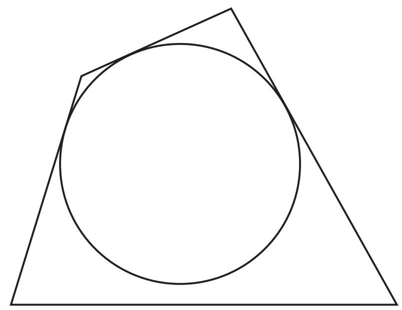
Guiding Questions for Share Phase, Questions 1 through 4

- What do you remember about the incenter of a triangle?
- How is locating the incenter of the quadrilateral helpful in this situation?
- How is the distance from the incenter to any side of the quadrilateral important in this situation?
- Are all four angle bisectors of a quadrilateral concurrent?
- In Quadrilateral *WXYZ*, which tangent segments are associated with point *W*?
- In Quadrilateral *WXYZ*, which tangent segments are associated with point *X*?
- In Quadrilateral *WXYZ*, which tangent segments are associated with point *Y*?

PROBLEM 3 Circumscribed Quadrilaterals

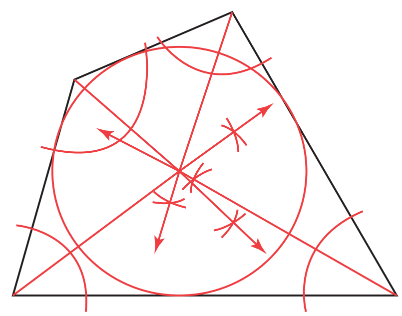
Ms. Rhombi asks her geometry class to draw a quadrilateral and to use a compass to draw a circle inside the quadrilateral such that the circle was tangent to each side of the quadrilateral.

Most of her students use a straightedge and a compass, draw a quadrilateral, and try to draw a circle inside the quadrilateral as shown.



One of her students has talked to a friend who happens to be in Mr. Scalene's class, so she knows exactly what to do if it was a triangle. Let's see if you are able to circumscribe the quadrilateral the same way you circumscribed the triangle.

1. Construct a circle inscribed in the quadrilateral shown. Explain your process.



I located the incenter of the quadrilateral by constructing the angle bisector of each angle. Then, I used the distance from the incenter to any sides of the quadrilateral as the radius of the circle.

- In Quadrilateral *WXYZ*, which tangent segments are associated with point *Z*?
- What do you know about tangent segments?
- What portion of the perimeter of Quadrilateral *WXYZ* does the sum $WX + YZ$ represent?
- Why does the sum $WX + YZ$ represent one half the perimeter of Quadrilateral *WXYZ*?
- Which theorems did Karl use to explain his reasoning?

2. Ms. Rhombi then writes a theorem on the blackboard:

A circle can be inscribed in a quadrilateral if and only if the angle bisectors of the four angles of the quadrilateral are concurrent.

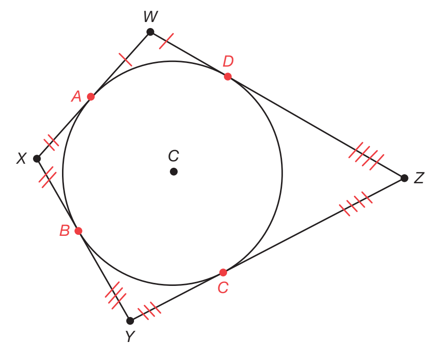
Using this theorem, how can you tell if it is possible to inscribe a circle in the quadrilateral in Question 1?

I need to construct the angle bisectors of the four interior angles of the quadrilateral to determine if they all intersect at the same point.

Is it possible to inscribe a circle in this quadrilateral?

Yes. It is possible because all four angle bisectors are concurrent.

3. Consider quadrilateral *WXYZ*, which is circumscribed about circle *C* as shown.



Given: $WX + YZ = 23$ centimeters

Determine the perimeter of quadrilateral *WXYZ*.

Using the Tangent Segment Theorem, we can conclude $WA = WD$, $XA = XB$, $YB = YC$, and $ZC = ZD$.

Using segment addition, we can conclude $WX = WA + XA$, $XY = XB + YB$, $YZ = YC + CZ$, and $ZW = ZD + WD$.

The perimeter of quadrilateral *WXYZ* is $WX + XY + YZ + ZW$.

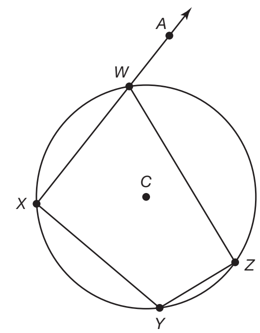
Using substitution, the perimeter of quadrilateral *WXYZ* is $(WA + XA) + (XB + YB) + (CY + ZC) + (ZD + WD)$.

Using substitution again, the perimeter of quadrilateral *WXYZ* is $2WA + 2XA + 2YB + 2ZC$ or $2(WA + XA + YC + ZC)$.

Given: $WX + YZ = 23$ centimeters or $WA + XA + YC + ZC = 23$ centimeters

Therefore, by substitution again, the perimeter of quadrilateral *WXYZ* is $2(WA + XA + YC + ZC) = 2(23 \text{ centimeters}) = 46$ centimeters.

4. Karl raises his hand and informs Ms. Rhombi that he has discovered another property related to the angles of an inscribed quadrilateral. Karl shows his teacher the diagram shown.



He claims that the measure of any exterior angle of the quadrilateral is equal to the measure of the opposite interior angle in the quadrilateral. In other words, $m\angle AWZ = m\angle Y$.

Explain Karl's reasoning.

Karl used the Inscribed Quadrilateral–Opposite Angles Theorem to conclude that $m\angle XWZ + m\angle Y = 180°$.

Next, he used the definition of a linear pair of angles to conclude that $m\angle XWZ + m\angle AWZ = 180°$.

Using substitution, Karl was able to conclude that $m\angle Y = m\angle AWZ$.

Be prepared to share your solutions and methods.

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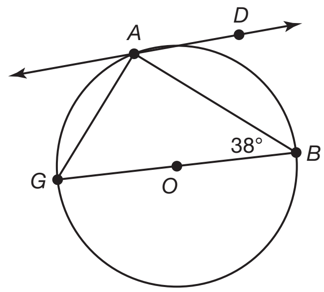
Check for Students' Understanding



$\overleftrightarrow{GB}$ is a diameter of circle O.

$\overleftrightarrow{AD}$ is tangent to circle O at point A.

$m\angle GBA = 38°$

1. Determine $m\angle A$.

   $m\angle A = 90°$

2. Determine $m\angle G$.

   $m\angle G = 52°$

   $180 - 90 - 38 = 52$

3. Determine $m\overset{\frown}{AG}$.

   $m\overset{\frown}{AG} = 76°$

   $38 \times 2 = 76$

4. Determine $m\overset{\frown}{AB}$.

   $m\overset{\frown}{AB} = 104°$

   $52 \times 2 = 104$

10.2

Gears

Arc Length

LEARNING GOALS

In this lesson, you will:

- Distinguish between arc measure and arc length.
- Use a formula to solve for arc length in degree measures.
- Distinguish between degree measure and radian measure.
- Use a formula to solve for arc length in radian measures.

KEY TERMS

- arc length
- radian

ESSENTIAL IDEAS

- Arc length is a portion of the circumference of a circle. Arc length is a linear measure.
- The arc length of $\overset{\frown}{AB} = \frac{m\overset{\frown}{AB}}{360}(2\pi r)$
- One radian is the measure of a central angle whose arc length is the same as the radius of the circle.
- To determine the radian measure of central angle θ, the intercepted arc length s, or the radius r of a circle use the equation $\theta = \frac{s}{r}$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-C Circles

Find arc lengths and areas of sectors of circles

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

G-MG Modeling with Geometry

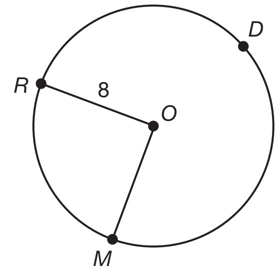
Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

Arc length is defined as a linear measurement. Arc length is differentiated from the degree measure of an arc and students determine the arc length by multiplying a fraction representing the portion of the circumference determined by the central angle and the circumference of the circle. The formula for determining the arc length is stated and students use it to solve problems. One radian is defined as the measure of a central angle whose arc length is the same as the radius of the circle $\left(\theta = \frac{s}{r}\right)$. Students covert radians to degrees by multiplying by $\frac{180°}{\pi}$ and convert degrees to radians by multiplying by $\frac{\pi}{180°}$

Warm Up



Circle O, with radius OR and OM.

$m\angle ROM = 90°$

$m\overline{OR} = 8$ cm

1. Determine the measure of $\overset{\frown}{RM}$.

   The measure of $\overset{\frown}{RM} = 90°$.

2. Calculate the circumference of circle O.

   The circumference of circle O is approximately 50.24 centimeters.

   $$\begin{aligned} \text{circumference} &= 2\pi r \\ &= 2\pi(8) \\ &= 16\pi \\ &\approx 50.24 \end{aligned}$$

3. Determine the length of $\overset{\frown}{RM}$.

   The length of $\overset{\frown}{RM}$ is approximately 12.56 centimeters.

   $$\left(\frac{90}{360}\right)(50.24) = 12.56$$

4. How did you determine the answer to Question 3?

   To determine the answer to Question 3, I took 90°, the measure of the arc and divided it by 360°, to conclude $\frac{1}{4}$ is the fraction of the circle the arc occupies. Then I took $\frac{1}{4}$ of the circumference to determine the length of the arc because the circumference is the distance around the entire circle.

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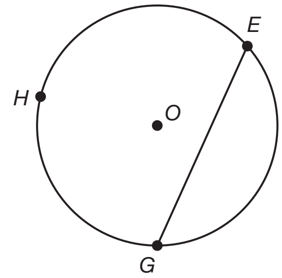


5. Using a piece of string and a ruler, how can you measure the length of major $\overset{\frown}{GHE}$?

To measure the length of arc $\overset{\frown}{GHE}$, you could use a piece of string and a ruler. Shape the string around the arc from point *G*, through point *H* to point *E*. Mark point *G* and point *E* on the string. Then straighten the out and use a ruler to measure the length of the string from point *G* to point *E*.

