Polynomial Expressions and Equations

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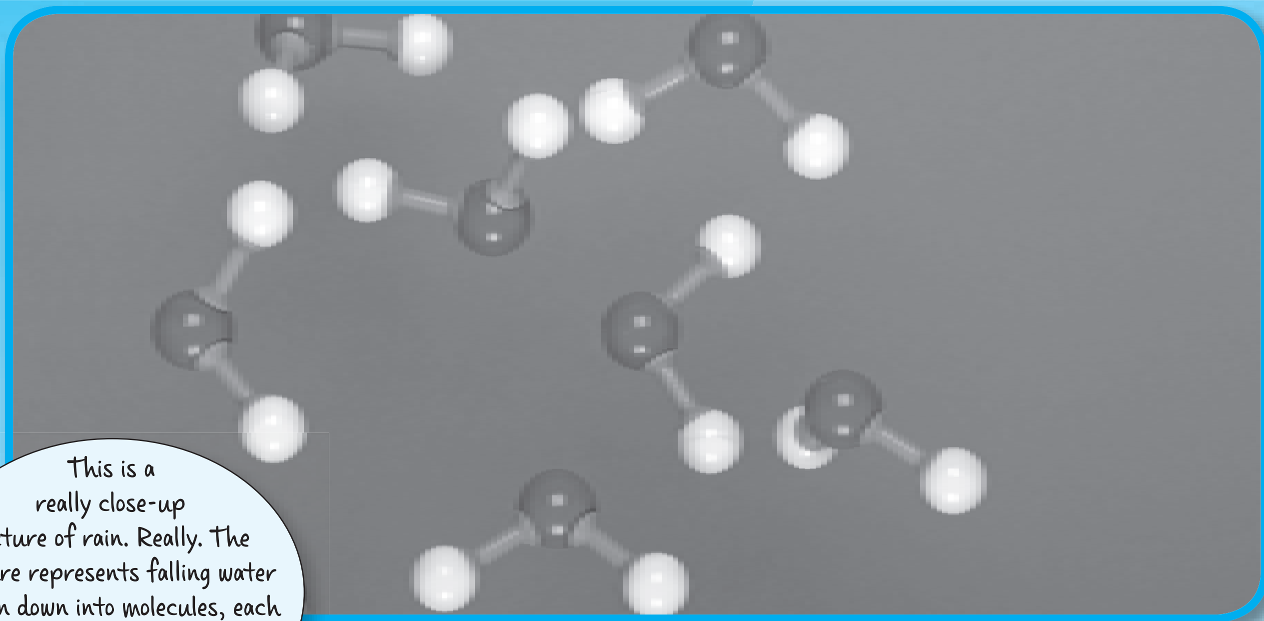

Chapter 6 Overview

This chapter presents opportunities for students to analyze, factor, solve, and expand polynomial functions. The chapter begins with an analysis of key characteristics of polynomial functions and graphs. Lessons then provide opportunities for students to divide polynomials using two methods and to expand on this knowledge in order to determine whether a divisor is a factor of the dividend. In addition, students will solve polynomial equations over the set of complex numbers using the Rational Root Theorem.

In the later part of the chapter, lessons provide opportunities for students to utilize polynomial identities to rewrite numeric expressions and identify patterns. Students will also explore Pascal's Triangle and the Binomial Theorem as methods to expand powers of binomials.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
6.1	Analyzing Polynomial Functions	A.SSE.1.a A.CED.3 A.REI.11 F.IF.4 F.IF.6	1	This lesson provides opportunities for students to analyze the key characteristics of polynomial functions in various problem situations. Questions ask students to answer questions related to a given graph and to determine the average rate of change for an interval.	x	x	x		
6.2	Polynomial Division	A.SSE.1.a A.SSE.3.a A.APR.1	2	This lesson reviews polynomial long division and synthetic division as methods to determine factors of polynomials. Questions lead students to determine factors of a polynomial from one or more zeros of a graph. Questions then ask students to calculate quotients as well as to write dividends as the product of the divisor and the quotient plus the remainder.		x	x		x

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Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
6.3	The Factor Theorem and Remainder Theorem	A.APR.2	1	This lesson presents the Remainder Theorem and Factor Theorem as methods for students to evaluate polynomial functions and to determine factors of polynomials respectively. Questions ask students to verify factors of polynomials and to determine unknown coefficients of functions.		x	x	x	
6.4	Factoring Higher Order Polynomials	N.CN.8 A.SSE.2 A.APR.3 F.IF.8.a	1	This lesson provides opportunities for students to utilize previous factoring methods to factor polynomials completely. Questions include factoring using the GCF, chunking, grouping, quadratic form, sums and differences of cubes, difference of squares, and perfect square trinomials.		x	x	x	
6.5	Rational Root Theorem	A.APR.2 F.IF.8.a	2	This lesson presents the Rational Root Theorem for students to explore solving higher order polynomials. Questions ask students to determine the possible rational roots of polynomials, to factor completely, and to solve polynomial equations over the set of complex numbers.			x		

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Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
6.6	Exploring Polynomial Identities	A.APR.4	2	This lesson provides opportunities for students to use polynomial identities to rewrite numeric expressions and identify patterns. Questions ask students to use polynomial identities such as Euclid's Formula to generate Pythagorean triples and to verify algebraic statements.		x	x		
6.7	Pascal's Triangle and the Binomial Theorem	A.APR.5	1	This lesson reviews Pascal's Triangle and the Binomial Theorem as methods for students to expand powers of binomials. Questions ask students to identify patterns in Pascal's Triangle, and to extend their understanding of the Binomial Theorem to include binomials with coefficients other than one.		x	x		

6

Skills Practice Correlation for Chapter 6

Lesson		Problem Set	Objectives
6.1	Analyzing Polynomial Functions		Vocabulary
		1 – 6	Answer questions in context using a graph
		7 – 12	Determine the average rate of change for given intervals of polynomial functions
		13 – 18	Solve equations using information from graphs
6.2	Polynomial Division		Vocabulary
		1 – 6	Write zeros which correspond to factors
		7 – 12	Write factors which correspond to zeros
		13 – 18	Determine whether given factors are factors of polynomials
		19 – 24	Determine quotients using polynomial long division and rewrite dividends as products
		25 – 30	Determine quotients using synthetic division and rewrite dividends as products
6.3	The Factor Theorem and Remainder Theorem		Vocabulary
		1 – 6	Determine function values using the Remainder Theorem
		7 – 12	Determine whether given expressions are factors of polynomials using the Factor Theorem
		13 – 18	Determine whether given functions are the factored form of other functions using the Factor Theorem
		19 – 24	Determine unknown coefficients given factors of functions using the Factor Theorem
6.4	Factoring Higher Order Polynomials	1 – 6	Factor expressions completely
		7 – 12	Factor expressions using the GCF
		13 – 18	Factor expressions completely using chunking
		19 – 24	Factor expressions completely using grouping
		25 – 30	Factor quartic expressions completely using quadratic form
		31 – 36	Factor binomials using sum or difference of perfect cubes
		37 – 42	Factor binomials completely over the set of real numbers using difference of squares
		43 – 48	Factor perfect square trinomials

Lesson		Problem Set	Objectives
6.5	Rational Root Theorem		Vocabulary
		1 – 6	Determine possible rational roots using the Rational Root Theorem
		7 – 14	List possible rational roots and solve polynomials completely using the Rational Root Theorem
6.6	Exploring Polynomial Identities		Vocabulary
		1 – 6	Use polynomial identities and number properties to perform calculations
		7 – 12	Determine whether sets of numbers are Pythagorean triples
		13 – 18	Generate Pythagorean triples using given numbers and Euclid's Formula
		19 – 24	Verify algebraic statements by transforming equations
6.7	Pascal's Triangle and the Binomial Theorem		Vocabulary
		1 – 6	Use Pascal's Triangle to expand binomials
		7 – 12	Perform calculations and simplify
		13 – 18	Perform calculations and simplify
		19 – 24	Use the Binomial Theorem and substitution to expand binomials

6

6.1

Don't Take This Out of Context

Analyzing Polynomial Functions

LEARNING GOALS

In this lesson, you will:

- Analyze the key characteristics of polynomial functions in a problem situation.
- Determine the average rate of change of a polynomial function.
- Solve equations and inequalities graphically.

KEY TERM

- average rate of change

ESSENTIAL IDEAS

- The average rate of change of a function is the ratio of the independent variable to the dependent variable over a specific interval.
- The formula for average rate of change is $\frac{f(b) - f(a)}{b - a}$ for an interval (a, b). The expression $a - b$ represents the change in the input of the function f. The expression $f(b) - f(a)$ represents the change in the function f as the input changes form a to b.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.
   a. Interpret parts of an expression, such as terms, factors, and coefficients.

A-CED Creating Equations

Create equations that describe numbers or relationships

3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

A-REI Reasoning with Equations and Inequalities

Represent and solve equations and inequalities graphically

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

F-IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

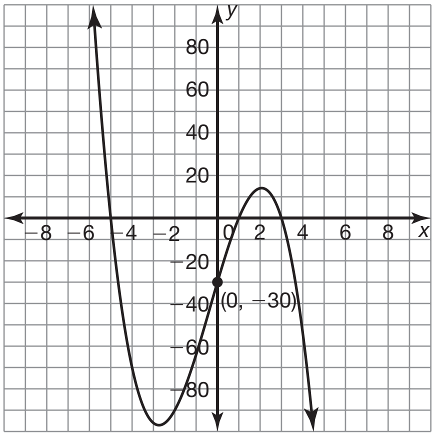
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

Overview

A cubic function is used to model the profit of a business over a period of time. In the first activity, the graph of a function appears on the coordinate plane without numbers. Students will analyze the graph using key characteristics, and then use the graph to answer questions relevant to the problem situation. In the second activity, the same graph now appears on the coordinate plane with numbers on the axes. Students use the graph to determine profit at different times. The average rate of change of a function is defined and a formula is provided. A worked example determines the average rate of change of the company's profit for a specified time interval. Students analyze this worked example so they are able to calculate an average rate of change over a different time interval.

Warm Up

1. Write an equation for the cubic function shown.



$f(x) = a(x + 5)(x - 1)(x - 3)$

$-30 = a(0 + 5)(0 - 1)(0 - 3)$

$-30 = 15a$

$a = -2$

$f(x) = -2(x + 5)(x - 1)(x - 3)$

$f(x) = -2(x^2 - 4x + 3)(x + 5)$

$f(x) = -2x^3 - 2x^2 + 34x - 30$

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6.1

Don't Take This Out of Context

Analyzing Polynomial Functions

LEARNING GOALS

In this lesson, you will:

- Analyze the key characteristics of polynomial functions in a problem situation.
- Determine the average rate of change of a polynomial function.
- Solve equations and inequalities graphically.

KEY TERM

- average rate of change

The *kill screen* is a term for a stage in a video game where the game stops or acts oddly for no apparent reason. More common in classic video games, the cause may be a software bug, a mistake in the program, or an error in the game design. A well-known kill screen example occurs in the classic game *Donkey Kong*. When a skilled player reaches level 22, the game stops just seconds into Mario's quest to rescue the princess. Game over even though the player did not do anything to end the game!

Video game technology has advanced dramatically over the last several decades, so these types of errors are no longer common. Games have evolved from simple movements of basic shapes to real-time adventures involving multiple players from all over the globe.

How do you think video games will change over the next decade?

Problem 1

A cubic function on a numberless graph is used to model a business plan for a video game company for the first couple of years. Students will label intervals on the graph of the function to match various descriptions of events taking place in the company. They then conclude that the cubic function is not the best model for this situation because the end behavior does not match the function.

Grouping

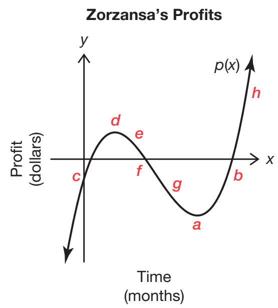
- Ask a student to read the introduction to the problem. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- What are the key characteristics of this graph?
- What is the significance of the zeros with respect to this problem situation?
- How many zeros are on this graph?
- What is the significance of the y-intercepts with respect to this problem situation?
- What is the end behavior of this function?
- Does the end behavior of this function make sense with respect to this problem situation?

PROBLEM 1 Play Is Our Work

The polynomial function $p(x)$ models the profits of Zorzansa, a video game company, from its original business plan through its first few years in business.



1. Label the portion(s) of the graph that model each of the memorable events in the company's history by writing the letter directly on the graph. Explain your reasoning.

   a. The Chief Executive Officer anxiously meets with her accountant.

   Answers will vary.

   Profits are decreasing or reaching a low point.

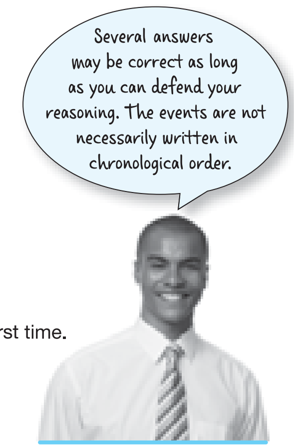
   b. The highly anticipated game, *Rage of Destructive Fury II*, is released.

   Answers will vary.

   Profits increase after the game is released.

   c. The company opens its doors for business for the first time.

   The y-intercept is the point at which the company opens its doors for business for the first time.



   d. The company reaches its first short-term sales goal just as the holiday shopping season ends.

   Answers will vary.

   The local maximum is the highest point over a given interval before profit declines.

- Can sales continually increase for an infinite period of time?
- Should profits increase or decrease immediately after the release of a new game?
- What is the significance of the local maximum with respect to this problem situation?
- Is the y-intercept or the x-intercepts associated with a zero profit?
- Which intervals of the graph indicate poor sales?
- Which intervals of the graph indicate good sales?

e. The company breaks even.

The *x*-intercepts represent when the profit is 0.

f. Members of the Board of Directors get in a heated debate over the next move the company should make.

Answers will vary.

The point at which the company is no longer breaking even.

g. The game design team is fired after their 2 game releases, *Leisurely Sunday Drive* and *Peaceful Resolution*, delight many parents but sell poorly.

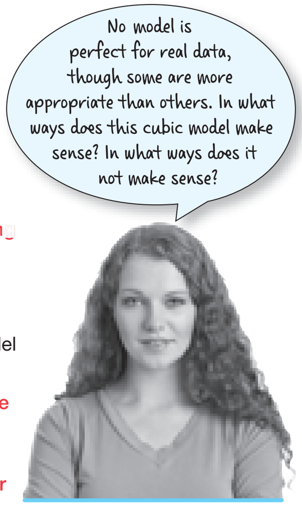
Answers will vary.

A point between the local maximum and local minimum represents when the two games sold poorly.

h. A large conglomerate buys the company.

Answers will vary.

The point at which profits are rising and continuing to rise.



2. Do you think this cubic function is an appropriate model for this scenario? Explain your reasoning.

The cubic function models the increase and decrease in profits. The end behavior does not match the function, because it's not realistic for a company's profits to continue approaching infinity at this rate for an extended period of time.

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Problem 2

The cubic function from Problem 1 is presented, but now appears on a numbered graph. Students use this graph to estimate when the company achieved various profits. They conclude the end behavior makes sense mathematically, but not relevant to this problem situation. The average rate of change of a function is defined and a formula is provided. A worked example determines the average rate of change of the company's profit for a specified time interval. Students then determine the average rate of change for a different time interval and discuss the success of the company over a period of time.

