Sequences and Series

8

8

Chapter 8 Overview

This chapter begins with a review of arithmetic and geometric sequences and their explicit and recursive formulas. Lessons provide opportunities for students to explore finite and infinite arithmetic series, and then finite and infinite geometric series are used to derive formulas to compute each type of series. Students will explore and analyze the common ratios of several infinite geometric series to understand under what conditions the series is either divergent or convergent. In the later part of the chapter, lessons provide opportunities for students to apply their understanding of geometric series to solve problems.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
8.1	Arithmetic and Geometric Sequences	F.BF.2	1	This lesson provides opportunities for students to review arithmetic and geometric sequences and their explicit and recursive formulas. Questions ask students to analyze several sequences and identify them as either arithmetic, geometric, or neither, and then as finite or infinite. They will also identify the common difference or common ratio when appropriate. Students then solve problems using the applicable formula.	x		x	x	
8.2	Finite Arithmetic Sequences	A.SSE.1.a A.CED.1 F.BF.2	1	This lesson provides opportunities for students to compute finite arithmetic series. Summation notation is introduced. Questions ask students to use summation notation to represent sums of finite arithmetic series and to compute finite arithmetic series. Students will use finite arithmetic series to solve problems.	x				

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
8.3	Geometric Series	A.SSE.1.a A.SSE.4 F.BF.2	1	This lesson provides opportunities for students to derive two formulas to compute any finite geometric series. Student work is used to provide an introduction to Euclid's Method to compute any finite geometric series. Questions then ask students to analyze and use the pattern generated from repeated polynomial long division to derive a second formula. Students will rewrite geometric series using summation notation, and compute geometric series. Finally, questions ask students to use the formulas to solve problems.		x	x		
8.4	Infinite Geometric Series	A.SSE.4	1	This lesson provides opportunities for students to explore convergent and divergent geometric series. Questions ask students to explore and analyze the common ratios of several infinite geometric series to understand under what conditions the series is either divergent or convergent. The Talk the Talk asks students to write the formulas they developed to compute each type of series: arithmetic series, geometric series, convergent geometric series, and divergent geometric series.	x		x	x	
8.5	Geometric Series Applications	A.SSE.4	1	This lesson provides opportunities for students to apply their understanding of geometric series to solve problems. Questions ask students to write explicit formulas and formulas for several geometric series to answer questions in a real-world situation.	x	x	x	x	x
8.6	Applications of Arithmetic and Geometric Series	A.SSE.4	1	This lesson provides opportunities to apply geometric series and explicit formulas to solve problems.	x				x

8

Skills Practice Correlation for Chapter 8

Lesson		Problem Set	Objectives
8.1	Arithmetic and Geometric Sequences		Vocabulary
		1 – 6	Identify whether sequences are arithmetic, geometric, or neither and determine common differences or common ratios
		7 – 12	Create sequences and include the first four terms
		13 – 18	Identify arithmetic or geometric sequences, write recursive formulas, and determine the next term
		19 – 24	Identify arithmetic or geometric sequences, write explicit formulas, and determine the 35^{th} term
8.2	Finite Arithmetic Series		Vocabulary
		1 – 6	Use sigma notation to rewrite finite series and calculate given sums
		7 – 12	Use Gauss's formula to calculate finite arithmetic series
		13 – 18	Write functions to calculate sums of the first n terms of arithmetic sequences and determine S_3
		19 – 24	Use given information to answer questions
8.3	Geometric Series		Vocabulary
		1 – 10	Use Euclid's Method to compute series
		11 – 16	Write geometric series in a given form and compute each
		17 – 22	Use Euclid's Method to solve problems
8.4	Infinite Geometric Series		Vocabulary
		1 – 10	Determine whether geometric series are convergent or divergent
		11 – 20	Determine whether geometric series are convergent or divergent and compute convergent series
		21 – 28	Identify formulas and compute each series
8.5	Geometric Series Applications	1 – 6	Write equations for the n^{th} term of geometric sequences that model problem situations
		7 – 14	Use geometric sequences and series to solve problems
8.6	Applications of Arithmetic and Geometric Series	1 – 8	Determine whether situations are best modeled with arithmetic or geometric series
		9 – 16	Use arithmetic and geometric series to solve problems

8.1

Sequence—Not Just Another Glittery Accessory

Arithmetic and Geometric Sequences

LEARNING GOALS

In this lesson, you will:

- Recognize patterns as sequences.
- Determine the next term in a sequence.
- Write explicit and recursive formulas for arithmetic and geometric sequences.
- Use formulas to determine unknown terms of a sequence.

KEY TERMS

- arithmetic sequence
- geometric sequence
- finite sequence
- infinite sequence

ESSENTIAL IDEAS

- An arithmetic sequence is a sequence of terms in which the difference between any two consecutive terms is a constant.
- A geometric sequence is a sequence of terms in which the ratio between any two consecutive terms is a constant.
- A finite sequence is a sequence that terminates.
- An infinite sequence is a sequence that continues forever.
- An explicit formula for a sequence is a formula used to calculate each term of the sequence using the index, a term's position in the sequence. For an arithmetic sequence; $a_n = a_1 + d(n - 1)$, where a_1 is the first term, d is the common difference, and n is the nth term in the sequence. For a geometric sequence; $g_n = g_1 \cdot r^{n-1}$, where g is the first term, and r is the common ratio.
- A recursive formula generates each new term of a sequence based on a preceding term of the sequence. For an arithmetic sequence, $a_n = a_{n-1} + d$, where a_{n-1} is the term previous to a_n, and d is the common difference. For a geometric sequence, $g_n = g_{n-1} \cdot r$ where g_{n-1} is the term previous to g_n and r is the common ratio.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-BF Building Functions

Build a function that models a relationship between two quantities

2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

8

Overview

The terms arithmetic sequence, geometric sequence, finite sequence, and infinite sequence are defined. Students will analyze several sequences by identifying each sequence as arithmetic or geometric, and finite or infinite. Explicit and recursive formulas for arithmetic and geometric sequences are reviewed. Students then use both formulas to determine specific terms in several sequences. They are given several terms in sequences and they identify the type of sequence and the 50^{th} term. Two situations are given and students will use either the recursive or explicit formula to solve the problem.

Warm Up

Analyze the table shown.

x	1	2	3	4	20
y	28	34	40		

1. Calculate the value of y when $x = 4$.
   $y = 46$

2. Calculate the value of y when $x = 20$.
   $y = 142$

3. Explain two different ways to calculate the y-value when $x = 20$.
   To calculate the y-value when $x = 20$, begin with the y-value at $x = 3$, which is 40, and add 17 6's to it. Use the equation of the line passing through the points listed in the table, $y = 6x + 22$, substitute the value of 20 for x, and evaluate the expression.

8

8.1

Sequence—Not Just Another Glittery Accessory

Arithmetic and Geometric Sequences

LEARNING GOALS

In this lesson, you will:

- Recognize patterns as sequences.
- Determine the next term in a sequence.
- Write explicit and recursive formulas for arithmetic and geometric sequences.
- Use formulas to determine unknown terms of a sequence.

KEY TERMS

- arithmetic sequence
- geometric sequence
- finite sequence
- infinite sequence

"Ostinato" is a musical term that indicates a repeating pattern of notes. A word that you might be familiar with that is related to "ostinato" is "obstinate," meaning "stubborn".

An ostinato is indeed a stubborn pattern. Musicians commonly use ostinati (the plural of ostinato) to underlay a particular feeling they want a certain song to portray. They may also use it to stabilize a variety of pitches to provide uniformity within a song.

A basso ostinato is a type of ostinato that is used to form a harmonic pattern and is repeated throughout a song. Some argue that the basso ostinato should be thought of more as a device than a form of music.

The term "riff" is the modern day ostinato for popular music. A riff is defined as a short series of notes that create a melody within a melody of a song. Unlike an ostinato, a riff does not need to be repeated throughout the whole song.

You may be familiar with Pachelbel's Canon in D, which features one of the most famous repeating patterns of all time.

Just as with ostinati, when dealing with sequences, you look to identify an underlying pattern. You try to identify what it is that is moving the pattern along, so that you may be able to determine what is coming next.

8

Problem 1

The terms arithmetic sequence, geometric sequence, finite sequence, and infinite sequence are defined. Students will analyze several sequences by identifying each sequence as arithmetic or geometric, and finite or infinite. They also identify the common difference or common ratio when appropriate. Students then create the first three terms of their own sequences that are arithmetic, geometric, and neither.

Grouping

Ask a student to read the information and definitions. Complete Question 1 as a class.

PROBLEM 1 I Spy With My Little Eye, A Pattern!

Patterns, both numerical and physical, can be defined as sequences. Recall, a sequence is a pattern involving an ordered arrangement of numbers, geometric figures, letters, or other objects called terms. An **arithmetic sequence** is a sequence of terms in which the difference between any two consecutive terms is a constant. A **geometric sequence** is a sequence of terms in which the ratio between any two consecutive terms is a constant. A sequence that is neither arithmetic or geometric has a pattern, but there is no common difference or ratio.

An ellipsis is 3 periods which means "and so on." Ellipses are used to represent infinite sequences.

Sequences can have a fixed number of terms, or they can continue forever. If a sequence terminates it is called a **finite sequence**. If a sequence continues forever it is called an **infinite sequence**.

1. Lisa and Ray give the next few terms in the sequence: 1, 1, 1, . . .

Lisa

2, 2, 2, 3, 3, 3

The sequence is writing each natural number three times.

Ray

1, 1, 1, 1, 1, 1, . . .

The sequence just repeats 1 forever.

Who is correct? Explain your reasoning.

Both Lisa and Ray are correct.

There is not enough information to determine how the sequence will continue.

It is important to recognize that when you are only given the first few terms in a sequence, you may not have enough information to determine the next term.

Grouping

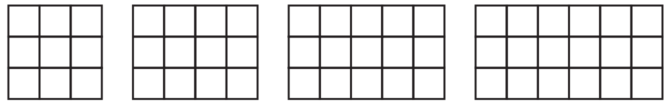
Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2, parts (a) through (c)

- How many tiles are in the first term? Second term? Third term? Fourth term?
- Is there a common difference or a common ratio between consecutive terms? How do you know?
- Is there a fifth term, a sixth term, and so on?
- How many toothpicks are in the first term? Second term? Third term?
- Is there a common difference or a common ratio between consecutive terms? How do you know?
- Is there a fourth term, a fifth term, and so on?
- How many rows are in the first term? Second term? Third term? Fourth term?
- Is there a common difference or a common ratio between consecutive terms? How do you know?
- Is there a fifth term, a sixth term, and so on?

2. Analyze each sequence and then circle the appropriate type of sequence. If the sequence is arithmetic, identify the common difference. If the sequence is geometric, identify the common ratio. Finally, circle whether the sequence is finite or infinite.

a. number of tiles



Arithmetic Sequence $d = 3$ Geometric Sequence ______ Neither

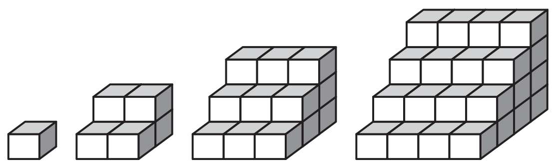
Infinite Sequence Finite Sequence

b. number of toothpicks

Arithmetic Sequence ______ Geometric Sequence ______ Neither

Infinite Sequence Finite Sequence

c. number of rows



Arithmetic Sequence $d = 1$ Geometric Sequence ______ Neither

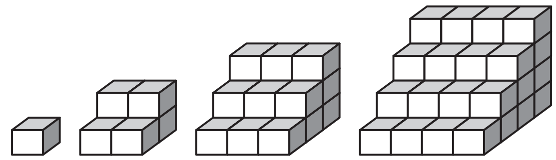
Infinite Sequence Finite Sequence

8

Guiding Questions for Share Phase, Question 2, parts (d) through (f)

- How many cubes are in the first term? Second term? Third term? Fourth term?
- Is there a common difference or a common ratio between consecutive terms? How do you know?
- Is there a fifth term, a sixth term, and so on?
- How many black triangles are in the first term? Second term? Third term? Fourth term?
- Is there a common difference or a common ratio between consecutive terms? How do you know?
- Is there a fifth term, a sixth term, and so on?
- How many white triangles are in the first term? Second term? Third term? Fourth term?
- Is there a common difference or a common ratio between consecutive terms? How do you know?
- Is there a fifth term, a sixth term, and so on?

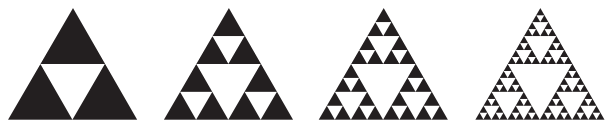
d. number of cubes



Arithmetic Sequence ________ Geometric Sequence ________ Neither

Infinite Sequence Finite Sequence

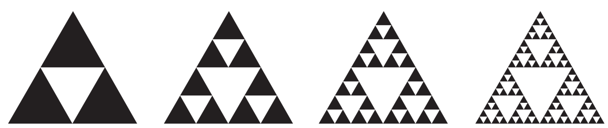
e. number of black triangles



Arithmetic Sequence ________ Geometric Sequence $r = 3$ Neither

Infinite Sequence Finite Sequence

f. number of white triangles



Arithmetic Sequence ________ Geometric Sequence ________ Neither

Infinite Sequence Finite Sequence

Guiding Questions for Share Phase, Question 2, parts (g) and (h)

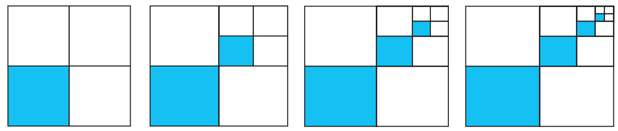
- What is the side length of the smallest shaded square in the first term? Second term? Third term? Fourth term?
- Is there a common difference or a common ratio between consecutive terms? How do you know?
- Is there a fifth term, a sixth term, and so on?
- What is the number of shaded squares in the first term? Second term? Third term? Fourth term?
- Is there a common difference or a common ratio between consecutive terms? How do you know?
- Is there a fifth term, a sixth term, and so on?

Guiding Question for Share Phase, Question 3

- How do you know the sequence you created is an arithmetic sequence?
- How do you know the sequence you created is a geometric sequence?
- How do you know the sequence you created is neither an arithmetic or a geometric sequence?

8

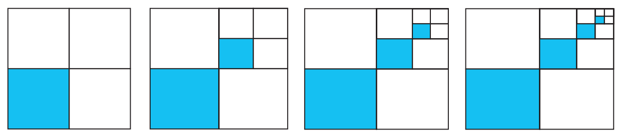
g. Side length of smallest shaded square within the unit square



Arithmetic Sequence ________ Geometric Sequence $r = 0.5$ Neither

Infinite Sequence Finite Sequence

h. Number of shaded squares



Arithmetic Sequence $d = 1$ Geometric Sequence ________ Neither

Infinite Sequence Finite Sequence

3. Create your own sequence given the type indicated. Include the first three terms.

a. Arithmetic Sequence

Answers will vary.

Student responses should include a sequence with a common difference.

b. Geometric Sequence

Answers will vary.

Student responses should include a sequence with a common ratio.

c. Neither Arithmetic or Geometric Sequence

Answers will vary.

Student responses should include a sequence with neither a common ratio nor difference.

8

Problem 2

Explicit and recursive formulas for arithmetic and geometric sequences are reviewed. Students will use both formulas to determine specific terms in a few sequences used in Problem 1. Next, they are given several terms in sequences and they identify the type of sequence and the 50th term. Two situations are given and students use either the recursive or explicit formula to solve the problem.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What is the term previous to the term a_5?
- What is the value of the term a_4 in this sequence?
- What is the common difference in this sequence?
- What is the value of the first term a_1 in this sequence?
- What is the common ratio in this sequence?
- How does the value of the 5th term determined using the recursive formula compare to the value of the 5th term determined using the explicit formula?

PROBLEM 2 Formula: Not Just for Babies

Previously, you learned the explicit and recursive formulas for arithmetic and geometric sequences. An explicit formula for a sequence is a formula used for calculating each term of the sequence using the index, a term's position in the sequence. A recursive formula generates each new term of a sequence based on a preceding term of the sequence.

	Arithmetic Sequence	Geometric Sequence
Explicit Formula	$a_n = a_1 + d(n - 1)$ where a_1 is the first term, d is the common difference, and n is the nth term in the sequence.	$g_n = g_1 \cdot r^{n-1}$ where g_1 is the first term, and r is the common ratio.
Recursive Formula	$a_n = a_{n-1} + d$ where a_{n-1} is the term previous to a_n, and d is the common difference.	$g_n = g_{n-1} \cdot r$ where g_{n-1} is the term previous to g_n, and r is the common ratio.

1. Consider the sequence in Problem 1, Question 1, part (a), *number of tiles*.

   a. Use the recursive formula to determine the 5th term.

   $a_n = a_{n-1} + d$

   $a_5 = a_4 + 3$

   $a_5 = 18 + 3$

   $a_5 = 21$

   b. Use the explicit formula to determine the 5th term.

   $a_n = a_1 + d(n - 1)$

   $a_5 = 9 + 3(5 - 1)$

   $a_5 = 9 + 12$

   $a_5 = 21$

2. Consider the sequence in Problem 1, Question 1, part (e), *number of black triangles*.

   a. Use the recursive formula to determine the 5th term.

   $g_n = g_{n-1} \cdot r$

   $g_5 = g_4 \cdot 3$

   $g_5 = 81 \cdot 3$

   $g_5 = 243$

   b. Use the explicit formula to determine the 5th term.

   $g_n = g_1 \cdot r^{n-1}$

   $g_5 = 3 \cdot 3^{5-1}$

   $g_5 = 3 \cdot 3^4$

   $g_5 = 243$

3. Which formula would you use if you wanted to determine the 95th term of either sequence? Explain your reasoning.

   Answers will vary.

   I would use the explicit formula because I do not need to determine the value of every term before determining the 95th term.

- What is the term previous to the term g_5?
- What is the value of the term g_4 in this sequence?
- What is the common ratio in this sequence?
- What is the value of the first term g_1 in this sequence?
- What is the common ratio in this sequence?
- How does the value of the 5th term determined using the recursive formula compare to the value of the 5th term determined using the explicit formula?
- Which formula uses the 94th term to determine the 95th term?

Grouping

Have students complete Questions 4 and 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 and 5

- Is there a common difference or a common ratio?
- Which formula requires the 49th term to determine the 50th term?
- What is the common difference in the sequence 180, 360, 540, . . .?
- How many sides and angles are in a decagon?
- What term in the sequence is associated with the degrees in a decagon?
- If the oven temperature increases 40% every 30 minutes, is this describing a common difference or a common ratio?
- What is the first term in this situation?
- Is 200 the first term in this situation?
- How did you determine the common ratio in this situation?
- Is this situation associated with a arithmetic sequence or a geometric sequence?

4. Identify each sequence as arithmetic, geometric, or neither. If possible, determine the 50th term of each sequence.

a. $-5, -1, 3, 7, 11, 15, 19, 23 \ldots$

Type of Sequence: arithmetic sequence

50th term: 191

$a_n = a_1 + d(n - 1)$

$a_{50} = -5 + 4(49)$

$a_{50} = 191$

b. 0, 1, 1, 2, 3, 5, 8, 13 . . .