0

10.2

Gears

Arc Length

LEARNING GOALS

In this lesson, you will:

- Distinguish between arc measure and arc length.
- Use a formula to solve for arc length in degree measures.
- Distinguish between degree measure and radian measure.
- Use a formula to solve for arc length in radian measures.

KEY TERMS

- arc length
- radian

Gears are used in many mechanical devices to provide torque, or the force that causes rotation. For instance, an electric screwdriver contains gears. The motor of an electric screwdriver can make the spinning components spin very fast, but the gears are needed to provide the force to push a screw into place. Gears can be very large or very small, depending on their application.

Problem 1

Gears are circular and used to investigate arc length. Using two different size gears, students explore the degree measures and arc lengths of the minor arcs associated with congruent central angles. Arc length is defined and a formula is stated. Students use the formula to solve problems.

Grouping

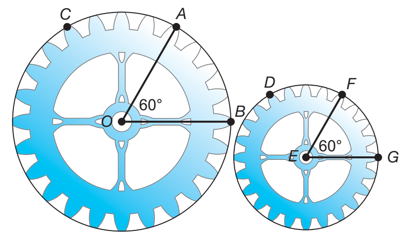
- Ask students to read Question 1. Discuss as a class.
- Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

- Are all circles similar?
- Are all radii of the same circle congruent?
- How do you determine the degree measure of the arc given the measure of the central angle?
- How many degrees are in every circle?
- How can the degree measure of both arcs be the same when one gear is larger than the other gear?
- Which arc looks longer?
- Is it possible for two arcs to have the same degree measure but different lengths?

PROBLEM 1 Large and Small Gears

1. Consider the large gear represented by circle O, containing a central angle; $\angle AOB$, whose measure is equal to 60°; a minor arc, $\overset{\frown}{AB}$; and a major arc, $\overset{\frown}{ACB}$, as shown.
Consider the small gear represented by circle E, containing a central angle; $\angle FEG$, whose measure is equal to 60°; a minor arc, $\overset{\frown}{FG}$; and a major arc, $\overset{\frown}{FDG}$, as shown.



a. Is the large gear similar to the small gear? Explain your reasoning.
Yes. The large gear is similar to the small gear because all circles are similar.

b. Is the length of the radii in the large gear proportional to the length of the radii in the small gear? Explain your reasoning.
Yes. The length of the radii in the large gear is proportional to the length of the radii in the small gear because all radii of the same circle are congruent.
$\frac{OA}{EF} = \frac{OB}{EG}$

c. Determine the degree measure of the minor arc in each circle.
$m\overset{\frown}{AB} = 60°$
$m\overset{\frown}{FG} = 60°$
Each minor arc is equal to the measure of the central angle associated with the arc.

d. What is the ratio of the degree measure of the minor arc to the degree measure of the entire circle for each of the two gears?
The ratio of the measure of the minor arc to the degree measure of the entire circle is $\frac{60}{360} = \frac{1}{6}$ for each gear.

e. The degree measure of the intercepted arc in the large gear is equal to the degree measure of the intercepted arc in the small gear, but do the two intercepted arcs appear to be the same length?
No. The intercepted arc in the large gear appears to be longer than the intercepted arc in the small gear.

- Is it possible for two arcs have the same lengths but different degree measures?
- If your answer is in terms of pi, is the answer an exact answer or an approximate answer?

2. Explain why Casey is incorrect.

> **Casey**
>
> The two minor arcs, $\overset{\frown}{AB}$ and $\overset{\frown}{FG}$, on the gears have the same measure, which is 60°. So, the two arcs are the same length.

The minor arcs, $\overset{\frown}{AB}$ and $\overset{\frown}{FG}$, have the same central angle measure of 60°, but circle *O* has a greater diameter than does circle *E*, so $\overset{\frown}{AB}$ must have a greater length than $\overset{\frown}{FG}$ does.

Arc length is a portion of the circumference of a circle. The *length* of an arc is different from the *degree measure* of the arc. Arcs are measured in degrees whereas arc lengths are linear measurements.

To determine the arc length of the minor arc, you need to work with the circumference of the circle, which requires knowing the radius of the circle.

3. If the length of the radius of the large gear, or line segment *OB* is equal to 4 centimeters, determine the circumference of circle *O*.

   The circumference of circle *O* is equal to $2\pi r$ or $(2)(\pi)(4) = 8\pi$ centimeters.

4. Use the circumference of circle *O* determined in Question 3 and the ratio determined in Question 1, part (d) to solve for the length of the minor arc.

   The length of the minor arc is equal to $\frac{1}{6} \cdot 8\pi = \frac{8}{6}\pi = \frac{4}{3}\pi$ centimeters.

5. If the length of the radius of the small gear, or line segment *EF*, is equal to 2 centimeters, determine the circumference of circle *E*.

   The circumference of circle *E* is equal to $2\pi r$ or $(2)(\pi)(2) = 4\pi$ centimeters.

6. Use the circumference of circle *E* determined in Question 5 and the ratio determined in Question 1, part (d) to solve for the length of the minor arc.

   The length of the minor arc is equal to $\frac{1}{6} \cdot 4\pi = \frac{4}{6}\pi = \frac{2}{3}\pi$ centimeters.

Grouping

Have students complete Questions 7 through 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 7 through 10

- If the central angle is 80°, what portion of the circle contains its intercepted arc?
- If the central angle is 120°, what portion of the circle contains its intercepted arc?
- How did doubling the length of the radius affect the arc length of the central angle?
- How is determining the circumference of the circle related to solving for the perimeter of the shaded region?
- Is the circumference of a circle the same as the arc length of the circle?
- Will the arc length of the minor arc in the large tree ring be 10 times longer than the arc length of the minor arc in the small tree ring?
- What portion of the circumference is the minor arc of the large tree ring?

You determined the arc length by multiplying a fraction that represents the portion of the circumference determined by the central angle and the circumference of the circle. So, the formula for determining arc length, s, can be written as follows.

$$s = \textit{circumference} \cdot \frac{\textit{measure of angle}}{360°}$$

$$s = 2 \cdot \pi \cdot r \cdot \frac{\textit{measure of angle}}{360°}$$

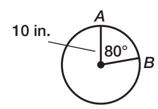
$$s = \frac{\textit{measure of angle}}{360°} \cdot 2\pi r$$

It is important to notice that this formula implies

- $\frac{s}{r} = m \cdot \frac{\pi}{180°}$, where m is the measure of the angle.
- $\frac{s}{r}$ is directly proportional to the measure of the central angle, m.

7. Calculate the arc length of each circle. Express your answer in terms of π.

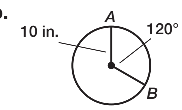
a.



Arc length of AB:

$$\frac{80°}{360°} \cdot 2\pi(10) = \frac{2}{9} \cdot 20\pi = \frac{40}{9}\pi$$

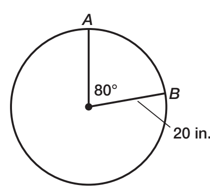
b.



Arc length of AB:

$$\frac{120°}{360°} \cdot 2\pi(10) = \frac{1}{3} \cdot 20\pi = \frac{20}{3}\pi$$

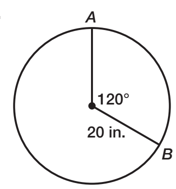
c.



Arc length of AB:

$$\frac{80°}{360°} \cdot 2\pi(20) = \frac{2}{9} \cdot 40\pi = \frac{80}{9}\pi$$

d.



Arc length of AB:

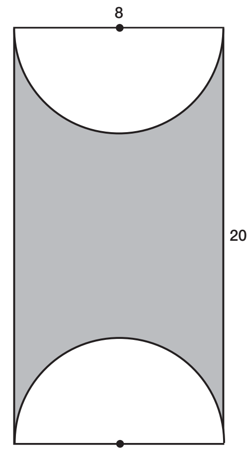
$$\frac{120°}{360°} \cdot 2\pi(20) = \frac{1}{3} \cdot 40\pi = \frac{40}{3}\pi$$

8. Look at Question 7, parts (a) and (c) as well as Question 7, parts (b) and (d). In each pair, the central angle is the same but the radius has been doubled. What effect does doubling the radius have on the length of the arc? Justify why this relationship exist?

If the radius of the circle is doubled but the central arc remains the same then the length of the arc also doubles.

Doubling the radius also doubles the circumference so the corresponding portion of the circle represented by the arc also doubles.

9. Two semicircular cuts were taken from the rectangular region shown. Determine the perimeter of the shaded region. Do not express your answer in terms of π.



The sum of the left and right sides of the rectangle: $2(20) = 40$

Length of one semicircle: $\frac{180^\circ}{360^\circ}(2\pi 4) = \frac{1}{2}(8\pi) = 4\pi$

Perimeter of shaded region: $40 + 8\pi \approx 65.13$ units

10. Use the diagram shown to answer each question.

a. The radius of a small tree ring (small circle) is *r*, and the radius of a larger tree ring (large circle) is 10*r*. How does the arc length of the minor arc in the small tree ring compare to the arc length of the minor arc in the large tree ring?

The arc length of the minor arc in the large tree ring will be 10 times longer than the arc length of the minor arc in the small tree ring.

b. If the arc length of the minor arc in the small tree ring is equal to 3 inches, what is the arc length of the minor arc in the large tree ring?

The arc length of the minor arc in the large tree ring is 30 inches.

c. If $m\angle A = 20°$, the length of the radius of the small tree ring is *r*, the length of the radius of the large tree ring is 10*r*, and the length of the minor arc of the small tree ring is 3 inches, determine the circumference of the large tree ring.

The minor arc of the large ring is $\frac{1}{18}$ of the circumference of the circle. The length of the minor arc of the large ring is 30 inches. So, the circumference of the large circle is $30(18) = 540$ inches.

Problem 2

Students use the Pythagorean Theorem and trigonometry to determine the length of a belt of a manufacturing machine.