Grouping

- Ask a student to read the introduction. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

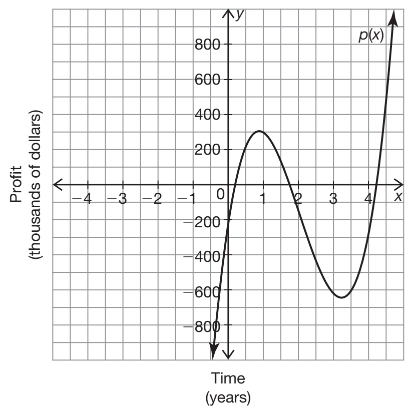
Guiding Questions for Share Phase, Question 1

- How is this graph different than the graph in the previous problem?
- How is this graph similar to the graph in the previous problem?
- What equation best represents a line on the graph where the profit is $800,000?

PROBLEM 2 There's Nothing "Average" About This Rate of Change

The cubic function $p(x)$ models Zorzansa's total profits over the first five years of business.

Zorzansa's Profits Over Years 0 – 5



1. Use the graph to estimate when Zorzansa's achieved each profit. Then explain how you determined your estimate.

   a. $800,000

   The profit was $800,000 at approximately 4.6 years. I drew a horizontal line at $y = 800$ and estimated the x-value at the point of intersection.

   b. $200,000

   The profit was $200,000 at approximately 0.5, 1.3, and 4.4 years. I drew a horizontal line at $y = 200$ and estimated the x-value at each point of intersection.

   c. greater than $200,000

   The profit was greater than $200,000 for the approximate intervals (0.5, 1.3) and (4.4, ∞). I located the x-values for all points above the horizontal line $y = 200$.

   d. the company is losing money

   The profit is negative for the approximate intervals (0.0, 0.2) and (1.75, 4.2). I located the x-values for all points below the x-axis.

   e. the company is making a profit.

   The company is making a profit for the time interval (0.2, 1.75) and (4.2, ∞).

- What equation best represents a line on the graph where the profit is $200,000?
- What do all of the points on the graph above the horizontal line $y = 200$ represent with respect to this problem situation?
- If the profit is a negative value, what does this mean with respect to this problem situation?
- If the profit is a positive value, what does this mean with respect to this problem situation?

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Guiding Questions for Share Phase, Question 2

- How does the domain of the function compare to the domain of this problem situation?
- How does the range of the function compare to the range of this problem situation?

2. Avi and Ariella disagree about the end behavior of the function.

Avi	Ariella
The end behavior is incorrect. As time increases, profit approaches infinity. It doesn't make sense that the profits are increasing before the company even opens.	The end behavior is correct. The function is cubic with a positive a-value. This means as x approaches infinity, y approaches infinity. As x approaches negative infinity, y also approaches negative infinity.

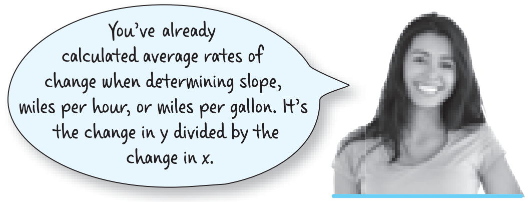
Who is correct? Explain your reasoning.

They are both correct. The end behavior makes sense mathematically, but not in terms of the problem context.

Grouping

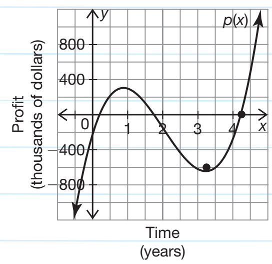
- Ask a student to read the information, example, and definitions. Discuss as a class.
- Complete Question 3 as a class.

The **average rate of change** of a function is the ratio of the change in the dependent variable to the change in the independent variable over a specific interval. The formula for average rate of change is $\frac{f(b) - f(a)}{b - a}$ for the interval (a, b). The expression $b - a$ represents the change in the input values of the function f. The expression $f(b) - f(a)$ represents the change in the output values of the function f as the input values change from a to b.



You can determine the average rate of change of Zorzansa's profit for the time interval (3.25, 4.25).

Zorzansa's Profits Over Years 0 – 5



Substitute the input and output values into the average rate of change formula.

$$\frac{f(b) - f(a)}{b - a} = \frac{f(4.25) - f(3.25)}{4.25 - 3.25}$$

Simplify the expression.

$$= \frac{0 - (-600)}{1}$$

$$= \frac{600}{1} = 600$$

The average rate of change for the time interval (3.25, 4.25) is approximately $600,000 per year.

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Guiding Questions for Discuss Phase, Question 3

- What is the independent variable in this problem situation?
- What is the dependent variable in this problem situation?
- What unit of measure is associated with the x-axis?
- What unit of measure is associated with the y-axis?
- How was the value for $f(4.25)$ determined in the worked example?
- How was the value for $f(3.25)$ determined in the worked example?
- If the slope over the interval is positive, should the average rate of change for this interval also be positive?
- If the slope over the interval is negative, should the average rate of change for this interval also be negative?
- What is the relationship between the slope of the interval and the average rate of change over the interval?

Grouping

Have students complete Questions 4 through 6 with a partner. Then have students share their responses as a class.

3. Analyze the worked example.

a. Explain why the average rate of change is \$600,000 per year, and not \$600 per year.

The y-values are measured in thousands of dollars.

b. Explain why the average rate of change is positive over this interval.

The rate of change is positive because the profit is increasing from 3.25 to 4.25 years.

c. What does the average rate of change represent in this problem situation?

It represents the average change in the profit over the time interval of 3.25 years to 4.25 years.

4. Determine the average rate of change of Zorzansa's profits for the time interval (1, 3).

$f(3) \approx -600$ and $f(1) \approx 300$

$$\frac{f(3) - f(1)}{3 - 1} = \frac{-600 - 300}{3 - 1}$$

$$= \frac{-900}{2} = -450$$

So the average rate of change of Zorzansa's profits from Year 1 to Year 3 is about $-$\$450,000.

Guiding Questions for Share Phase, Question 4

- How was the value for $f(1)$ determined?
- How was the value for $f(3)$ determined?

Guiding Questions for Share Phase, Questions 5 and 6

- Did Sam add or subtract the numbers in this situation?
- Should the values add to 0 when the rate of change is a negative number?
- Did the profit decrease over the entire interval?
- Is Zorzansa headed in the right direction? How so?
- What happened to the profit over Year 1 to Year 2.5?

5. Sam has a theory about the average rate of change.

> **Sam**
>
> *I can quickly estimate the average rate of change for intervals that are above and below the x-axis because they add to zero. For example, at year 1, the profit is about \$300,000 and at year 2.25 the profit is about −\$300,000. Therefore, the average rate of change for the time interval (1, 2.25) is approximately \$0.*

Describe the error in Sam's reasoning.

Sam calculated the sum of the profits for Years 1 and 2.25. He should have calculated the change in the profits from year 1 to 2.25 and divided that change by the change in the number of years. The profit decreased over the entire interval, so it doesn't make sense that the average rate of change would be 0.

6. After 4.5 years, would you consider Zorzansa a successful business? Explain your reasoning.

Answers will vary.

After 4.5 years, Zorzansa is making a profit and headed in the right direction. However, between years 1 and 2.25, Zorzansa lost as much as it made.

Be prepared to share your solutions and methods.

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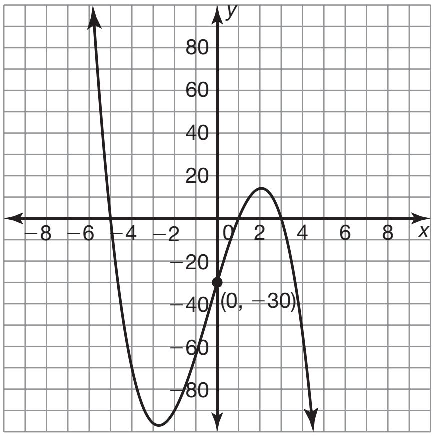
Check for Students' Understanding

Determine the average rate of change for the time interval (0, 3).



$f(3) = 0$

$f(0) = -30$

$$\frac{f(3) - f(0)}{3 - 0} = \frac{0 - (-30)}{3 - 0}$$

$$= \frac{30}{3}$$

$$= 10$$

6

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The Great Polynomial Divide

6.2

Polynomial Division

LEARNING GOALS

In this lesson, you will:

- Describe similarities between polynomials and integers.
- Determine factors of a polynomial using one or more roots of the polynomial.
- Determine factors through polynomial long division.
- Compare polynomial long division to integer long division.

KEY TERMS

- polynomial long division
- synthetic division

ESSENTIAL IDEAS

- A polynomial equation of degree n has n roots and can be written as the product of n factors of the form $(ax + b)$.
- Factors of polynomials divide into a polynomial without a remainder.
- Polynomial long division is an algorithm for dividing one polynomial by another of equal or lesser degree.
- When a polynomial is divided by a factor, the remainder is zero.
- Synthetic division is a shortcut method for dividing a polynomial by a linear factor of the form $(x - r)$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.
   a. Interpret parts of an expression, such as terms, factors, and coefficients.

Write expressions in equivalent forms to solve problems.

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
   a. Factor a quadratic expression to reveal the zeros of the function it defines.

A-APR Arithmetic with Polynomials and Rational Expressions

Perform arithmetic operations on polynomials

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Overview

A cubic function is graphed and students will determine the factors of the function. Two factors are imaginary and one factor is real. A table of values is used to organize data and reveal observable patterns. Identifying key characteristics helps students identify the quadratic function that is a factor of the cubic function and using algebra, they are able to identify its imaginary factors. A worked example of polynomial long division is provided. Students then use the algorithm to determine the quotient in several problems. Tables, graphs, and long division are all used to determine the factors of different functions. A worked example of synthetic division is provided. Students use the algorithm to determine the quotient in several problems. They also use a graphing calculator to compare the graphs and table of values for pairs of functions to determine if they are equivalent. A graphing calculator is used to compare the graphical representations of three functions and identify the key characteristics.

Warm Up

Use the long division algorithm to determine each quotient.

1. $3\overline{)375}$

$$\begin{array}{r}
125 \\
3\overline{)375} \\
\underline{3} \\
7 \\
\underline{6} \\
15 \\
\underline{15} \\
0
\end{array}$$

2. $6\overline{)852}$

$$\begin{array}{r}
142 \\
6\overline{)852} \\
\underline{6} \\
25 \\
\underline{24} \\
12 \\
\underline{12} \\
0
\end{array}$$

3. $15\overline{)328}$

$$\begin{array}{r}
21 \\
15\overline{)328} \\
\underline{30} \\
28 \\
\underline{15} \\
13
\end{array}$$

Remainder

6

6

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The Great Polynomial Divide

Polynomial Division

6.2

LEARNING GOALS

In this lesson, you will:

- Describe similarities between polynomials and integers.
- Determine factors of a polynomial using one or more roots of the polynomial.
- Determine factors through polynomial long division.
- Compare polynomial long division to integer long division.

KEY TERMS

- polynomial long division
- synthetic division

Did you ever notice how little things can sometimes add up to make a huge difference? Consider something as small and seemingly insignificant as a light bulb. For example, a compact fluorescent lamp (CFL) uses less energy than "regular" bulbs. Converting to CFLs seems like a good idea, but you might wonder: how much good can occur from changing one little light bulb? The answer is a lot—especially if you convince others to do it as well. According to the U.S. Department of Energy, if each home in the United States replaced one light bulb with a CFL, it would have the same positive environmental effect as taking 1 million cars off the road!

If a new product such as the CFL can have such a dramatic impact on the environment, imagine the effect that other new products can have. A group of Canadian students designed a car that gets over 2,500 miles per gallon, only to be topped by a group of French students whose car gets nearly 7,000 miles per gallon! What impacts on the environment can you describe if just 10% of the driving population used energy efficient cars? Can all of these impacts be seen as positive?

Problem 1

Students will determine the factors from one or more zeros of a polynomial from a graph. The function is cubic with one real zero and two imaginary zeros. A table of values is used to organize values of the real factor, the quadratic function, and the polynomial function for different values of *x*. The table reveals patterns which help students determine the key characteristics of the quadratic function, which leads to the equation of the quadratic function. Once the quadratic equation is identified, students are able to determine the two imaginary factors. Steps to determine a regression equation of a set of data on a graphing calculator are provided in this problem.

Grouping

- Ask a student to read the introduction to the problem. Discuss as a class.
- Have students complete Question 1 with a partner. Then have students share their responses as a class.

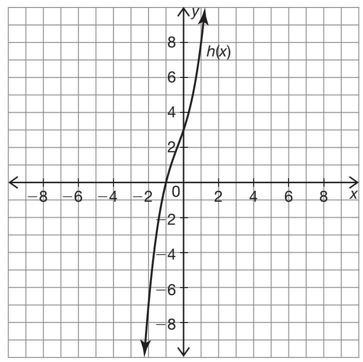
Guiding Questions for Share Phase, Question 1

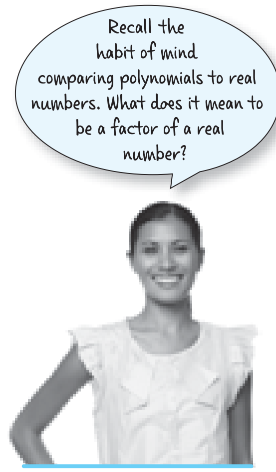
- How many zeros are associated with a cubic function?
- If the graph of the cubic function has exactly one *x*-intercept, how many of its zeros are real numbers?

PROBLEM 1 A Polynomial Divided . . .

The previous function-building lessons showed how the factors of a polynomial determine its key characteristics. From the factors, you can determine the type and location of a polynomial's zeros. Algebraic reasoning often allows you to reverse processes and work backwards. Specifically in this problem, you will determine the factors from one or more zeros of a polynomial from a graph.

1. Analyze the graph of the function $h(x) = x^3 + x^2 + 3x + 3$.





   a. Describe the number and types of zeros of $h(x)$.

   The function $h(x)$ is cubic. From the Fundamental Theorem of Algebra, I know this function has 3 zeros. There is 1 real zero and 2 imaginary zeros.

   b. Write the factor of $h(x)$ that corresponds to the zero at $x = -1$.

   The factor is $(x + 1)$ because $x = -1$ is a solution to $x + 1 = 0$

   c. What does it mean to be a factor of $h(x) = x^3 + x^2 + 3x + 3$?

   It means that $(x + 1)$ divides into $h(x) = x^3 + x^2 + 3x + 3$ without a remainder.

   d. How can you write any zero, *r*, of a function as a factor?

   A zero *r* can be written as a factor $(x - r)$.

- If the *x*-intercept of the cubic function is $(-1, 0)$, how is this root written as a factor?
- If $(x + 1)$ is a factor of the cubic function, does it divide into the function without a remainder?
- What is the difference between a zero and a factor?
- What is the difference between a root and a zero?
- What is the difference between a factor and a root?

Grouping

Have students complete Questions 2 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- How did you determine $q(x)$ when $x = -3$?
- How did you determine $q(x)$ when $x = -1$?
- How did you determine $q(x)$ when $x = 0$?
- How many outputs are associated with each input for any function?
- If two polynomials are multiplied together, is the result always a polynomial?
- What degree function is $q(x)$?
- Considering the vertex and line of symmetry associated with the quadratic function, how does using the pattern in the table of values help determine the output value for $q(-1)$?