Type of Sequence: neither

50th term: no solution using the formula

c. $27, 9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$

Type of Sequence: geometric sequence

50th term: $\approx 1.12 \cdot 10^{-22}$

$g_n = g_1 \cdot r^{n-1}$

$g_{50} = 27 \cdot \left(\frac{1}{3}\right)^{49}$

$g_{50} \approx 1.12 \cdot 10^{-22}$

5. Use either the recursive or explicit formula to determine each answer.

a. The sum of the interior angles in a triangle is 180°, in a quadrilateral is 360°, and in a pentagon is 540°. How many degrees are in a decagon?

$a_n = a_1 + d(n - 1)$

$a_8 = 180 + 180(7)$

$a_8 = 1440$

b. The employees at Franco's Pizza Shack turn the pizza ovens down to 200° overnight. When the workers open the shop in the morning, they turn the ovens up to 550°. The temperature of each oven increases by 40% every 30 minutes. Will the ovens reach the required 550° in 1.5 hours?

$g_4 = 200 \cdot r^3$

$g_4 = 200 \cdot (1.4)^3$

$g_4 = 200 \cdot 2.744$

$g_4 = 548.8$

The ovens will be 548.8° after 1.5 hours. It will have not met the required temperature of 550°.

8

Talk the Talk

Students will complete a table listing several sequences, the type of the sequence, the recursive formula, and the explicit formula.

Grouping

Have students complete the table with a partner. Then have students share their responses as a class.

Talk the Talk

Complete each row in the table using the given information in that row.

	Sequences	Type of Sequence	Recursive Formula	Explicit Formula
A	4, 12, 36, 108, . . .	geometric	$g_n = 3(g_{n-1})$ $g_1 = 4$	$g_n = 4 \cdot 3^{n-1}$ $n = 1, 2, 3 \ldots$
B	320, 80, 20, 5, . . .	geometric	$g_n = \frac{1}{4}(g_{n-1})$ $g_1 = 320$	$g_n = 320 \cdot \left(\frac{1}{4}\right)^{n-1}$ $n = 1, 2, 3 \ldots$
C	20, 30, 40, 50 . . .	arithmetic	$a_n = a_{n-1} + 10$ $a_1 = 20$	$a_n = 20 + 10(n - 1)$ $n = 1, 2, 3 \ldots$
D	10, 50, 250, 1250 . . .	geometric	$g_n = 5(g_{n-1})$ $g_1 = 10$	$g_n = 10 \cdot 5^{n-1}$ $n = 1, 2, 3 \ldots$
E	3, 11, 19, 27 . . .	arithmetic	$a_n = a_{n-1} + 8$ $a_1 = 3$	$a_n = 3 + 8(n - 1)$ $n = 1, 2, 3 \ldots$
F	5, 25, 45, 65 . . .	arithmetic	$a_n = a_{n-1} + 20$ $a_1 = 5$	$a_n = 5 + 20(n - 1)$ $n = 1, 2, 3 \ldots$

Be prepared to share your solutions and methods.

Check for Students' Understanding

The U.S. Postal Service increased the cost of mailing a first-class letter weighing up to one ounce to \$0.41 in 2007. The cost for mailing a letter increased \$0.17 for each additional ounce up to 13 ounces.

a. Using the notation for an arithmetic sequence, let a_n represent the cost in dollars for first-class postage for a letter that weighs n ounces, where $n \geq 1$. Write an explicit equation for a_n.

$a_n = 0.17(n - 1) + 0.41$

b. Write a recursive formula to model the same situation.

$a_n = a_{n-1} + 0.17$

c. What is the domain of the arithmetic sequence?

The domain is the set of whole numbers from 1 to 13.

d. Based on the explicit formula you have written, what is the cost of a letter that weights 9 ounces?

$a_n = 0.17(9 - 1) + 0.41 = 1.77$

The cost to mail a letter that weighs 9 ounces is \$1.77.

e. If the postage is \$2.45, what is the weighs of the letter?

$2.45 = 0.17(n - 1) + 0.41$

$2.04 = 0.17(n - 1)$

$12 = n - 1$

$13 = n$

The weight of the letter is 13 ounces.

This Is Series(ous) Business

Finite Arithmetic Series

8.2

LEARNING GOALS

In this lesson, you will:

- Compute a finite series.
- Use sigma notation to represent a sum of a finite series.
- Use Gauss's method to compute finite arithmetic series.
- Write a function to represent the sum of a finite arithmetic series.
- Use finite arithmetic series to solve real-world problems.

KEY TERMS

- tessellation
- series
- finite series
- infinite series
- arithmetic series

ESSENTIAL IDEAS

- A tessellation is created when a geometric shape is repeated over a two-dimensional plane such that there are no overlaps and no gaps.
- A series is the sum of terms in a given sequence. The sum of the first n terms of a sequence is denoted by S_n.
- $S_n = \sum_{i=1}^{n} a_i$, which is read as "the sum of all a_i for i from 1 to n."
- A finite series is the sum of a finite number of terms.
- An infinite series is the sum of an infinite number of terms.
- Gauss's rule to compute the first n terms of an arithmetic series is $S_n = \frac{n(a_1 + a_n)}{2}$, where a_1 is the first term, where a_n is the last term, and n is the number of terms in the series.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.
   a. Interpret parts of an expression, such as terms, factors, and coefficients.

A-CED Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems.

8

F-BF Building Functions

Build a function that models a relationship between two quantities

2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

Overview

A tessellation is used to generate a sequence. The terms finite series and infinite series ad defined. Students will use summation notation to rewrite indicated sums of various finite series and determine the total number of toothpicks needed to complete 5 rows of the tessellation. Students derive Gauss's rule to calculate the sum of any finite arithmetic series and use it to answer questions related to the Toothpick Tessellation. Students write an explicit formula to calculate any term of a sequence of odd whole numbers and any term of a sequence of even whole numbers, and then use Gauss's rule to calculate the sum of the first 20 terms of each sequence. Another solution method is explored and results in a quadratic function which is used to verify answers and calculate the sum of the first 100 odd whole numbers. A situation is introduced involving a seating arrangement for a concert. Students will sketch a seating chart and use an explicit formula and Gauss's rule to solve problems related to the situation.

Warm Up

1. Write the first 20 terms of the arithmetic sequence determined by $a_{n+1} = a_n + 2$ and $a_1 = 1$.

   1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39

2. Calculate the sum of the sequence in Question 1.

   The sum of the sequence is 400.

3. What method did you use to calculate the sum of the sequence?

   Answers will vary.

   Grouping or mental math skills can be used to make the process more efficient.

8

8.2

This Is Series(ous) Business

Finite Arithmetic Series

LEARNING GOALS

In this lesson, you will:

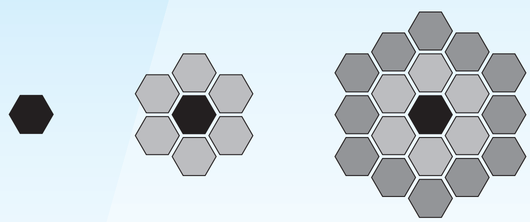
- Compute a finite series.
- Use sigma notation to represent a finite series.
- Use Gauss's method to compute finite arithmetic series.
- Write a function to represent a finite arithmetic series.
- Use finite arithmetic series to solve real-world problems.

KEY TERMS

- tessellation
- series
- finite series
- infinite series
- arithmetic series

Honey bees are fascinating little creatures. Did you know that honey bees are the only insects that produce food that humans eat? They also identify members of their colony by a unique smell.

Another amazing aspect of the honey bee is how they build their hive. Honey bees build hexagonal honey cells from a single cell. Layers of honey cells are then built around the edges, as shown.



Could you sketch the next figure in the sequence? Could you predict how many total hexagons would be in the next term of the sequence? How does this pattern translate to an arithmetic sequence?

Problem 1

The term tessellation is defined and students create a tessellation using toothpicks. The first rows of the tessellation are drawn and students will draw the next several rows. They then complete a table listing the number of additional toothpicks used to create each row and associate the tessellation with an arithmetic sequence. Students write an explicit formula for the sequence and use it to determine the number of additional toothpicks needed for the 18th row. The terms finite series, infinite series, and summation notation are introduced. Students use sigma notation to rewrite indicated sums of various finite series and determine the total number of toothpicks needed to complete 5 rows of the tessellation.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

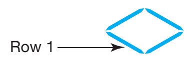
Guiding Questions for Share Phase, Questions 1 and 2

- How many toothpicks were used to create the diamond in row 1?
- If 4 toothpicks were used to create the diamond in row 1, why weren't 8 toothpicks needed to create the 2 diamonds in row 2?
- Once the first diamond is created, will there ever be another diamond created using 4 additional toothpicks?
- How many toothpicks are needed to create each diamond at the beginning and end of each row?
- How many toothpicks are needed to create each diamond between the first and last diamond in each row?

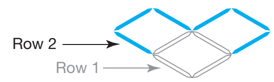
PROBLEM 1 Project: Toothpick Tessellation

Josephine is helping her little brother Pauley with his latest art project. He is using toothpicks to create a *tessellation*. A **tessellation** is created when a geometric shape is repeated over a two-dimensional plane such that there are no overlaps and no gaps.

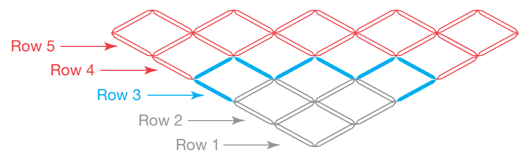
Pauley starts his tessellation project by gluing toothpicks to a large piece of poster board to make a single diamond shape. This is the first row.



Then, he places additional toothpicks parallel to the first row to create the second row. The second row consists of two diamond shapes.



He continues to place toothpicks in this manner, so that each row has one more diamond shape than the previous row. The first three rows of Pauley's tessellation are shown.



1. Sketch the next two rows of the tessellation on the previous diagram.

   See sketch above Question 1.

2. Complete the table to show the number of additional toothpicks used to create each row.

Row	Number of Additional Toothpicks Used to Create the Row
1	4
2	6
3	8
4	10
5	12

Guiding Questions for Share Phase, Questions 3 through 6

- Does the number of toothpicks increase by a constant amount for each additional row? Why does this imply about the sequence?
- What is the explicit formula for an arithmetic sequence?
- What does *n* represent in the explicit formula for an arithmetic sequence?
- How does knowing the number of toothpicks in the 18^{th} row help determine the number of toothpicks in the 17^{th} row?
- How does knowing the number of toothpicks in the 18^{th} row help determine the total number of toothpicks needed to complete an 18 row tessellation?

3. Can this tessellation be represented by an arithmetic or geometric sequence? Explain how you know.

This tessellation can be represented by an arithmetic sequence. I know because the number of toothpicks increases by a constant amount for each additional row in the tessellation.

4. Write an explicit formula for this sequence. Let *n* represent the row number, and let a_n represent the number of additional toothpicks used to create that row.

$a_n = a_1 + (n - 1)d$

$a_n = 4 + (n - 1)2$

$a_n = 4 + 2n - 2$

$a_n = 2n + 2$

5. Suppose that Pauley knows that he wants his tessellation to include 18 rows. How many additional toothpicks will he need for the 18^{th} row? Explain how you determined your answer.

$a_n = 2n + 2$

$a_{18} = 2(18) + 2$

$a_{18} = 36 + 2$

$a_{18} = 38$

Pauley will need 38 additional toothpicks for the 18^{th} row. I determined my answer by substituting 18 for *n* in the explicit formula.

6. Describe how to calculate the total number of toothpicks that Pauley needs for a tessellation that includes 18 rows. (Do not actually perform the calculation.)

Answers will vary.

Student responses could include drawing a diagram that includes 18 rows and then counting all of the toothpicks, or extending the table to include 18 rows and then adding all of the numbers in the toothpick column.

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Grouping

- Ask a student to read the information and definitions. Discuss as a class.
- Have students complete Questions 7 and 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 7 and 8

- How many terms are in the series?
- The sigma notation describes the sum of how many terms?
- How many toothpicks are needed to determine the first row? Second row? Third row? Fourth row? Fifth row?

You know how to determine the *n*th term of a sequence. However, sometimes it is necessary to determine the *sum* of the terms in a sequence.

A **series** is the sum of terms in a given sequence. The sum of the first n terms of a sequence is denoted by S_n. For example, S_3 is the sum of the first three terms of a sequence.

There is a special notation for the summation of terms using a capital sigma, Σ:

upper bound of summation

$$S_n = \sum_{i=1}^{n} a_i$$

an indexed variable representing each successive term in the series

index of summation

lower bound of summation

This expression means sum the values of a, starting at a_1 and ending with a_n.

In other words, $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$.

A series can be *finite* or *infinite*. A **finite series** is the sum of a finite number of terms. An **infinite series** is the sum of an infinite number of terms. For example, the sum of all of the even integers from 1 to 100 is a finite series, and the sum of all of the even whole numbers is an infinite series.

Think about it . . . what is the sum of an infinite arithmetic series with a negative common difference? What is the sum of an infinite arithmetic series with a positive common difference?

7. Use sigma notation to rewrite each finite series, and then compute.

a. $5 + 9 + 13 + 17 + 21$

$S_2 = \sum_{i=1}^{2} a_i = 5 + 9 = 14$

$S_5 = \sum_{i=1}^{5} a_i = 5 + 9 + 13 + 17 + 21 = 65$

b. $3 + 6 + 12 + 24 + 48 + 96 + 192$

$S_1 = \sum_{i=1}^{1} a_i = 3$

$S_7 = \sum_{i=1}^{7} a_i = 3 + 6 + 12 + 24 + 48 + 96 + 192 = 381$

8. Use sigma notation to represent the total number of toothpicks Pauley needs to complete 5 rows of his tessellation. Then, use your table in Question 2 to calculate this amount.

$$S_5 = \sum_{i=1}^{5} a_i = 4 + 6 + 8 + 10 + 12$$
$$= 40$$

Pauley needs a total of 40 toothpicks to complete 5 rows of his tessellation.

Problem 2

Students will derive Gauss's rule to compute any finite arithmetic series and use it to answer questions related to the Toothpick Tessellation in Problem 1.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- What is the difference between an arithmetic series and an arithmetic sequence?
- What is a partial sum in this situation?
- How do you determine a partial sum?
- How do you know there are 100 partial sums in this situation?
- How do you determine the sum of the partial sums?
- How did you determine the sum of the first 100 terms?
- Is there another way to solve this problem situation? How?

PROBLEM 2 Gauss's Method to the Rescue!

Remember that an arithmetic sequence is a sequence of numbers in which the difference between any two consecutive terms is a constant. An **arithmetic series** is the sum of an arithmetic sequence.

You can compute a finite arithmetic series by adding each individual term, but this can take a lot of time. A famous mathematician named Carl Friedrich Gauss developed another way to compute a finite arithmetic series.

As the story goes, when Gauss was in elementary school, his teacher asked the class to calculate the sum of the first 100 positive integers. Apparently, Gauss determined the answer in a matter of seconds! How did Gauss determine his answer so quickly?

1. Complete the steps and answer the questions to see how Gauss was able to calculate the sum of the first 100 positive integers so quickly.

   a. The series S_{100} is shown. The same series in descending order is shown beneath it. Add the series by computing the sum of each pair of vertical, or partial sums.

$$\begin{array}{rcccccccccccc} S_{100} = & 1 & + & 2 & + & 3 & + \cdots + & 98 & + & 99 & + & 100 \\ +S_{100} = & 100 & + & 99 & + & 98 & + \cdots + & 3 & + & 2 & + & 1 \\ \hline 2S_{100} = & \underline{101} & + & \underline{101} & + & \underline{101} & + \cdots + & \underline{101} & + & \underline{101} & + & \underline{101} \end{array}$$

   b. What do you notice about each partial sum?

   Each partial sum is the same. Each partial sum is 101.

   c. How many partial sums are there in this series?

   There are 100 partial sums.

   d. Write the sum of the partial sums.

   $2S_{100} = \underline{100(101)}$

   e. To arrive at the total in part (d), you actually added each term of the series twice. How could you calculate the correct total from the sum of the partial sums, or S_{100}?

   Divide 100(101) by 2.

   f. What is S_{100}?

   $S_{100} = \underline{5050}$

   $\frac{100(101)}{2} = 50(101) = 5050$

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Grouping

Complete Question 2 as a class.

Guiding Questions for Discuss Phase

- Is the common difference added to each term in the sequence?
- How is the common difference expressed in each term of the sequence?
- What happens to the common differences when determining each partial sum?
- Are all of the partial sums equal?
- Does the product of (the number of terms) and (the sum of the first and last term) determine the sum of the finite series? Why not?

Gauss's method can be generalized for any finite arithmetic series.

2. Consider a finite arithmetic series S_n written as a sum of its terms.

$$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n$$

Complete the steps shown to determine Gauss's formula to compute any finite arithmetic series.

a. First, write S_n in terms of a_1, a_n, and the common difference d. Remember that for an arithmetic sequence, $a_n = a_1 + d(n - 1)$.

$$S_n = a_1 + (\underline{a_1 + d}) + (a_1 + 2d) + \cdots + (a_n - 2d) + (\underline{a_n - d}) + \underline{a_n}$$

b. Then, write S_n in reverse order.

$$S_n = \underline{a_n} + (a_n - d) + (\underline{a_n - 2d}) + \cdots + (a_1 + 2d) + (\underline{a_1 + d}) + a_1$$

c. Add the series, keeping the "+" and "=" signs vertically aligned.

$$\begin{aligned} S_n &= a_1 + \underline{(a_1 + d)} + \underline{(a_1 + 2d)} + \cdots + \underline{(a_n - 2d)} + \underline{(a_n - d)} + a_n \\ +S_n &= a_n + \underline{(a_n - d)} + \underline{(a_n - 2d)} + \cdots + \underline{(a_1 + 2d)} + \underline{(a_1 + d)} + a_1 \\ \hline 2S_n &= \underline{(a_1 + a_n)} + \underline{(a_1 + a_n)} + \underline{(a_1 + a_n)} + \cdots + \underline{(a_1 + a_n)} + \underline{(a_1 + a_n)} + \underline{(a_1 + a_n)} \end{aligned}$$

d. Identify each partial sum.

Each partial sum is $a_1 + a_n$.

e. Fill in the blanks to show the sum of the partial sums.

$$2S_n = \underline{n}(a_1 + \underline{a_n})$$

f. Fill in the blanks to write the formula for S_n.

$$S_n = \frac{\underline{n}(a_1 + \underline{a_n})}{2}$$

g. Describe Gauss' rule to compute any finite arithmetic series by completing the sentence.

Add the ___first___ term and the ___last___ term of the series, multiply the sum by the number of ___terms___ of the series, and divide by ___2___.

Grouping

Have students complete Question 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

- What do you need to know to use Gauss's rule?
- What is the first and last term in this situation?
- What is the number of terms in this situation?
- How did you determine the first and last term in this situation?