Grouping

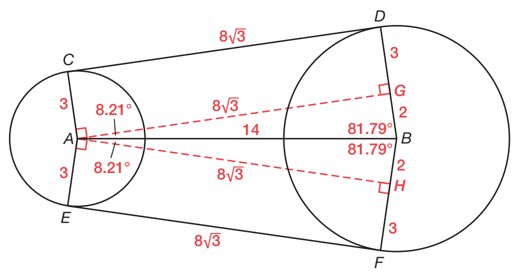
- Ask students to read introduction. Discuss as a class.
- Have students complete Questions 1 through 11 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 11

- Explain why the Pythagorean Theorem can be used to determine the length of segment *AG*.
- Which sides of a right triangle are used for the cosine ratio?
- What type of calculation is required to determine the angle measure given the ratio of the adjacent side and the hypotenuse of a right triangle?

PROBLEM 2 Belts? For Wheels?

Amy is a summer intern at a manufacturing plant. One of the machines at the plant consists of two wheels with a belt connecting the two wheels. The radius of the smaller wheel is 3 centimeters. The radius of the larger wheel is 5 centimeters. The centers of the wheels are 14 centimeters apart.



Amy's first task for her new internship is to calculate the length of the belt that will fit snugly around the two wheels. She notices that the belt can be divided into four sections. Help Amy calculate the length of the belt.

1. Describe the four sections that Amy noticed.

   The belt can be divided into four sections:

   - The arc along the smaller wheel
   - The arc along the larger wheel
   - The two tangent segments

2. Draw a segment from point *A* perpendicular to radius $\overline{DB}$. Label the point of intersection point *G*. Draw a segment from point *A* perpendicular to radius $\overline{BF}$. Label the point of intersection point *H*.

   See diagram.

3. Classify quadrilateral *ACDG*. Explain your reasoning.

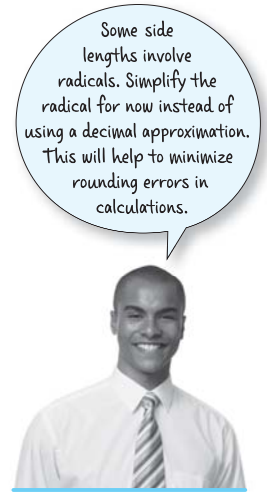
   I know that the radius of a circle is perpendicular to the tangent. So, $\angle ACD$ and $\angle GDC$ are right angles. I drew $\overline{AG}$ perpendicular to side $\overline{DB}$ so $\angle AGD$ is also a right angle. The sum of the angles of a quadrilateral is 360 degrees so $\angle CAG$ is also a right angle.

   Quadrilateral *ACGD* has four right angles so it is a rectangle.

1

4. Classify triangle *AGB*. Explain your reasoning.

I drew $\overline{AG}$ perpendicular to side $\overline{DB}$ so $\angle AGD$ is a right angle. So, triangle *AGB* is a right triangle.



5. Calculate the side lengths of triangle *ABG*. Label each side length in the diagram.

I know the centers of the wheels are 14 centimeters apart so the length of $\overline{AB}$ is 14 centimeters.

I know radius $\overline{DB}$ is 5 centimeters. I also know that the length of $\overline{DG}$ is 3 centimeters because opposite sides of a rectangle are congruent. So, the length of $\overline{GB}$ is $5 - 3$, or 2 centimeters.

I can use the Pythagorean Theorem to calculate the length of $\overline{AG}$.

$AG^2 + 2^2 = 14^2$

$AG^2 + 4 = 196$

$AG^2 = 192$

$AG = \sqrt{192}$

$AG = 8\sqrt{3}$

6. Calculate the length of $\overline{CD}$.

Opposite sides of a rectangle are congruent so the length of $\overline{CD}$ is $8\sqrt{3}$ centimeters.

7. The length of the belt along each circle is an arc length. Describe what information is needed to calculate an arc length.

To calculate an arc length I need to know the circumference of the circle and the measure of the arc.

8. Calculate the measure of major arc *DF* by first calculating the measure of the central angle for major arc *DF*.

To calculate the measure of the central angle, subtract the measure of $\angle DBF$ from 360°. The measure of $\angle DBF$ is twice the measure of $\angle AGB$.

$$\cos(m\angle ABG) = \frac{2}{14}$$
$$m\angle ABG = \cos^{-1}\left(\frac{1}{7}\right)$$
$$m\angle ABG \approx 81.79°$$

The measure of $\angle DBF$ is 163.58°. The measure of the central angle for major arc *DB* is 360° − 163.58°, or 196.42°. So, the measure of major arc *DF* is 196.42°.

9. Calculate the length of major arc *DF*.

$$\text{arc length} = 2\pi(5)\left(\frac{196.42}{360}\right) \approx 17.14$$

The length of major arc *DF* is 17.14 centimeters.

10. Calculate the length of minor arc *CE*.

To calculate the measure of the central angle of minor arc *CE*, subtract the measure of $\angle CAG$, $\angle GAB$, $\angle BAH$, and $\angle HAE$ from 360°.

Angle *CAG* and angle *HAE* are right angles so their measures are 90°.

$$\sin(m\angle GAB) = \frac{2}{14}$$
$$m\angle GAB = \sin^{-1}\left(\frac{1}{7}\right)$$
$$m\angle GAB \approx 8.21°$$

$$m\angle CAE = 360° - (m\angle CAG + m\angle GAB + m\angle BAH + m\angle HAE)$$
$$= 360° - (90° + 8.21° + 8.21° + 90°)$$
$$= 163.58°$$

$$\text{arc length} = 2\pi(3)\left(\frac{163.58}{360}\right) \approx 8.57$$

The length of minor arc *CE* is 8.57 centimeters.

11. Calculate the total length of the belt.

$$\text{length of belt} = 17.14 + 8.57 + 2(8\sqrt{3}) \approx 53.42$$

The total length of the belt is approximately 53.42 centimeters.

Problem 3

One radian is defined as the measure of a central angle whose arc length is the same as the radius of the circle. Students practice converting degree measures to radians and converting radians to degree measures.

Grouping

- Ask students to read introduction and worked example. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

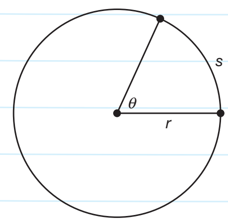
- How many degrees are in a circle?
- How many radians are in a circle?
- Why is 180° equivalent to π radians?
- When converting degrees to radians, what is the multiplier?
- When converting radians to degrees, what is the multiplier?
- In two different circles with equivalent central angles, is the ratio of the arc length to the radius the same?
- When determining the arc length, when should the formula $\theta = \frac{s}{r}$ be used?

PROBLEM 3 **Radian Measure**

So far you have described the measures of angles and arcs using degrees. A *radian* is another unit that can be used to measure angles and arcs.

One **radian** is defined as the measure of a central angle whose arc length is the same as the radius of the circle.

Let r represent the length of the radius of the circle, θ represent the measure of the central angle in radians, and s represent the length of the intercepted arc. Note that $\frac{s}{r}$ is equal to θ.



$\theta = \frac{s}{r}$

The circle has a radius of length r. The circumference of the circle is $2\pi r$. There are 360° in a circle.

Because 360° is equivalent to $\frac{2\pi r}{r} = 2\pi$ radians, it follows that 180° is equivalent to π radians.

Therefore, $\frac{\textit{radian measure}}{\textit{degree measure}} = \frac{\pi}{180°}$ and $\frac{\textit{degree measure}}{\textit{radian measure}} = \frac{180°}{\pi}$.

When converting degrees to radians, multiply a degree measure by $\frac{\pi}{180°}$.

When converting radians to degrees, multiply a degree measure by $\frac{180°}{\pi}$.

1. If $\theta = \frac{\pi}{2}$ and $r = 4$, solve for the length of the intercepted arc.

   $\theta = \frac{s}{r}$

   $\frac{\pi}{2} = \frac{s}{4}$

   $4\pi = 2s$

   $s = \frac{4\pi}{2} = 2\pi$

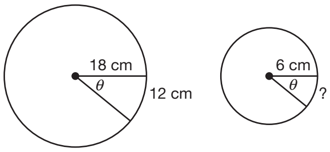
2. If $r = 2$ and the intercepted arc length is 5, what is the measure of the central angle?

   $\theta = \frac{s}{r}$

   $\theta = \frac{5}{2}$ radians

- When determining the arc length, when should the formula $\widehat{AB} = \frac{m\widehat{AB}}{360}(2\pi r)$ be used?

3. At the same central angle θ, if the radius is 6 centimeters, what is the arc length of the intercepted arc?



For the same central angle, the ratio of the arc length to the radius is the same. Because 6 centimeters is one-third of 18 centimeters, the arc length is one-third of 12 centimeters. The arc length is 4 centimeters.

4. The measure of a central angle is 120°. The length of the radius is 20 centimeters.

a. Determine the arc length using the formula:

$\frac{\textit{measure of angle}}{360°} \cdot 2\pi r$

$\textit{Arc length} = \frac{\textit{measure of angle}}{360°} \cdot 2\pi r$

$\textit{Arc length} = \frac{120°}{360°} \cdot 2\pi(20)$

$\textit{Arc length} = \frac{1}{3} \cdot 40\pi = \frac{40\pi}{3}$

b. Determine the arc length using the formula:

$\theta = \frac{s}{r}$

First, convert 120° to radians.

$\theta = 120 \cdot \frac{\pi}{180} = \frac{2\pi}{3}$ radians

$\theta = \frac{s}{r}$

$\frac{2\pi}{3} = \frac{s}{20}$

$3s = 40\pi$

$s = \frac{40\pi}{3}$

c. Compare your answers in part (a) and part (b).

They are the same.

Talk the Talk

Students describe the difference between radians and degrees.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

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Talk the Talk

1. Describe the similarities and differences between radians and degrees.

   Answers will vary.

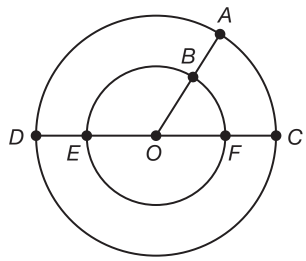
   Radians and degrees are both units used to measure angles. There are 2π radians in a circle, whereas there are 360° in a circle.

2. Which unit do you prefer to use? Explain your reasoning.

   Answers will vary.

Be prepared to share your solutions and methods.