In Question 1 you determined that $(x + 1)$ is a factor of $h(x)$. One way to determine another factor of $h(x)$ is to analyze the problem algebraically through a table of values.

2. Analyze the table of values for $d(x) \cdot q(x) = h(x)$.

x	$d(x) = (x + 1)$	$q(x)$	$h(x) = x^3 + x^2 + 3x + 3$
-3	-2	12	-24
-2	-1	7	-7
-1	0	4	0
0	1	3	3
1	2	4	8
2	3	7	21

a. Complete the table of values for $q(x)$. Explain your process to determine the values for $q(x)$.

See table of values.

I divided $h(x)$ by $d(x)$.

b. Two students, Tyler and McCall, disagree about the output $q(-1)$.

Tyler

The output value $q(-1)$ can be any integer. I know this because $d(-1) = 0$. Zero times any number is 0, so I can complete the table with any value for $q(-1)$.

McCall

I know $q(x)$ is a function so only one output value exists for $q(-1)$. I have to use the key characteristics of the function to determine that exact output value.

Who is correct? Explain your reasoning, including the correct output value(s) for $q(-1)$.

McCall is correct.

Polynomials are closed under multiplication, so $q(x)$ is a function with only one possible output value for $q(-1)$. I know that $q(x)$ is a quadratic function. This guarantees a vertex and line of symmetry. From the pattern in the table of values I know that 4 must be the output.

c. How can you tell from the table of values that $d(x)$ is a factor of $h(x)$?

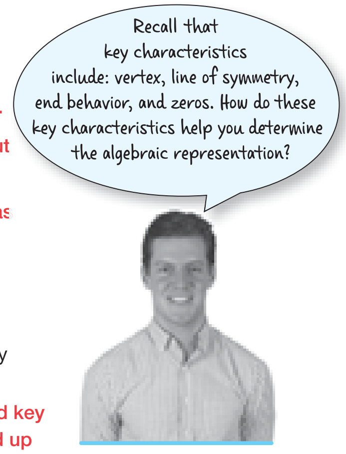
All of the points divide evenly without a remainder.

Guiding Questions for Share Phase, Questions 3 and 4

- Do all of the points associated with $d(x)$ divide evenly without a remainder into $h(x)$?
- Does the graph of the quadratic function open upward or downward? How do you know?
- What are the coordinates of the vertex of the graph of the quadratic function?
- Where is the line of symmetry for $q(x)$?
- What is the end behavior of the quadratic function?
- Using the location of the vertex and the key points, how would you describe the transformation of the power function x^2?
- If the power function x^2 was shifted up 3 units, how is this captured algebraically?
- If $x^2 = -3$, what are the values of x?
- If $x^2 = -3$, are the values of x real or imaginary?
- What is the product of $\sqrt{3}i$ and $-\sqrt{3}i$?

3. Describe the key characteristics of $q(x)$. Explain your reasoning.

- The function $q(x)$ is quadratic.
- The graph of $q(x)$ is a parabola that opens up.
- The graph of the function has symmetry about $x = 0$ with vertex (0, 3).
- The output values approach positive infinity as x approaches positive and negative infinity.



4. What is the algebraic representation for $q(x)$? Verify algebraically that $d(x) \cdot q(x)$ is equivalent to $h(x)$.

The function is $q(x) = x^2 + 3$. From the vertex and key points I know that it is the power function shifted up 3 units.

$(x + 1)(x^2 + 3)$

$x^3 + 3x + x^2 + 3$

$x^3 + x^2 + 3x + 3$

You can use a graphing calculator to determine the quadratic, cubic, or quartic regression of a set of data.

Step 1: Press **ENTER** to return to the Plot Menu. Scroll down to **7:QUIT**. Press **ENTER**.

Step 2: You will see on the screen Lists 1 through 4 (**L1-L4**). The data needed to determine the quadratic, cubic, or quartic regression is located in Lists 1 and 2 which measures time and distance.

Step 3: Press **STAT**. Scroll right to **CALC**. Choose **5:QuadReg**. Press **ENTER**.

Step 4: Select **L1**, **L2**. Press **ENTER**.

Step 5: The information shows you the standard form of a quadratic, cubic, or quartic equation and the values of a, b, c, and r2.

5. Determine the zeros of $q(x)$. Then rewrite $h(x)$ as a product of its factors.

Solving $x^2 + 3 = 0$, I determined that the zeros are $x = i\sqrt{3}$ and $x = -i\sqrt{3}$. Written as a product of its factors, $h(x) = (x + 1)(x + i\sqrt{3})(x - i\sqrt{3})$.

6

Problem 2

A worked example of polynomial long division is provided. In the example, integer long division is compared to polynomial long division and each step of the process is described verbally. Students analyze the worked example comparing both algorithms and then they use the algorithm to determine the quotient in several problems. Students identify factors of functions by division. If the remainder is not zero, then the divisor cannot be a factor of the dividend or function. Tables, graphs, and long division are all used to determine the factors of different functions.

Grouping

Ask a student to read the information, definition, and worked example. Discuss as a class.

PROBLEM 2 Long Story Not So Short

The Fundamental Theorem of Algebra states that every polynomial equation of degree n must have n roots. This means that every polynomial can be written as the product of n factors of the form $(ax + b)$. For example, $2x^2 - 3x - 9 = (2x + 3)(x - 3)$.

You know that a factor of an integer divides into that integer with a remainder of zero. This process can also help determine other factors. For example, knowing 5 is a factor of 115, you can determine that 23 is also a factor since $\frac{115}{5} = 23$. In the same manner, factors of polynomials also divide into a polynomial without a remainder. Recall that $a \div b$ is $\frac{a}{b}$, where $b \neq 0$.

Polynomial long division is an algorithm for dividing one polynomial by another of equal or lesser degree. The process is similar to integer long division.

Notice in the dividend of the polynomial example, there is a gap in the degrees of the terms; every power must have a placeholder. The polynomial $8x^3 - 12x - 7$ does not have an x^2 term.

Integer Long Division	Polynomial Long Division	Description
$4027 \div 12$ or $\frac{4027}{12}$	$(8x^3 - 12x - 7) \div (2x + 3)$ or $\frac{8x^3 - 12x - 7}{2x + 3}$	
$\begin{array}{r} 335 \\ 12\overline{)4027} \\ -36 \\ \hline 42 \\ -36 \\ \hline 67 \\ -60 \\ \hline 7 \end{array}$ Remainder	$\begin{array}{r} 4x^2 - 6x + 3 \\ 2x + 3\overline{)8x^3 + 0x^2 - 12x - 7} \\ -(8x^3 + 12x^2) \\ \hline -12x^2 - 12x \\ -(-12x^2 - 18x) \\ \hline 6x - 7 \\ -(6x + 9) \\ \hline \end{array}$ Remainder -16	A. Rewrite the dividend so that each power is represented. Insert $0x^2$. B. Divide $\frac{8x^3}{2x} = 4x^2$. C. Multiply $4x^2(2x + 3)$, and then subtract. D. Bring down $-12x$. E. Divide $\frac{-12x^2}{2x} = -6x$. F. Multiply $-6x(2x + 3)$, and then subtract. G. Bring down -7. H. Divide $\frac{6x}{2x} = 3$. I. Multiply $3(2x + 3)$, and then subtract.
$\frac{4027}{12} = 335 \text{ R } 7$	$\frac{8x^3 - 12x - 7}{2x + 3} =$ $4x^2 - 6x + 3 \text{ R } -16$	Rewrite
$4027 = (12)(335) + 7$	$8x^3 - 12x - 7 =$ $(2x + 3)(4x^2 - 6x + 3) - 16$	Check

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Why are zero coefficients used as placeholders for gaps in any degrees of the terms of the polynomial dividend?
- When $f(x) = 8x^3 - 12x - 7$ is divided by $2x + 3$, is there a remainder? What does this imply?

Grouping

Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 2 and 3

- Which quotients do not have a remainder? What does this imply?
- Which quotients have a remainder? What does this imply?
- If the term $0x^2$ was not included in the dividend in part (a), what might have happened?
- Why does a polynomial divided by one of its factors always have a remainder of zero?

6

1. Analyze the worked example that shows integer long division and polynomial long division.

   a. In what ways are the integer and polynomial long division algorithms similar?

   Answers will vary.

   The steps of the algorithm are basically the same. Complex division is broken down into a series of smaller division problems. You divide, multiply, and subtract. Also, zero coefficients are used as placeholders for gaps in any degrees of the terms of the polynomial dividend.

To determine another factor of $x^3 + x^2 + 3x + 3$ in Problem 1, you completed a table, divided output values, and then determined the algebraic expression of the result. Polynomial Long Division is a more efficient way to calculate.

   b. Is $2x + 3$ a factor of $f(x) = 8x^3 - 12x - 7$? Explain your reasoning.

   No. The expression $2x + 3$ is not a factor because the remainder is not 0.

2. Determine the quotient for each. Show all of your work.

   a. $x\overline{)4x^3 - 0x^2 + 7x}$

$$
\begin{array}{r}
4x^2 - 0x + 7 \\
x\overline{)4x^3 - 0x^2 + 7x + 0} \\
\underline{4x^3 } \\
0 - 0x^2 \\
\underline{- 0x^2} \\
0 + 7x \\
\underline{- 7x} \\
0
\end{array}
$$

   $4x^2 + 7$

   b. $x - 4\overline{)x^3 + 2x^2 - 5x + 16}$

$$
\begin{array}{r}
x^2 + 6x + 19 \\
x - 4\overline{)x^3 + 2x^2 - 5x + 16} \\
\underline{x^3 - 4x^2} \\
6x^2 - 5x \\
\underline{6x^2 - 24x} \\
19x + 16 \\
\underline{19x - 76} \\
92
\end{array}
$$

   $x^2 + 6x + 19$ R 92

   c. $(4x^4 + 5x^2 - 7x + 9) \div (2x - 3)$

$$
\begin{array}{r}
2x^3 + 3x^2 + 7x + 7 \\
2x - 3\overline{)4x^4 + 0x^3 + 5x^2 - 7x + 9} \\
\underline{4x^4 - 6x^3} \\
6x^3 + 5x^2 \\
\underline{6x^3 - 9x^2} \\
14x^2 - 7x \\
\underline{14x^2 - 21x} \\
14x + 9 \\
\underline{14x - 21} \\
30
\end{array}
$$

   $2x^3 + 3x^2 + 7x + 7$ R 30

   d. $(9x^4 + 3x^3 + 4x^2 + 7x + 2) \div (3x + 2)$

$$
\begin{array}{r}
3x^3 - x^2 + 2x + 1 \\
3x + 2\overline{)9x^4 + 3x^3 + 4x^2 + 7x + 2} \\
\underline{9x^4 + 6x^3} \\
-3x^3 + 4x^2 \\
\underline{-3x^3 - 2x^2} \\
6x^2 + 7x \\
\underline{6x^2 + 4x} \\
3x + 2 \\
\underline{3x + 2} \\
0
\end{array}
$$

   $3x^3 - x^2 + 2x + 1$

3. Consider Question 2 parts (a) through (d) to answer each.

   a. Why was the term $0x^2$ included in the dividend in part (a)? Why was this necessary?

   There was a gap in the degrees of the dividend. The dividend was rewritten so that each power was represented. This is necessary to help keep the powers aligned.

   b. When there was a remainder, was the divisor a factor of the dividend? Explain your reasoning.

   No. If the remainder is not zero, then the divisor cannot be a factor of the dividend.

   c. Describe the remainder when you divide a polynomial by a factor.

   The remainder is 0.

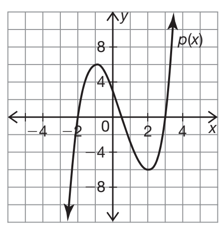
4. Determine whether $m(x) = 2x + 1$ is a factor of each function. Explain your reasoning.

   a. $j(x) = 2x^3 + 3x^2 + 7x + 5$

   No. The function $j(x)$ does not divide evenly without a remainder; therefore $m(x)$ is not a factor.

$$
\begin{array}{r l}
 & x^2 + x + 3 \text{ R } 2 \\
2x + 1 \,\big) & 2x^3 + 3x^2 + 7x + 5 \\
 & \underline{2x^3 + x^2} \\
 & 2x^2 + 7x \\
 & \underline{2x^2 + x} \\
 & 6x + 5 \\
 & \underline{6x + 3} \\
 & 2
\end{array}
$$

   b.



   No. The graph of $p(x)$ does not have a zero at $-\frac{1}{2}$. So, $m(x)$ is not a factor of $p(x)$.

Grouping

Have students complete Question 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- If $m(x)$ divides into $j(x)$ with no remainder, is $m(x)$ always a factor of $j(x)$?
- If $2x + 1$ is a factor, what zero is associated with this factor?
- Is the zero associated with the factor $2x + 1$ on the graph of the function?
- If the zero associated with the factor $2x + 1$ is not on the graph of the function, what does this imply?
- If x has a value of -2, what is the value of $m(x)$? What is the value of $k(x) + m(x)$?
- If x has a value of -1, what is the value of $m(x)$? What is the value of $k(x) + m(x)$?
- If x has a value of 0, what is the value of $m(x)$? What is the value of $k(x) + m(x)$?
- Do all of the outputs divide evenly? What does this imply?
- Does the table of values represent the function $k(x) = 2x^2 + 11x + 5$?
- Does the function $m(x)$ divide into $k(x)$ without a remainder?

c.

x	$k(x)$
-2	-9
-1	-4
0	5
1	18
2	35

$m(x)$	$k(x) \div m(x)$
-3	3
-1	4
1	5
3	6
5	7

Yes. The function $m(x)$ is a factor of $k(x)$ because all of the outputs divide evenly. The table represents the function $k(x) = 2x^2 + 11x + 5$. The function $m(x)$ divides into $k(x)$ without a remainder.

Grouping

Have students complete Question 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- How did you solve for x in this situation?
- Why won't the product of $(x + 3)$ and $(3x^2 + 14x + 15)$ be equivalent to $p(x)$?
- How was the distributive property used to solve for x?
- Looking at the graph of $p(x)$, describe the behavior when x has the value of -3.
- Looking at the equation of the function $p(x)$, why is the function undefined when x has the value of -3?

5. Determine the unknown in each.

a. $\frac{x}{7} = 18$ R 2. Determine x.

The value for x is $(18)(7) + 2 = 128$.

b. $\frac{p(x)}{x + 3} = 3x^2 + 14x + 15$ R 3. Determine the function $p(x)$.

The function is $p(x) = (x + 3)(3x^2 + 14x + 15) + 3 = 3x^3 + 23x^2 + 57x + 48$.

c. Describe the similarities and differences in your solution strategies.

The solution strategies are basically the same. Working with polynomials, I had to use the distributive property and combine like terms, but the processes are the same.

d. Use a graphing calculator to analyze the graph and table of $\frac{p(x)}{x + 3}$ over the interval $(-10, 10)$. What do you notice?

The table shows an error at $x = -3$. The graph shows a vertical line at $x = -3$.