Problem 3

Students write an explicit formula to calculate any term of a sequence of odd whole numbers, and then use Gauss's rule to calculate the sum of the first 20 odd whole numbers. A third method is explored and results in a quadratic function which is used to verify answers and calculate the sum of the first 100 odd whole numbers. A similar activity uses a sequence of even whole numbers. Students then compare the sequences and conclude that each successive even number is one more than each successive odd number, so the sum of the first n-even numbers will be $1(n)$, or n, more than the sum of n-odd numbers.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

So, Gauss's formula to compute the first n terms of an arithmetic series is shown.

$$S_n = \frac{n(a_1 + a_n)}{2}$$

3. Use the *Toothpick Tessellation* problem situation to answer each question.

a. Use Gauss's formula to calculate the total number of toothpicks Pauley needs to complete 5 rows of his tessellation. Show your work.

$$S_5 = \frac{5(a_1 + a_5)}{2}$$
$$= \frac{5(4 + 12)}{2}$$
$$= \frac{5(16)}{2}$$
$$= 5(8) = 40$$

Pauley needs a total of 40 toothpicks to complete 5 rows of his tessellation.

b. Remember that Pauley wanted his tessellation to include a total of 18 rows. If he has a box of 350 toothpicks, does he have enough? Explain why or why not.

$$S_{18} = \frac{18(a_1 + a_{18})}{2}$$
$$= \frac{18(4 + 38)}{2}$$
$$= 9(42) = 378$$

No. A box of 350 toothpicks is not enough. Pauley needs 378 toothpicks to complete a total of 18 rows of his tessellation.

PROBLEM 3 Human Calculator, or Inspiration from Gauss?

In the previous problem, you learned a way to compute the first n-terms of any finite arithmetic series. Now, you will take a closer look at some special series of numbers.

1. Consider a sequence of odd whole numbers.

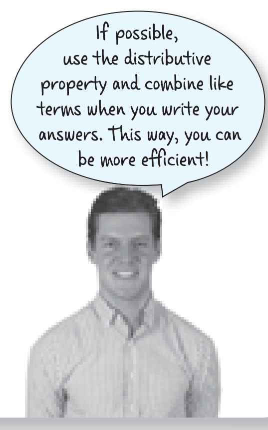
a. Write an explicit formula to determine any term of the sequence.

$a_n = a_1 + (n - 1)d$, with $a_1 = 1$ and $d = 2$

$a_n = 1 + (n - 1)2$

$a_n = 1 + 2n - 2$

$a_n = 2n - 1$



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Guiding Questions for Share Phase, Questions 1 through 3

- What is the common difference in this sequence?
- What is the first odd number?
- What information is needed to use this formula?
- When is it appropriate to use this formula?
- Is this explicit formula needed to use Gauss's rule?
- How is this explicit formula used in combination with Gauss's rule?
- How did Emma use Gauss's formula differently?
- What degree is Emma's function?
- Using the quadratic rule, what is the sum of the first 10 terms in this sequence? Can this be verified using Gauss's rule differently? How so?

b. Use Gauss's formula to calculate the sum of the first 20 odd whole numbers. What information did you need to use Gauss's formula?

$a_{20} = 2(20) - 1 = 39$

$$S_{20} = \frac{20(a_1 + a_{20})}{2} = \frac{20(1 + 39)}{2} = 10(40) = 400$$

I needed to know the first term and the last term of the series.

Emma claims that she can calculate the sum of the first 20 odd whole numbers using a different method. Emma's method does not require her to calculate the last term of the series.

2. Let's determine how Emma can perform this calculation.

a. Substitute the known value of a_1 and the algebraic expression for a_n into Gauss's formula.

$$S_n = \frac{n(a_1 + a_n)}{2} = \frac{n(1 + 2n - 1)}{2} = \frac{n(2n)}{2} = \frac{2n^2}{2} = n^2$$

b. Write your answer from part (a) using function notation. Describe the function.

$f(n) = n^2$

This is a quadratic function.

c. Use your function from part (b) to calculate the sum of the first 20 odd whole numbers. Verify that your result the same as your result in Question 1.

$f(20) = 20^2 = 400$

My result is the same as in Question 1.

3. Calculate the sum of the first 100 odd whole numbers. Then, verify your result using Gauss's formula.

When I square 100, I get 10,000. I can verify using Gauss's formula:

$a_{100} = 2(100) - 1 = 199$

$$S_{100} = \frac{100(a_1 + a_{100})}{2} = \frac{100(1 + 199)}{2} = 50(200) = 10{,}000$$

Grouping

Have students complete Questions 4 and 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 and 5

- How is the explicit formula to calculate the sum of the even whole numbers different than the formula derived to calculate the sum of the odd whole numbers?
- How do the common differences in the two explicit formulas compare to each other?
- How do the first terms in the two explicit formulas compare to each other?
- Did the alternate method result in another quadratic rule?
- How is this quadratic rule different than the previous quadratic rule that dealt with odd whole numbers?
- Can the mathematics to determine the sum of the first 100 even whole numbers be done mentally?
- How does each successive even number compare to each successive odd number?
- What is the sum of the first n-odd numbers?
- What is the sum of the first n-even numbers?

8

What about even numbers? You can use a similar process to compute the series of even whole numbers.

4. Follow the given steps to calculate the sum of the first 100 even whole numbers.

a. Write an explicit formula to calculate any term of the sequence of even whole numbers.

$a_n = a_1 + (n - 1)d$, with $a_1 = 2$ and $d = 2$

$a_n = 2 + (n - 1)2$

$a_n = 2 + 2n - 2$

$a_n = 2n$

b. Substitute the known value of a_1 and the algebraic expression for a_n into Gauss's formula.

$$S_n = \frac{n(a_1 + a_n)}{2}$$

$$= \frac{n(2 + 2n)}{2}$$

$$= \frac{2n(n + 1)}{2}$$

$$= n(n + 1) = n^2 + n$$

c. Write your answer from part (b) using function notation.

$f(n) = n^2 + n$

d. Use your function from part (c) to calculate the first 100 even whole numbers. Then, verify using Gauss's formula.

When I square 100, I get 10,000. Then, I add 100 to get 100,100.

I can verify using Gauss's formula:

$a_{100} = 2(100) = 200$

$$S_{100} = \frac{100(a_1 + a_{100})}{2}$$

$$= \frac{100(2 + 200)}{2}$$

$$= 50(202) = 10{,}100$$

5. Compare the function for the series of even whole numbers with the function for the series of odd whole numbers. What makes them different? Explain why you think the difference exists.

Answers will vary.

Student responses should mention that each successive even number is one more than each successive odd number. So, the sum of the first n-even numbers will be $1(n)$, or n, more than the sum of n-odd numbers.

Problem 4

Information is given that enables students to determine a seating arrangement for a concert. They will sketch a seating chart and use the explicit formula and Gauss's rule to determine the number of chairs in the first five rows of concert venue. Given 500 chairs, students then determine how many rows will be needed and the depth of the seating area. Then with no row having more than 40 chairs, they determine the maximum number of rows possible, and the maximum number of people that can be seated.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Is the total seating area an important factor when determining the number of rows of chairs? Why?
- Is the number of people attending the concert an important factor when determining the number of rows of chairs? Why?
- Is the audience's view from the chairs an important factor when determining the number of rows of chairs? Why?
- What is the width of the stage?
- How much space is needed between each chair?
- What is the width of a chair?

PROBLEM 4 "Chair"-ity Case

You are in charge of setting up for your high school band's annual Spring concert. The concert will be held outdoors on the school soccer field, and one of your duties is to arrange the seating for the show.

You have gathered the following information.

- The stage is 20 feet wide.
- The first row of chairs will be about the same width as the stage.
- Each successive row of chairs will have three more chairs than the previous row. This way, the chairs are offset so that each person does not have a chair directly in front of them for better viewing.
- Each chair is 1.5 feet wide and 1.5 feet deep.
- There needs to be 0.5 foot of spacing in between the chairs within a row so that the audience can sit comfortably.
- In order to have enough room for people to walk through the rows, there needs to be 4 feet of space in between each row, from the back of one chair to the back of the other chair.

Use the given information to answer each question.

1. What factors do you need to consider when determining how many rows of chairs there could be?

   Answers will vary.

   Sample answers may include, but are not limited to, the following: total seating area, number of people attending the concert, total number of chairs available to you, the audience's view from the chairs, etc.

2. How many chairs are in the first row? Explain your reasoning.

   The stage is 20 feet wide, so the first row of seats should take up a width of about 20 feet. There needs to be 0.5 feet of spacing between the chairs, and each chair is 1.5 feet wide. So, the first row will include 10 chairs, for a total width of $10(1.5) + 9(0.5) = 19.5$ feet.

Guiding Questions for Share Phase, Question 3

- How many chairs are in the second row?
- How many rows of chairs are in your sketch?
- Do your classmates have the same number of rows of chairs in their sketches?
- Is there more than one correct sketch?

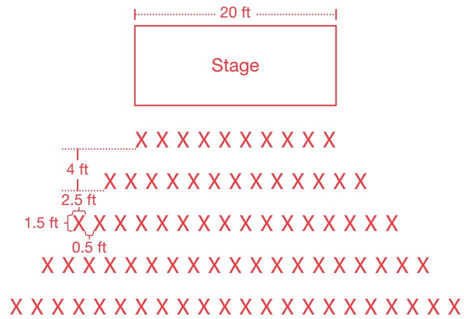
Grouping

Have students complete Questions 4 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- Which formula is used to determine the number of chairs in the gold circle section?
- Why is an explicit formula needed to use Gauss's rule?
- What is the first term in the explicit formula?
- What is the common difference in the explicit formula?
- What expression is determined by the explicit formula?
- Why do you need to know the number of chairs in the fifth row?
- What is the first term used in Gauss's rule?
- What is the last term used in Gauss's rule?

3. Sketch a seating chart that includes the given information and dimensions.
Note: The number of rows of chairs will vary.



Answer each question based on the additional given information. Show all your work.
Note: Solving methods and answers will vary.

4. Suppose that the first 5 rows of chairs make up the "gold circle" section.

a. How many chairs are in the gold circle section?

An explicit formula for this sequence is shown.

$a_n = a_1 + (n - 1)d$

$a_n = 10 + (n - 1)3$

$a_n = 10 + 3n - 3$

$a_n = 3n + 7$

The number of chairs in the fifth row is shown.

$a_5 = 3(5) + 7$

$= 15 + 7 = 22$

Then, I can use Gauss's formula to calculate the sum of the chairs in the first 5 rows.

$S_n = \frac{n(a_1 + a_n)}{2}$

$S_5 = \frac{5(a_1 + a_5)}{2}$

$= \frac{5(10 + 22)}{2} = \frac{5(32)}{2} = 5(16) = 80$

So, 80 chairs are in the gold circle section.

b. How many feet deep is the gold circle section?

There needs to be 4 feet of space from the back of the chair of one row to the back of the chair in the next row. There are 5 rows in the gold circle section, so it is 5(4) = 20 feet deep from the back of the first row to the back of the fifth row. Each chair is 1.5 feet deep, so I need to add this amount to account for the depth of the first row.

So, the gold circle section is a total of 20 + 1.5 = 21.5 feet deep.

- How much space needs to be between the back of the chair of one row to the back of the chair in the next row?
- How many rows are in the gold circle section?
- How much space is between the back of the first row to the back of the fifth row?
- How deep is each chair?

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Guiding Questions for Share Phase, Question 5

- What formula is used to determine the number of rows needed to accommodate 500 chairs?
- Why does Gauss's rule result in a quadratic equation in this situation?
- How did you solve the quadratic equation? Is it factorable?
- Does a negative answer make sense in this situation?
- How can you create 15.64 rows of chairs?
- When determining the depth, do you need to account for 15 or 16 rows of chairs?

5. Suppose that you need a total of 500 chairs for the concert.

a. How many rows will you need with this number of chairs?

$S_n = \frac{n(a_1 + a_n)}{2}$, where $S_n = 500$, $a_1 = 10$, and $a_n = 3n + 7$

$$500 = \frac{n(10 + 3n + 7)}{2}$$

$$500 = \frac{10n + 3n^2 + 7n}{2}$$

$$1000 = 10n + 3n^2 + 7n$$

$$3n^2 + 17n - 1000 = 0$$

$$n = \frac{-17 \pm \sqrt{17^2 - 4(3)(-1000)}}{2(3)}$$

$$n = \frac{-17 \pm \sqrt{12{,}289}}{6}$$

$$n \approx 15.64, -21.31$$

The number of rows of chairs cannot be negative, so $n \approx 15.64$.

This means that I can arrange 15 full rows and create a partial 16th row.

b. How deep is the seating area with this number of chairs?

When determining the depth, I need to account for 16 rows.

There are 16 rows, so the seating area is 16(4), or 64 feet deep from the back of the first row to the back of the 16th row. Each chair is 1.5 feet deep, so I need to add this amount to account for the depth of the first row.

So, the seating area is a total of 64 + 1.5 = 65.5 feet deep.

Guiding Questions for Share Phase, Question 6

- What inequality best represents this situation?
- Is $3n + 7 < 40$ or is $3n + 7 \leq 40$? How do you know?
- Which row has 40 chairs?
- What is the first and last term used in Gauss's rule in this situation?

8

6. Suppose that no row can have more than 40 chairs.

a. What is the maximum number of rows possible?

The number of chairs in n-rows is given by the explicit formula, $a_n = 3n + 7$. To determine the maximum number of rows that are possible if no row can have more than 40 chairs, solve the following inequality.

$$3n + 7 \leq 40$$
$$3n \leq 33$$
$$n \leq 11$$

So, the maximum number of rows possible is 11, because the 11th row has 40 chairs.

b. What is the maximum number of people that can be seated?

The maximum number of rows possible is 11, and there are 40 chairs in the 11th row. The total number of chairs in 11 rows is the sum of the series when $n = 11$.

$$S_{11} = \frac{11(10 + 40)}{2}$$
$$= \frac{11(50)}{2}$$
$$= \frac{550}{2}$$
$$= 275$$

A maximum of 275 people can be seated.

Be prepared to share your solutions and methods.

8

Check for Students' Understanding

A sequence is defined as:

$a_n = 8(n - 1) + 20$

$n = 1, 2, 3 \ldots$

Determine the sum of the first 15 terms of this sequence.

This is an arithmetic sequence.

The 15th term is $8(15 - 1) + 20 = 132$.

$$S_n = \frac{n(a_1 + a_n)}{2}$$

$$S_{15} = \frac{15(20 + 132)}{2} = 1140$$

The sum of the first 15 terms is 1140.

8.3

I Am Having a Series Craving (For Some Math)!

Geometric Series

LEARNING GOALS

In this lesson, you will:

- Generalize patterns to derive the formula for the sum of a finite geometric series.
- Compute a finite geometric series.

KEY TERM

- geometric series

ESSENTIAL IDEAS

- A geometric series is the sum of the terms of a geometric sequence.
- The formula to compute any geometric series is $S_n = \frac{g_n(r) - g_1}{r - 1}$, where g_n is the last term, r is the common ratio, and g_1 is the first term.
- Another method to compute any geometric series is $S_n = \frac{g_1(r^n - 1)}{r - 1}$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.
   a. Interpret parts of an expression, such as terms, factors, and coefficients.

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

F-BF Building Functions

Build a function that models a relationship between two quantities

2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

8

Overview

The term geometric series is defined. Students will explore different methods to compute any geometric series. Worked examples are provided and Euclid's Method is introduced. Next students use the pattern generated from repeated polynomial long division to write a formula for the sum of any geometric series, $1 + r + r^2 + r^3 + \cdots + r^{n-1}$, where n is the number of terms in the series, r is the common ratio, and g_1 is 1: $S_n = \frac{r^n - 1}{r - 1}$. Student work is presented to show that any series where g_1 does not equal 1 can be rewritten by factoring out a greatest common factor. A second formula to compute any geometric series is derived, $S_n = \frac{g_1(r^n - 1)}{r - 1}$. A worked example is provided to verify the two formulas are equivalent. Students then rewrite geometric series using summation notation, and compute geometric series. Finally, they use formulas to compute geometric series in a problem situation that involves salary comparisons and a situation that involves basketballs used in a tournament.

Warm Up

1. Write the first 6 terms of the geometric sequence determined by $r = -8$ and $a_1 = 1$.

1, −8, 64, −512, 4096, −32,768

2. Calculate the sum of the sequence in Question 1.

The sum of the sequence is −29,127.

3. What method did you use to calculate the sum of the sequence?

Answers will vary.

Grouping or mental math skills can be used to make the process more efficient.

8

8.3

I Am Having a Series Craving (For Some Math)!

Geometric Series

LEARNING GOALS

In this lesson, you will:

- Generalize patterns to derive the formula for the sum of a finite geometric series.
- Compute a finite geometric series.

KEY TERM

- geometric series

The art that is produced in a culture often reflects the peoples' social values, struggles, and important events over a given time period. While it is generally not considered one of the great art forms of our time, television drama is an art that regularly reflects current events and social issues.

Consider *Mission Impossible*, a spy series which brought millions of viewers the secret assignments of a group of government agents battling dictators around the globe. It's no accident that this series was hugely popular in the 1960's, a time of heightened Cold War anxieties. During the 1980s, a time when more women entered the work force, *Cagney and Lacey* featured a career-focused, single mother battling crime. During the 2000s, *West Wing* focused on political scandals, terrorism, and other foreign affairs issues that were in the news during that period.

What are some of the pressing current events right now? Are they reflected in any popular television series you watch?

Problem 1

The term geometric series is defined. Through worked examples and related questions, students arrive at the formula to compute the sum of any geometric series. Next, they prove Euclid's Method, $S_n = \frac{g(r^n - 1)}{r - 1}$ for a geometric series in the form $gr^0 + gr^1 + gr^2 + gr^3 + \cdots + gr^{n-1}$ where g is a constant, r is the common ratio, and n is the number of terms. The formula is used to determine several series.

Grouping

- Ask a student to read the definition and information. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What is the common ratio in Paul's series?
- What is the common ratio in Stella's series?
- What is the common ratio in Julian's series?
- What is the common ratio in Henry's series?
- Did Theresa multiply the last term in each series by the common ratio?

PROBLEM 1 Geometric Series Episode 1: The Rise of Euclid

A **geometric series** is the sum of the terms of a geometric sequence. Recall, that the sequence 1, 3, 9, 27, 81 is a geometric sequence because the ratio of any two consecutive terms is constant. Adding the terms creates the geometric series 1 + 3 + 9 + 27 + 81.

Theresa raises her hand and claims that she has a "trick" for quickly calculating the sum of any geometric series. She asks members of the class to write any geometric series on the board. She boasts that she can quickly tell them how to determine the sum without adding all of the terms. Several examples are shown.

Paul: "OK, so prove it! What is the sum of 1 + 3 + 9 + 27 + 81 + 243 + 729?"	**Theresa:** "Multiply 729(3) and subtract 1. Then divide by 2."
Stella: "What is 5 + 20 + 80 + 320 + 1280 + 5120?"	**Theresa:** "I will have the answer if I multiply 5120(4), subtract 5, and then divide by 3."
Julian: "Let me see . . . How about 10 + 50 + 250 + 1250?"	**Theresa:** "No problem. Multiply 1250(5), subtract 10, and then divide by 4."
Henry: "Hmmm . . . I bet I can stump you with 10 + (−20) + 40 + (−80) + 160."	**Theresa:** "Pretty sneaky with the negatives, Henry, but the method still works. Multiply 160(−2) and subtract 10. This time divide by −3."