Check for Students' Understanding



- The two circles have the same center at O.
- $\overline{OB}$ is a radius of the small circle. $\overline{OA}$ is a radius of the large circle.
- $\overleftrightarrow{EF}$ is a diameter of the small circle. $\overleftrightarrow{DC}$ is a diameter of the large circle.
- $m\overline{OB} = 4$ inches, $m\overline{AB} = 5$ inches
- $m\angle BOF = 50°$

1. Determine $m\angle AOD$.

   $m\angle AOD = 130°$

   $180 - 50 = 130$

2. Determine $m\overset{\frown}{AD}$.

   $m\overset{\frown}{AD} = 130°$

3. Determine the length of $\overset{\frown}{AD}$.

   The length of $\overset{\frown}{AD}$ is approximately 11.34 inches.

   $\frac{130}{360} \cdot 2\pi(5) \approx 11.34$

4. Determine $m\angle BOE$.

   $m\angle BOE = 130°$

5. Determine $m\overset{\frown}{BE}$.

   $m\overset{\frown}{BE} = 130°$

6. Determine the length of $\overset{\frown}{BE}$.

   The length of $\overset{\frown}{BE}$ is approximately 9.08 inches.

   $\frac{130}{360} \cdot 2\pi(4) \approx 9.08$

7. Compare the measures and lengths of $\overset{\frown}{AD}$ and $\overset{\frown}{BE}$.

   $\overset{\frown}{AD}$ and $\overset{\frown}{BE}$ have the same measure, but the length of $\overset{\frown}{AD}$ is approximately 11.34 inches and the length of $\overset{\frown}{BE}$ is approximately 9.08 inches.

10.3

Playing Darts

Sectors and Segments of a Circle

LEARNING GOALS

In this lesson, you will:

- Determine the area of sectors of a circle.
- Derive the formula for the area of a sector.
- Determine the area of segments of a circle.

KEY TERMS

- concentric circles
- sector of a circle
- segment of a circle

ESSENTIAL IDEAS

- Concentric circles are circles that share the same center point.
- A sector of a circle is a region of the circle bounded by two radii and the included arc.
- The formula for calculating the area of a sector is $A = \frac{m\overset{\frown}{AB}}{360}(\pi r^2)$, where A is the area of the sector, $\overset{\frown}{AB}$ is the arc that bounds the sector, and r is the length of the radius of the circle.
- The segment of a circle is a region of the circle bounded by a chord and the included arc. Each segment of a circle can be associated with a sector of the circle.
- The strategy for calculating the area of a segment of a circle is to calculate the area of the sector associated with the segment and from that, subtract the area of the triangle within the sector formed by the two radii and the chord. The formula can be expressed as $A = \frac{m\overset{\frown}{AB}}{360}(\pi r^2) - \frac{1}{2}bh$, where A is the area of the segment, $\frac{m\overset{\frown}{AB}}{360}(\pi r^2)$ is the area of the sector, and $\frac{1}{2}bh$ is the area of the triangle.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-C Circles

Find arc lengths and areas of sectors of circles

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

G-MG Modeling with Geometry

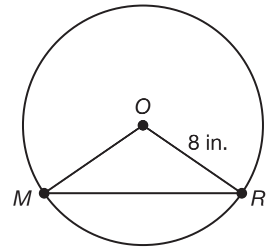
Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

Overview

The terms concentric circle, sector of a circle, and segment of a circle are defined. Students explore and describe methods for determining the area of a sector and the area of a segment of a circle. The formula for each is stated and students apply them to solve problem situations.

Warm Up



Circle O, with radius OR and OM. $m\angle ROM = 90°$ $m\overline{OR} = 8$ in.

1. How can the area of triangle *MOR* be determined?

Triangle *MOR* is a 45-45-90 triangle because it is an isosceles right triangle. To determine the area of triangle *MOR*, draw the bisector of angle *MOR* connecting point *O* to segment *MR* which forms two congruent 45-45-90 triangles, because the bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to the base. The base and height of triangle *MOR* can be calculated and then the area of triangle *MOR* determined.

2. Determine the area of triangle *MOR*.

The hypotenuse of a 45-45-90 triangle is equal to 8 inches and the hypotenuse is equal to the length of the leg times the square root of 2.

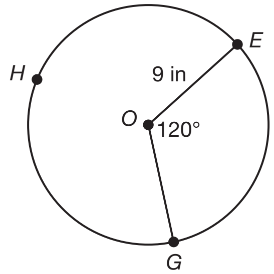
$h = l\sqrt{2}$

$8 = l\sqrt{2}$

$l = \frac{8}{\sqrt{2}} = 4\sqrt{2}$

The height of triangle *MOR* is equal to $4\sqrt{2}$ inches and the base is equal to $8\sqrt{2}$ inches.

The area of triangle *MOR*: $A = \frac{1}{2}(8\sqrt{2})(4\sqrt{2}) = 32$ square inches.



3. How can the area of the region bound by radii *OE*, *OG* and $\overset{\frown}{GE}$ be determined?

To determine the area of the region bound by radii *OE*, *OG*, and $\overset{\frown}{GE}$, first calculate the area of circle *O* and divide the area by 3, because this region occupies $\frac{1}{3}$ of the circle, or three of these regions comprise the entire circle.

4. Determine the area of the region bound by radii *OE*, *OG* and $\overset{\frown}{GE}$.

The area of the circle is 81π square inches, so the area of the region is 27π square inches.

5. Determine the area of the region bound by radii *OE*, *OG* and $\overset{\frown}{GHE}$.

The area of the circle is 81π square inches, so the area of the region is 54π square inches.

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10.3

Playing Darts

Sectors and Segments of a Circle

LEARNING GOALS

In this lesson, you will:

- Determine the area of sectors of a circle.
- Derive the formula for the area of a sector.
- Determine the area of segments of a circle.

KEY TERMS

- concentric circles
- sector of a circle
- segment of a circle

The game of darts has a lot of interesting language associated with it. Players have given special names to certain types of scores. For example, a "Bed and Breakfast" happens when a player scores a 20, a 5, and a 1 on his or her turn. When a player lands on a triple score, a double score, and a single score all in the same sector of the dartboard, that's called a "Shanghai." And trying to end a game by getting a double score of 2 is apparently so frustrating that it has been called the "Madhouse."

The term "hat trick" is also used in darts. Can you guess what a hat trick is?

Problem 1

A dartboard is divided into several regions shaped like sectors of a circle and are coded with different colors relating to different point values. The terms concentric circles and sector of a circle are defined. Students explore the sectors of the dartboard and calculate the area of one sector by computing the total area of the circle and the fraction of the circle the sector occupies. Students write and compare expressions for calculating the area of a sector and calculating the length of an arc.

Grouping

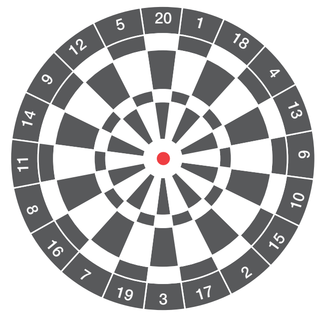
- Ask students to read introduction. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- Which formula is used to calculate the area of the dartboard?
- How did you determine the number of sectors on the dartboard?
- How did you determine the measure of the central angle associated with each sector?
- How do you determine the arc length associated with each sector?

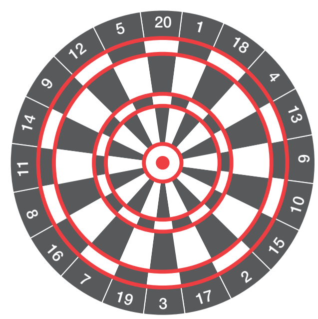
PROBLEM 1 Hitting the Bull's-Eye of a Circle

A standard dartboard is shown. Each section of the board is surrounded by wire, and the numbers indicate scoring for the game. For a single throw, the highest possible score can be achieved by landing a dart at the very center or the bull's-eye, of the dartboard.



Concentric circles are circles that share the same center point.

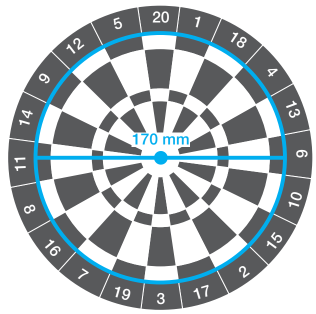
1. How many concentric circles do you see in the dartboard shown, not including the dartboard itself? Draw these circles.



There are 6 concentric circles on the dartboard.

- Is there another way to determine the area of each sector?
- Is the circumference formula associated with determining the area of a sector or arc length?
- Is the area of a circle formula associated with determining the area of a sector or arc length?

2. The diameter of the outermost circle is 170 millimeters. Calculate its area. Express your answer in terms of π.

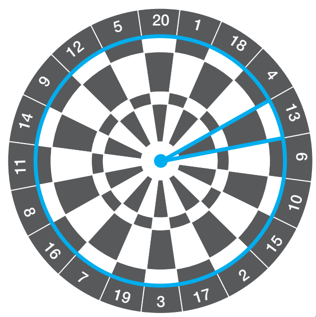


$A = \pi r^2$

$A = \pi(85)^2 = 7225\pi$ square millimeters

A **sector of a circle** is a region of the circle bounded by two radii and the included arc.

3. The dartboard can be divided into congruent sectors.



a. Determine the number of sectors contained in the outermost circle.

There are 20 sectors in the outermost circle.

b. Determine the measure of the central angle and the measure of the intercepted arc formed by each sector.

The measure of each central angle and its intercepted arc forming every section is $\frac{360°}{20} = 18°$.

c. Determine the ratio of the length of each intercepted arc to the circumference.

The ratio of the length of each intercepted arc to the circumference is $\frac{1}{20}$.

1

d. Determine the ratio of the area of each section to the area of the circle.

The ratio of the area of each section to the area of the circle is $\frac{1}{20}$.

e. Determine the area contained by each of these sectors of the circle. Express your answer in terms of π. Explain how you determined the area.

$A = \left(\frac{1}{20}\right)\pi r^2$

$A = \frac{1}{20}(7225\pi) = \frac{1445}{4}\pi$

The area contained by each section is $\frac{1445}{4}\pi$ square millimeters.