6

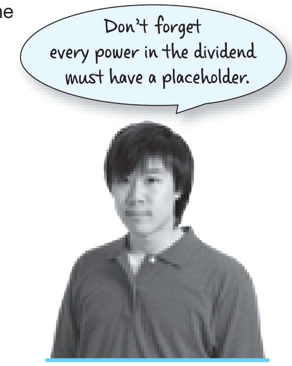
Grouping

Have students complete Questions 6 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6, parts (a) and (b)

- What is the degree of the quotient when a quadratic function is divided by a linear function?
- What is the degree of the quotient when a cubic function is divided by a linear function?

6. Calculate the quotient using long division. Then write the dividend as the product of the divisor and the quotient plus the remainder.



a. $f(x) = x^2 - 1$

$g(x) = x - 1$

Calculate $\frac{f(x)}{g(x)}$.

$$\begin{array}{r} x+1 \\ x-1\overline{)x^2+0x-1} \\ \underline{x^2-1x} \\ x-1 \\ \underline{x-1} \\ 0 \end{array}$$

$\frac{f(x)}{g(x)} = x + 1$

$x^2 - 1 = (x - 1)(x + 1)$

b. $f(x) = x^3 - 1$

$g(x) = x - 1$

Calculate $\frac{f(x)}{g(x)}$.

$$\begin{array}{r} x^2+x+1 \\ x-1\overline{)x^3+0x^2+0x-1} \\ \underline{x^3-x^2} \\ x^2+0x \\ \underline{x^2-x} \\ x-1 \\ \underline{x-1} \\ 0 \end{array}$$

$\frac{f(x)}{g(x)} = x^2 + x + 1$

$x^3 - 1 = (x - 1)(x^2 + x + 1)$

Guiding Questions for Share Phase, Question 6, parts (c) and (d)

- What is the degree of the quotient when a quartic function is divided by a linear function?
- What is the degree of the quotient when a quintic function is divided by a linear function?

c. $f(x) = x^4 - 1$

$g(x) = x - 1$

Calculate $\frac{f(x)}{g(x)}$.

$$\begin{array}{rl} & x^3 + x^2 + x + 1 \\ x - 1 \!\!\!& \overline{)\,x^4 + 0x^3 + 0x^2 + 0x - 1} \\ & \underline{x^4 - x^3} \\ & x^3 + 0x^2 \\ & \underline{x^3 - x^2} \\ & x^2 + 0x \\ & \underline{x^2 - x} \\ & x - 1 \\ & \underline{x - 1} \\ & 0 \end{array}$$

$\frac{f(x)}{g(x)} = x^3 + x^2 + x + 1$

$x^4 - 1 = (x - 1)(x^3 + x^2 + x + 1)$

d. $f(x) = x^5 - 1$

$g(x) = x - 1$

Calculate $\frac{f(x)}{g(x)}$.

$$\begin{array}{rl} & x^4 + x^3 + x^2 + x + 1 \\ x - 1 \!\!\!& \overline{)\,x^5 + 0x^4 + 0x^3 + 0x^2 + 0x - 1} \\ & \underline{x^5 - x^4} \\ & x^4 + 0x^3 \\ & \underline{x^4 - x^3} \\ & x^3 + 0x^2 \\ & \underline{x^3 - x^2} \\ & x^2 + 0x \\ & \underline{x^2 - x} \\ & x - 1 \\ & \underline{x - 1} \\ & 0 \end{array}$$

$\frac{f(x)}{g(x)} = x^4 + x^3 + x^2 + x + 1$

$x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$

6

Guiding Questions for Share Phase, Questions 7 and 8

- When $p(x)$ is divided by $q(x)$, do the output values divide evenly? What does this imply?
- How did you determine the remainder for each x value?
- Is the quotient of two polynomials always a polynomial?

Problem 3

A worked example of synthetic division is provided. In the example, polynomial long division is compared to synthetic division. Students analyze the worked example comparing both algorithms and then they use the algorithm to determine the quotient in several problems. Students will use a graphing calculator to compare the graphs and table of values for pairs of functions to determine if they are equivalent or how they are different. Before performing synthetic division, students factor out a constant, in order to rewrite the divisor in the form $x - r$. They then determine the quotient of several functions using synthetic division. A graphing calculator is used to compare the graphical representations of three functions and identify the key characteristics.

7. Analyze the table of values. Then determine if $q(x)$ is a factor of $p(x)$. If so, explain your reasoning. If not, determine the remainder of $\frac{p(x)}{q(x)}$. Use the last column of the table to show your work.

x	$q(x)$	$p(x)$	$\frac{p(x)}{q(x)}$
0	1	5	5
1	2	7	3.5
2	3	9	3
3	4	11	2.75
4	5	13	2.6

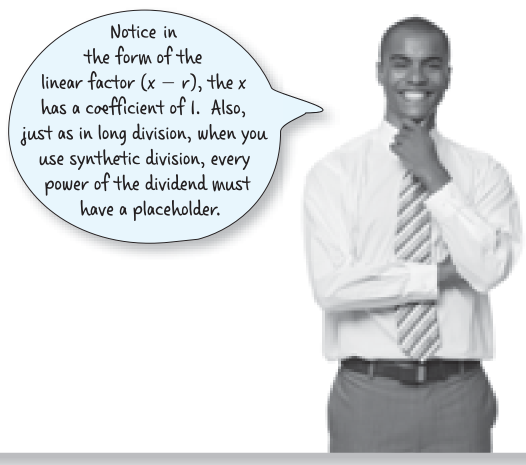
The function $q(x)$ is not a factor of $p(x)$ since the output values do not divide evenly.

8. Look back at the various polynomial division problems you have seen so far. Do you think polynomials are closed under division? Explain your reasoning.

Polynomials are not closed under division.

PROBLEM 3 Improve Your Efficiency Rating

Although dividing polynomials through long division is analogous to integer long division, it can still be inefficient and time consuming. **Synthetic division** is a shortcut method for dividing a polynomial by a linear factor of the form $(x - r)$. This method requires fewer calculations and less writing by representing the polynomial and the linear factor as a set of numeric values. After the values are processed, you can then use the numeric outputs to construct the quotient and the remainder.



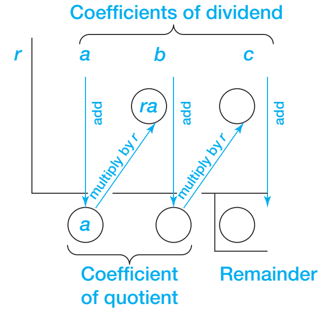
Grouping

- Ask a student to read the information, definition, and worked example. Discuss as a class.
- Complete Question 1 as class.

Guiding Questions for Discuss Phase

- How is synthetic division similar to polynomial long division?
- How is synthetic division different than polynomial long division?
- Where is the dividend located in this example of synthetic division?
- What is the relationship between the divisor and the number that appears in the extreme left column outside the box in the example of synthetic division?

To use synthetic division to divide a polynomial $ax^2 + bx + c$ by a linear factor $x - r$, follow this pattern.



You can use synthetic division in place of the standard long division algorithm to determine the quotient for $(2x^2 - 3x - 9) \div (x - 3)$.

Long Division	Synthetic Division
$2x + 3$ $x - 3\overline{)2x^2 - 3x - 9}$ $\underline{2x^2 - 6x}$ $3x - 9$ $\underline{3x - 9}$ 0	$r = 3$ $3 \mid 2 \quad -3 \quad -9$ $6 \quad 9$ $2 \quad 3 \quad 0$ add, multiply by r
$(2x^2 - 3x - 9) \div (x - 3) = 2x + 3$	

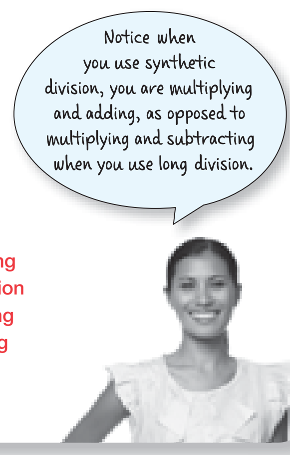
1. Analyze the worked example.

   a. Write the dividend as the product of its factors.

   $2x^2 - 3x - 9 = (2x + 3)(x - 3)$

   b. Why does the synthetic division algorithm work?

   Synthetic division follows the same process as long division. It works because it parallels integer division and polynomial long division. It saves time by using only the coefficients and quick methods for adding and multiplying the terms.



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Grouping

Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

Guiding Question for Share Phase, Question 2

How do you know the degree of the first term in the quotient?

2. Two examples of synthetic division are provided. Perform the steps outlined for each problem:

i. Write the dividend.

ii. Write the divisor.

iii. Write the quotient.

iv. Write the dividend as the product of the divisor and the quotient plus the remainder.

a.

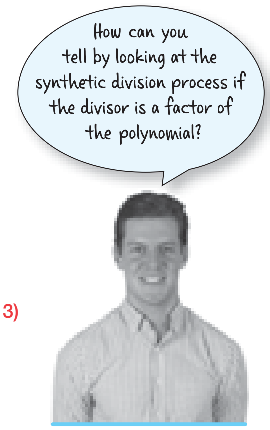
$$\begin{array}{r|rrrr|r} 2 & 1 & 0 & -4 & -3 & 6 \\ & & 2 & 4 & 0 & -6 \\ \hline & 1 & 2 & 0 & -3 & 0 \end{array}$$

i. $x^4 + 0x^3 - 4x^2 - 3x + 6$

ii. $x - 2$

iii. $x^3 + 2x^2 + 0x - 3$

iv. $x^4 + 0x^3 - 4x^2 - 3x + 6 = (x - 2)(x^3 + 2x^2 + 0x - 3)$



b.

$$\begin{array}{r|rrrr|r} -3 & 2 & -4 & -4 & -3 & 6 \\ & & -6 & 30 & -78 & 243 \\ \hline & 2 & -10 & 26 & -81 & 249 \end{array}$$

i. $2x^4 - 4x^3 - 4x^2 - 3x + 6$

ii. $x + 3$

iii. $(2x^3 - 10x^2 + 26x - 81) + \frac{249}{x + 3}$

iv. $2x^4 - 4x^3 - 4x^2 - 3x + 6 = (x + 3)\left((2x^3 - 10x^2 + 26x - 81) + \frac{249}{x + 3}\right)$

6

Guiding Questions for Share Phase, Question 3

- What is the degree of the quotient when a cubic function is divided by a linear function?
- What pattern seems to be emerging as each cubic function and linear function changes?
- If there were a part (e) to Question 3, using the same pattern, how would you describe $g(x)$, $r(x)$, and the quotient of the two functions?

6

3. Calculate each quotient using synthetic division. Then write the dividend as the product of the divisor and the quotient plus the remainder.

   a. $g(x) = x^3 + 1$

   $r(x) = x + 1$

   Calculate $\frac{g(x)}{r(x)}$.

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

   $\frac{g(x)}{r(x)} = x^2 - x + 1$

   $x^3 + 1 = (x + 1)(x^2 - x + 1)$

   b. $g(x) = x^3 + 8$

   $r(x) = x + 2$

   Calculate $\frac{g(x)}{r(x)}$.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

   $\frac{g(x)}{r(x)} = x^2 - 2x + 4$

   $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$

   c. $g(x) = x^3 + 27$

   $r(x) = x + 3$

   Calculate $\frac{g(x)}{r(x)}$.

$$\begin{array}{r|rrrr} -3 & 1 & 0 & 0 & 27 \\ & & -3 & 9 & -27 \\ \hline & 1 & -3 & 9 & \boxed{0} \end{array}$$

   $\frac{g(x)}{r(x)} = x^2 - 3x + 9$

   $x^3 + 27 = (x + 3)(x^2 - 3x + 9)$

   d. $g(x) = x^3 + 64$

   $r(x) = x + 4$

   Calculate $\frac{g(x)}{r(x)}$.

   $\frac{g(x)}{r(x)} = x^2 - 4x + 16$

   $x^3 + 64 = (x + 4)(x^2 - 4x + 16)$

Grouping

Have students complete Question 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

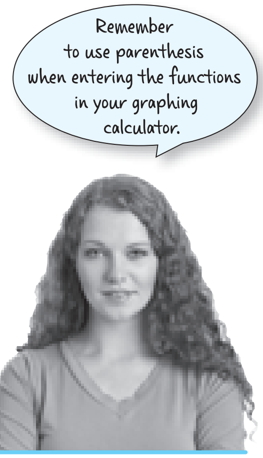
- Looking at the equation of the function $g(x)$, what value of x causes this function to be undefined in each group?
- If the function is undefined for a value of x, what happens on the graph of the function at this value?
- What happens in the table of values when the function is undefined at a specific x-value?
- When two functions are equivalent, must the output be the same for every input value? Is this the case in this situation?

4. Use a graphing calculator to compare the graphs and table of values for each pair of functions.

Group 1: $g(x) = \frac{x^3 + 1}{x + 1}$ and $j(x) = x^2 - x + 1$

Group 2: $g(x) = \frac{x^3 + 8}{x + 2}$ and $j(x) = x^2 - 2x + 4$

Group 3: $g(x) = \frac{x^3 + 27}{x + 3}$ and $j(x) = x^2 - 3x + 9$



a. Describe the similarities and differences in the graphs and tables of values within each pair of functions.

Group 1: The graphs and table of values are the same, except for an error at $x = -1$.

Group 2: The graphs and table of values are the same, except for an error at $x = -2$.

Group 3: The graphs and table of values are the same, except for an error at $x = -3$.

b. Are the functions within each pair equivalent? Explain your reasoning.

No. The functions are not equivalent because of the one input value that results in an error. For two functions to be equivalent, the output values must be the same for every input value.

Grouping

- Ask a student to read the information and worked example. Discuss as a class.
- Complete Question 5 as a class.

Guiding Questions for Discuss Phase

- If the divisor is written in the form $2x - 5$, how must it be rewritten to perform synthetic division?
- If the divisor is written in the form $3x - 2$, how must it be rewritten to perform synthetic division?
- How do you account for the remainder when writing the dividend as the product of the divisor and the quotient?
- If $(3x - 2)(x^2 + x + 1) = 3x^3 + x^2 + x - 2$, should there be a remainder when one factor is divided into the product?

Synthetic division works only for linear divisors in the form $x - r$. If the divisor has a leading coefficient other than 1, you may need to factor out a constant in order to rewrite the divisor in the form $x - r$.

You can use synthetic division to determine the quotient of $\frac{2x^3 - 6x^2 + 4x + 2}{2x - 3}$. Since the divisor is not in the form $x - r$, you can rewrite $2x - 3$ as $2\left(x - \frac{3}{2}\right)$.

$r = \frac{3}{2}$

$$\begin{array}{c|cccc}
\frac{3}{2} & 2 & -6 & 4 & 2 \\
 & & 3 & -\frac{9}{2} & -\frac{3}{4} \\
\hline
 & 2 & -3 & -\frac{1}{2} & \boxed{\frac{5}{4}}
\end{array}$$

The numbers in the last row become the coefficients of the quotient.

$2x^2 - 3x - \frac{1}{2}$ R $\frac{5}{4}$

You can write the dividend as the product of the divisor and the quotient plus the remainder.

$2x^3 - 6x^2 + 4x + 2 = \left(x - \frac{3}{2}\right)\left(2x^2 - 3x - \frac{1}{2}\right) + \frac{5}{4}$

5. Verify $(3x - 2)(x^2 + x + 1) = 3x^3 + x^2 + x - 2$ using synthetic division. Show all work and explain your reasoning.

$$\begin{array}{c|cccc}
\frac{2}{3} & 3 & 1 & 1 & -2 \\
 & & 2 & 2 & 2 \\
\hline
 & 3 & 3 & 3 & \boxed{0}
\end{array}$$

The remainder is 0, so I know that $(3x - 2)$ is a factor of $3x^3 + x^2 + x - 2$.