1. Verify that Theresa is correct for each series.

   I added each series and the result was equal to Theresa's calculations.

2. What is Theresa's "trick"? Describe in words how to calculate the sum of any geometric sequence.

   Theresa multiplied the last term by the common ratio. She then subtracted the value of the first term. Finally, she divided by 1 less than the common ratio.

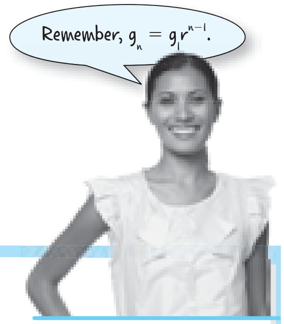
- Did Theresa subtract the first term from the product of the last term and common ratio in each calculation?
- What is the relationship between the number Theresa divided by and the common ratio in each calculation?

Grouping

Ask a student to read the information and worked example. Complete Question 4 as a class.

3. Use Theresa's "trick" to calculate $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128$. Show all work and explain your reasoning.

The common ratio is 2. The first term is 1. The last term is 128. Using Theresa's "trick," the sum is $\frac{(2)(128) - 1}{2 - 1} = 255$



Theresa's "trick" really isn't a trick. It is known as Euclid's Method. An example of this method, along with a justification for each step, is shown.

Compute $\sum_{i=1}^{5} 3^{i-1}$.

$S_5 = 1 + 3 + 9 + 27 + 81$

- The common ratio is 3.

$$3S_5 = \quad 3 + 9 + 27 + 81 + 243$$
$$S_5 = 1 + 3 + 9 + 27 + 81$$

- Write $3S_n$ above the original series. Multiply each term of the original series by the common ratio. Line up each product above the original series.

$$3S_5 = \quad 3 + 9 + 27 + 81 + 243$$
$$-S_5 = -(1 + 3 + 9 + 27 + 81)$$

- Subtract to determine $3S_n - S_n = 2S_n$.

$$2S_5 = -1 + 243$$
$$\frac{2S_5}{2} = \frac{242}{2}$$
$$S_5 = 121$$

- Divide by 2.

In all of the examples, Theresa knew that she could calculate each sum by first multiplying the last term by the common ratio and subtracting the first term. Then she could divide that quantity by one less than the common ratio.

In other words, $S_n = \frac{(\text{Last Term})(\text{Common Ratio}) - (\text{First Term})}{(\text{Common Ratio} - 1)}$.

4. Analyze the worked example.

a. In the worked example, why multiply both sides of the equation by 3? Does the algorithm still work if you multiply by a different number? Explain your reasoning.

To use Euclid's Method, the algorithm must multiply by the common ratio because that predicts the next term. This makes it possible for the terms to subtract to zero. It will not work if you multiply by another constant because the terms will not be the same.

8

Grouping

Have students complete Question 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- What is the first term in the series?
- What is the common ratio?
- What is the last term?
- How did you use Euclid's Method to compute the geometric series?

Problem 2

Students will use the pattern generated from repeated polynomial long division to write a formula for the sum of any geometric series, $1 + r + r^2 + r^3 + \cdots + r^{n-1}$, where n is the number of terms in the series, r is the common ratio, and g_1 is 1: $S_n = \dfrac{r^n - 1}{r - 1}$. Student work is presented to show that any series where g_1 does not equal 1 can be rewritten by factoring out a greatest common factor. A second formula to compute any geometric series is derived, $S_n = \dfrac{g_1(r^n - 1)}{r - 1}$. A worked example is provided to verify the two formulas are equivalent. Students will rewrite geometric series using summation notation, and compute geometric series.

b. Why do you always divide by one less than the common ratio?

After multiplying each term in the series by the common ratio, it must be subtracted from the original series. This results in $(r - 1)S_n$. Dividing by $r - 1$ isolates S_n.

The formula to compute any geometric series becomes $S_n = \dfrac{g_n(r) - g_1}{r - 1}$, where g_n is the last term, r is the common ratio, and g_1 is the first term.

5. Apply Euclid's Method to compute each.

a. $1 + 10 + 100 + \cdots + 1{,}000{,}000$

$S_n = \dfrac{(1{,}000{,}000)(10) - 1}{10 - 1}$

$= 1{,}111{,}111$

b. $10 + 20 + 40 + 80 + 160 + 320$

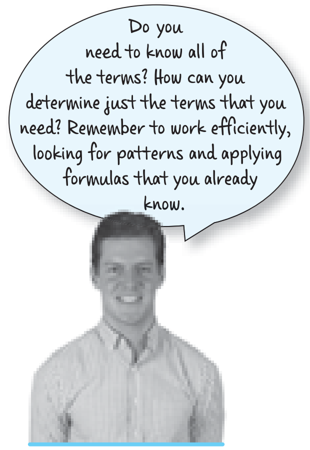
$S_6 = \dfrac{(320)(2) - 10}{2 - 1}$

$= 630$

c. $\displaystyle\sum_{k=1}^{8} 5^{k-1}$

$S_8 = \dfrac{5^7 \cdot 5 - 5^0}{5 - 1} = \dfrac{5^8 - 1}{4}$

$= 97{,}656$



d. A sequence with 9 terms, a common ratio of 2, and a first term of 3.

The 9th term can be calculated using the explicit formula $g_9 = 3 \cdot 2^{9-1} = 768$.

$S_9 = \dfrac{768 \cdot 2 - 3}{2 - 1} = 1533$.

PROBLEM 2 Return of Long Division: The Pattern Strikes Back

Recall previously you used long division to determine each quotient:

Polynomial Long Division	→	Rewritten Using the Reflexive and Commutative Properties of Equality
Example 1		
$\dfrac{r^3 - 1}{r - 1} = r^2 + r + 1$	→	$1 + r + r^2 = \dfrac{r^3 - 1}{r - 1}$
Example 2		
$\dfrac{r^4 - 1}{r - 1} = r^3 + r^2 + r + 1$	→	$1 + r + r^2 + r^3 = \dfrac{r^4 - 1}{r - 1}$
Example 3		
$\dfrac{r^5 - 1}{r - 1} = r^4 + r^3 + r^2 + r + 1$	→	$1 + r + r^2 + r^3 + r^4 = \dfrac{r^5 - 1}{r - 1}$

Grouping

Ask a student to read the information and discuss as a class.

Grouping

- Ask a student to read the information and discuss as a class.
- Have students complete Questions 1 through 3 as a class and discuss the worked example.

8

Each Example represents a geometric series, where r is the common ratio and $g_1 = 1$. Each geometric series can be written in summation notation.

Example 1: $n = 3$ $\sum_{i=1}^{3} r^{i-1}$ or $\sum_{i=0}^{2} r^{i}$

Example 2: $n = 4$ $\sum_{i=1}^{4} r^{i-1}$ or $\sum_{i=0}^{3} r^{i}$

1. For each Example, explain why the power of the common ratio in the summation notation is different, yet still represents the series.

In Example 1, the indexes for both summation notations represent $n = 3$. When $i = 1$, the power of the common ratio must be $i - 1$ to produce $g_1 = 1$. When $i = 0$, the power of the common ratio must be i to produce $g_1 = 1$.

In Example 2, the indexes for both summation notations represent $n = 4$. When $i = 1$, the power of the common ratio must be $i - 1$ to produce $g_1 = 1$. When $i = 0$, the power of the common ratio must be i to produce $g_1 = 1$.

2. Identify the number of terms in the series in Example 3, and then write the series in summation notation.

$n = 5$ $\sum_{i=1}^{5} r^{i-1}$ or $\sum_{i=0}^{4} r^{i}$

3. Use the pattern generated from repeated polynomial long division to write a formula to compute any geometric series $1 + r + r^2 + r^3 + \cdots + r^{n-1}$ where n is the number of terms in the series, r is the common ratio, and $g_1 = 1$.

$\sum_{i=0}^{n} r^{i} = \underline{\dfrac{r^n - 1}{r - 1}}$

You can show a proof of $S_n = \dfrac{r^n - 1}{r - 1}$ where S_n is a series in the form $r^0 + r^1 + r^2 + r^3 + \cdots + r^{n-1}$ with n-terms and a common ratio r.

$$S_n = r^0 + r^1 + r^2 + r^3 + \cdots + r^{n-1}$$

$$\begin{aligned} rS_n &= \quad\; r^1 + r^2 + r^3 + \cdots + r^{n-1} + r^n \\ S_n &= r^0 + r^1 + r^2 + \cdots + r^{n-2} + r^{n-1} \end{aligned}$$

$\times r$

- Write rS_n above the original series. Multiply each term by r. Line up each product above the original series.

$$\begin{aligned} rS_n &= \quad\quad\; \cancel{r^1} + \cancel{r^2} + \cancel{r^3} + \cdots + \cancel{r^{n-1}} + r^n \\ -\;\; S_n &= -(1 + \cancel{r^1} + \cancel{r^2} + \cancel{r^3} + \cdots + \cancel{r^{n-1}}) \\ \hline rS_n - S_n &= -1 + r^n \\ S_n(r - 1) &= r^n - 1 \end{aligned}$$

- Subtract $rS_n - S_n$. Eliminate terms that subtract to 0.

$$\frac{S_n(r-1)}{(r-1)} = \frac{(r^n - 1)}{(r - 1)}$$

- Divide by $(r - 1)$.

$$S_n = \frac{r^n - 1}{r - 1}$$

8

Grouping

Have students complete Question 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- Which formula did you use to compute the series? Why?
- What is the same for each series?

4. Identify the number of terms, the common ratio, and g_1 for each series. Then compute each.

a. $1 + 2^1 + 2^2 + 2^3 + 2^4$

$n = 5$

$r = 2$

$g_1 = 1$

$S_5 = \frac{2^5 - 1}{2 - 1} = 31$

b. $1 + 5 + 25 + 125 + 625$

$n = 5$

$r = 5$

$g_1 = 1$

$S_5 = \frac{5^5 - 1}{5 - 1}$

$= 781$

c. $1 + (-2) + 4 + (-8) + 16 + (-32)$

$n = 6$

$r = -2$

$g_1 = 1$

$S_6 = \frac{(-2)^6 - 1}{(-2) - 1}$

$= -21$

Grouping

Have students complete Questions 5 and 6 as a class and discuss the worked example.

Guiding Questions for Discuss Phase

- What is Angus's solution?
- What is Perry's solution?
- Did Angus and Perry arrive at the same solution?
- Which method do you prefer to compute the series?
- How did Angus use summation notation?
- How did Perry use summation notation?
- What is the same about each representation?
- What is different about each representation?
- Which way do you prefer to use summation notation?

8

5. Angus and Perry each wrote the geometric series 7 + 14 + 28 + 56 + 112 + 224 + 448 + 896 in summation notation and then computed the sum.

Angus

I know that $g_n = g_1r^{n-1}$. The number of terms is 8, the common ratio is 2, and the first term is 7, so I can write the series as $\sum_{i=1}^{8} 7 \cdot 2^{i-1}$.

I know the last term is 896, so I can use Euclid's Method to compute the sum.

$\frac{896 \cdot 2 - 7}{2 - 1}$

Perry

I can rewrite the series as

$7(1 + 2 + 4 + 8 + 16 + 32 + 64 + 128)$.

I know the common ratio is 2, so I can rewrite the series using powers as

$7(2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7)$.

The number of terms is 8, so I can write the series in summation notation as

$7\sum_{i=1}^{8} 2^{i-1}$.

Then, I can compute the series as $7\left(\frac{2^8 - 1}{2 - 1}\right)$.

Verify that both methods produce the same sum.

$\frac{896 \cdot 2 - 7}{2 - 1} = 1785$

$7\left(\frac{2^8 - 1}{2 - 1}\right) = 1785$

The formula to compute a geometric series that Perry used is $S_n = \frac{g_1(r^n - 1)}{r - 1}$.

Recall Euclid's Method to compute a geometric series is $S_n = \frac{g_n(r) - g_1}{r - 1}$.

You can use the fact that $g_n = g_1r^{n-1}$ to verify that these two formulas are equivalent.

$S_n = \frac{g_n(r) - g_1}{r - 1}$	Given Euclid's Method.
$= \frac{g_1r^{n-1}(r) - g_1}{r - 1}$	Substitute $g_n = g_1r^{n-1}$.
$= \frac{g_1r^n - g_1}{r - 1}$	Perform multiplication.
$= \frac{g_1(r^n - 1)}{r - 1}$	Factor out g_1.

8

Grouping

Have students complete Questions 7 and 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 7 and 8

- How many terms are in the series?
- How did you decide which indexes to use?
- How did your choice of indexes affect the power of your common ratio?
- Is there more than one way to represent each series using summation notation?
- How does the summation notation tell you how many terms are in the series?
- What is the common ratio?
- Which formula did you use to compute the series? Why?

6. When is it appropriate to use each formula?

To use $S_n = \frac{g_n(r) - g_1}{r - 1}$, I need to know the first term, last term, and the common ratio of the series.

To use $S_n = \frac{g_1(r^n - 1)}{r - 1}$, I need to know the number of terms, the first term, and the common ratio of the series.

7. Rewrite each series using summation notation.

a. 4 + 12 + 36 + 108 + 324

$4\sum_{i=0}^{4} 3^i$ or $\sum_{i=1}^{5} 4 \cdot 3^{i-1}$

b. 64 + 32 + 16 + 8 + 4 + 2 + 1

$64\sum_{i=0}^{6} \frac{1}{2}^i$

or

I can rewrite the pattern in ascending order and write the series as $\sum_{i=0}^{6} 2^i$.

8. Compute each geometric series.

a. $\sum_{i=1}^{4} 6^{i-1}$

$g_1 = 1; r = 6; n = 4$

$S_4 = \frac{6^4 - 1}{6 - 1}$

$= \frac{1295}{5}$

$= 259$

b. $10\sum_{i=0}^{4} 3^i$

$g_1 = 10; r = 3; n = 5$

$S_5 = 10\left(\frac{3^5 - 1}{5 - 1}\right)$

$= 10\left(\frac{242}{4}\right)$

$= 605$

c. $6\sum_{i=0}^{4} \left(\frac{1}{3}\right)^i$

$g_1 = 6; r = \frac{1}{3}; n = 5$

$S_5 = 6\left(\frac{\left(\frac{1}{3}\right)^5 - 1}{\frac{1}{3} - 1}\right)$

$= 6\left(\frac{\frac{1}{243} - 1}{\frac{-2}{3}}\right)$

$= 6\left(\frac{-242}{243} \cdot \frac{-3}{2}\right)$

$= 6\left(\frac{121}{81}\right)$

$= 8\frac{26}{27}$

Grouping

Have students complete Question 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 9

- When completing the table, is it easier to determine the terms of the geometric series or to be given the terms of the geometric series?
- When completing the table, is it easier to determine the summation notation of the geometric series or to be given the summation notation of the geometric series?
- How do the output values compare to each other?
- Is there a common difference?
- How did you determine $f(5)$?
- What method was used to determine the 10th term of the geometric sequence?
- What method was used to determine the sum of the geometric series?
- Do any two consecutive terms in a geometric sequence contain a factor of r?
- When the ratio of any two consecutive terms in a geometric sequence is determined, is the r value factored out?

8

9. Analyze the table of values.

x	$f(x)$	$\frac{f(x+1)}{f(x)}$
0	3	1.5
1	4.5	1.5
2	6.75	1.5
3	10.125	1.5
4	15.1875	1.5
5	22.78125	1.5

a. Complete the table.

See the table.

b. Describe any patterns that you notice.

The output values increase by a factor of 1.5.

The column $\frac{f(x+1)}{f(x)}$ is always 1.5.

c. Assume the geometric sequence continues, determine $f(0) + f(1) + \cdots + f(9)$. Show all work and explain your reasoning

The 10th term is $3 \cdot 1.5^9 \approx 115.33$

The sum is $\frac{(115.33)(1.5) - 3}{1.5 - 1} \approx 339.99$

d. Explain why the ratio of any two consecutive terms in a geometric sequence is always a constant.

Answers will vary.

I must multiply a term in the sequence by the common ratio, r, to determine the next term. Reversing this operation and dividing will result in the common ratio.

8

Problem 3

Students use formulas to compute geometric series in a problem situation that involves salary comparisons and a situation that involves basketballs used in a tournament.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1, part (a)

- Are the sequences in this situation geometric?
- What is the common ratio associated with Nothing's Shocking salaries?
- What is the common ratio associated with High Voltage salaries?
- When determining the salary in Year 10 at Nothing's Shocking, why is \$40,000 multiplied by 1.06^9 rather than 1.06^{10}?
- When determining the salary in Year 10 at High Voltage, why is \$46,000 multiplied by 1.04^9 rather than 1.04^{10}?

PROBLEM 3 Making Choices

1. Jane analyzes the salary schedule for the same position at two different electrical engineering companies, Nothing's Shocking and High Voltage. The salary schedules for the first 5 years are provided with promises from each company that the rate of salary increase will be the same over time.

Time (years)	Nothing's Shocking Salary (\$)	High Voltage Salary (\$)
1	40,000	46,000
2	42,400	47,840
3	44,944	49,754
4	47,641	51,744
5	50,499	53,814

a. What is the salary in year 10 for each company? Show all work and explain your reasoning.

Nothing's Shocking offers a salary increase of 6% every year based on the salary at the end of each year.

Nothing's Shocking pays $\$40,000 \cdot 1.06^9 \approx \$67,579$.

High Voltage offers a salary increase of 4% every year based on the salary at the end of each year.

High Voltage pays $\$46,000 \cdot 1.04^9 \approx \$65,472$.

Guiding Questions for Share Phase, Question 1, parts (b) and (c)

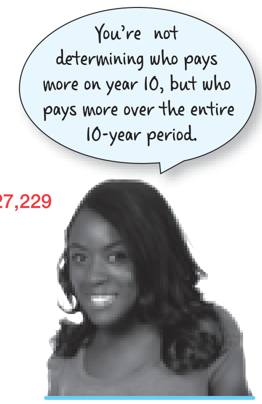
- What formula or method is used to determine the better salary over a 10-year period?
- What formula or method is used to determine the better salary over a 30-year period?

b. Assuming all other factors are equal, which company offers the better salary over a 10-year period? Show all work and explain your reasoning.

The High Voltage salary is the better choice because it pays more over a 10-year career. The sum over the 10-year period is:

Nothing's Shocking pays $\dfrac{(67{,}579)(1.06) - 40{,}000}{1.06 - 1} = \$527{,}229$

High Voltage pays $\dfrac{(65{,}472)(1.04) - 46{,}000}{1.04 - 1} = \$552{,}272$



c. What would be the difference in total career salary if you choose one company over the other? Assume a 30-year career. Show all work and explain your reasoning.

Nothing's Shocking pays \$582,428 more over a 30-year career.

On year 30 the pay is:

Nothing's Shocking pays $\$40{,}000 \cdot 1.06^{29} \approx \$216{,}736$.