To determine the area of the sector of a circle, I multiplied the fraction representing the portion of the circumference bounded by the sector by the area of the entire circle.

The formula for determining the area of a sector can be written as follows:

$$A = \text{area of circle} \cdot \frac{\text{measure of angle}}{360°}$$

$$= \pi \cdot r^2 \cdot \frac{\text{measure of angle}}{360°}$$

$$= \frac{\text{measure of angle}}{360°} \cdot \pi r^2$$

4. How does the formula for determining the area of a sector compare to the formula for determining the arc length.

Both formulas use the same fraction to represent the portion of the circumference cut out by the central angle and its intercepted arc. The difference between the formulas is that to determine the area of a sector, I use the product of this fraction and the area of the circle, whereas to determine the arc length, I use the product of this fraction and the circumference of the circle.

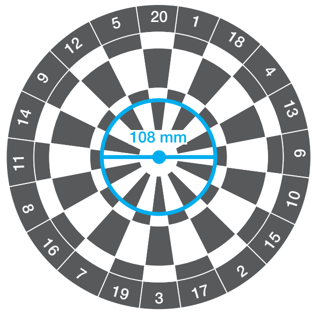
Grouping

Have students complete Question 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- What are the advantages of using Darcy's method?
- What are the disadvantages of using Darcy's method?
- What are the advantages of using Mike's method?
- What are the disadvantages of using Mike's method?
- Whose method do you prefer?

The innermost circle of the dartboard has a diameter of 108 millimeters and is divided into 20 congruent sectors as shown.



5. Darcy and Mike notice that half of the sectors of the innermost circle on the dartboard are the same color. Mike says that to calculate the total area of all the sectors of the same color, he could calculate the area of half the circle. Darcy says to, instead, calculate the area of one sector and multiply that area by 10.

Who's correct? Explain your reasoning.

Darcy and Mike are both correct.

$A = \pi r^2$

$= \pi(54)^2$

$= 2916\pi$

$\frac{2916\pi}{2} = 1458\pi$

The area of all of the sectors of the same color is 1458π square millimeters.

$\frac{1458\pi}{10} = 145.8\pi$

The area of one sector is 145.8π square millimeters.

1

Problem 2

The segment of a circle is the area of the circle bounded by a chord. It is important for students to understand that each segment of a circle is associated with and contained by a specific sector. The sector is divided into two regions when the endpoints of the included arc are connected. Connecting the endpoints of the included arc form a chord and that chord defines the segment of the circle. Students describe a method for calculating a segment of a circle and apply the method to solve problems.

Grouping

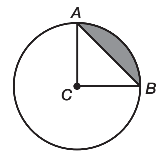
Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What is the difference between a sector of a circle and a segment of a circle?
- If the central angle determining the sector was not a right angle, how would you calculate the area of the triangle within the sector?
- When answers are expressed in terms of pi, are the answers exact or approximate?

PROBLEM 2 Segment of a Circle

A **segment of a circle** is a region of the circle bounded by a chord and the included arc.



1. Name the chord and the arc that bound the shaded segment of the circle.

   The chord that bounds the segment of the circle is $\overline{AB}$, and the arc that bounds the segment of the circle is $\overset{\frown}{AB}$.

2. Describe a method to calculate the area of the segment of the circle.

   One method for calculating the area of the segment would be to calculate the difference between the area of the sector and the area of the triangle formed inside the sector.

3. If the length of the radius of circle C is 8 centimeters and $m\angle ACB = 90^\circ$, use your method to determine the area of the shaded segment of the circle. Express your answer in terms of π. Then, rewrite your answer rounded to the nearest hundredth.

   Area of sector: $\frac{m\overset{\frown}{AB}}{360^\circ}(\pi r^2) = \frac{90^\circ}{360^\circ}(\pi 8^2) = \frac{1}{4}(64\pi) = 16\pi$

   Area of triangle: $\frac{1}{2}bh = \frac{1}{2}(8)(8) = \frac{1}{2}(64) = 32$

   Area of segment: $16\pi - 32$

   The area of the segment is $16\pi - 32$ square centimeters, or approximately 18.24 square centimeters.

- Is it easier to solve for the area of a sector, or work backwards and solve for the length of the radius? Why?
- What strategy was used to calculate the area of the shaded region?

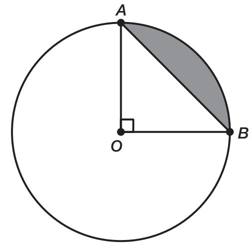
Grouping

Have students complete Questions 4 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 6

- What is the reason for only using 9π in the solution, instead of $9\pi - 18$? Explain.
- Can you use an equation that contains $9\pi - 18$ to answer the question? Explain.
- What is similar about Questions 4 and 5?
- What is different about Questions 4 and 5?

4. The area of the segment shown is $9\pi - 18$ square feet. Calculate the radius of circle O.



$9\pi = \frac{90°}{360°}(\pi r^2)$

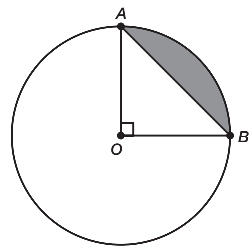
$9\pi = \frac{1}{4}(\pi r^2)$

$36 = r^2$

$r = 6$

The length of the radius is 6 feet.

5. The area of the segment is 10.26 square feet. Calculate the radius of circle O.



$10.26 = \frac{90°}{360°}(\pi r^2) - \frac{1}{2}bh$

$10.26 = \frac{1}{4}(\pi r^2) - \frac{1}{2}rr$

$10.26 = \frac{1}{4}\pi r^2 - \frac{1}{2}r^2$

$10.26 = r^2\left(\frac{1}{4}\pi - \frac{1}{2}\right)$

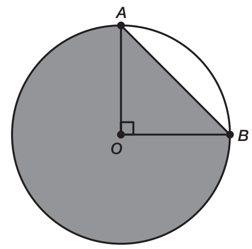
$10.26 = r^2(0.285)$

$36 = r^2$

$6 = r$

The length of the radius is 6 feet.

6. The length of the radius is 10 inches. Calculate the area of the shaded region of circle O. Express your answer in terms of π.



Area of shaded region: Area of circle − Area of segment

Area of the segment: Area of sector − Area of triangle

Area of circle: 100π square inches

Area of triangle: 50 square inches

Area of sector:

$\frac{90°}{360°}(\pi 10^2) = \frac{1}{4}(100\pi) = 25\pi$

Area of segment: $25\pi - 50$

Area of shaded region:

$100\pi - (25\pi - 50) = 75\pi + 50$ square inches

The area of the shaded region of circle O is $75\pi + 50$ square inches.

Problem 3

Students use their knowledge of sectors and segments to answer questions about a piece of pie topped with whipped cream.

Grouping

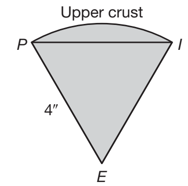
Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 6

- If the entire pie was cut into 4 equal sized pieces, what would be the measure of the central angle of one piece?
- If the entire pie was cut into equal pieces and the central angle of one piece of was was 30 degrees, how many pieces of pie exist?

PROBLEM 3 Is It Time For Dessert?

James covers the entire top of his piece of pie with whipped cream, but he doesn't like to eat the upper portion of the pie crust where it is thickest.



$\overline{EP}$ and $\overline{EI}$ are radii of a circle.

$m\overline{EP} = 4''$

$m\angle PEI = 60°$

1. What in this situation relates to a sector of a circle?

   One piece of pie can be associated with a sector of a circle.

2. What in this situation relates to a segment of a circle?

   The upper pie crust can be associated with a segment of a circle.

3. If each piece of pie is the same size, how many pieces are in the pie?

   There are six pieces in the pie.

4. Determine the area to be covered with whipped cream before James removes the upper crust.

   $A = \frac{60}{360}(\pi)(4^2) = \frac{1}{6}(16\pi) = \frac{8}{3}\pi \approx 8.4$ square inches

   The area to be covered with whipped cream before James removes the upper crust is approximately 8.4 square inches.

5. Determine the area to be covered with whipped cream on only the upper crust.

   $A = \frac{60}{360}(\pi)(4^2) - \frac{1}{2}(4)(2\sqrt{3}) = \frac{8}{3}\pi - (2)(2\sqrt{3}) \approx 1.45$ square inches

   The area to be covered with whipped cream on only the upper crust is approximately 1.45 square inches.

6. Determine the length of the upper crust.

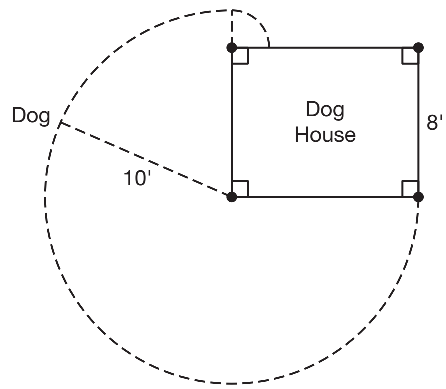
   The length of the upper crust is

   $\frac{60}{360}(2\pi \cdot 4) = \frac{1}{6}(8\pi) = \frac{4}{3}\pi \approx 4.2$ inches

   The length of the upper crust is approximately 4.2 inches.

Be prepared to share your solutions and methods.

Check for Students' Understanding



A dog is tethered to the corner of his rectangular dog house with 10′ chain. The chain keeps the dog within 10′ of the corner.

1. Determine the area of the sector determined by the 10′ radius.

The area of the sector determined by the 10 foot radius is approximately 235.62 square feet.

$$A = \frac{270}{360}(\pi)(10^2)$$
$$= \frac{3}{4}(100\pi)$$
$$= 75\pi$$
$$\approx 235.62$$

2. What is the radius of the smaller sector?

The radius of the smaller sector is 2 feet because the chain is 2 feet longer than the width of the dog house.

3. Determine the total area or space the dog has to play.

The total area of space the dog has to play is approximately 238.76 square feet.

I determined the answer by calculating the area of the sector with the 2 foot radius and adding that to the area of the sector with the 10 foot radius.

$$A = \frac{90}{360}(\pi)(2^2)$$
$$= \frac{1}{4}(4\pi)$$
$$= \pi$$
$$\approx 3.14 \text{ square feet}$$

3.14 + 235.62 = 238.76 square feet.