$\left(x - \frac{2}{3}\right)(3x^2 + 3x + 3)$

$3\left(x - \frac{2}{3}\right)(x^2 + x + 1)$

$(3x - 2)(x^2 + x + 1) = 3x^3 + x^2 + x - 2$

6

Grouping

Have students complete Questions 6 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 6, parts (a) and (b)

- Which of the quotients have remainders?
- Why is $h(x) = \frac{g(x)}{2}$?
- Why is $j(x) = \frac{g(x)}{3}$?
- Why are the zeros the same for $g(x)$, $h(x)$, and $j(x)$?
- Do all of the problems use the same divisor?
- What divisor is the same in each problem?

6. Analyze each division problem given $f(x) = x^3 - 3x^2 - x + 3$.

$$g(x) = \frac{f(x)}{x - 1} \qquad h(x) = \frac{f(x)}{2x - 2} \qquad j(x) = \frac{f(x)}{3x - 3}$$

a. Determine the quotient of each function.

$$g(x) = \frac{x^3 - 3x^2 - x + 3}{x - 1} = x^2 - 2x - 3$$

$$\begin{array}{r|rrrr} 1 & 1 & -3 & -1 & 3 \\ & & 1 & -2 & -3 \\ \hline & 1 & -2 & -3 & \boxed{0} \end{array}$$

$$h(x) = \frac{x^3 - 3x^2 - x + 3}{2x - 2}$$

$$= \frac{x^3 - 3x^2 - x + 3}{2(x - 1)}$$

$$= \frac{1}{2}\left(\frac{x^3 - 3x^2 - x + 3}{x - 1}\right)$$

$$= \frac{1}{2}(x^2 - 2x - 3) \text{ or } \frac{x^2 - 2x - 3}{2}$$

$$j(x) = \frac{x^3 - 3x^2 - x + 3}{3x - 3}$$

$$= \frac{x^3 - 3x^2 - x + 3}{3(x - 1)}$$

$$= \frac{1}{3}\left(\frac{x^3 - 3x^2 - x + 3}{x - 1}\right)$$

$$= \frac{1}{3}(x^2 - 2x - 3) \text{ or } \frac{x^2 - 2x - 3}{3}$$

b. Use function notation to write $h(x)$ and $j(x)$ in terms of $g(x)$.

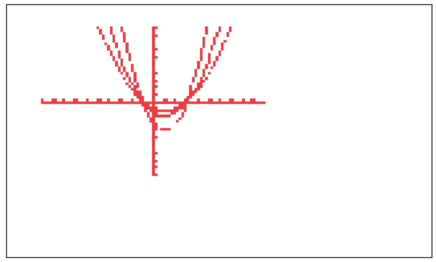
$$h(x) = \frac{g(x)}{2}$$

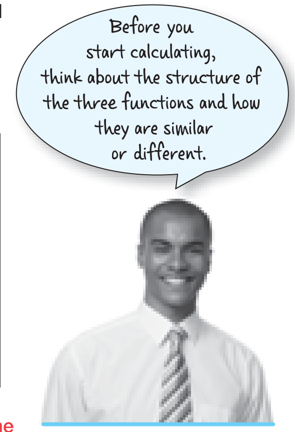
$$j(x) = \frac{g(x)}{3}$$

Guiding Questions for Share Phase, Question 6, part (c) through Question 7

- How do the location of the vertices compare to each other?
- Does the axis of symmetry remain the same?
- Does division cause a horizontal translation?
- Does division change the constant and the output values?
- Is the graph dilated?
- Is the graph vertically shifted?
- If the function $q(x)$ is not a factor, what is the remainder?

c. Use a graphing calculator to compare the graphical representations of $g(x)$, $h(x)$, and $j(x)$. What are the similarities and differences in the key characteristics? Explain your reasoning.





- All problems used the same divisor, $x - 1$, so the zeros remain the same.
- The vertex shifted vertically.
- Dividing does not cause a horizontal shift, so the axis of symmetry remains unchanged.

d. Given the function $g(x)$, describe the transformation(s) that occurred to produce $h(x)$ and $j(x)$.

Division changed the constant and the output values. Therefore, the graph is vertically shifted and dilated.

7. Is the function $q(x) = (x + 2)$ a factor of the function $p(x) = (x + 2)(x - 4)(x + 3) + 1$? Show all work and explain your reasoning.

No. It is not a factor. The remainder is 1.

8. The lesson opener discussed efficiency. Describe patterns and algorithms learned in this lesson that made your mathematical work more efficient.

Answers will vary.

Synthetic division is an efficient algorithm for polynomial long division.

Be prepared to share your solutions and methods.

6

Check for Students' Understanding

1. Use the long division algorithm to divide $x^3 - 9x^2 + 8x + 60$ by $x - 5$.

$$
\begin{array}{r}
x^2 - 4x - 12 \\
x - 5\overline{)\,x^3 - 9x^2 + 8x + 60} \\
\underline{-(x^3 - 5x^2)} \\
-4x^2 + 8x \\
\underline{-(-4x^2 + 20x)} \\
-12x + 60 \\
\underline{-(-12x + 60)} \\
0
\end{array}
$$

$x^3 - 4x - 12$

2. Use the synthetic division algorithm to divide $x^3 - 9x^2 + 8x + 60$ by $x - 5$.

$$
\begin{array}{r|rrrr}
5 & 1 & -9 & 8 & 60 \\
 & & 5 & -20 & -60 \\
\hline
 & 1 & -4 & -12 & \boxed{0}
\end{array}
$$

$x^3 - 4x - 12$

6

6.3

The Factors of Life

The Factor Theorem and Remainder Theorem

LEARNING GOALS

In this lesson, you will:

- Use the Remainder Theorem to evaluate polynomial equations and functions.
- Use the Factor Theorem to determine if a polynomial is a factor of another polynomial.
- Use the Factor Theorem to calculate factors of polynomial equations and functions.

KEY TERMS

- Remainder Theorem
- Factor Theorem

ESSENTIAL IDEAS

- The Remainder Theorem states that when any polynomial equation or function $f(x)$ is divided by a linear factor $(x - r)$, the remainder is $R = f(r)$ or the value of the equation or function when $x = r$.
- The Factor Theorem states that a polynomial has a linear polynomial as a factor if and only if the remainder is zero; $f(x)$ has $(x - r)$ as a factor if and only if $f(r) = 0$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-APR Arithmetic with Polynomials and Rational Expressions

Understand the relationship between zeros and factors of polynomials

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

Overview

The Remainder Theorem and the Factor Theorem are stated. Worked examples using both theorems are provided and students will use the theorems to verify factors of various functions. A graphing calculator, polynomial long division, or synthetic division can also be used to verify factors of functions.

6

Warm Up

1. Use the synthetic division algorithm to divide $f(x) = 3x^5 - 4x^4 + 2x^3 - x^2 + 5x - 3$ by $x - 2$.

$$\begin{array}{c|cccccc} 2 & 3 & -4 & 2 & -1 & 5 & -3 \\ & & 6 & 4 & 12 & 22 & 54 \\ \hline & 3 & 2 & 6 & 11 & 27 & \boxed{51} \end{array}$$

$3x^4 + 2x^3 + 6x^2 + 11x + 27$ R 51

2. Evaluate $f(x) = 3x^5 - 4x^4 + 2x^3 - x^2 + 5x - 3$ when $x = 2$.

$f(2) = 3(2)^5 - 4(2)^4 + 2(2)^3 - (2)^2 + 5(2) - 3$

$= 51$

3. What do you notice about the remainder that resulted from synthetic division in Question 1 and the value of $f(2)$ in Question 2?

The remainder and $f(2)$ are the same value.

6

The Factors of Life

The Factor Theorem and Remainder Theorem

6.3

LEARNING GOALS

In this lesson, you will:

- Use the Remainder Theorem to evaluate polynomial equations and functions.
- Use the Factor Theorem to determine if a polynomial is a factor of another polynomial.
- Use the Factor Theorem to calculate factors of polynomial equations and functions.

KEY TERMS

- Remainder Theorem
- Factor Theorem

When you hear the word *remainder*, what do you think of? Leftovers? Fragments? Remnants?

The United States, as a country, produces a great deal of its own "leftovers." The amount of paper product leftovers per year is enough to heat 50,000,000 homes for 20 years. The average household disposes of over 13,000 pieces of paper each year, most coming from the mail. Some studies show that 2,500,000 plastic bottles are used every hour, most being thrown away, while 80,000,000,000 aluminum soda cans are used every year. Aluminum cans that have been disposed of and not recycled will still be cans 500 years from now.

There are certain things you can do to help minimize the amount of leftovers you produce. For example, recycling one aluminum can save enough energy to watch TV for three hours. Used cans can be recycled into "new" cans in as little as 60 days from when they are recycled. If just $\frac{1}{10}$ of the daily newspapers were recycled, 25,000,000 trees could be saved per year. Recycling plastic uses half the amount of energy it would take to burn it.

6

Problem 1

Any polynomial $p(x)$ can be written in the form $\frac{p(x)}{\textit{linear factor}} = \textit{quotient} + \frac{\textit{remainder}}{\textit{linear factor}}$. Students verify the equation $p(x) = (x - r)\, q(x) + R$, where $(x - r)$ is a linear factor of the function, $q(x)$ is the quotient, and R represents the remainder. They discuss why $p(r)$, where $(x - r)$ is a linear factor, will always equal the remainder R. Students conclude that any polynomial evaluated at r will equal the remainder when the polynomial is divided by the linear factor $(x - r)$. The Remainder Theorem is stated, and students will use the theorem to solve problems.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- What algorithm was used to multiply $(x - r)$ and $q(x)$?
- How is $p(3)$ determined?
- What is the relationship between $p(3)$ and the remainder in part (a)?
- If the product of $(r - r)$ and $q(x)$ is 0, will $p(r)$ always equal R? Why?

PROBLEM 1 You Have the Right to the Remainder Theorem

You learned that the process of dividing polynomials is similar to the process of dividing integers. Sometimes when you divide two integers there is a remainder, and sometimes there is not a remainder. What does each case mean? In this lesson, you will investigate what the remainder means in terms of polynomial division.

Remember from your experiences with division that:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

or

$$\text{dividend} = (\text{divisor})\,(\text{quotient}) + \text{remainder}.$$

It follows that any polynomial, $p(x)$, can be written in the form:

$$\frac{p(x)}{\text{linear factor}} = \text{quotient} + \frac{\text{remainder}}{\text{linear factor}}$$

or

$$p(x) = (\text{linear factor})\,(\text{quotient}) + \text{remainder}.$$

Generally, the linear factor is written in the form $(x - r)$, the quotient is represented by $q(x)$, and the remainder is represented by R, meaning:

$$p(x) = (x - r)q(x) + \text{R}.$$

1. Given $p(x) = x^3 + 8x - 2$ and $\frac{p(x)}{(x - 3)} = x^2 + 3x + 17\ \text{ R } 49$.

   a. Verify $p(x) = (x - r)q(x) + \text{R}$.

   $x^3 + 8x - 2 = (x - 3)(x^2 + 3x + 17) + 49$

   $x^3 + 8x - 2 = x^3 + 3x^2 + 17x - 3x^2 - 9x - 51 + 49$

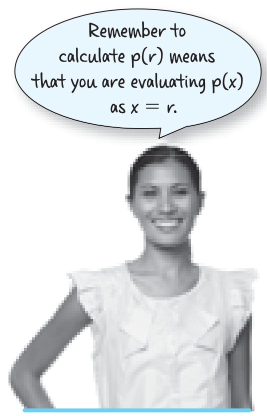
   $x^3 + 8x - 2 = x^3 + 8x - 2$

   b. Given $x - 3$ is a linear factor of $p(x)$, evaluate $p(3)$.

   $p(3) = 3^3 + 8(3) - 2$

   $= 27 + 24 - 2$

   $= 49$



2. Given $p(x) = (x - r)q(x) + \text{R}$, calculate $p(r)$.

   $p(r) = (r - r) \cdot q(x) + \text{R}$

   $p(r) = 0 \cdot q(x) + \text{R}$

   $p(r) = \text{R}$

3. Explain why $p(r)$, where $(x - r)$ is a linear factor, will always equal the remainder R, regardless of the quotient.

   When I substitute the value r into the expression $(x - r)$ the result will always be 0. Then when I multiply 0 times any quotient $q(x)$, that product will also be 0. The sum of 0 and the remainder R is always R. Therefore, $p(r)$ will always equal R.

- Will any polynomial evaluated at r always equal the remainder when the polynomial is divided by the linear factor $(x - r)$?

6

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 5 and 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 and 6

- If $(x - r) = (x - 2)$, is the value of r equal to 2 or -2?
- Using the factor $(x - 2)$, why is the r-value 2 instead of -2?
- If the function $p(x)$ is divided by the factor $(x - 2)$ and the remainder is 30, what does the Remainder Theorem tell you about $p(2)$?
- What does $f(r)$ equal?
- What does $f(2r)$ equal?
- If $0 = 2r(6r + 1)$, how do you solve for the value(s) of r?

4. What conclusion can you make about any polynomial evaluated at r?

Any polynomial evaluated at r will equal the remainder when the polynomial is divided by the linear factor $(x - r)$.

The **Remainder Theorem** states that when any polynomial equation or function, $f(x)$, is divided by a linear factor $(x - r)$, the remainder is $R = f(r)$, or the value of the equation or function when $x = r$.

5. Given $p(x) = x^3 + 6x^2 + 5x - 12$ and $\frac{p(x)}{(x - 2)} = x^2 + 8x + 21$ R 30,

Rico says that $p(-2) = 30$ and Paloma says that $p(2) = 30$.

Without performing any calculations, who is correct? Explain your reasoning.

Paloma is correct. The Remainder Theorem states that the remainder is the value of the function when $x = r$ of the linear factor $(x - r)$. The r-value is 2; Rico used the incorrect r-value.

6. The function, $f(x) = 4x^2 + 2x + 9$ generates the same remainder when divided by $(x - r)$ and $(x - 2r)$, where r is not equal to 0. Calculate the value(s) of r.

If the linear factors $(x - r)$ and $(x - 2r)$ generate the same remainder, then by the Remainder Theorem $f(r) = f(2r)$.

$$f(r) = 4r^2 + 2r + 9$$
$$f(2r) = 4(2r)^2 + 2(2r) + 9$$
$$= 16r^2 + 4r + 9$$

Therefore,

$$4r^2 + 2r + 9 = 16r^2 + 4r + 9$$
$$0 = 12r^2 + 2r$$
$$0 = 2r(6r + 1)$$

$r = 0$ and $r = -\frac{1}{6}$

According to the problem statement, r is not equal to 0, so $r = -\frac{1}{6}$.

6

Problem 2

The Factor Theorem is stated. Students will analyze an example that shows how the Factor Theorem is used to solve a problem. A worked example continues to factor the quadratic expression with the linear factor and students use the Factor Theorem to prove each factor shown in the worked example is correct. A graphing calculator, polynomial long division, or synthetic division can also be used to verify the factors are correct. Students then use the Factor Theorem to prove the product of two imaginary roots written as factors is in fact the factored form of a given quadratic function. In the last activity, two polynomial functions are written with a missing coefficient in one term. They are given one linear factor of the function and asked to determine the unknown coefficient.