High Voltage pays $\$46{,}000 \cdot 1.04^{29} \approx \$143{,}458$

The sum over 30 years is:

Nothing's Shocking pays $\dfrac{(216{,}736)(1.06) - 40{,}000}{1.06 - 1} \approx \$3{,}162{,}336$

High Voltage pays $\dfrac{(143{,}458)(1.04) - 46{,}000}{1.04 - 1} \approx \$2{,}579{,}908$

Guiding Questions for Share Phase, Question 2

- How is the sequence written for the basketball situation?
- What is the first term of the sequence? What does it mean with respect to this problem situation?
- What is the last term of the sequence? What does it mean with respect to this problem situation?
- What is the common ratio associated with this problem situation?
- How many terms are in this sequence?

2. A single elimination basketball tournament begins with 128 games in the first round. Each round eliminates half of the teams until an overall winner is decided. The tournament sponsor needs to purchase a new ball for every game that will be played throughout the tournament. How many basketballs must the sponsor purchase? Explain your reasoning.

The sequence is written 128, 64, 32, 16, 8, 4, 2, 1. The common ratio is $\frac{1}{2}$. The number of terms is 8. Therefore, the sum is $\dfrac{128\left(\left(\frac{1}{2}\right)^8 - 1\right)}{\frac{1}{2} - 1} = 255$.

The formula says that there will be 255 games played in the tournament. However, the last term of my sequence is 1 which doesn't make sense in terms of this situation. So, the sponsor must purchase 255 − 1, or 254 total basketballs.

Be prepared to share your solutions and methods.

8

Check for Students' Understanding

Compute the geometric series.

$3 + 15 + 75 + 375 + 1875 + 9375 + 46{,}875$

$$S_n = \frac{g_n(r) - g_1}{r - 1}$$

$$S_7 = \frac{46{,}875(5) - 3}{5 - 1}$$

$$S_7 = \frac{234{,}372}{4}$$

$$S_7 = 58{,}593$$

8.4

These Series Just Go On . . . And On . . . And On . . .

Infinite Geometric Series

LEARNING GOALS

In this lesson, you will:

- Write a formula for an infinite geometric series.
- Compute an infinite geometric series.
- Draw diagrams to model infinite geometric series.
- Determine whether series are convergent or divergent.
- Use a formula to compute a convergent infinite geometric series.

KEY TERMS

- convergent series
- divergent series

ESSENTIAL IDEAS

- A convergent series is an infinite series that has a finite sum.
- A divergent series is an infinite series that does not have a finite sum. The sum is infinity.
- The formula to compute a convergent geometric series is $S = \frac{g_1}{1 - r}$, where S represents the sum of the series, r is the common ratio, and g_1 is the first term.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

8

Overview

The terms convergent series and divergent series are introduced. A formula to compute a convergent geometric series is given. Students will explore several infinite geometric series and determine the sum of the series. They conclude that when the common ratio is greater than 1, the sum of the infinite geometric series is infinite and the series is divergent. In addition, students conclude that when the common ratio is between 0 and 1, the sum of the infinite geometric series is finite and the series is convergent.

Warm Up

1. Take an 8.5″ by 11″ sheet of paper and hold it so it has a landscape orientation.
   a. Fold it in half making a vertical fold, fold it in half again making a horizontal fold, and continue alternating between vertical and horizontal folds three more times.
   b. Open up the paper and label a section of the paper to represent each of the following fractions: $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, and $\frac{1}{32}$.

      Note that labeled sections should not overlap one another. Draw boundaries along the folds to delineate the fractional parts.

2. The fractions you labeled form a sequence. Do they have a common difference or a common ratio?
   The fractions have a common ratio. The common ratio is $\frac{1}{2}$.

3. Do the fractions you labeled represent an arithmetic sequence or a geometric sequence?
   The fractions represent a geometric sequence.

4. What are the next two terms of this sequence?
   The next two terms of the sequence are $\frac{1}{64}$ and $\frac{1}{128}$.

8

8.4

These Series Just Go On . . . And On . . . And On . . .

Infinite Geometric Series

LEARNING GOALS

In this lesson, you will:

- Write a formula for an infinite geometric series.
- Compute an infinite geometric series.
- Draw diagrams to model infinite geometric series.
- Determine whether series are convergent or divergent.
- Use a formula to compute a convergent infinite geometric series.

KEY TERMS

- convergent series
- divergent series

Infinity is a concept that philosophers and mathematicians have struggled with for centuries. Infinity is a very abstract idea. How can something be limitless? What does it mean for something to go on forever?

The following quote is from an Indian philosophical text dating back to the 4^{th} or 3^{rd} century B.C.

If you remove a part from infinity or add a part to infinity, still what remains is infinity.

How can this be so?

What does infinity mean to you?

8

Problem 1

The first three terms of an infinite sequence are given. Students will sketch the next two terms to model the sequence and identify the sequence as geometric, writing an explicit formula to represent the sequence. They then complete a table listing the term number of the sequence, the sum of the first *n* terms as a fraction, and the sum of the first *n* terms as a decimal. Students conclude as the term number increases, the sum of the terms approach the value of 1. Students show two different formulas used to calculate the first *n* terms of the sequence are equivalent. They then conclude that this infinite geometric sequence has a finite sum.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

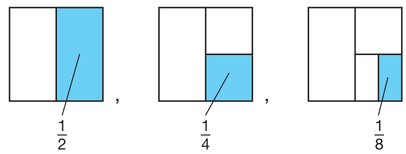
Guiding Questions for Share Phase, Questions 1 and 2

- How was the third term used to sketch the fourth term?
- How was the fourth term used to sketch the fifth term?
- Does this sequence have a common difference or a common ratio?

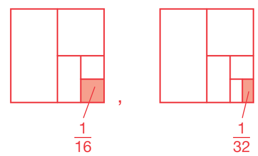
PROBLEM 1 Hang On Bessie, We're Almost There

Previously, you calculated sums of finite series. What if a series was infinite? Let's see if there is a way to calculate the sum of an infinite series.

The first three terms of an infinite sequence are represented by the figures shown. In this sequence, each square represents a unit square, and the shaded part represents area.



1. Sketch the next two figures to model this sequence, and write the numbers that correspond to each term.



2. Is this sequence arithmetic, geometric, or neither? Explain how you know. If possible, write an explicit formula for the sequence.

   This sequence is geometric. I know because each term is multiplied by $\frac{1}{2}$ to determine the next term.

$$g_n = g_1 \cdot r^{n-1}$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$

$$= \left(\frac{1}{2}\right)^{1+n-1}$$

$$= \left(\frac{1}{2}\right)^{n}$$

- What is the common ratio in this geometric sequence?
- What information is needed to write an explicit formula to represent this sequence?

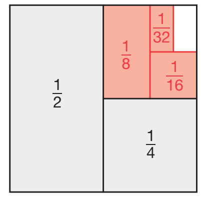
Grouping

Have students complete Questions 3 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 through 5

- As additional terms are shaded in the unit square, will the square be eventually filled?
- Is the value of the shaded region approaching the value of 1 as the number of terms increase?
- How were the decimal equivalences of the fractions determined in the table?
- Is the explicit formula needed to determine the sum of the first 2, 3, 4, or 5 terms?
- What algebraic expression best represents the denominator of each fraction with respect to the term number, *n*?
- What algebraic expression best represents the numerator of each fraction with respect to the term number, *n*?
- How was the sum of the first 10 terms determined?
- How was the sum of the first 25 terms determined?

3. Consider the series, or sum, of the first two terms of this infinite sequence. The sum of the first two terms can be modeled with a diagram, as shown.



Continue shading the diagram to represent the sum of the first five terms of the series. What happens to the total area that is shaded every time you shade another piece of the unit square?

See figure.

As I keep shading pieces of the square, the total amount of the square that is shaded becomes closer to being filled. Since this is a unit square, the total area becomes closer to 1 unit.

4. In the table shown, n represents the term number of the series, and S_n represents the sum of the first n terms of the series. Use the sequence from the unit square in Question 3 to answer each question.

n	1	2	3	4	5	10	25
S_n as a Fraction	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	$\frac{31}{32}$	$\frac{1023}{1024}$	$\frac{33,554,431}{33,554,432}$
S_n as a Decimal	0.5	0.75	0.875	0.9375	0.96875	0.99902 . . .	0.99999 . . .

a. Complete the table for $n = 1$ through $n = 5$ to show the sum of the series that corresponds to the previous diagram. Write each sum as a fraction and as a decimal.

See table.

b. Describe the pattern you see in the table.

The denominator of the fraction is equal to 2^n, and the numerator is 1 less than 2^n.

c. Use the pattern to complete the table for the final two columns.

See table.

5. What value does the series approach as n gets greater?

As n gets larger, the series gets closer to 1.

8

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Question 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- Which operations were used to show the formulas are equivalent?
- Which algebraic properties were used to show the formulas are equivalent?

In the figure that models this series, each additional part of the unit square is one half of the previous part. If you could continue to add these parts forever, the unit square would eventually be filled.

Likewise, in the table of values that models this series, the sums get closer and closer to 1 as n gets greater. Therefore, you can say that this infinite geometric series is equal to 1.

6. Miley and Damian determined formulas they could use to compute the first n-terms of the series.

Miley

I noticed that when each sum is written as a fraction, the denominator is equal to 2^n and the numerator is one less than the denominator.

So, I can calculate the first n-terms of the series by using the formula shown.

$$S_n = \frac{2^n - 1}{2^n}$$

Damian

I know that $S_n = \frac{g_n \cdot r - g_1}{r - 1}$ can be used to compute the first n-terms of any geometric series.

For this series, $g_n = \left(\frac{1}{2}\right)^n$, $g_1 = \frac{1}{2}$, and $r = \frac{1}{2}$.

Substituting these values gives:

$$S_n = \frac{\left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right) - \frac{1}{2}}{\frac{1}{2} - 1}$$

Show that both representations for S_n are equivalent.

$\frac{2^n - 1}{2^n}$	$\frac{\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)}{\frac{1}{2} - 1} = \frac{\left(\frac{1}{2}\right)^{n+1} - \frac{1}{2}}{-\frac{1}{2}}$
$1 - \frac{1}{2^n}$	$= \frac{\left(\frac{1}{2}\right)^{n+1} - \frac{1}{2}}{-\frac{1}{2}} \cdot \frac{-2}{-2}$
$-\frac{1}{2^n} + 1$	$= -2\left(\frac{1}{2}\right)^{n+1} + 1 = -2(2)^{-n-1} + 1$
	$= -2^{-n} + 1$
	$= -\frac{1}{2^n} + 1$

Problem 2

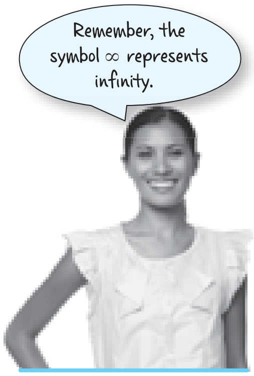
Students will explore if any infinite geometric series have a finite sum. Several formulas are accompanied with diagrams and students determine if the sum, in each situation is infinite or finite. They then conclude that when the common ratio was greater than 1, the sum of the infinite geometric series was infinite, and when the common ratio was between 0 and 1, the sum of the infinite geometric series was finite. The terms convergent series and divergent series are given. The formula for the sum of a convergent geometric series is provided. Students will identify all infinite geometric series in this problem as convergent or divergent and use the convergent formula to determine the sum of the series when appropriate. Other infinite geometric series are given and students compute sums, if it is not infinite. In the last activity, students conclude all infinite arithmetic series are divergent due to the common difference.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

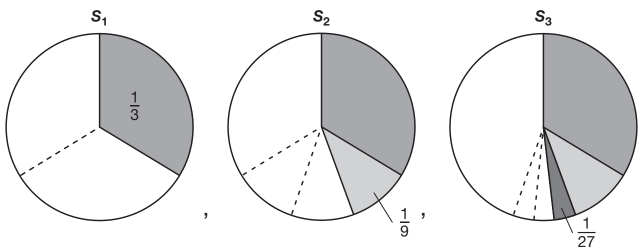
PROBLEM 2 To Infinity, and Beyond!

In the previous problem, you saw how an infinite geometric series can have a finite sum. Let's see if this is the case for *any* infinite geometric series.



1. Examine the given formula and accompanying diagram for each infinite geometric series. Identify both r and g_1 for each series. Then, determine if the sum is infinite or finite. If the sum is finite, estimate it.

   a. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^i$



$r =$ $\frac{1}{3}$

$g_1 =$ $\frac{1}{3}$

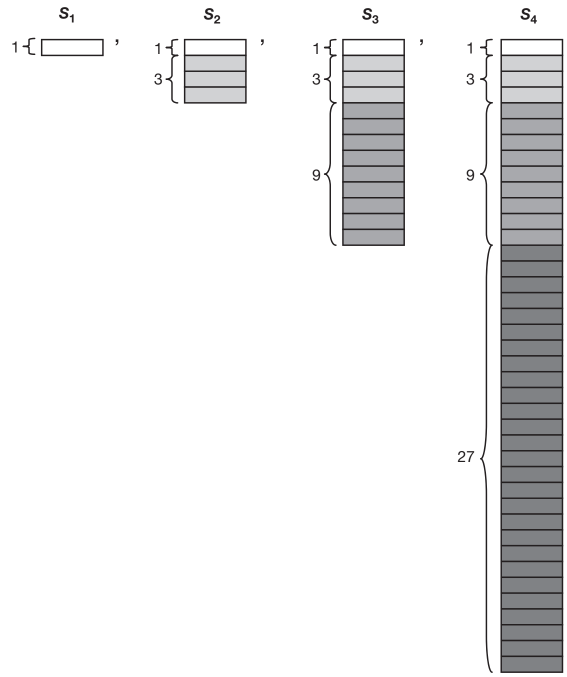
$S =$ $\frac{1}{2}$

8

Guiding Questions for Share Phase, Questions 1 and 2

- What is the common ratio in this situation?
- How was the sum of the series determined?
- Can the sum of the series be greater than $\frac{1}{2}$? How do you know?
- If the sum of the series approaches the value of $\frac{1}{2}$ but never actually reaches the value, is it still considered a finite sum?
- Is the common ratio greater than 1 or less than 1 in this situation?
- Which infinite geometric series have common ratios that are greater than 1? Are the sums of these series infinite?
- Which infinite geometric series have common ratios that are less than 1? Are the sums of these series finite?

b. $1 + 3 + 9 + 27 + \cdots = \sum_{i=1}^{\infty} \frac{1}{3}(3)^i$

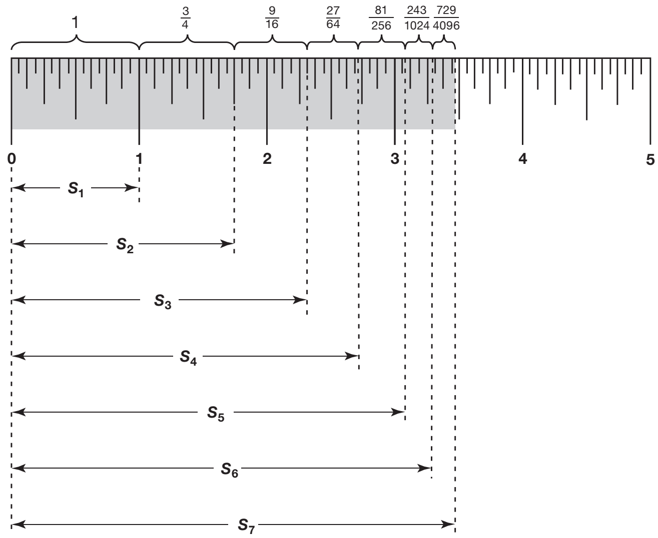


$r =$ 3

$g_1 =$ 1

$S =$ Infinite sum

c. $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \frac{81}{256} + \frac{243}{1024} + \frac{729}{4096} + \cdots = \sum_{i=1}^{\infty} \frac{4}{3}\left(\frac{3}{4}\right)^i$

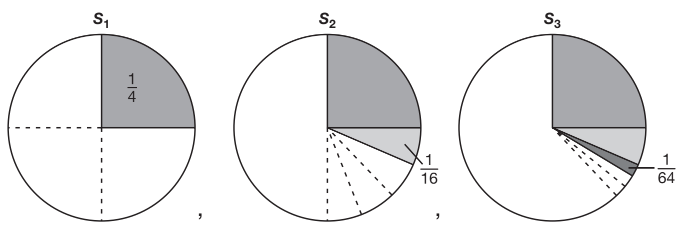


r = $\frac{3}{4}$

g_1 = 1

S = 4

d. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i$



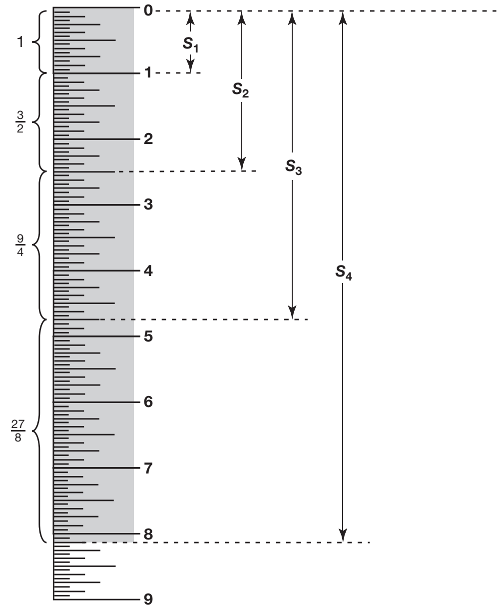
r = $\frac{1}{4}$

g_1 = $\frac{1}{4}$

S = $\frac{1}{3}$

8

e. $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots = \sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{3}{2}\right)^i$



$r =$ $\frac{3}{2}$

$g_1 =$ 1

$S =$ Infinite sum

2. Analyze the common ratio for each series in Question 1.

a. What do you notice about the series with infinite sums?

The common ratio is greater than 1 for the series with infinite sums.

b. What do you notice about the series with finite sums?

The common ratio is between 0 and 1 for the series with finite sums.

Grouping

- Ask a student to read the definitions and information. Discuss as a class.
- Have students complete Question 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

- What is the difference between a convergent series and a divergent series?
- How is the notation S_n different than the notation S?
- Is the common ratio in this infinite geometric series between 0 and 1?
- If the common ratio in this infinite geometric series between 0 and 1, does this imply the series in convergent or divergent?
- Is the common ratio in this infinite geometric series greater than 1?
- If the common ratio in this infinite geometric series is greater than 1, does this imply the series in convergent or divergent?

A **convergent series** is an infinite series that has a finite sum. A **divergent series** is an infinite series that does not have a finite sum. If a series is divergent, the sum is infinity.

The formula to compute a convergent geometric series S is shown.

$$S = \frac{g_1}{1 - r}$$

Notice that S denotes the sum of an *infinite* series. This notation should not be confused with S_n, which represents the sum of the *n*th term of a series.

3. Consider each infinite geometric series from Question 1. Determine whether each series is convergent or divergent, and explain how you know. If a series is convergent, use the formula to compute the sum. If a series is divergent, write infinity.

a. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^i$

Convergent or divergent? convergent

Explanation: The series is convergent because the common ratio is between 0 and 1.

$$\frac{g_1}{1 - r} = \frac{\frac{1}{3}}{1 - \frac{1}{3}}$$

$$= \frac{1}{2}$$

$S =$ ______

b. $1 + 3 + 9 + 27 + \cdots = \sum_{i=1}^{\infty} \frac{1}{3}(3)^i$

Convergent or divergent? divergent

Explanation: The series is divergent because the common ratio is greater than 1.