10.4

Circle K. Excellent!

Circle Problems

LEARNING GOALS

In this lesson, you will:

- Use formulas associated with circles to solve problems.
- Use theorems associated with circles to solve problems.
- Use angular velocity and linear velocity to solve problems.

KEY TERMS

- linear velocity
- angular velocity

ESSENTIAL IDEAS

- The arc length of $\widehat{AB} = \frac{m\widehat{AB}}{360}(2\pi r)$
- The formula for calculating the area of a sector is $A = \frac{m\widehat{AB}}{360}(\pi r^2)$, where A is the area of the sector, $\widehat{AB}$ is the arc that bounds the sector, and r is the length of the radius of the circle.
- The strategy for calculating the area of a segment of a circle is to calculate the area of the sector associated with the segment and from that, subtract the area of the triangle within the sector formed by the two radii and the chord. The formula can be expressed as $A = \frac{m\widehat{AB}}{360}(\pi r^2) - \frac{1}{2}bh$, where A is the area of the segment, $\frac{m\widehat{AB}}{360}(\pi r^2)$ is the area of the sector, and $\frac{1}{2}bh$ is the area of the triangle.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

G-MG Modeling with Geometry

Apply geometric concepts in modeling situations

1. Use geometric shapes, their measures, and their properties to describe objects.

3. Apply geometric methods to solve design problems.

Overview

Formulas and theorems related to circles are used to solve problems in various problem situations.

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Warm Up

Identify each formula.

1. $A = \frac{m\widehat{AB}}{360}(\pi r^2)$

   This formula determines the area of a sector.

2. $\widehat{AB} = \frac{m\widehat{AB}}{360}(2\pi r)$

   This formula determines arc length.

3. $A = \frac{m\widehat{AB}}{360}(\pi r^2) - \frac{1}{2}bh$

   This formula determines the area of a segment of a circle.

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10.4

Circle K. Excellent!

Circle Problems

LEARNING GOALS

In this lesson, you will:

- Use formulas associated with circles to solve problems.
- Use theorems associated with circles to solve problems.
- Use angular velocity and linear velocity to solve problems.

KEY TERMS

- linear velocity
- angular velocity

A pendulum is a freely swinging weight, usually attached to a string or wire which is fixed at one of its ends. You can discover something really interesting about a pendulum by conducting a simple experiment:

While keeping the string or wire tight and fixed, move the weight to an angle of about 10 degrees to vertical. Let go, and time how long it takes for the weight to make 10 swings up and back. Then, repeat this process with the weight starting at about 20 degrees to vertical.

What you should find is that the times are almost exact! This is why pendulums are so useful for keeping time.

Problem 1

Students determine the length of the steel arcs connecting each seat on a Ferris Wheel.

Grouping

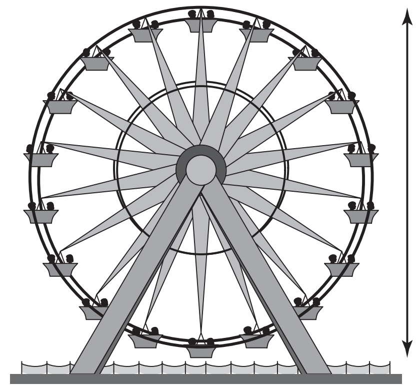
Have students complete the problem with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase

- What is the diameter of the Ferris Wheel?
- What is the circumference of the Ferris Wheel?
- How many passenger cars are on the Ferris Wheel?
- How many arcs separate the passenger cars on the Ferris Wheel?

PROBLEM 1 Ferris Wheel

This Ferris wheel is 50 meters tall. The actual wheel is constructed with steel arcs that connect one passenger car to the next passenger car.



Determine the length of each steel arc connecting one passenger car to the next passenger car.

The length of the diameter of the wheel is 50 meters.
Circumference of the wheel $= d\pi = (50)(\pi) = 50\pi$ meters

There are 18 arcs separating the 18 passenger cars on the wheel.

$\frac{50\pi}{18} = \frac{25}{9}\pi \approx 8.7$

The length of each steel arc connecting one passenger car to the next passenger car is approximately 8.7 meters.

Problem 2

Students determine arc length, the area of a sector, and the area of a segment on the World's Largest Clock Face.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- What is the circumference of the clock face?
- How many arcs separate the numbers of the clock face?
- What is the area of the entire clock face?
- How many sectors are on the clock face that are equivalent to the sector formed by the minute hand and hour hand when it is 1:00?
- What is the measure of the central angle formed by the minute hand and the hour hand at 3:00?
- What fraction of the circle is represented by this central angle?
- What is the area of the triangle in the sector?
- Is the triangle in the sector a right triangle? Explain.
- What is the measure of the central angle formed by the minute hand and the hour hand at 4:00?

PROBLEM 2 World's Largest Clock Face

Abraj Al Bait Towers clock in Mecca, Saudi Arabia, has a clock face with a diameter of 43 meters.

1. Determine the length of an arc connecting any two numbers on the clock face.

   Circumference of the clock $= d\pi = (43)(\pi) = 43\pi$ meters

   There are 12 arcs separating the 12 numbers on the clock face.

   $\frac{43\pi}{12} \approx 11.3$

   The length of each arc connecting any two numbers on the clock face is approximately 11.3 meters.

2. Determine the area of the sector formed by the minute hand and the hour hand when the time is 1:00.

   $A = \pi r^2$

   $A = \pi(21.5)^2 = 462.25\pi$

   $\frac{462.25\pi}{12} \approx 38.5\pi \approx 121$

   The area of the sector is approximately 121 square meters.

- What fraction of the circle is represented by this central angle?

3. Determine the area of the sector formed by the minute hand and the hour hand when the time is 3:00.

$A = \pi r^2$

$A = \pi(21.5)^2 = 462.25\pi$

$\frac{462.25\pi}{4} \approx 115.6\pi \approx 363.3$

The area of the sector is approximately 363.3 square meters.

The area of the triangle is $\frac{1}{2}(21.5)(21.5) = 231.125$ square meters.

The area of the segment is $363.3 - 231.125$, or 132.175, square meters.

4. Determine the area of the sector formed by the minute hand and the hour hand when the time is 4:00.

$A = \pi r^2$

$A = \pi(21.5)^2 = 462.25\pi$

$\frac{462.25\pi}{3} \approx 154.1\pi \approx 484.1$

The area of the sector is approximately 484 square meters.

Problem 3

Students are given formulas for linear velocity and angular velocity. The formulas are used to determine the linear velocity of a point on a circle given distance traveled and time, and the angular velocity of a point that rotates through given distance in radians and time. Students solve a problem situation where they determine the distance traveled by a tire working with both types of velocities.

Grouping

- Ask students to read introduction. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What is the arc length in each situation?
- What is the time in each situation?
- What is the unit in each situation? How is the velocity measured?
- How many seconds are in a minute?
- If there are six radians per second, how many radians per minute?

PROBLEM 3 Velocities in Circular Motion

Two types of circular velocity are *linear velocity* and *angular velocity*.

Linear velocity can be described as an amount of distance over a specified amount of time.

Linear velocity can be expressed as $v = \frac{s}{t}$ where

v = velocity (mph, ft/s)

s = arc length (m, ft)

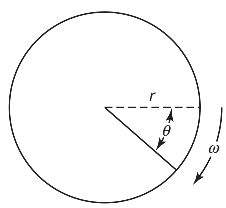
t = time (s, min, hr)

1. Use the formula to determine the linear velocity of a point on a circle that travels 24 centimeters in 8 seconds.

$v = \frac{s}{t}$

$v = \frac{24 \text{ cm}}{8 \text{ sec}} = 3 \text{ cm/sec}$

Angular velocity can be described as an amount of angle movement (in radians) over a specified amount of time.



Angular velocity can be expressed as $\omega = \frac{\theta}{t}$ where

ω = angular velocity (radians/s, radians/min)

θ = angular measurement (radians)

t = time (s, min, hr)

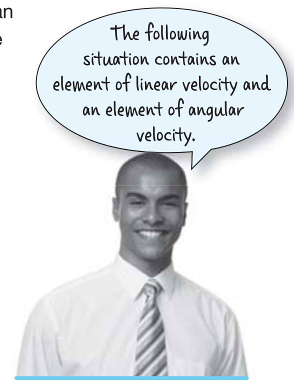
- How many radians per five minutes?
- What is the radius of the car tire?

2. Use the formula to determine the angular velocity of a point that rotates through $\frac{3\pi}{4}$ radians in 6 seconds.

$\omega = \frac{\theta}{t}$

$\omega = \frac{\frac{3\pi}{4}}{6} = \frac{\pi}{8}$ rad/sec

3. A car tire with a diameter of 60 centimeters turns with an angular velocity of 6 radians per second. Determine the distance traveled by the tire in 5 minutes.



$\omega = \frac{\theta}{t}$

$\omega = \frac{6 \text{ rad}}{\text{sec}}$

1 minute = 60 seconds

1 minute = 60 · 6 radians = 360 radians

5 minutes = 360 · 5 = 1800 radians

So, θ = 1800 radians.

$V = \frac{s}{t}$

$\theta = \frac{s}{r}$

$s = \theta \cdot r$

s = 1800 · 30 = 54,000 centimeters = 540 meters.

The tire will travel 540 meters in 5 minutes.

Problem 4

A square is inscribed in a circle and a circle is inscribed in square. Students determine the area of shaded regions in each situation using area formulas and the Pythagorean Theorem.

Grouping

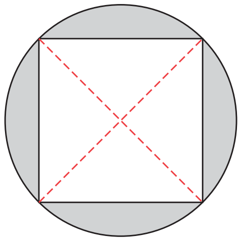
Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Is the diagonal of the square the diameter of the circle?
- What theorem is helpful when determining the length of the diagonal of the square?
- If the length of the diagonal of the square is $2\sqrt{2}$ centimeters, what is the length of the radius?
- What is the area of the circle?
- What is the area of the square?
- What strategy was used to determine the area of the shaded region?
- Is the length of the diameter of the circle also the length of the side of the square?
- What is the area of the circle?
- What is the area of the square?
- What strategy was used to determine the area of the shaded region?