Grouping

Ask a student to read the information and complete Question 1 as a class.

6

PROBLEM 2 Factors to Consider

Consider the factors of 24: 1, 2, 3, 4, 6, 8, 12, 24.

Notice that when you divide 24 by any of its factors the remainder is 0. This same principle holds true for polynomial division.

The **Factor Theorem** states that a polynomial has a linear polynomial as a factor if and only if the remainder is zero; or, in other words, $f(x)$ has $(x - r)$ as a factor if and only if $f(r) = 0$.

1. Haley and Lillian each prove that $(x - 7)$ is a factor of the polynomial $f(x) = x^3 - 10x^2 + 11x + 70$.

Haley

$(x^3 - 10x^2 + 11x + 70) \div (x - 7)$

$$
\begin{array}{r}
x^2 - 3x - 10 \\
x - 7 \overline{) x^3 - 10x^2 + 11x + 70} \\
\underline{x^3 - 7x^2 } \\
-3x^2 + 11x \\
\underline{-3x^2 + 21x } \\
-10x + 70 \\
\underline{-10x + 70} \\
0
\end{array}
$$

Lillian

$f(x) = x^3 - 10x^2 + 11x + 70$

$f(7) = 7^3 - 10(7)^2 + 11(7) + 70$

$f(7) = 343 - 490 + 77 + 70$

$f(7) = 0$

Explain why each student's method is correct.

Haley is correct because she proved that $(x - 7)$ is a factor of $f(x) = x^3 - 10x^2 + 11x + 70$ by showing that $f(x)$ is divisible by $(x - 7)$ with no remainder. If $(x - 7)$ is a linear factor, then $f(7)$ must equal zero according to the Factor Theorem.

Lillian is correct because she showed that $f(7) = 0$. According to the Remainder Theorem, when any polynomial equation or function is divided by a linear factor $(x - r)$, the remainder is the value of the function when $x = r$. In this case, the remainder is 0, meaning that $(x - 7)$ is a linear factor of $f(x)$. Therefore $f(7)$ must equal zero by the Factor Theorem.

Grouping

Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 2 and 3

- If $f(x)$ is written in factor form, what are the r-values determined by each factor?
- What is the value of $f(7)$?
- If $(x - 7)$ is a factor of the function, what is the value of $f(7)$?
- What is the value of $f(-2)$?
- If $(x + 2)$ is a factor of the function, what is the value of $f(-2)$?
- What is the value of $f(5)$?
- If $(x - 5)$ is a factor of the function, what is the value of $f(5)$?
- Could a graphing calculator be used to verify that the factors shown in the worked example are correct?

You can continue to factor the polynomial $f(x) = x^3 - 10x^2 + 11x + 70$.

From Haley and Lillian's work, you know that $f(x) = (x - 7)(x^2 - 3x - 10)$.

The quadratic factor can also be factored.

$$f(x) = (x - 7)(x^2 - 3x - 10)$$
$$f(x) = (x - 7)(x + 2)(x - 5)$$

2. Use the Factor Theorem to prove each factor shown in the worked example is correct.

$f(7) = 7^3 - 10(7)^2 + 11(7) + 70$
$f(7) = 343 - 490 + 77 + 70$
$f(7) = 0$

$f(-2) = (-2)^3 - 10(-2)^2 + 11(-2) + 70$
$f(-2) = -8 - 40 - 22 + 70$
$f(-2) = 0$

$f(5) = 5^3 - 10(5)^2 + 11(5) + 70$
$f(5) = 125 - 250 + 55 + 70$
$f(5) = 0$

3. What other method(s) could you use to verify that the factors shown in the worked example are correct?

Answers will vary.

I could use a graphing calculator to calculate the zeros. I could also use polynomial long division or synthetic division to show that the remainder when $f(x)$ is divided by each linear factor is zero.

- Could polynomial long division be used to verify that the factors shown in the worked example are correct?
- Could synthetic division be used to verify that the factors shown in the worked example are correct?

Grouping

Have students complete Questions 4 and 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 and 5

- What is the product of $3i$ and $3i$?
- What is the value of i^2?
- What is the value of $9i^2$?
- Is the value of $f(1 + 3i)$ equal to 0? What does this imply?
- Is the value of $f(1 - 3i)$ equal to 0? What does this imply?
- If $(x - 3)$ is a factor of the function $f(x)$, what is the value of $f(3)$?
- If the value of $f(3)$ is equal to $57 - 3a$, and $f(3)$ equals 0, what is the value of a?
- If $(x + 1)$ is a factor of the function $f(x)$, what is the value of $f(-1)$?
- If the value of $f(-1)$ is equal to $a - 6$, and $f(-1)$ equals 0, what is the value of a?

4. Use the Factor Theorem to prove that $f(x) = (x + 1 - 3i)(x + 1 + 3i)$ is the factored form of $f(x) = x^2 + 2x + 10$.

$$\begin{aligned} f(-1 + 3i) &= (-1 + 3i)^2 + 2(-1 + 3i) + 10 \\ &= (-1 + 3i)(-1 + 3i) - 2 + 6i + 10 \\ &= 1 - 3i - 3i + 9i^2 + 6i + 8 \\ &= 1 - 6i - 9 + 6i + 8 \\ &= 0 \end{aligned}$$

Therefore by the Factor Theorem, $(-1 + 3i)$ must be a factor.

$$\begin{aligned} f(-1 - 3i) &= (-1 - 3i)^2 + 2(-1 - 3i) + 10 \\ &= (-1 - 3i)(-1 - 3i) - 2 - 6i + 10 \\ &= 1 + 3i + 3i + 9i^2 - 6i + 8 \\ &= 1 + 6i - 9 - 6i + 8 \\ &= 0 \end{aligned}$$

Therefore by the Factor Theorem, $(-1 - 3i)$ must be a factor.

5. Determine the unknown coefficient, a, in each function.

a. $f(x) = 2x^4 + x^3 - 14x^2 - ax - 6$ if $(x - 3)$ is a linear factor.

If $(x - 3)$ is a factor, then by the Factor Theorem:
$f(3) = 0$.

$$\begin{aligned} f(3) &= 2(3)^4 + (3)^3 - 14(3)^2 - a(3) - 6 \\ &= 162 + 27 - 126 - 3a - 6 \\ &= 57 - 3a \end{aligned}$$

By the Transitive Property of Equality, $57 - 3a = 0$.

$$\begin{aligned} 57 - 3a &= 0 \\ -3a &= -57 \\ a &= 19 \end{aligned}$$

b. $f(x) = ax^4 + 25x^3 + 21x^2 - x - 3$ if $(x + 1)$ is a linear factor.

If $(x + 1)$ is a factor, then by the Factor Theorem:
$f(-1) = 0$.

$$\begin{aligned} f(-1) &= a(-1)^4 + 25(-1)^3 + 21(-1)^2 - (-1) - 3 \\ &= a - 25 + 21 + 1 - 3 \\ &= a - 6 \end{aligned}$$

By the Transitive Property of Equality, $a - 6 = 0$.

$$\begin{aligned} a - 6 &= 0 \\ a &= 6 \end{aligned}$$

6

Talk the Talk

Students are given $p(x)$, $p(x) \div (x + 4)$, and the remainder. They will determine if several statements regarding the evaluation of the function or a specific zero of the function are true or false based on the given information.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- What does the Remainder Theorem imply?
- What does the Factor Theorem imply?
- In a division problem, what does the product of the divisor and the quotient plus the remainder always equal?

Talk the Talk

Given the information:

$$p(x) = x^3 + 6x^2 + 11x + 6, \text{ and}$$
$$p(x) \div (x + 4) = x^2 + 2x + 3 \text{ R } -6$$

Determine whether each statement is true or false. Explain your reasoning.

1. $p(-4) = 6$

This statement is true because the Remainder Theorem states that the remainder is the value of the function when $x = r$ of the linear factor $(x - r)$.

2. $p(x) = (x + 4)(x^2 + 2x + 3) - 6$

This statement is true because as in integer division, (divisor) (quotient) + remainder = dividend.

3. -4 is not a zero of $p(x)$

This is true because the Factor Theorem states that $(x - r)$ is a factor of $p(x)$ if and only if the value r is a zero of the related polynomial function. In this case, $p(-4) = -6$, so -4 is not a zero of $p(x)$.

4. -2 is a zero of $p(x)$

This is true because $p(-2) = 0$, so -2 is a zero of $p(x)$. The Factor Theorem states that $f(x)$ has $(x - r)$ as a factor if and only if r is a zero of $p(x)$.

Be prepared to share your solutions and methods.

6

Check for Students' Understanding

1. Is $(x - 4)$ a factor of $f(x) = x^3 - x^2 - 22x + 40$? Explain.

   If $(x - 4)$ is a factor, then by the Factor Theorem $f(4) = 0$.

   $f(4) = (4)^3 - (4)^2 - 22(4) + 40 = 64 - 16 - 88 + 40 = 0$

   Yes, it is a factor.

2. Is $(x - 3)$ a factor of $f(x) = x^2 + 9$? Explain.

   If $(x - 3)$ is a factor, then by the Factor Theorem $f(3) = 0$.

   $f(3) = (3)^2 + 9 = 9 + 9 = 18$

   No, it is not a factor.

3. Explain the difference between the Remainder Theorem and the Factor Theorem.

   Answers will vary.

6

Break It Down

6.4

Factoring Higher Order Polynomials

LEARNING GOAL

In this lesson, you will:

- Factor higher order polynomials using a variety of factoring methods.

ESSENTIAL IDEAS

- The graphs of all polynomials that have a monomial GCF that includes a variable will pass through the origin.
- Chunking is a method of factoring a polynomial in quadratic form that does not have common factors is all terms. Using this method, the terms are rewritten as a product of 2 terms, the common term is substituted with a variable, and then it is factored as is any polynomial in quadratic form.
- Factoring by grouping is a method of factoring a polynomial that has four terms in which not all terms have a common factor. The terms can be first grouped together in pairs that have a common factor, and then factored.
- Factoring by using quadratic form is a method of factoring a 4th degree polynomial in the form, $ax^4 + bx^2 + c$.
- The factoring formula for the difference of two cubes is in the form: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
- The factoring formula for the sum of two cubes is in the form: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
- Factoring the difference of squares is in the form: $a^2 - b^2 = (a + b)(a - b)$.
- Factoring a perfect square trinomial can occur in two forms:
  $a^2 - 2ab + b^2 = (a - b)^2$
  $a^2 + 2ab + b^2 = (a + b)^2$

COMMON CORE STATE STANDARDS FOR MATHEMATICS

N-CN The Complex Number System

Use complex numbers in polynomial identities and equations.

8. Extend polynomial identities to the complex numbers.

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions

2. Use the structure of an expression to identify ways to rewrite it.

A-APR Arithmetic with Polynomials and Rational Expressions

Understand the relationship between zeros and factors of polynomials

3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

F-IF Interpreting Functions

Analyze functions using different representations

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

Overview

Several different methods of factoring polynomials are introduced such as factoring out the the GCF, chunking, factoring by grouping, factoring in quadratic form, sum and difference of cubes, difference of squares, and perfect square trinomials. Students will use these methods to factor polynomials.

6

Warm Up

Consider the following statement to answer each question.

The expression $(x - 2)$ is a factor of $f(x) = x^5 - 3x^4 + x^3 + 5x^2 - x - 10$.

1. What can be concluded using the Fundamental Theorem of Algebra?

The Fundamental Theorem of Algebra states that this 5th degree polynomial has 5 roots.

2. What can be concluded using the Factor Theorem?

The Factor Theorem states that $f(2) = 0$.

3. Using synthetic division, divide $f(x) = x^5 - 3x^4 + x^3 + 5x^2 - x - 10$ by $(x - 3)$.

$$\begin{array}{r|rrrrrr} 3 & 1 & -3 & 1 & 5 & -1 & -10 \\ & & 3 & 0 & 3 & 24 & 69 \\ \hline & 1 & 0 & 1 & 8 & 23 & \boxed{59} \end{array}$$

$$x^5 - 3x^4 + x^3 + 5x^2 - x - 10 = (x - 3)(x^4 + x^2 + 8x + 23) + \frac{59}{x - 3}$$

4. What can be concluded using the Remainder Theorem?

The Remainder Theorem states that $f(3) = 59$.

6

6

6.4

Break It Down

Factoring Higher Order Polynomials

LEARNING GOAL

In this lesson, you will:

- Factor higher order polynomials using a variety of factoring methods.

Factoring in mathematics is similar to the breakdown of physical and chemical properties in chemistry.

For example, the chemical formula of water is H_2O. This formula means that 2 molecules of hydrogen (H) combined with one molecule of oxygen (O) creates water. The formula for water gives us insight into its individual parts or factors.

Although the general idea is the same between factoring in mathematics, and the breakdown of chemicals, there are some big differences. When factoring polynomials, the factored form does not change any of the characteristics of the polynomial; they are two equivalent expressions. The decomposition of chemicals, however, can sometimes cause an unwanted reaction.

If you have ever taken prescription medication, you might have read the warning labels giving specific directions on how to store the medication, including temperature and humidity. The reason for these directions: if the temperature is too hot or too cold, or if the air is too humid or too dry, the chemicals in the medication may begin to decompose, thus changing its properties.

What other reasons might people want to break down the chemical and physical components of things? How else can these breakdowns be beneficial to people?

6

Problem 1

Students factor polynomials by factoring out the greatest common factor. They then use the factors to sketch a graph the polynomial. Students compare the graphs of the polynomials and conclude that the graph of all polynomials that have a GCF that includes a variable will pass through the origin.

Grouping

- Ask a student to read the information aloud. Discuss as a class.
- Complete Questions 1 and 2 as a class.

Guiding Questions for Discuss Phase, Questions 1 and 2

- Is each term in the polynomial a multiple of 3?
- Does each term in the polynomial contain the variable x?
- What is the greatest common factor in this polynomial?
- How many zeros are associated with this polynomial?
- Are all of the zeros associated with this polynomial real numbers?
- What are the zeros of this polynomial?

6

PROBLEM 1 There's More Than One Way to Parse a Polynomial

In this lesson, you will explore different methods of factoring. To begin factoring any polynomial, always look for a greatest common factor (GCF). You can factor out the greatest common factor of the polynomial, and then factor what remains.

Remember, a greatest common factor can be a variable, constant, or both.

1. Ping and Shalisha each attempt to factor $3x^3 + 12x^2 - 36x$ by factoring out the greatest common factor.

Ping's Work

$3x^3 + 12x^2 - 36x$

$3x(x^2 + 4x - 12)$

Shalisha's Work

$3x^3 + 12x^2 - 36x$

$3(x^3 + 4x^2 - 12x)$

Analyze each student's work. Determine which student is correct and explain the inaccuracy in the other student's work.

Ping is correct. Shalisha is incorrect because, although she factored out a common factor, she did not factor out the greatest common factor.