$S =$ infinity

c. $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \frac{81}{256} + \frac{243}{1024} + \frac{729}{4096} + \cdots = \sum_{i=1}^{\infty} \frac{4}{3}\left(\frac{3}{4}\right)^i$

Convergent or divergent? convergent

Explanation: The series is convergent because the common ratio is between 0 and 1.

$$\frac{g_1}{1-r} = \frac{1}{1-\frac{3}{4}}$$

$S =$ $= 4$

d. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i$

Convergent or divergent? convergent

Explanation: The series is convergent because the common ratio is between 0 and 1.

$$\frac{g_1}{1-r} = \frac{\frac{1}{4}}{1-\frac{1}{4}}$$

$S =$ $= \frac{1}{3}$

e. $1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots = \sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{3}{2}\right)^i$

Convergent or divergent? divergent

Explanation: The series is divergent because the common ratio is greater than 1.

$S =$ infinity

8

Grouping

Have students complete Questions 4 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 6

- Is the common ratio in Zoe's infinite geometric series greater than 1 or between 0 and 1?
- Is Zoe's infinite geometric series convergent or divergent?
- Is the formula for the sum of the infinite geometric series used if the series is divergent?
- Is the common ratio in each of the infinite geometric series greater than 1 or between 0 and 1? What does this imply about the series?
- Suppose the common difference of an infinite arithmetic series is a positive number, why will the sum of the series always approach positive infinity?
- Suppose the common difference of an infinite arithmetic series is a negative number, why will the sum of the series always approach negative infinity?

4. Zoe computed the infinite geometric series.

$\frac{5}{8} + \frac{25}{32} + \frac{125}{128} + \frac{625}{512} + \cdots = \sum_{i=1}^{\infty} \frac{1}{2}\left(\frac{5}{4}\right)^i$

Zoe

The formula to compute the series is $S = \frac{g_1}{1-r}$.
In this series, $g_1 = \frac{5}{8}$ and $r = \frac{5}{4}$.

So, $S = \frac{\frac{5}{8}}{1-\frac{5}{4}} = \frac{\frac{5}{8}}{-\frac{1}{4}} = \frac{5}{8}\left(-\frac{4}{1}\right) = -\frac{5}{2}$.

Explain why Zoe is incorrect, and then determine the correct sum.

Zoe is incorrect because she used the formula for the sum of a convergent series, but the series is divergent. Because the series is divergent, it has no finite sum. The sum is infinite.

5. Compute each infinite geometric series, if possible.

a. $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10{,}000} + \frac{9}{100{,}000} + \cdots$

The common ratio is $\frac{1}{10}$. Because the common ratio is between 0 and 1, the series is convergent and I can use the formula to calculate the sum.

$$S = \frac{g_1}{1-r}$$
$$= \frac{\frac{9}{10}}{1-\frac{1}{10}} = 1$$

b. $0.9 + 0.09 + 0.009 + 0.0009 + \cdots$

The common ratio is 0.1. Because the common ratio is between 0 and 1, the series is convergent and I can use the formula to calculate the sum.

$$S = \frac{g_1}{1-r}$$
$$= \frac{0.9}{1-0.1} = 1$$

c. 0.9999999 . . .

The number 0.9999999 . . . is an equivalent form of the series in part (b). So, the common ratio is 0.1. Because the common ratio is between 0 and 1, the series is convergent and I can use the formula to calculate the sum.

$$S = \frac{g_1}{1-r}$$
$$= \frac{0.9}{1-0.1} = 1$$

Talk the Talk

Students will write formulas for: the sum of the first n terms of an arithmetic series, the sum of the first n terms of a geometric series, the sum of a convergent geometric series, and the sum of a divergent geometric series.

Grouping

Have students complete the formulas with a partner. Then have students share their responses as a class.

So far in this lesson, you have only seen infinite *geometric* series. What about infinite *arithmetic* series?

6. Consider the statements made by Ronald and Jeremiah about the infinite arithmetic series.

Ronald

Some infinite arithmetic series are convergent, and some are divergent; it all depends on the common difference.

Jeremiah

All infinite arithmetic series are divergent.

Who is correct? Explain your reasoning.

Jeremiah is correct. In any infinite arithmetic series, a constant amount is added or subtracted from each term. So, the sum of any infinite arithmetic series will always be positive infinity (if the common difference is positive) or negative infinity (if the common difference is negative).

Talk the Talk

Write the formula to compute each type of series.

The First n-Terms of an Arithmetic Series:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

The First n-Terms of a Geometric Series:

$$S_n = \frac{g_n \cdot r - g_1}{r - 1} \text{ or } \frac{g_1(r^n - 1)}{r - 1}$$

A Convergent Geometric Series:

$$S = \frac{g_1}{1 - r}$$

A Divergent Geometric Series:

$S =$ infinity

Be prepared to share your solutions and methods.

Check for Students' Understanding

Compute the infinite geometric series.

$8 + 2 + \frac{1}{2} + \frac{1}{8} + \cdots$

$r = \frac{1}{4}$

$S = \frac{g_1}{1 - r}$

$S = \frac{8}{1 - \frac{1}{4}}$

$S = \frac{32}{3}$

8

8.5

The Power of Interest (It's a Curious Thing)

Geometric Series Applications

LEARNING GOALS

In this lesson you will:

- Apply your understanding of series to problem situations.
- Write the formula for a geometric series representing a problem situation.

ESSENTIAL IDEA

- Apply geometric series and explicit formulas to problem situations.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

Overview

Students will discover making credit card purchases at high interest rates while making minimum payments is a costly proposition. They will write explicit formulas to determine details surrounding the payback over different periods of time. Students then create formulas for geometric series to determine final payback figures and the total amount of time needed to pay off the credit card purchase. Four different scenarios of credit card debt are presented and students explore each situation comparing the amounts paid back in interest throughout the life of the loan.

Warm Up

Jean charges $1800 on her credit card to pay for the work on her car needed to pass inspection. Her credit card company requires a minimum monthly payment of the greater of:

- 3% of the balance on the card, or
- $50.00.

When making a monthly payment, 70% of the minimum payment goes towards interest and the remaining portion of the minimum monthly payment goes towards the principal.

Jean can only afford to make the minimum monthly payment.

1. Calculate Jean's first monthly payment.

   Jean's first monthly payment is $54.

   $0.03(1800) = 54$

2. What portion of Jean's first monthly payment goes toward interest?

   $37.80 of Jean's first monthly payment goes toward interest.

   $0.7(54) = 37.80$

3. What portion of Jean's first monthly payment goes toward the principal?

   $16.20 of Jean's first monthly payment goes toward the principal.

   $0.3(54) = 16.20$

8.5

The Power of Interest (It's a Curious Thing)

Geometric Series Applications

LEARNING GOALS

In this lesson you will:

- Apply your understanding of series to problem situations.
- Write the formula for a geometric series representing a problem situation.

Imagine walking up to the counter of a major electronics store to purchase a new flat screen television for $999. When the salesperson rings up your purchase, she says: "After fees and interest charges, your total is $1950!" Would you still buy the television?

In another scenario, imagine walking into the tuition office at a local university. You are interested in taking 2 classes this fall while working part-time. The cost of the 6 credits is supposed to be $5000, but after discussing a particular payment option with the financial aid officer, you realize that the classes will cost more than $10,000 over the next 12 years. Would you still accept that payment option?

It may surprise you, but many people accept these terms for their purchases every day and don't even realize it. You may wonder how this could happen. The answer lies in the mathematics behind credit cards. When used wisely, credit cards can be convenient and flexible. They allow consumers to purchase expensive or necessary items and pay for them at a later date. On the other hand, if used unwisely, consumers waste a lot of money on interest charges.

How do credit cards work? How could you end up paying twice the amount for a television, or make monthly payments and still carry a balance on two college classes that you took over a decade ago?

8

Problem 1

Students will determine how long it will take to pay off a credit card purchase when paying the minimum balance. A worked example shows them how to calculate the amount of each payment that goes toward principal and the amount that goes toward interest to determine the new monthly balance so they are able to calculate the monthly payment details for the first 12 months of minimum payments. Using all of the information, students write explicit formulas that represent: the balance before monthly payment, the minimum payment, the amount paid toward interest, the amount paid toward principal, and the balance after the minimum payment. Students also write formulas for the geometric series that represent: the total monthly payment, the total payment towards interest over time, and the total payment towards principal over time. The formulas are used to calculate the amounts paid toward principal and interest over a period of 2 years and 5 years. Students conclude the payback figure for the credit card purchase is over 4 times the initial amount borrowed if the minimum payments are made over a period of 21.6 years.

Grouping

Ask a student to read the information and worked example. Complete Question 1 as a class.

PROBLEM 1 I Don't Want Credit For This

Vince wants to purchase a laptop with high screen resolution for his gaming hobby. He charges the $1000 purchase to a credit card with 19% interest.

The credit card company requires a minimum monthly payment of the greater of:

- 2% of the balance on the card, or
- $15.00.

To determine how long it will take to pay off the credit card when paying the minimum balance, Vince calls the company. He learns that when making a monthly payment, 75% of the minimum payment goes toward interest and the remaining portion of the minimum monthly payment goes toward the principal.

1. Determine the percent of the payment that is paid toward interest and principal for each monthly payment.

   a. Monthly payment is 2% of balance.

      1.5% of the balance goes toward interest.

      0.5% of the balance goes toward principal.

   b. Monthly payment is 10% of balance.

      7.5% of the balance goes toward interest.

      2.5% of the balance goes toward principal.

   c. Monthly payment is 25% of balance.

      18.75% of the balance goes toward interest.

      6.25% of the balance goes toward principal.

Let's consider Vince's monthly payment.

In order to calculate the minimum monthly payment, you will need to calculate 2% of the balance.

$$\begin{aligned} \text{minimum payment} &= (0.02)(\text{balance}) \\ &= 0.02(1000) \\ &= 20 \end{aligned}$$

If the credit card balance is \$1000.00, the minimum monthly payment would be \$20.00.

The amount paid toward interest is 1.5% of the balance.

$$\begin{aligned} \text{amount paid toward interest} &= 0.015(\text{balance}) \\ &= (0.015)(1000) \\ &= 15 \end{aligned}$$

If the minimum monthly payment is \$20.00, the amount paid toward interest would be \$15.00.

The amount paid toward principal is 0.5% of the balance.

$$\begin{aligned} \text{amount paid toward principal} &= (0.005)(\text{balance}) \\ &= (0.005)(1000) \\ &= 5 \end{aligned}$$

If the minimum monthly payment is \$20.00, the amount paid toward principal would be \$5.00.

The remaining balance on the credit card will be the current balance minus the amount paid toward principal.

$$\begin{aligned} \text{remaining balance} &= (\text{current balance}) - (\text{amount paid toward principal}) \\ &= 1000 - 5 \\ &= 995 \end{aligned}$$

So, if Vince makes a monthly payment of \$20.00 on the \$1000.00 balance, the balance after the monthly payment is \$995.00.

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Grouping

Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

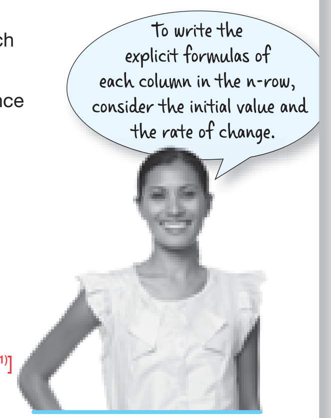
Guiding Questions for Share Phase, Questions 2 and 3

- As the months increase, what happens to the amount applied toward the principal?
- As the months increase, what happens to the amount applied toward the interest?
- How was each monthly payment calculated?
- How was the amount paid toward principal determined?
- How was the amount paid toward interest determined?
- How was the balance after monthly payment determined?
- How was the balance before monthly payment determined?
- What explicit formula is associated with a geometric sequence?
- How is the explicit formula used to determine the minimum payment formula?
- How is the explicit formula used to determine the amount paid toward interest formula?
- How is the explicit formula used to determine the amount paid toward principal formula?

2. Calculate the monthly payment details for the first 12 months of minimum payments. The first row of the table reflects the calculations from the worked example.

Number of Months (n)	Balance Before Monthly Payment ($)	Minimum Monthly Payment ($)	Amount Paid Toward Principal ($)	Amount Paid Toward Interest ($)	Balance After Monthly Payment ($)
1	1000.00	20.00	5.00	15.00	995.00
2	995.00	19.90	4.98	14.92	990.02
3	990.02	19.80	4.95	14.85	985.07
4	985.07	19.70	4.93	14.77	980.14
5	980.14	19.60	4.90	14.70	975.24
6	975.24	19.50	4.88	14.62	970.36
7	970.36	19.41	4.85	14.56	965.51
8	965.51	19.31	4.83	14.48	960.68
9	960.68	19.21	4.80	14.41	955.88
10	955.88	19.12	4.78	14.34	951.10
11	951.10	19.02	4.76	14.26	946.34
12	946.34	18.93	4.73	14.20	941.61
n	$1000(0.995)^{(n-1)}$	$20(0.995)^{(n-1)}$	$5(0.995)^{(n-1)}$	$15(0.995)^{(n-1)}$	$1000(0.995)^{n}$

3. Consider the worked example and the answers to each part to complete the n-row of the table.



a. Write the explicit formula for the geometric sequence represented in the “Balance Before Monthly Payment” column.

The formula that represents “Balance Before Monthly Payment” is $1000(0.995)^{(n-1)}$.

b. Write the formula to calculate the minimum monthly payment.

$$\begin{aligned}\text{minimum monthly payment} &= (0.02)(\text{balance})\\ &= 0.02[1000(0.995)^{(n-1)}]\\ &= 20(0.995)^{(n-1)}\end{aligned}$$

The formula that represents the minimum payment is $20(0.995)^{(n-1)}$.

Grouping

Have students complete Questions 4 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- Which formula is used to determine the total monthly payment over a period of 12 months? What information is needed to use the formula?
- What information is needed to use the formula? What information is needed to use the formula?
- Which formula is used to determine the total interest payment over a period of 12 months? What information is needed to use the formula?
- Which formula is used to determine the total principal payment over a period of 12 months? What information is needed to use the formula?

8

c. Write the formula to calculate the amount paid toward interest.

amount paid toward interest $= (0.015)(\text{balance})$

$= [0.015(1000(0.995)^{(n-1)}]$

$= 15(0.995)^{(n-1)}$

The formula to calculate amount paid toward interest is $15(0.995)^{(n-1)}$.

d. Write the formula to calculate the amount paid toward principal.

amount paid toward principal $= (0.005)(\text{balance})$

$= (0.005)[1000(0.995)^{(n-1)}]$

$= 5(0.995)^{(n-1)}$

The formula to calculate amount paid toward principal is $5(0.995)^{(n-1)}$.

e. Write the formula to calculate the balance after minimum payment.

remaining balance $= 1000(0.995)^{n}$

I calculate this the same way I calculate the current balance, but instead of the previous term ($n - 1$), I need to use the next term (n).

Vince knows how much he is paying each month, but it would be helpful if he knew how much he had paid in both interest and principal over a certain amount of time instead of on any given month.

4. Calculate each.

a. the total monthly payment in the first 12 months

Students could add up the appropriate column in the table, use the table function of the calculator, or use the formula for the series.

$$S_{12} = \frac{20(0.995^{12} - 1)}{0.995 - 1}$$

Vince spent $233.51 in minimum payments.

b. the amount paid toward principal in the first 12 months

Students could add up the appropriate column in the table, use the table function of the calculator, or use the formula for the series.

$$S_{12} = \frac{5(0.995^{12} - 1)}{0.995 - 1}$$

Vince spent $58.38 in interest.

Guiding Questions for Share Phase, Questions 5 and 6

- What n-value is used when calculating the total amount of interest paid over a 2-year period?
- What n-value is used when calculating the total amount of interest paid over a 5-year period?

Grouping

Have students complete Questions 7 through 11 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 7 and 8

- How can a graphing calculator be used to determine how long it will take to pay off the credit card completely if the company does not require a monthly payment of $15?
- Is it reasonable to take 2% of an infinite period of time?

c. the amount paid toward interest in the first 12 months

Students could add up the appropriate column in the table, use the table function of the calculator, subtract the answer from part (b) from part (a), or use the formula for the series.

$$S_{12} = \frac{15(0.995^{12} - 1)}{0.995 - 1}$$

Vince spent $175.13 in interest.

5. Write a formula for the geometric series that represents:

a. the total monthly payment.

$$S_n = \frac{20(0.995^{12} - 1)}{0.995 - 1}$$

b. the total payment toward principal over time.

$$S_n = \frac{5(0.995^{12} - 1)}{0.995 - 1}$$

c. the total payment toward interest over time.

$$S_n = \frac{15(0.995^{12} - 1)}{0.995 - 1}$$

6. Use the formulas you created in Question 5 to complete the table.

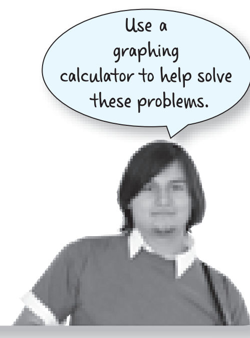
	Total Amount Paid Toward Principal	Total Amount Paid Toward Interest
2 years	$S_{24} = \frac{5(0.995^{24} - 1)}{0.995 - 1} \approx \$113.35$	$S_{24} = \frac{15(0.995^{24} - 1)}{0.995 - 1} \approx \$340.04$
5 years	$S_{60} = \frac{5(0.995^{60} - 1)}{0.995} \approx \$259.74$	$S_{60} = \frac{15(0.995^{60} - 1)}{0.995 - 1} \approx \$779.22$

7. Assume the credit card company does not require a monthly payment of $15.00. Determine how long will it take to pay off the credit card completely.

It would take over 100 years to pay off the credit card debt.

8. Does your answer to Question 7 seem reasonable? Explain your reasoning.

No. It does not sound reasonable to pay off the credit card balance longer than 100 years. Eventually the minimum payment will become so low that you just keep taking 2% of a really small number. Thus, the number of payments approaches infinity, but in reality it could be paid off with a minimum payment.



Guiding Questions for Share Phase, Questions 9 and 10

- How many years must Vince pay 2% of the balance as a monthly payment?
- How can the balance on Vince's credit card be determined when the minimum payment became $15?
- What percent of each minimum payment goes toward interest?
- What percent of each minimum payment goes toward principal?
- Will Vince end up paying more than 4 times the cost of the original laptop when he is done making his payments? Is this reasonable?

9. Determine the amount of time it will take to pay off the credit card balance, taking into account the minimum payment of $15.00.

a. After how many months will the minimum monthly payment become $15.00?

I need to determine when 2% of the balance is the same as $15.00.

$20(0.995)^{(n-1)} = 15$

$n \approx 58.392$

In Month 59, the minimum monthly payment will become $15.00.

b. What will be the balance on Vince's credit card when the minimum monthly payment becomes $15.00?

$1000(0.995)^{(59-1)} = 747.72$

The balance on Vince's credit card will be $747.72 when the minimum monthly payment becomes $15.00.

c. How much will Vince pay toward principal with every $15.00 payment?