PROBLEM 4 Circles and Squares

1. A square is inscribed in a circle. The length of each side of the square is 2 centimeters. Determine the area of the shaded region.



$a^2 + b^2 = c^2$

$2^2 + 2^2 = c^2$

$c^2 = 8$

$c = \sqrt{8} = 2\sqrt{2}$

The length of the diagonal of the square, which is also the diameter of the circle, is $2\sqrt{2}$ centimeters.

The radius of the circle is $\sqrt{2}$ centimeters.

$A = \pi r^2$

$= (\pi)(\sqrt{2})^2$

$= 2\pi$

The area of the circle is 2π.

The area of the square is (2)(2) = 4 square centimeters.

The area of the shaded region is $2\pi - 4$ square centimeters.

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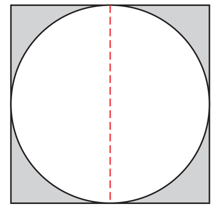
2. A circle is inscribed in a square. The length of the diameter of the circle is $2\sqrt{2}$ centimeters. Determine the area of the shaded region.



$A = \pi r^2$

$= (\pi)(\sqrt{2})^2$

$= 2\pi$

The area of the circle is 2π.

The length of the diameter of the circle is also the length of the side of the square.

The area of the square is $(2\sqrt{2})(2\sqrt{2}) = 8$ square centimeters.

The area of the shaded region is $8 - 2\pi$ square centimeters.

Problem 5

Students use their knowledge of sectors and segments to answer questions about watering a lawn using the most effective sprinkle placement.

Grouping

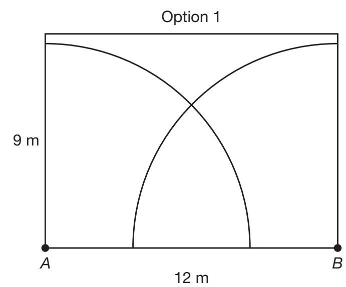
- Ask students to read introduction. Discuss as a class.
- Have students complete Questions 1–2 with a partner. Then have students share their responses as a class.

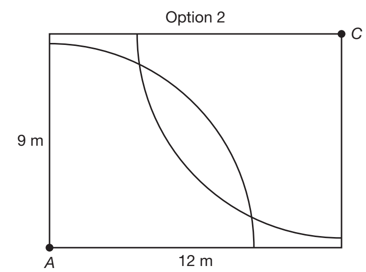
Guiding Questions for Share Phase, Questions 1 and 2

- Describe the sprinkler positions for each option.
- Which option do you predict will water more of the lawn?
- Why is cosine the appropriate trigonometric ratio for calculating the angle measures in Question 2? Explain.

PROBLEM 5 **Lawn Care**

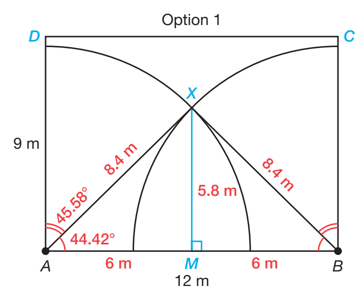
Angela owns a lawn care business. One client has a rectangular lawn that is 9 meters by 12 meters. He wants to install sprinklers to water the lawn. He is willing to pay for two sprinklers and suggests the following placements for the sprinklers. Each sprinkler rotates in quarter circles and sprays water 8.4 meters.

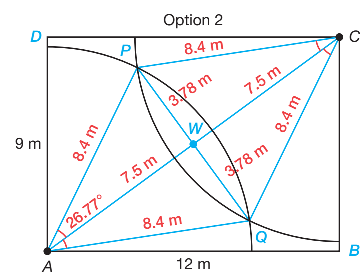




Angela must prepare a report for the client that includes her recommendation for how to best water his lawn.

1. Angela begins by analyzing the suggestions made by the client. She begins by drawing some additional lines to decompose the area of the lawn that is covered by the sprinklers. Describe the shapes that make up each composed area.





The first option consists of an isosceles triangle (or two right triangles) and two congruent sectors.

The second option consists of a rhombus (or two congruent isosceles triangles or four congruent right triangles) and two pairs of congruent sectors.

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2. Angela then calculates the area of the lawn covered by the sprinklers in the first option.

a. Determine the lengths of $\overline{AX}$, $\overline{BX}$, $\overline{MX}$, $\overline{AM}$, and $\overline{BM}$.

Both $\overline{AX}$ and $\overline{BX}$ are radii of the quarter-circles so $AX = 8.4$ meters and $BX = 8.4$ meters.

The perpendicular segment drawn from the vertex of an isosceles triangle bisects the opposite side so $AM = 6$ meters and $BM = 6$ meters.

I can use the Pythagorean Theorem to calculate XM.

$XM^2 + 6^2 = 8.4^2$

$XM^2 + 36 = 70.56$

$XM^2 = 34.56$

$XM = \sqrt{34.56}$

$XM \approx 5.88$

b. Calculate the area of triangle AXB.

$A = \frac{1}{2}(12)(5.88)$

$= 35.28$

The area of triangle AXB is approximately 35.28 square meters.

c. Calculate the measures of $\angle DAX$, $\angle BAX$, $\angle CBX$, and $\angle ABX$.

$\cos(m\angle BAX) = \frac{6}{8.4}$

$m\angle BAX = \cos^{-1}\left(\frac{6}{8.4}\right)$

$m\angle BAX \approx 44.42°$

Base angles of an isosceles triangle are congruent so $m\angle CBX \approx 44.42°$.

Angle DAB is a right angle because quadrilateral $ABCD$ is a rectangle. So, $m\angle DAX = 90° - 44.42° = 45.58°$.

Similarly, $m\angle CBX = 90° - 44.42° = 45.58°$.

d. Calculate the area of the two sectors.

Area of one sector $= \pi(8.4)^2\left(\frac{45.58}{360}\right)$

≈ 28.07

The area of each sector is about 28.07 square meters so the area of both sectors is about 56.14 square meters.

e. Calculate the total area of the lawn covered by the sprinklers in the first option.

$A = 35.28 + 56.14 = 91.42$

The total area of the lawn covered by the sprinklers in the first option is approximately 91.42 square meters.

Grouping

Have students complete Questions 3 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 through 6

- How do you calculate the area of a rhombus?
- Did you use different geometric figures to model the the watering area in both options? Explain.
- Are there any disadvantages to using the single sprinkler that waters more than the entire area of the lawn?
- Which of the 3 options is the wisest choice for watering the lawn? Is there a wiser option using 1 or 2 sprinklers?

3. Angela then calculates the area of the lawn covered by the sprinklers in the second option.

a. Calculate the lengths of $\overline{AC}$ and $\overline{PQ}$.

Line segment *AC* is a diagonal of rectangle *ABCD*. I can use the Pythagorean Theorem to calculate the length of $\overline{AC}$.

$AC^2 = 9^2 + 12^2$

$AC^2 = 81 + 144$

$AC^2 = 225$

$AC = \sqrt{225}$

$AC = 15$

The length of $\overline{AC}$ is 15 meters.

The diagonals of a rhombus bisect each other so $AW = 7.5$ meters and $CW = 7.5$ meters.

I can use the Pythagorean Theorem to calculate the length of $\overline{PW}$.

$AP^2 = AW^2 + PW^2$

$8.4^2 = 7.5^2 + PW^2$

$70.56 = 56.25 + PW^2$

$14.31 = PW^2$

$PW = \sqrt{14.31}$

$PW \approx 3.78$

The length of $\overline{PW}$ is about 3.78. Similarly, the length of $\overline{QW}$ is also about 3.78 meters. So, $\overline{PQ}$ is about 7.56 meters.

b. Calculate the area of rhombus *APCQ*.

$A = \frac{1}{2}(AC)(PQ)$

$= \frac{1}{2}(15)(7.56)$

$= 56.7$

The area of rhombus *APCQ* is about 56.7 square meters.

c. Calculate $m\angle DAP + m\angle BAQ$ and $m\angle DCP + m\angle BCQ$.

$\cos(m\angle PAW) = \frac{AW}{AP}$

$\cos(m\angle PAW) = \frac{7.5}{8.4}$

$m\angle PAW = \cos^{-1}\left(\frac{7.5}{8.4}\right)$

$m\angle PAW \approx 26.77°$

The diagonals of a rhombus bisect the angles of the rhombus, so $m\angle QAW = 26.77°$. Opposite angles of a rhombus are congruent, so $m\angle PCW = 26.77°$ and $m\angle QCW = 26.77°$.

Angle *DAB* and angle *DCB* are right angles because quadrilateral *ABCD* is a rectangle.

$m\angle DAP + m\angle BAQ = 90° - (m\angle PAW + m\angle QAW)$

$= 90° - (26.77° + 26.77°)$

$= 36.46°$

Similarly, $m\angle DCP + m\angle BCQ = 36.46°$

d. Calculate the area of the two sectors for each sprinkler.

Area of sectors for one sprinkler $= \pi(8.4)^2\left(\frac{36.46}{360}\right)$

≈ 22.45

The area of the sectors for one sprinkler is about 22.45 square meters so the area of all four sectors is about 44.90 square meters.

e. Calculate the total area of the lawn covered by the sprinklers in the second option.

$A = 56.7 + 44.90 = 101.60$

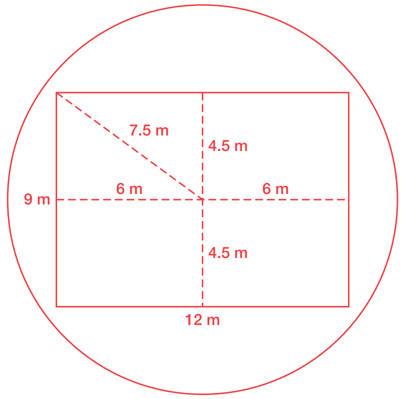
The total area of the lawn covered by the sprinklers in the second option is approximately 101.60 square meters.

4. Which option covered a larger percentage of the lawn?

The first option covered 91.42 square meters. The second option covered 101.60 square meters.

So, the second option covered a larger percentage of the lawn.