2. If possible, completely factor the expression that Ping and Shalisha started.

$3x(x^2 + 4x - 12)$

$3x(x + 6)(x - 2)$

Grouping

Have students complete Questions 3 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 through 5

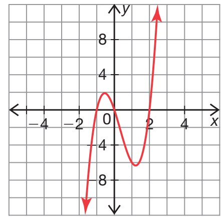
- What degree is this polynomial?
- What is the greatest common factor in this polynomial?
- Is the quadratic expression factorable?
- How many zeros are associated with this polynomial?
- Are all of the zeros associated with this polynomial real numbers?
- What are the zeros of this polynomial?
- How are the zeros of this polynomial used to sketch a graph?
- What is the end behavior of the polynomial? How do you know?
- Which sketches pass through the origin? What is their GCF?
- What do you notice about the graphs that pass through the origin?
- Which sketches do not pass through the origin? What is their GCF?
- Do all polynomials passing through the origin have a GCF containing a variable?

3. Factor each expression over the set of real numbers. Remember to look for a greatest common factor first. Then, use the factors to sketch the graph of each polynomial.

a. $3x^3 - 3x^2 - 6x$

$3x(x^2 - x - 2)$

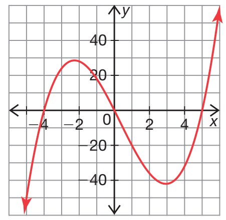
$3x(x + 1)(x - 2)$



b. $x^3 - x^2 - 20x$

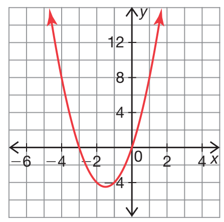
$x(x^2 - x - 20)$

$x(x + 4)(x - 5)$



c. $2x^2 + 6x$

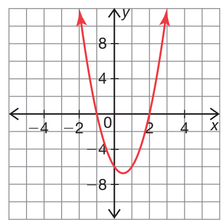
$2x(x + 3)$



d. $3x^2 - 3x - 6$

$3(x^2 - x - 2)$

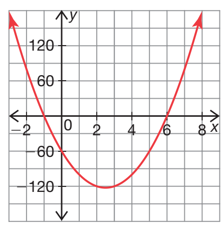
$3(x + 1)(x - 2)$



e. $10x^2 - 50x - 60$

$10(x^2 - 5x - 6)$

$10(x - 6)(x + 1)$



6

Problem 2

Polynomials in quadratic form have common factors in some of the terms, but not all terms. Students will practice a method called chunking, where they write the terms as a product of 2 terms, then substitute the common term with a variable, *z*, and factor as they would any polynomial in quadratic form. A worked example is provided. Using polynomials with four terms, students pair terms together that have a common term and then factor. This method is called factoring by grouping. Another method introduced is factoring by using quadratic form. When a 4^{th} degree polynomial is written as a trinomial and looks very similar to a quadratic, in the form, $ax^4 + bx^2 + c$, the polynomial can be factored using the same methods used to factor a quadratic function. A worked example is provided. Students then use this method to factor polynomial expressions over the set of complex numbers.

Grouping

- Ask a student to read the information and worked example. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

6

4. Analyze the factored form and the corresponding graphs in Question 3. What do the graphs in part (a) through part (c) have in common that the graphs of part (d) and part (e) do not? Explain your reasoning.

The graphs of polynomials in part (a) through part (c) all pass through the origin. The graphs of part (d) and part (e) do not. This is because the GCF of each polynomial in parts (a) through (c) is the product of a constant and a variable. When you set this type of factor equal to 0 and solve for the variable, you get 0. The GCF of polynomials in part (d) and part (e) do not contain a variable, only a constant, so they do not have a zero of 0.

5. Write a statement about the graphs of all polynomials that have a monomial GCF that contains a variable.

All polynomials that have a monomial GCF that includes a variable will pass through the origin.

PROBLEM 2 Continue Parsing

Some polynomials in quadratic form may have common factors in some of the terms, but not all terms. In this case, it may be helpful to write the terms as a product of 2 terms. You can then substitute the common term with a variable, *z*, and factor as you would any polynomial in quadratic form. This method of factoring is called *chunking*.

You can use chunking to factor $9x^2 + 21x + 10$.

Notice that the first and second terms both contain the common factor, $3x$.

$9x^2 + 21x + 10 = (3x)^2 + 7(3x) + 10$	Rewrite terms as a product of common factors.
$= z^2 + 7z + 10$	Let $z = 3x$.
$= (z + 5)(z + 2)$	Factor the quadratic.
$= (3x + 5)(3x + 2)$	Substitute $3x$ for z.

The factored form of $9x^2 + 21x + 10$ is $(3x + 5)(3x + 2)$.

1. Use chunking to factor $49x^2 + 35x + 6$.

$(7x)^2 + 5(7x) + 6$

Let $z = 7x$.

$z^2 + 5z + 6$

$(z + 2)(z + 3)$

$(7x + 2)(7x + 3)$

Guiding Questions for Share Phase, Question 1

- What factor do 49 and 35 have in common?
- What factor do x and x^2 have in common?
- What factor do $49x^2$ and $35x$ have in common?

Guiding Questions for Share Phase, Question 2

- What is the value of z or the GCF in this situation?
- Will the polynomial $(2x - 3)(2x + 5)$ or $4x^2 + 4x - 15$ meet the criteria?
- Will the polynomial $(3x - 3)(3x + 5)$ or $9x^2 + 6x - 15$ meet the criteria?

Grouping

Have students complete Questions 3 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- What did Colt factor out of the first two terms?
- What did Colt factor out of the last two terms?
- What binomial did Colt factor out?
- How was the difference of two squares used in this situation?
- What is factored out of the first two terms?
- What is factored out of the last two terms?
- What binomial is factored out?
- How was the difference of two squares used in this situation?

2. Given $z^2 + 2z - 15 = (z - 3)(z + 5)$, write another polynomial in standard form that has a factored form of $(z - 3)(z + 5)$ with different values for z.

Answers will vary.

Student responses should include a quadratic in the form $z^2 + 2z - 15$ for any value of z.

Using a similar method of factoring, you may notice, in polynomials with 4 terms, that although not all terms share a common factor, pairs of terms might share a common factor. In this situation, you can *factor by grouping*.

3. Colt factors the polynomial expression $x^3 + 3x^2 - x - 3$.

Colt

$x^3 + 3x^2 - x - 3$

$x^2(x + 3) - 1(x + 3)$

$(x + 3)(x^2 - 1)$

$(x + 3)(x + 1)(x - 1)$

Explain the steps Colt took to factor the polynomial expression.

$x^3 + 3x^2 - x - 3$

$x^2(x + 3) - 1(x + 3)$ Step 1: Colt factored out an x^2 from the first 2 terms and (-1) from the last 2 terms.

$(x + 3)(x^2 - 1)$ Step 2: Colt factored out the binomial $(x + 3)$.

$(x + 3)(x + 1)(x - 1)$ Step 3: Colt used the difference of two squares to completely factor the polynomial.

4. Use factor by grouping to factor the polynomial expression $x^3 + 7x^2 - 4x - 28$.

$x^3 + 7x^2 - 4x - 28$

$x^2(x + 7) - 4(x + 7)$

$(x^2 - 4)(x + 7)$

$(x + 2)(x - 2)(x + 7)$

6

Guiding Questions for Share Phase, Question 5

- Did Braxton or Kenny factor over the set of complex numbers? How do you know?
- Did Braxton or Kenny factor over the set of real numbers? How do you know?

Grouping

Ask a student to read the information aloud. Discuss as a class.

5. Braxton and Kenny both factor the polynomial expression $x^3 + 2x^2 + 4x + 8$.

Braxton

$x^3 + 2x^2 + 4x + 8$

$x^2(x + 2) + 4(x + 2)$

$(x^2 + 4)(x + 2)$

Kenny

$x^3 + 2x^2 + 4x + 8$

$x^2(x + 2) + 4(x + 2)$

$(x^2 + 4)(x + 2)$

$(x + 2i)(x - 2i)(x + 2)$

Analyze the set of factors in each student's work. Describe the set of numbers over which each student factored.

Braxton factored over the set of real numbers.

Kenny factored over the set of imaginary numbers.

Recall that the Fundamental Theorem of Algebra states that any polynomial equation of degree n must have exactly n complex roots or solutions. Also, the Fundamental Theorem of Algebra states that every polynomial function of degree n must have exactly n complex zeros.

This implies that any polynomial function of degree n must have exactly n complex factors:

$f(x) = (x - r_1)(x - r_2) \ldots (x - r_n)$ where $r \in$ {*complex numbers*}.

Some 4th degree polynomials, written as a trinomial, look very similar to quadratics as they have the same form, $ax^4 + bx^2 + c$. When this is the case, the polynomial may be factored using the same methods you would use to factor a quadratic. This is called *factoring by using quadratic form*.

Factor the quartic polynomial by using quadratic form.

$x^4 - 29x^2 + 100$	• Determine whether you can factor the given trinomial into 2 factors.
$(x^2 - 4)(x^2 - 25)$	• Determine if you can continue to factor each binomial.
$(x - 2)(x + 2)(x - 5)(x + 5)$	

6

Grouping

Have students complete Question 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- What is factored out of the first two terms?
- What is factored out of the last two terms?
- What binomial is factored out?
- How was the difference of two squares used in this situation?

Problem 3

Students will factor polynomial functions over the set of real numbers given a graph or an equation and table of values. The factoring formula for the difference of two cubes is given and students derive the factoring formula for the sum of two cubes by dividing $a^3 + b^3$ by $a + b$. Students use the difference of squares to factor binomials in the form $a^2 - b^2$. The factoring formulas for perfect square trinomials are reviewed, and students identify perfect square trinomials and write them as a sum or difference of squares.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

6. Factor each polynomial expression over the set of complex numbers.

a. $x^4 - 4x^3 - x^2 + 4x$

$x^3(x - 4) - x(x - 4)$

$(x^3 - x)(x - 4)$

$x(x^2 - 1)(x - 4)$

$x(x - 1)(x + 1)(x - 4)$

b. $x^4 - 10x^2 + 9$

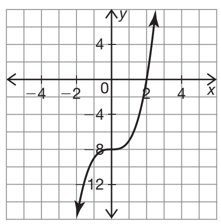
$(x^2 - 9)(x^2 - 1)$

$(x + 3)(x - 3)(x + 1)(x - 1)$

PROBLEM 3 Still Parsing

1. Factor each polynomial function over the set of real numbers.

a. $f(x) = x^3 - 8$



From the graph, 2 appears to be a root. I will use synthetic division to determine that 2 is a root.

2	1	0	0	−8
		2	4	8
	1	2	4	0

$f(x) = (x - 2)(x^2 + 2x + 4)$

6

Guiding Questions for Share Phase, Question 1, part (a)

- Can a zero or factor be identified from the graph of the function?
- Can the factor identified in the graph of the function be used in combination with synthetic division to solve for the remaining zeros or factors?

Guiding Questions for Share Phase, Question 1, part (b)

- Can a root be identified using the table of values? How?
- Can the root identified from the table of values be used in combination with synthetic division to solve for the resulting quadratic?
- Can the resulting quadratic be factored?

Grouping

- Ask a student to read the information and worked example. Discuss as a class.
- Complete Question 2 as a class.

b.

x	$f(x) = x^3 + 27$
-4	-37
-3	0
-2	19
-1	26
0	27
1	28

From the table, I can determine there is a root at $x = -3$. I will use synthetic division to determine the resulting quadratic.

$$\begin{array}{r|rrrr} -3 & 1 & 0 & 0 & 27 \\ & & -3 & 9 & -27 \\ \hline & 1 & -3 & 9 & \boxed{0} \end{array}$$

$f(x) = (x + 3)(x^2 - 3x + 9)$

You may have noticed that all the terms in the polynomials from Question 1 are perfect cubes. You can rewrite the expression $x^3 - 8$ as $(x)^3 - (2)^3$, and $x^3 + 27$ as $(x)^3 + (3)^3$. When you factor sums and differences of cubes, there is a special factoring formula you can use, which is similar to the difference of squares for quadratics.

To determine the formula for the difference of cubes, generalize the difference of cubes as $a^3 - b^3$.

To determine the factor formula for the difference of cubes, factor out $(a - b)$ by considering $(a^3 - b^3) \div (a - b)$.

$$\begin{array}{rl} & a^2 + ab + b^2 \\ a - b \,\big)\!\!\!\! & a^3 + 0 + 0 - b^3 \\ & \underline{a^3 - a^2b} \\ & \phantom{a^3 -{}} a^2b + 0 \\ & \phantom{a^3 -{}} \underline{a^2b - ab^2} \\ & \phantom{a^3 - a^2b -{}} ab^2 - b^3 \\ & \phantom{a^3 - a^2b -{}} \underline{ab^2 - b^3} \\ & \phantom{a^3 - a^2b - ab^2 -{}} 0 \end{array}$$

Therefore, the difference of cubes can be rewritten in factored form:
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

6

Guiding Questions for Discuss Phase, Question 2

- Is there a remainder when $a^3 + b^3$ is divided by $(a + b)$?
- Is $(a + b)$ a factor of $a^3 + b^3$? How do you know?

Grouping

Have students complete Question 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

- What is the a-value in the binomial?
- What is the b-value in the binomial?

2. Determine the formula for the sum of cubes by dividing $a^3 + b^3$ by $(a + b)$.

$$
\begin{array}{r|l}
 & a^2 - ab + b^2 \\
\hline
a + b & a^3 + 0 + 0 + b^3 \\
 & \underline{a^3 + a^2b} \\
 & -a^2b + 0 \\
 & \underline{-a^2b - ab^2} \\
 & ab^2 + b^3 \\
 & \underline{ab^2 + b^3} \\
 & 0
\end{array}
$$

So $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

Remember that you can factor a binomial that has perfect square a- and c-values and no middle value using the difference of squares.

You can use the difference of squares when you have a binomial of the form $a^2 - b^2$.

The binomial $a^2 - b^2 = (a + b)(a - b)$.

3. Use the difference of squares to factor each binomial over the set of real numbers.

a. $x^2 - 64$

$(x + 8)(x - 8)$

b. $x^4 - 16$

$(x^2 + 4)(x^2 - 4)$

$(x^2 + 4)(x - 2)(x + 2)$

c. $x^8 - 1$

$(x^4 + 1)(x^4 - 1)$

$(x^4 + 1)(x^2 + 1)(x^2 - 1)$

$(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)$

d. $x^4 - y^4$

$(x^2 + y^2)(x^2 - y^2)$

$(x^2 + y^2)(x + y)(x - y)$

6

Grouping

- Ask a student to read the information aloud. Discuss as a class.
- Have students complete Question 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- Can the *c*-term be negative in a perfect square trinomial? Why not?
- Is the *b*-term of a perfect square trinomial be equal to the product *ac*?
- How do you know which form is associated with the perfect square trinomial?
- How are the signs of the terms helpful in determining which form is associated with the perfect square trinomial?
- Which perfect square trinomial can be written as a difference of squares?
- Which perfect square trinomial can be written as a sum of squares?

Talk the Talk

Students will complete a table by matching several polynomials with a method of factoring from a given list. They then explain their method of choice and write each polynomial in factored form over the set of real numbers.

6

Another special form of polynomial is the perfect square trinomials. Perfect square trinomials occur when the polynomial is a trinomial, and where the first and last terms are perfect squares and the middle term is equivalent to 2 times the product of the first and last term's square root.