$15 \times 0.25 = 3.75$

The amount paid toward principal for each $15.00 monthly payment is $3.75.

d. How much will Vince pay toward interest with every $15.00 payment?

$15 - 3.75 = 11.25$

The amount paid toward interest for each $15.00 monthly payment is $11.25.

e. Suppose Vince continues to make $15.00 payments until the end of the loan. How many months will Vince have to make the $15.00 minimum payment to pay off the remaining balance?

$\frac{747.72}{3.75} = 199.39$ months

Vince will need to make a $15.00 monthly payment for 200 months to pay off the entire balance.

f. How many years will it take Vince to pay off the entire balance?

$200 + 59 = 259$

$\frac{259}{12} = 21.6$ years

It will take Vince approximately 21.6 years to pay off the balance.

10. How much money will Vince end up spending on his $1000.00 laptop?

Total of 2% payments + Total of $15.00 payments

$= \frac{20(0.995^{58} - 1)}{0.995 - 1} + (15)(200)$

$= 1009.12 + 3000$

$= 4009.12$

Vince will end up paying approximately $4009.12 for his laptop.

Problem 2

Using the same scenario from the previous problem, students will determine the details of paying the credit card debt at a rate of 10% of the balance every month as opposed to the required minimum monthly payment. Using much of the same formulas and answering questions similar to the previous problem, students conclude that the total payback figure is much more reasonable that making the required minimum monthly payments.

Grouping

Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- As the months increase, what happens to the amount applied toward the principal?
- As the months increase, what happens to the amount applied toward interest?

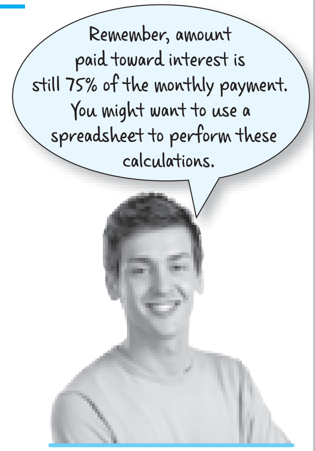
11. How much money will Vince spend in interest to pay for his $1000.00 laptop?

4009.12 − 1000 = 3009.12

Vince will pay $3009.12 in interest.

PROBLEM 2 A Little Less Interest-ing

After realizing how long it would take to pay his entire credit card bill when only paying the minimum amount, Vince decides that he should pay more than the minimum amount every month. Vince determines that he can pay 10% of the balance every month.



1. Complete the table to represent this information for 12 months.

Number of Months (n)	Balance Before Monthly Payment ($)	10% Monthly Payment ($)	Amount Paid Toward Principal ($)	Amount Paid Toward Interest ($)	Balance After Monthly Payment ($)
1	1000.00	0.10(1000) = 100	25.00	75.00	975.00
2	975.00	97.50	24.38	73.13	950.63
3	950.63	95.06	23.77	71.30	926.86
4	926.86	92.69	23.17	69.51	903.69
5	903.69	90.37	22.59	67.78	881.10
6	881.10	88.11	22.03	66.08	859.07
7	859.07	85.91	21.48	64.43	837.59
8	837.59	83.76	20.94	62.82	816.65
9	816.65	81.67	20.42	61.25	796.24
10	796.24	79.62	19.91	59.72	776.33
11	776.33	77.63	19.41	58.22	756.92
12	756.92	75.69	18.92	56.77	738.00
n	$1000(0.975)^{(n-1)}$	$100(0.975)^{(n-1)}$	$25(0.975)^{(n-1)}$	$75(0.975)^{(n-1)}$	$1000(0.975)^{n}$

Guiding Questions for Share Phase, Questions 2 through 5

- Which formula is used to determine the total monthly payment over a period of 12 months? What information is needed to use the formula?
- Which formula is used to determine the total interest payment over a period of 12 months? What information is needed to use the formula?
- Which formula is used to determine the total principal payment over a period of 12 months? What information is needed to use the formula?
- What *n*-value is used when calculating the total amount of interest paid over a 1.5 year period?
- What *n*-value is used when calculating the total amount of interest paid over a 5 year period?
- How does the total interest payback in this situation compare to the total interest payback in the previous problem?

2. Write the formula for the geometric series that represents:

a. the total monthly payment.

$S_n = \frac{1000(0.975^n - 1)}{0.975 - 1}$

b. the total payment toward principal over time.

$S_n = \frac{25(0.975^n - 1)}{0.975 - 1}$

c. the total payment toward interest over time.

$S_n = \frac{75(0.975^n - 1)}{0.975 - 1}$

3. Use the formulas you created in Question 2 to complete the table.

	Total Money Spent in Principal	Total Money Spent in Interest
1.5 years	$S_{18} = \frac{25(0.975^{18} - 1)}{0.975 - 1} \approx \$366.01$	$S_{18} = \frac{75(0.975^{18} - 1)}{0.975 - 1} \approx \$1098.03$
2 years	$S_{24} = \frac{25(0.975^{24} - 1)}{0.975 - 1} \approx \$455.36$	$S_{24} = \frac{75(0.975^{24} - 1)}{0.975 - 1} \approx \$1366.08$

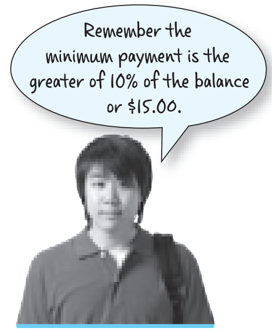
4. After how many months will the minimum monthly payment become $15.00?

I need to determine when 10% of the balance is the same as $15.00.

$(0.01)(1000)(0.975)^{(n-1)} = 15$

$n \approx 12.363$

If Vince continues to pay 10% of the balance each month, in Month 76 the minimum monthly payment will become $15.00.



5. What will the balance on his credit card, when the minimum payment becomes $15.00?

$1000(0.975)^{(76-1)} = 149.74$

In Month 76, the balance on his credit card will be approximately $149.74.

8 Guiding Questions for Share Phase, Questions 6 and 7

- Which method of payback would you choose? Why?
- What are the advantages and disadvantages to paying the credit card debt using a 10% balance to determine each monthly payment?

6. Vince decides that once he gets to the minimum payment, he will pay off the rest of the balance in one lump sum.

   a. How long will it take Vince to pay off the entire balance?

   It will take Vince 76 months years to pay off the entire balance.

   b. How much money will Vince end up spending on his $1000.00 laptop?

   Total of 10% payments + Lump sum payment

   $$\frac{100(0.975^{76} - 1)}{0.975 - 1} + 149.74 = 3401.03 + 149.74$$
   $$= 3251.29$$

   Vince will spend approximately $3251.29 on his laptop.

   c. How much money will Vince spend in interest to pay for his $1000.00 laptop?

   $3251.29 - 1000 = 2251.29$

   Vince will spend $2251.29 in interest.

7. How much money will Vince save by paying 10% instead of the required 2% minimum payment?

   $4009.12 - 2251.29 = 1757.83$

   Vince will save $1757.83.

Problem 3

Using the same scenario from the two previous problems, students will determine the details of paying the credit card debt at a flat rate of $100 every month. Students once again complete a table listing the details of the credit card payments over a period of 12 months.

Grouping

Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

PROBLEM 3 Enough of These Interest Payments

Vince is still concerned about paying so much money in interest on his credit card, and decides that he can afford to pay a flat amount of $100 each month.

1. Sean calculates how long it will take Vince to pay off his credit card this way.

   Sean

   $\frac{\$1000}{100} = 10$

   It will take Vince 10 months to pay off his credit card bill.

   What is wrong with Sean's work? Explain your reasoning.

   Sean does not calculate interest into the payments. He assumes the entire $100.00 will be going toward the principal.

Guiding Question for Share Phase, Question 1

Using Sean's method, how much of each monthly payment was interest and how much was principal?

Guiding Questions for Share Phase, Questions 2 through 7

- Will it take Vince more or less than 10 months to pay his debt?
- How was the amount paid toward principal determined?
- How was the amount paid toward interest determined?
- How was the balance after monthly payment determined?
- How was the balance before monthly payment determined?
- As the months increase, what happens to the amount applied toward the principal?
- As the months increase, what happens to the amount applied toward the interest?

2. How long do you think it will take Vince to pay off his credit card this way? Explain your reasoning.

Answers will vary.

Student answers should include a number over 10 months because they know that 10 months would be the amount if none of the money went toward interest.

3. Complete the table to represent this information for 12 months.

Number of Months (n)	Balance Before Monthly Payment ($)	$100 Monthly Payment ($)	Amount Paid Toward Principal ($)	Amount Paid Toward Interest ($)	Balance After Monthly Payment ($)
1	1000.00	100.00	25.00	75.00	975.00
2	975.00	100.00	25.00	75.00	950.00
3	950.00	100.00	25.00	75.00	925.00
4	925.00	100.00	25.00	75.00	900.00
5	900.00	100.00	25.00	75.00	875.00
6	875.00	100.00	25.00	75.00	850.00
7	850.00	100.00	25.00	75.00	825.00
8	825.00	100.00	25.00	75.00	800.00
9	800.00	100.00	25.00	75.00	775.00
10	775.00	100.00	25.00	75.00	750.00
11	750.00	100.00	25.00	75.00	725.00
12	725.00	100.00	25.00	75.00	700.00
n	$1000 - 25(n - 1)$	100.00	25.00	75.00	$975 - 25(n - 1)$

4. What pattern do you notice in this table that is different from the tables in Problems 1 and 2?

This situation represents an arithmetic sequence as opposed to the geometric sequences in Problems 1 and 2.

5. How long will it take Vince to pay off the entire balance?

$$1000 - 25(n - 1) = 0$$
$$-25(n - 1) = -1000$$
$$n - 1 = 40$$
$$n = 41$$

It will take Vince 41 months or $3\frac{5}{12}$ years to pay off his credit card.

6. How much money will Vince end up spending on his $1000.00 laptop?

$100(41) = 4100$

Vince will end up paying $4100 in interest.

7. How much money will Vince spend in interest to pay for his $1000.00 laptop?

$75(41) = 3075$

Vince will spend $3075 in interest.

8

Problem 4

Using the same scenario from the three previous problems, students will determine the details of paying a credit card debt of $1000 at a rate of 10% of the balance every month. This time, the credit card offers the first 6 months interest free. Students once again complete a table listing the details of the credit card payments over a period of 12 months. Students will then analyze the information and make a recommendation to Vince about paying off his credit card.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

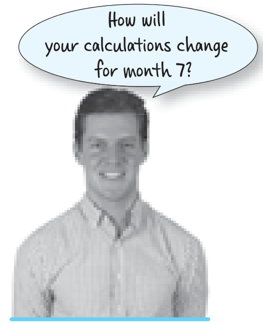
Guiding Questions for Share Phase, Questions 1 through 3

- How is this table different from the previous tables?
- How was each monthly payment determined?
- How was the amount paid toward the principal determined?
- Did the balance decrease faster in the first 6 months or the second 6 months?

PROBLEM 4 Interest Free? Whoopee!

Now that Vince has become more educated about credit card finances and proven that he can be responsible for paying off his debt, he applies for a new credit card that offers the first 6 months interest free for any purchases. Like his other credit card, this new credit card requires a minimum monthly payment of the greater of:

- 2% of the balance on the card, or
- $15.00.



Vince is approved for the card, and charges $1000 for a flat screen TV. He still decides to pay 10% of the balance after noticing how much money that saved him the last time.

1. Complete the table to show 12 months of payments.

Number of Months (n)	Balance Before Monthly Payment ($)	10% Monthly Payment ($)	Amount Paid Toward Principal ($)	Amount Paid Toward Interest ($)	Balance After Monthly Payment ($)
1	1000.00	0.10(1000) = 100	100.00	0.00	900.00
2	900.00	90.00	90.00	0.00	810.00
3	810.00	81.00	81.00	0.00	729.00
4	729.00	72.90	72.90	0.00	656.10
5	656.10	65.61	65.61	0.00	590.49
6	590.49	59.05	59.05	0.00	531.44
Interest begins on the 7th month.					
7	531.44	53.14	13.29	39.86	518.15
8	518.15	51.82	12.96	38.86	505.20
9	505.20	50.52	12.63	37.89	492.57
10	492.57	49.25	12.32	36.94	480.25
11	480.26	48.02	12.01	36.02	468.25
12	468.25	46.82	11.71	35.12	456.54
n	$531.44(0.975)^{(n-7)}$	$53.14(0.975)^{(n-7)}$	$13.29(0.975)^{(n-7)}$	$39.86(0.975)^{(n-7)}$	$518.15(0.975)^{(n-7)}$

2. Describe the change in Vince's balance in the first 6 months compared to the last 6 months.

Answers will vary.

In the first 6 months the balance was paid down much quicker than in the following 6 months.

3. Analyze the 12 months of payments. What recommendations would you give Vince about paying off his credit card? Include the number of months he would be paying on the balance, the total amount that he would pay for the flat screen TV, and the amount of total interest he would pay.

Answers will vary.

8

Talk the Talk

The four credit card payment scenarios in this lesson are summarized. Students will answer questions related to each of the situations.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Talk the Talk

Consider the four credit card payment scenarios Vince explored:

- **a.** Paying minimum payment
- **b.** Paying 10% of balance until minimum payment and then paying off in one lump sum
- **c.** Paying $100.00 a month
- **d.** Paying 10% interest free for 6 months and then continuing to pay 10%

1. Why might payment method (c) not be an option for some people?

   Answers will vary.

   Student responses could include some people may not have enough money to contribute $100.00 per month to their bill.

2. Which method of payment do you consider the best option? Explain your reasoning.

   Answers will vary.

   Student responses could include thinking method (c) is best because the least amount of interest in being paid.

3. When considering applying for a credit card, what details should you look for? How would you plan to pay off your bill?

   Answers will vary.

   Student responses could include:

   - interest
   - how much a person is charging to the credit card
   - minimum payment
   - how much you can afford to pay per month

Be prepared to share your solutions and methods.

Check for Students' Understanding

Jean charges $1800 on her credit card to pay for the work on her car needed to pass inspection. Her credit card company requires a minimum monthly payment of the greater of:

- 3% of the balance on the card, or
- $50.00.

When making a monthly payment, 70% of the minimum payment goes toward interest and the remaining portion of the minimum monthly payment goes toward the principal.

Jean decides to make the minimum monthly payment.

1. Calculate the monthly payment details for the first 12 months of minimum payments.

Number of Months (*n*)	Balance Before Monthly Payment ($)	Minimum Monthly Payment ($)	Amount Paid Toward Principal ($)	Amount Paid Toward Interest ($)	Balance After Monthly Payment ($)
1	1800.00	54	16.20	37.80	1783.80
2	1783.80	53.51	16.05	37.46	1767.75
3	1767.75	53.03	15.91	37.12	1751.84
4	1751.84	52.56	15.77	36.79	1736.07
5	1736.07	52.08	15.62	36.45	1720.46
6	1720.44	51.61	15.48	36.12	1704.98
7	1704.96	51.15	15.34	35.80	1689.64
8	1689.62	50.69	15.21	35.48	1674.41
9	1674.41	50.23	15.07	35.16	1659.34
10	1659.34	50.00	15.00	35.00	1644.34
11	1644.34	50.00	15.00	35.00	1629.34
12	1629.34	50.00	15.00	35.00	1614.34

2. Jean decides that once she gets to the $50 minimum payment, she will pay off the rest of the balance in one lump sum. How much money will Jean end up spending for the $1800 charge on her credit card?

Total of 3% payments + Lump sum payment

$468.86 + $1659.34 = $2128.20

Jean's total payback for the $1800 charge on her credit card is $2128.20.

A Series of Fortunate Events

8.6

Applications of Arithmetic and Geometric Series

LEARNING GOALS

In this lesson, you will:

- Apply your understanding of series to problem situations.
- Determine whether a situation is best modeled by a geometric or arithmetic series.

ESSENTIAL IDEA

- Apply geometric series and explicit formulas to problem situations.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

8

Overview

Students will identify the nature of different situations as geometric or arithmetic. They then use appropriate formulas to determine the total amount of study time, the better monetary investment, and the company that pays the better salary over a period of time.

Warm Up

Ricky's job pays $16 an hour and he receives a yearly raise of $1.25 an hour.

1. Complete the table.

Year	Hourly Wage ($)
1	16
2	17.25
3	18.50
4	19.75
5	21.00

2. Is this situation an example of an arithmetic sequence or a geometric sequence? Explain your reasoning.

This situation is an example of an arithmetic sequence. The common difference is $1.25.

8

8.6

A Series of Fortunate Events

Applications of Arithmetic and Geometric Series

LEARNING GOALS

In this lesson, you will:

- Apply your understanding of series to problem situations.
- Determine whether a situation is best modeled by a geometric or arithmetic series.

Have you ever heard the expression “money can’t buy happiness”? Do you think it’s true? People spend a lot of time and mental energy dreaming about having a lot of money or material possessions. It’s interesting to think about whether winning the lottery or suddenly acquiring a lot of fancy things would actually make you a happier person. Researchers at universities across the globe have studied this question, and some of the results of the studies may surprise you.

- Lottery winners often become less satisfied with life’s simple pleasures over time.
- Once earnings surpass the ability to purchase essential items (food, clothing, shelter, etc.), additional money generally doesn’t lead to an increase in happiness.
- Wealthy people tend to relish positive life experiences much less than people who aren’t wealthy.

That isn’t to say that money isn’t important. Making sound financial decisions can save you a lot of headaches and put you in a position where you aren’t worrying about money. However, research says that having a lot of money won’t necessarily make you a happier person.

What financial decisions have you made so far in your life? What important financial decisions are coming up?

8

Problem 1

Three scenarios are given and students associate each situation with a geometric or arithmetic series. They will answer questions and make decisions that lead to the best possible outcomes.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Does this situation have a common difference or a common ratio?
- Is the amount of study time increasing at a constant rate each day?
- What formula can be used to calculate the total amount of study time?
- What information is needed to use the formula to calculate the total amount of study time?
- How many hours is 1305 minutes?

PROBLEM 1 A Time of Serious Financial Decisions

Some of the most important financial decisions often occur during the years following the completion of high school or college. This is a time when young adults usually face their first serious choices about things such as a career, buying a car, assuming a mortgage for a house, or investing money in the bank.

1. Benjamin is anxious. After finishing his undergraduate degree he must take the GRE in order to get into graduate school, but the amount of information that he needs to cover is overwhelming. "I only have a month to prepare!" he exclaims. Sally tells him to calm down and start slowly. She recommends studying just 15 minutes the first day, but adding 3 minutes every day to his study time.

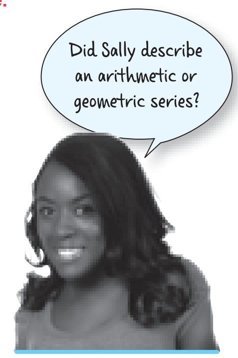
   "But a friend told me that you have to study at least 20 hours to get ready for this thing! I think I need a different plan."

   Will Sally's plan lead to enough study time? Show all work and explain your reasoning.

   Yes. Sally's plan will lead to more than 20 hours of study time.