5. Angela is disappointed that neither option waters the entire lawn. She wonders if she can water the entire lawn if she positions the sprinklers differently, perhaps in the interior of the lawn. Draw a recommendation that includes where to position the sprinklers and a justification that the entire lawn will be watered. If it is not possible to water the entire lawn using two sprinklers, explain why not.



If I place a single sprinkler that sprays water 8.4 meters at the center of the lawn, then I can water the entire lawn. The only disadvantage is that the spray of the water will extend beyond the 12 × 9 lawn.

I know that the entire lawn will be covered because the points of the lawn that are furthest from the center are the vertices of the rectangle. I can use the Pythagorean Theorem to calculate the distance from the center of the rectangle to a vertex.

$d^2 = 6^2 + 4.5^2$

$d^2 = 36 + 20.25$

$d^2 = 56.25$

$d = \sqrt{56.25}$

$d = 7.5$

The distance from the center to a vertex is 7.5 meters which is within the spray radius of 8.4 meters.

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6. Write a letter to the client with your recommendation for the best option to water his entire lawn.

Dear Mr. Fitsioris,

I have reviewed and analyzed your suggestions for sprinkler placement for your lawn. The total area of your lawn is 108 square meters. If you place sprinklers on adjacent corners, then you will cover 91.42 square meters, or about 85% of your lawn. If you place sprinklers on opposite corners, then you will cover 101.60 square meters, or about 94% of your lawn.

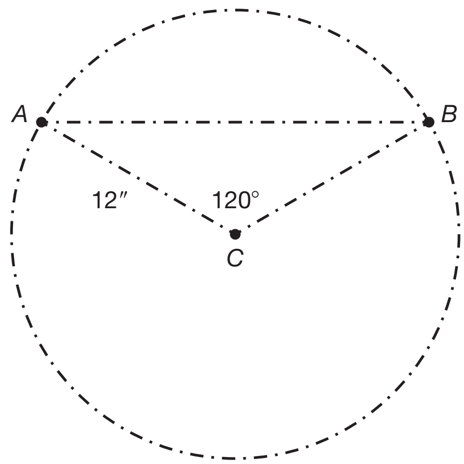
I have also looked into an alternate placement of the sprinklers. If you place a single sprinkler at the exact center of your lawn, then you can cover the entire lawn. The only disadvantage is that the spray of the water will extend beyond the 12×9 lawn.

Please review this information and let me know how you would like to proceed. I look forward to hearing your response.

Thank you.

Be prepared to share your solutions and methods.

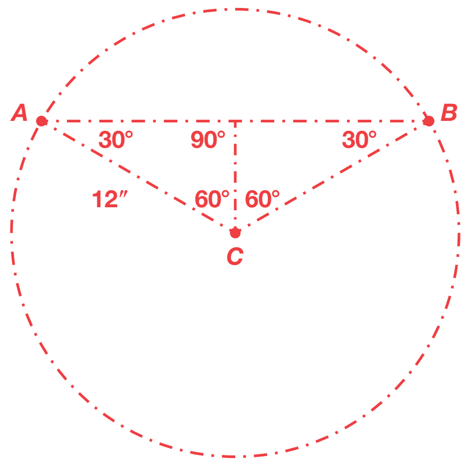
Check for Students' Understanding



$AO = 12$ inches.

$m\angle AOB = 120°$

Determine the area of segment AB.



The area of segment AB is exactly $48\pi - 36\sqrt{3}$ square inches or approximately 88.44 square inches.

Area of sector AOB:

$$A = \frac{120}{360}(\pi)(12^2) = \frac{1}{3}(144\pi) = 48\pi \approx 150.8 \text{ square inches}$$

The height of triangle AOB is half the hypotenuse (12″) or 6″.

The length of the base of triangle AOB is $2x$ half the hypotenuse times $\sqrt{3}$ or $12\sqrt{3}$

Area of triangle AOB:

$$A = \frac{1}{2}bh = \frac{1}{2}(12\sqrt{3})(6) = 36\sqrt{3}$$

Area of segment AB:

$48\pi - 36\sqrt{3}$ square inches

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Chapter 10 Summary

KEY TERMS

- inscribed polygon (10.1)
- circumscribed polygon (10.1)
- arc length (10.2)
- radian (10.2)
- concentric circles (10.3)
- sector of a circle (10.3)
- segment of a circle (10.3)
- linear velocity (10.4)
- angular velocity (10.4)

POSTULATES AND THEOREMS

- Inscribed Right Triangle–Diameter Theorem (10.1)
- Inscribed Right Triangle–Diameter Converse Theorem (10.1)
- Inscribed Quadrilateral–Opposite Angles Theorem (10.1)

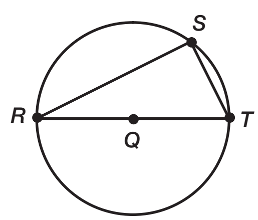
10.1 Using the Inscribed Right Triangle–Diameter Theorem and the Inscribed Right Triangle–Diameter Converse Theorem

The Inscribed Right Triangle–Diameter Theorem states: "If a triangle is inscribed in a circle such that one side of the triangle is a diameter of the circle, then the triangle is a right triangle."

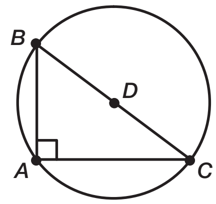
The Inscribed Right Triangle–Diameter Converse Theorem states: "If a triangle is inscribed in a circle then the hypotenuse is a diameter of the circle."

Examples

Triangle *RST* is inscribed in circle *Q*. Because $\overline{RT}$ of $\triangle RST$ is a diameter of circle *Q*, $\triangle RST$ is a right triangle.



Triangle *ABC* is inscribed in circle *D*. Because $\triangle ABC$ is a right triangle, the hypotenuse of $\triangle ABC$, $\overline{BC}$, is a diameter of circle *D*.



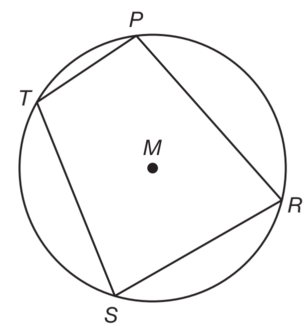
10.1 Using the Inscribed Quadrilateral–Opposite Angles Theorem

The Inscribed Quadrilateral–Opposite Angles Theorem states: "If a quadrilateral is inscribed in a circle, then the opposite angles are supplementary."

Example

Quadrilateral *PRST* is inscribed in circle *M*.

So, $m\angle P + m\angle S = 180°$ and $m\angle T + m\angle R = 180°$.



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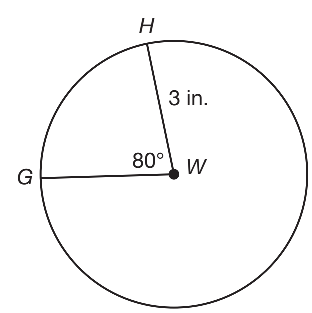
10.2 Determining Arc Length using Degrees

Arc length is a portion of the circumference of a circle. The length of an arc is different from the degree measure of the arc. Arcs are measured in degrees while arc lengths are measured in linear measurements. To determine arc length, *s*, using degrees, use the following formula:

$$s = \frac{\text{measure of angle}}{360} \cdot 2\pi r$$

Example

In circle *W*, the length of the radius is 3 inches and $\overline{GH}$ has a measure of 80°. So, the arc length of *GH* is $\frac{80}{360} \cdot 2\pi(3) = \frac{2}{9} \cdot 6\pi = \frac{4}{3}\pi$.



10.2 Determining Arc Length using Radians

A radian is another unit of measure for lengths of arcs. One radian is the measure of a central angle whose arc length is the same as the radius of the circle. To determine arc length using radians, use the following formula: $\theta = \frac{s}{r}$ where θ is the measure of the central angle in radians, *s* is the length of the intercepted arc, and *r* is the length of the radius.

Example

In circle *P*, $\theta = \frac{\pi}{4}$, and $r = 8$ m. So, $\frac{\pi}{4} = \frac{s}{8}$, and $8\pi = 2s$, or $s = 2\pi$.

10.3 Determining the Area of a Sector Using the Formula

The sector of a circle is a region of the circle bounded by two radii and the included arc. To determine the area of a sector, use the following formula:

Area of a sector $= \dfrac{\text{measure of angle}}{360} \cdot \pi r^2$ where r is the length of the radius of the circle.

Example

Determine the area of a sector bounded by a 40° inscribed angle in a circle with a radius of 5 cm.

Area of sector $= \dfrac{40}{360} \cdot \pi(5)^2 = \dfrac{1}{9} \cdot 25\pi = \dfrac{25}{9}\pi$ cm.

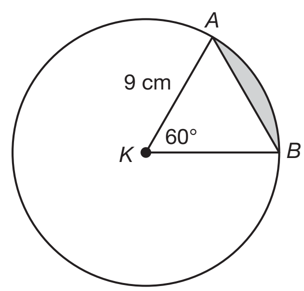
10.3 Determining the Area of a Segment

The segment of a circle is a region of the circle bounded by a chord and the included arc.

Example

Determine the area of the segment in circle K.

Area of sector $= \dfrac{40}{360} \cdot \pi(5)^2 = \dfrac{1}{9} \cdot 25\pi = \dfrac{25}{9}\pi$ cm.



10.4 Determining Linear Velocity

Linear velocity is the amount of distance an object travels over a specified amount of time. Linear velocity is expressed using the formula $v = \dfrac{s}{t}$, where v is the velocity, s is the arc length, and t is time.

Example

A point on a circle travels 38 inches in 10 seconds. So, the velocity of the point is

$v = \dfrac{24 \text{ in.}}{10 \text{ sec}} = 2.4$ in./sec.

10.4 Determining Angular Velocity

Linear velocity is the amount of angle movement in radians an object travels over a specified amount of time. Linear velocity is expressed using the formula: $\omega = \frac{\theta}{t}$, where ω is the angular velocity, θ is the angular measurement in radians, and t is time.

Example

A point on a circle rotates through $\frac{\pi}{4}$ radians in 8 seconds. So, the angular velocity of the point is $\omega = \frac{\frac{\pi}{4}}{8} = \frac{\pi}{32}$ rad/sec.