   Her plan is an arithmetic series. The first term is 15 and the last term is $15 + 3(30 - 1) = 102$. The total amount of study time is $\frac{30(15 + 102)}{2} = \frac{30(117)}{2} = 1755$ minutes.

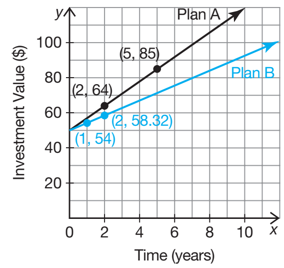
   I must divide 1755 minutes by 60 min/hour to convert to 29.25 hours.



Guiding Questions for Share Phase, Question 2

- Is Investment Plan A linear or exponential? How do you know?
- Is Investment Plan B linear or exponential? How do you know?
- Which plan do you predict will be better for a long term investment, a linear plan or an exponential plan?
- What is the common difference in Plan A?
- What function best represents Plan A?
- What is the common ratio in Plan B?
- What function best represents Plan B?
- How many years into the investment do both Plan A and Plan B pay the same amount?
- How can each plan be evaluated for a 20-year investment period?

2. Carlos meets up with Jake after a visit to the bank. He has a confused look. Carlos said, "I wanted to determine the best way to invest my money, but everybody at the bank was busy. I found this graph showing how \$50 increases over time from two of their investment options."



"So what's the problem?" Jake asks.

Carlos explains his dilemma: "I want to invest my money in a plan and keep it there for 20 years. This brochure was ripped and only shows the first few years. I need to know which plan is a better long-term investment.

a. Are the investment plans arithmetic or geometric? Explain your reasoning.

The situation is both arithmetic and geometric.

Plan A is linear and therefore arithmetic. Plan B is exponential and therefore geometric.

b. Determine the better investment for Carlos. Show all work and explain your reasoning.

Plan B is the better option.

Plan A is linear with a rate of increase of $\frac{65-64}{5-2} = \frac{64-50}{2-0} = 7$. The function $f(x) = 50 + 7x$ represents the value of the investment over time.

Plan B is increasing exponentially with a common ratio between years. The common ratio is $\frac{58.32}{54} = \frac{54}{50} = 1.08$. Therefore the function $g(x) = 50(1.08)^x$ represents the value of the investment over time. I graphed the functions and determined that the plans pay the same amount after approximately 14 years. For an amount of time beyond 14 years, Plan B is better.

I also substituted 20 into the equations to determine the investment value after 20 years. Plan A pays $50 + 7(20) = 190$ dollars. Plan B pays $50(1.08)^{20} = 233$ dollars.

8

Guiding Questions for Share Phase, Question 3

- Is the salary for the Range of Motion position associated with a common ratio or a common difference?
- What is the common difference in the Range of Motion situation?
- Is the salary for the Mobility, Inc. position associated with a common ratio or a common difference?
- What is the common ratio in the Mobility, Inc. situation?
- What function best represents the Range of Motion situation?
- What function best represents the Mobility, Inc. situation?
- Will graphing the functions be helpful when determining the years Range of Motion pays more than Mobility, Inc.?
- What does the point of intersection on the graph represent with respect to this problem situation?
- Which company pays more before the point of intersection?
- Which company pays more after the point of intersection?
- What equation is used to calculate the total earnings after n years for the Range of Motion position?

3. Rhonda is considering two different physical therapist positions.
   - Range of Motion offers an initial salary of $50,000 per year with annual increases of $1,500 per year.
   - Mobility, Inc. offers an initial salary of $42,000 with a guaranteed 4% increase in salary every year.

   a. Is this situation arithmetic or geometric? Explain your reasoning.

   The situation is both arithmetic and geometric.

   Range of Motion is arithmetic because it increases by a constant amount of $1500 each year. Mobility, Inc. is geometric because it is increasing by a percentage of the previous amount.

   b. Determine the years for which Range of Motion pays more than Mobility, Inc. Show all work and explain your reasoning.

   Range of Motion will pay more for approximately the first 12.6 years.

   I graphed the functions $y = 50{,}000 + 1500x$ and $y = 42{,}000(1.04)^x$. Range of Motion pays more at first so I need to determine when Mobility, Inc. pays the same amount. Based on the intersection point, I determined that the two jobs will pay the same amount of money after about 12.6 years. This means that after 12.6 years Mobility, Inc. pays more money each year.

   c. Determine which company pays more salary over a 30-year career. Show all work and explain your reasoning.

   Range of Motion increases arithmetically while Mobility, Inc. increases geometrically. The equations for total earnings after n-years are:

   Range of Motion Total Earnings $= \frac{n}{2}(100{,}000 + 1500(n - 1))$

   Mobility, Inc. Total Earnings $= 42{,}000\left(\frac{1.04^n - 1}{1.04 - 1}\right)$

   I graphed the two equations and determined that after 23 years the total earnings for Mobility, Inc. are greater. For a 30-year career, the earnings at Mobility, Inc. are approximately $210,000 more than Range of Motion.

- What equation is used to calculate the total earnings after n years for the Mobility, Inc. position?
- Do you think the geometric situation will always, in the long run, be more than the arithmetic situation?

Problem 2

A stomach virus and cold virus are spreading rapidly. Information regarding each virus is given and students will answer related questions and use formulas to determine the total number of people infected at specified times. They then estimate the costs involved in producing a vaccine and predict when the virus will be eliminated.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- What function best represents the number of new people infected by the stomach virus?
- Is the function linear or exponential? How do you know?
- What is the difference between the questions in part (a) and part (b)?
- What function best represents the total number of people infected on the 10^{th} day?
- Is the cold virus situation linear or exponential?
- What function best represents the number of new cases reported?

PROBLEM 2 If This Wordplay Doesn't End, I Might "Series-ly" Get Sick!

1. A stomach virus spreads rapidly through a town. Initially only 12 people were infected, but the virus spreads quickly, increasing the number of people infected by 15% every day.
   a. How many new people are infected on the 10th day? Show all work and explain your reasoning.
   The number of new people infected is $f(x) = 12(1.15)^9 = 42$
   b. How many total people were infected on the 10th day? Show all work and explain your reasoning.
   The total number of people infected on the 10th day was $12\left(\frac{1.15^{10} - 1}{1.15 - 1}\right) = 243$.

2. A total of 123,000 cases of a different cold virus were reported throughout the country in a particular year. The production and distribution of a vaccine is projected to decrease the number of reported cases by 26% every year.
   a. Approximately how many new cases will be reported in 15 years? Show all work and explain your reasoning.
   The number of new cases will be $f(x) = 123{,}000(0.74)^{14} = 1{,}816$ in 15 years.
   b. A company reports that vaccine production will cost approximately $9 per person. Estimate the total cost of production for the next 15 years.
   The total cost is $4,211,171.
   I need to determine the total number of cases over 15 years. The total number of cases is $123{,}000\left(\frac{0.74^{15} - 1}{0.74 - 1}\right) = 467{,}908$ cases. At $9 per person, the total cost is $(467{,}908)(9) = \$4{,}211{,}171$.
   c. Will the virus be eliminated? If so, when? Show all work and explain your reasoning.
   Answers will vary.
   The output value never reaches 0, but the number of new cases drops below 1 after 38 years. Therefore, the virus is realistically eliminated in approximately 40 years.

Be prepared to share your solutions and methods.

- What equation is used to determine the total cost of producing the vaccine for the next 15 years?
- What does it mean when the output value reaches 0?
- When does the output value drop below 1? What does this mean with respect to this problem situation?

8

Check for Students' Understanding

Molly breeds guppies and sells them to the local pet store.

The table shows how many guppies she sells to the pet store over the course of 6 weeks.

Week	Number of Guppies
1	20
2	40
3	80
4	160
5	320
6	640

The number of guppies sold to the local pet store is a geometric sequence.

I can use Euclid's Method to determine how many guppies Molly sold to the pet store.

$S_6 = \frac{g_6(r) - g_1}{r - 1}$, where $n = 6$, $g_6 = 640$, $r = 2$, and $g_1 = 20$

$S_6 = \frac{640(2) - 20}{2 - 1}$

$S_6 = 1260$

Molly sold a total of 1260 guppies to the local pet store over the 6 weeks.

Chapter 8 Summary

KEY TERMS

- arithmetic sequence (8.1)
- geometric sequence (8.1)
- finite sequence (8.1)
- infinite sequence (8.1)
- tessellation (8.2)
- series (8.2)
- finite series (8.2)
- infinite series 8.2)
- arithmetic series (8.2)
- geometric series (8.3)
- convergent series (8.4)
- divergent series (8.4)

8.1 Identifying Whether a Sequence is Arithmetic, Geometric, or Neither

An arithmetic sequence is a sequence of numbers in which the difference between two consecutive terms is a constant. A geometric sequence is a sequence of numbers in which the ratio between two consecutive terms is a constant. A sequence that is neither arithmetic nor geometric has no common difference or ratio between two consecutive terms.

Example

The sequence 5, 11, 17, 23, 29 is arithmetic because the difference between consecutive terms is 6.

The sequence $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}$ is geometric because the ratio between two consecutive terms is $\frac{1}{3}$.

The sequence $-2, 0, 4, 10$ is neither arithmetic nor geometric because there is no common difference or ratio between two consecutive terms.

8.1 Writing an Explicit Formula and/or Recursive Formula for Determining a Term of an Arithmetic or Geometric Sequence

An explicit formula for a sequence is a formula for calculating each term of the sequence using the index, which is a term's position in the sequence. A recursive formula expresses each new term of a sequence based on a preceding term of the sequence.

	Arithmetic Sequence	**Geometric Sequence**
Explicit Formula	$a_n = a_1 + d(n - 1)$ where a_1 is the first term, d is the common difference, and n is the nth term in the sequence.	$g_n = g_1 \cdot r^{n-1}$ where g_1 is the first term and r is the common ratio.
Recursive Formula	$a_n = a_{n-1} + d$ where a_{n-1} is the term previous to a_n and d is the common difference.	$g_n = g_{n-1} \cdot r$ where g_{n-1} is the term previous to g_n and r is the common ratio.

Example

The Explicit Formula for the arithmetic sequence 3, 7, 11, 15, 19 where $a_1 = 3$ and $d = 4$ is as shown.

$$a_n = a_1 + d(n - 1)$$
$$= 3 + 4(n - 1)$$
$$= 3 + 4n - 4$$
$$a_n = 4n - 1$$

The Recursive Formula for the geometric sequence $-2, -4, -8, -16$ where $g_1 = -2$ and $r = 2$ is as shown.

$$g_n = g_{n-1} \cdot r$$
$$g_n = g_{n-1} \cdot 2$$

8.1 Using an Explicit Formula and/or Recursive Formula for Determining a Term of an Arithmetic or Geometric Sequence

Deciding whether to use an explicit formula or recursive formula to determine a term in a sequence depends on what information is known. To determine the nth term in an arithmetic sequence use an explicit formula if the first term and common difference are known or use a recursive formula if the $(n - 1)$th term and common difference are know. To determine the nth term in an geometric sequence use an explicit formula if the first term and common ratio are known or use a recursive formula if the $(n - 1)$th term and common ratio are know.

Example

To determine the term a_7 of an arithmetic sequence given $a_6 = 15$ and $d = 3$ use a recursive formula.

$$a_n = a_{n-1} + d$$

$$\begin{aligned} a_7 &= a_6 + 3 \\ &= 15 + 3 \\ &= 18 \end{aligned}$$

To determine the term g_7 of a geometric sequence given $g_1 = 2$ and $r = 4$ use an explicit formula.

$$g_n = g_1 \cdot r^{n-1}$$

$$\begin{aligned} g_7 &= 2 \cdot 4^6 \\ &= 8192 \end{aligned}$$

8.2 Using Sigma Notation to Rewrite the Sum of a Series

Sigma notation is a convenient way to write the sum, S_n, of the series $a_1 + a_2 + a_3 + \cdots + a_n$.

The notation is as follows: $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

Example

Series: $2 + 7 + 12 + 17$

$$\begin{aligned} S_4 &= \sum_{i=1}^{4} a_i \\ &= 2 + 7 + 12 + 17 \end{aligned}$$

8

8.2 Using Gauss's Formula to Compute the First *n* Terms of an Arithmetic Series

To compute S_n, of the first n terms of an arithmetic series Gauss's Formula,

$S_n = \frac{n(a_1 + a_n)}{2}$, can be used providing the first term, a_1, and the last term, a_n, are known.

Example

Series: $-\frac{1}{2} + 0 + \frac{1}{2} + 1 + \frac{3}{2}$

$n = 5, a_1 = -\frac{1}{2}, a_5 = \frac{3}{2}$

$$S_n = \frac{n(a_1 + a_n)}{2}$$

$$S_5 = \frac{5\left(-\frac{1}{2} + \frac{3}{2}\right)}{2}$$

$$= \frac{5}{2}$$

8.2 Determining a Function that Computes the First *n* Terms of an Arithmetic Series

The explicit formula, $a_n = a_1 + d(n - 1)$, together with Gauss's Formula, $S_n = \frac{n(a_1 + a_n)}{2}$, can be used to determine a function that computes the first n terms of an arithmetic series.

Example

Series: $-1 + (-3) + (-5) + (-7) + (-9) + \cdots$

$a_1 = -1, d = -2$

$$a_n = a_1 + d(n - 1)$$
$$= -1 + (-2)(n - 1)$$
$$= -1 - 2n + 2$$
$$a_n = 1 - 2_n$$

$$S_n = \frac{n(a_1 + a_n)}{2}$$
$$= \frac{n[-1 + (1 - 2n)]}{2}$$
$$= \frac{n[-2n]}{2}$$
$$= \frac{-2n^2}{2}$$
$$S_n = -n^2$$

8.2 Solving Real-World Problems Using Finite Arithmetic Series

Identify what the problem is asking. Then determine what information is provided so that the correct approach to the solution can be used. Choose the appropriate formula(s) need, make the correct substitution(s), and solve the problem. Check to see that the question asked has been answered.

Example

Barry planted sunflowers in his garden. In the first row he planted 4 plants, in the second row he planted 10 plants, and in the third row he planted 16 plants. If this pattern continued, how many plants did Barry place in the sixth row?

The problem asks how many sunflower plants are placed in the sixth row. The information provided is that $n = 6$, $a_1 = 4$, and $d = 6$.

$a_n = a_1 + d(n - 1)$

$a_6 = 4 + 6(6 - 1)$

$= 34$ sunflowers

Barry placed 34 sunflower plants in the sixth row.

8.3 Using Euclid's Method to Determine the Sum of a Finite Geometric Series

Euclid's Method can be used to compute a finite geometric series provided the first term, g_1, the last term, g_n, and the common ratio, r, are known. In which case, the formula $S_n = \dfrac{g_n(r) - g_1}{r - 1}$ can be used.

Example

$-5 + (-10) + (-20) + (-40)$

$g_1 = -5, n = 4, g_4 = -40, r = 2$

$S_n = \dfrac{g_n(r) - g_1}{r - 1}$

$S_4 = \dfrac{-40(2) - (-5)}{2 - 1}$

$= -75$

8.3 Using the Formula $S_n = \dfrac{g_1(r^n - 1)}{r - 1}$ to Compute a Finite Geometric Series

The formula, $S_n = \frac{g_1(r^n - 1)}{r - 1}$, can be used to compute a finite geometric series provided the first term, g_1, the common ratio, r, and the number of terms, n, are known.

Example

$2 + 6 + 18 + 54 + 162$

$g_1 = 2, r = 3, n = 5$

$$S_n = \frac{g_1(r^n - 1)}{r - 1}$$

$$S_5 = \frac{2(3^5 - 1)}{3 - 1}$$

$$= 242$$

8.4 Determining Whether an Infinite Geometric Series Converges or Diverges

A convergent series is an infinite series that has a finite sum. A divergent series is an infinite series that does not have a finite sum. If a series is divergent, it is said that the sum is infinity. An infinite geometric series converges providing the common ratio, r, is greater than zero and less than 1.

Example

$$\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \cdots$$

The common ratio for the infinite geometric series is $\frac{1}{3}$ which is greater than zero and less than 1. The series converges.

8.4 Computing an Infinite Convergent Geometric Series

The formula to compute S of a convergent geometric series with a first term g_1 and common ratio r is $S = \frac{g_1}{1 - r}$ provided $0 < r < 1$.

Example

$5 + \frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \cdots$

Observe that $g_1 = 5$ and $r = \frac{1}{2}$. Since r is greater than zero and less than 1, the infinite geometric series converges.

$$S = \frac{g_1}{1 - r}$$

$$S = \frac{5}{1 - \frac{1}{2}}$$

$$= 10$$

8.5 Solving Real-World Problems Involving Geometric Sequences or Series

Identify what the problem is asking. Then determine what information is provided so that the correct approach to the solution can be used. Choose the appropriate formula(s) need, make the correct substitution(s), and solve the problem. Check to see that the question asked has been answered.

Example

Every Saturday Chloe places money in her piggy bank. On the first Saturday she placed \$0.02 in the piggy bank, on the second Saturday she placed \$0.06 in the piggy bank, and on the third Saturday she place \$0.18 in the piggy bank. If Chloe continues to place money in her piggy bank using the same pattern, how much money will she place in her piggy bank on the seventh Saturday?

The problem is asking how much money Chloe places in her piggy bank on the seventh Saturday. The information indicates that Chloe is using a geometric pattern where the first term, g_1, is \$0.02 and the common ratio, r, is 3. The formula needed to answer the question is $g_n = g_1 \cdot r^{n-1}$ where $n = 7$.

$$g_n = g_1 \cdot r^{n-1}$$

$$g_7 = 0.02 \cdot 3^6$$

$$= \$14.58$$

The amount of money Chloe places in her piggy bank on the seventh Saturday is \$14.58.

8.6 Determining Whether a Situation Is Best Modeled by a Arithmetic or Geometric Series

When solving a problem involving a series it is important to determine whether the given situation is arithmetic or geometric. If consecutive terms of the series have a common difference then the series is arithmetic. If consecutive terms of the series have a common ratio then the series is geometric.

Example

Gia applied for a job that initially pays her \$13 per hour but she is guaranteed a yearly raise of \$1.50 per hour.

The progression in her yearly salary is arithmetic with a common difference of \$1.50.

8.6 Solving Real-World Problems Involving Arithmetic or Geometric Sequences or Series

Identify what the problem is asking. Then determine what information is provided so that the correct approach to the solution can be found. Choose the appropriate formula(s) need, make the correct substitution(s), and solve the problem. Check to see that the question asked has been answered.

Example

Belinda likes to write poetry. The table shows how many poems she wrote over the course of 5 weeks. Determine how many poems Belinda wrote.

Week	Number of Poems Written
1	4
2	8
3	16
4	32
5	64

The problem asks for the number of poems written by Belinda. The number of poems written forms a geometric series. Using Euclid's Method, $S_n = \frac{g_n(r) - g_1}{r - 1}$, where $n = 5$, $g_5 = 64$, $r = 2$, and $g_1 = 4$ the number of poems written by Belinda can be determined.

$$S_5 = \frac{g_5(r) - g_1}{r - 1}$$
$$= \frac{64(2) - 4}{2 - 1}$$
$$= 124 \text{ poems}$$

Belinda wrote 124 poems during the 5 week period.