Factoring a perfect square trinomial can occur in two forms:

$$a^2 - 2ab + b^2 = (a - b)^2$$
$$a^2 + 2ab + b^2 = (a + b)^2$$

4. Determine which of the polynomial expression(s) is a perfect square trinomial and write it as a sum or difference of squares. If it is not a perfect square trinomial, explain why.

a. $x^4 + 14x^2 - 49$

This is not a perfect square trinomial because the *c*-term is negative.

b. $16x^2 - 40x + 100$

This is not a perfect square trinomial because the *b*-term is (*ac*) not 2(*ac*).

c. $64x^2 - 32x + 4$

This is a perfect square trinomial. Written as a difference of squares it is $(8x - 2)^2$.

d. $9x^4 + 6x^2 + 1$

This is a perfect square trinomial. Written as a sum of squares it is $(3x^2 + 1)^2$.

Talk the Talk

You have used many different methods of factoring:

- Factoring Out the Greatest Common Factor
- Chunking
- Factoring by Grouping
- Factoring in Quadratic Form
- Sum or Difference of Cubes
- Difference of Squares
- Perfect Square Trinomials

Grouping

Have students complete the table. Then have students share their responses as a class.

1. Based on the form and characteristics, match each polynomial with the method of factoring you would use from the bulleted list given. Every method from the bulleted list should be used only once. Explain why you choose the factoring method for each polynomial. Finally, write the polynomial in factored form over the set of real numbers.

Polynomial	Method of Factoring	Reason	Factored Form
$3x^4 + 2x^2 - 8$	Quadratic Form	You can factor the polynomial just as you would a quadratic: $(3x^2 - 4)(x^2 + 2)$.	$(x^2 + 2)(3x^2 - 4)$
$9x^2 - 16$	Difference of Squares	All terms of the polynomial are perfect squares and the operation is subtraction.	$(3x + 4)(3x - 4)$
$x^2 - 12x + 36$	Perfect Square Trinomial	The *a* and *c* terms are perfect squares, the *b* term is twice the product of the *a* and *c* term.	$(x - 6)^2$
$x^3 - 64$	Difference of Cubes	All terms of the polynomial are perfect cubes and the operation is subtraction.	$(x - 4)(x^2 + 4x + 16)$
$x^3 + 2x^2 + 7x + 14$	Factor by Grouping	There are four terms in the polynomial and pairs of terms both contain the factor $(x + 2)$.	$x^2(x + 2) + 7(x + 2)$ $(x^2 + 7)(x + 2)$
$25x^2 - 30x - 7$	Chunking	The x^2 term is a perfect square and the first 2 terms can be written as something times $5x$, allowing for *z*-substitution.	$(5x)^2 - 6(5x) - 7$ Let $z = 5x$. $z^2 - 6z - 7$ $(z - 7)(z + 1)$ $(5x - 7)(5x + 1)$
$2x^4 + 10x^3 + 12x^2$	Greatest Common Factor	The term $2x^2$ can be factored out of all 3 terms in the polynomial.	$2x^2(x^2 + 5x + 6)$ $2x^2(x + 3)(x + 2)$

Be prepared to share your solutions and methods.

6

Check for Students' Understanding

Complete the chart.

Polynomial	Method of Factoring	Factored Form
$x^2 - 10x + 25$	Perfect Square Trinomial	$(x - 5)^2$
$x^3 - 125$	Difference of Cubes	$(x - 5)(x^2 + 5x + 25)$
$16x^2 - 25$	Difference of Squares	$(4x + 5)(4x - 5)$
$x^4 - 13x^2 + 36$	Quadratic Form	$(x^2 - 4)(x^2 - 9)$ $= (x - 2)(x + 2)(x - 3)(x + 3)$
$5x^4 + 55x^3$	Greatest Common Factor	$5x^3(x + 11)$
$x^3 + 5x^2 + 5x + 25$	Factor by Grouping	$x^2(x + 5) + 5(x + 5)$ $= (x^2 + 5)(x + 5)$

6

6.5

Getting to the Root of It All

Rational Root Theorem

LEARNING GOALS

In this lesson, you will:

- Use the Rational Root Theorem to determine possible roots of a polynomial.
- Use the Rational Root Theorem to factor higher order polynomials.
- Solve higher order polynomials.

KEY TERMS

- Rational Root Theorem

ESSENTIAL IDEAS

- The Rational Root Theorem states that a rational root of a polynomial

  $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0x^0$

  with integer coefficients of the form $\frac{p}{q}$ where p is a factor of the constant term, a_0, and q is a factor of the leading coefficient, a_n.
- To determine the roots or solutions of a polynomial equation, determine the possible rational roots, use synthetic division to determine one of the roots, determine the possible rational roots of the quotient, and repeat the process until all the rational roots are determined. Factor the remaining polynomial to determine any irrational or complex roots.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-APR Arithmetic with Polynomials and Rational Expressions

Understand the relationship between zeros and factors of polynomials

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

F-IF Interpreting Functions

Analyze functions using different representations

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

Overview

The Rational Root Theorem is stated and used to determine possible rational roots of a polynomial function. Next, synthetic division is used to determine which of the possible roots are actual roots. Once the first root is determined, the entire process is repeated to determine the remaining roots of the polynomial.

6

Warm Up

Consider the polynomial, $f(x) = x^3 - 2x^2 - 5x + 6$.

The Possible rational zeros of the function are ±1, ±2, ±3, and ±6.

Use synthetic division to determine the zeros.

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -5 & 6 \\ & & -1 & 3 & 2 \\ \hline & 1 & -3 & -2 & \boxed{8} \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & -5 & 6 \\ & & 2 & 0 & -10 \\ \hline & 1 & 0 & -5 & \boxed{-4} \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -5 & 6 \\ & & -2 & 8 & -6 \\ \hline & 1 & -4 & 3 & \boxed{0} \end{array}$$

$$\begin{array}{r|rrrr} 3 & 1 & -2 & -5 & 6 \\ & & 3 & 3 & -6 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$x^3 - 2x^2 - 5x + 6 = (x - 1)(x + 2)(x - 3)$

The roots are -2, 1, and 3.

6

6

6.5

Getting to the Root of It All

Rational Root Theorem

LEARNING GOALS

In this lesson, you will:

- Use the Rational Root Theorem to determine possible roots of a polynomial.
- Use the Rational Root Theorem to factor higher order polynomials.
- Solve higher order polynomials.

KEY TERM

- Rational Root Theorem

Out of the many vegetables there are to eat, root vegetables are unique. Root vegetables are distinguishable because the root is the actual vegetable that is edible, not the part that grows above ground. These roots would provide the plant above ground the nourishment they need to survive, just like the roots of daisies, roses, or trees; however, we pull up the roots of particular plants from the ground to provide our own bodies with nourishment and vitamins. Although root vegetables should only pertain to those edible parts below the ground, the category of root vegetables includes corms, rhizomes, tubers, and any vegetable that grows underground. Some of the most common root vegetables are carrots, potatoes, and onions.

Root vegetables were a very important food source many years ago before people had the ability to freeze and store food at particular temperatures. Root vegetables, when stored between 32 and 40 degrees Fahrenheit, will last a very long time. In fact, people had root cellars to house these vegetable types through cold harsh winters. In fact, some experts believe people have been eating turnips for over 5000 years! Now that's one popular root vegetable! So, what other root vegetables can you name? What root vegetables do you like to eat?

Problem 1

A table lists several polynomials, the roots, the product of the roots, and the sum of the roots. Students will use the table to answer questions which focus on the relationship between the coefficients of specific terms in each polynomial and the sum or product of its roots.

Grouping

- Ask a student to read the information in the table. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What are the first two coefficients in the polynomial?
- What is the ratio of the second coefficient to the first or leading coefficient?
- What is the negative of the ratio of the second coefficient to the first or leading coefficient? How does this value compare to the sum of the roots?
- What are the first and last coefficients of the odd degree polynomial?
- What is the ratio of the last coefficient to the first or leading coefficient?

6

PROBLEM 1 Ideas Taking Root

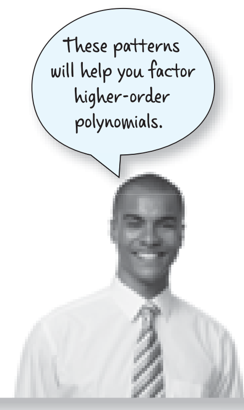
Consider the product and sum of each set of roots.

Polynomial	Roots	Product of Roots	Sum of Roots
$x^2 + 4x - 1 = 0$	$-2 \pm \sqrt{5}$	-1	-4
$x^3 + 2x^2 - 5x - 6 = 0$	$-1, 2, -3$	6	-2
$2x^3 + 5x^2 - 8x - 20 = 0$	$\pm 2, -\frac{5}{2}$	10	$-\frac{5}{2}$
$4x^3 - 3x^2 + 4x - 3 = 0$	$\pm i, \frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$36x^3 + 24x^2 - 43x + 86 = 0$	$\frac{2}{3} \pm \frac{\sqrt{3}}{2} i, -2$	$-\frac{43}{18}$	$-\frac{2}{3}$
$4x^4 - 12x^3 + 13x^2 - 2x - 6 = 0$	$1 \pm i, -\frac{1}{2}, \frac{3}{2}$	$-\frac{3}{2}$	3

1. Compare the sums of the roots to the first two coefficients of each polynomial equation. What conclusion can you draw?
   The sum of the roots is equal to the negative of the ratio of the second coefficient to the first or leading coefficient.

2. Compare the products of the roots to the first and last coefficients of each odd degree polynomial equation. What conclusion can you draw?
   The product of the roots of an odd degree polynomial equation is equal to the negative of the ratio of the last coefficient to the first leading coefficient.

3. Compare the products of the roots to the first and last coefficients of each even degree polynomial equation. What conclusion can you draw?
   The product of the roots of an even degree polynomial equation is the ratio of the last coefficient to the leading coefficient.



- What is the negative of the ratio of the last coefficient to the first or leading coefficient? How does this value compare to the product of the roots?
- What are the first and last coefficients of the even degree polynomial?
- What is the ratio of the last coefficient to the leading coefficient? How does this value compare to the product of the roots?

Grouping

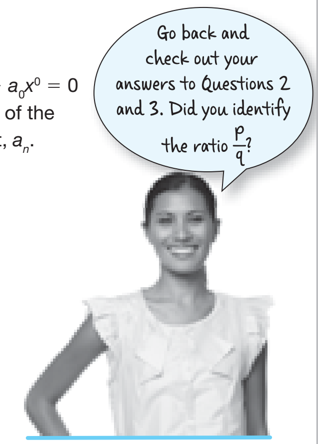
- Ask a student to read the information and theorem. Discuss as a class.
- Have students complete Question 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- What are the constant and coefficient of the leading term in Beyonce's polynomial?
- How did Beyonce factor the constant?
- How did Beyonce factor the coefficient of the leading term?
- What are the constant and coefficient of the leading term in Ivy's polynomial?
- How did Ivy factor the constant?
- How did Ivy factor the coefficient of the leading term?
- Did Ivy include all of the possible factors?

Up until this point, in order to completely factor a polynomial with a degree higher than 2, you needed to know one of the factors or roots. Whether that was given to you, taken from a table, or graph and verified by the Factor Theorem, you started out with one factor or root. What if you are not given any factors or roots? Should you start randomly choosing numbers and testing them to see if they divide evenly into the polynomial? This is a situation when the *Rational Root Theorem* becomes useful.

The **Rational Root Theorem** states that a rational root of a polynomial equation $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0x^0 = 0$ with integer coefficients is of the form $\frac{p}{q}$, where p is a factor of the constant term, a_0, and q is a factor of the leading coefficient, a_n.



4. Beyonce and Ivy each list all possible rational roots for the polynomial equation they are given.

Beyonce

$4x^4 - 2x^3 + 5x^2 + x - 10 = 0$

p could equal any factors of -10, so $\pm1, \pm2, \pm5, \pm10$

q could equal any factors of 4, so $\pm1, \pm2, \pm4$

Therefore, possible zeros are

$\frac{p}{q} = \pm1, \pm2, \pm5, \pm10,$
$\pm\frac{1}{2}, \pm\frac{5}{2}, \pm\frac{1}{4}, \pm\frac{5}{4}.$

Ivy

$6x^5 - 2x^3 + x^2 - 3x - 15 = 0$

p could equal any factors of -15, so $\pm1, \pm3, \pm5, \pm15$

q could equal any factors of 6, so $\pm2, \pm3, \pm6$

Therefore, possible zeros are

$\frac{p}{q} = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2},$
$\pm\frac{1}{3}, \pm1, \pm\frac{5}{3}, \pm5,$
$\pm\frac{1}{6}, \pm\frac{5}{6}.$

Explain why Ivy is incorrect and correct her work.

Ivy is incorrect because she forgot to include ±1 as a factor of $q = 6$. To correct her work, she needs to include ±3 and ±15 as possible rational roots.

6

Grouping

Have students complete Questions 5 and 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 and 6

- What degree is this polynomial?
- How many possible p-values for the original polynomial?
- How many possible q-values for the original polynomial?
- How many possible rational roots for the original polynomial?
- If the remainder in the synthetic division problem is zero, what does this imply about the divisor?
- After the linear factor is removed, what degree is the remaining polynomial?
- How many possible p-values for the remaining polynomial?
- How many possible q-values for the remaining polynomial?
- How many possible rational roots for the remaining polynomial?
- After the second linear factor is removed, what degree is the remaining polynomial?
- Is the quadratic easily factored?
- Did you identify four roots for the original polynomial function?

6

5. Complete each step to factor and solve $x^4 + x^3 - 7x^2 - x + 6 = 0$.

a. Determine all the possible rational roots.

$p = \pm 1, \pm 2, \pm 3, \pm 6$

$q = \pm 1$

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6$

b. Use synthetic division to determine which of the possible roots are actual roots.

Answers may vary.

$$\begin{array}{r|rrrrr} 1 & 1 & 1 & -7 & -1 & 6 \\ & & 1 & 2 & -5 & -6 \\ \hline & 1 & 2 & -5 & -6 & 0 \end{array}$$

R = 0, so 1 is a root of the equation.

c. Rewrite the polynomial as a product of its quotient and linear factor.

Answers may vary.

$(x - 1)(x^3 + 2x^2 - 5x - 6) = 0$

d. Repeat steps $a-c$ for the cubic expression.

$p = \pm 1, \pm 2, \pm 3, \pm 6$

$q = \pm 1$

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6$

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

R = 0, so −1 is a root of the equation.

$(x - 1)(x + 1)(x^2 + x - 6) = 0$

e. Factor completely and solve.

$(x - 1)(x + 1)(x^2 + x - 6) = 0$

$(x - 1)(x + 1)(x - 2)(x + 3) = 0$

$x = -1, 1, 2, -3$

6. Determine all roots for $x^4 - 7x^2 - 18 = 0$.

a. Determine the possible roots.

$p = \pm1, \pm2, \pm3, \pm6, \pm9, \pm18$

$q = \pm1$

$\frac{p}{q} = \pm1, \pm2, \pm3, \pm6, \pm9, \pm18$

b. Use synthetic division to determine one of the roots.

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & -7 & 0 & -18 \\ & & 1 & 1 & -6 & -6 \\ \hline & 1 & 1 & -6 & -6 & -24 \end{array}$$

R = −24, so 1 is not a root of the equation.

$$\begin{array}{r|rrrrr} 2 & 1 & 0 & -7 & 0 & -18 \\ & & 2 & 4 & -6 & -12 \\ \hline & 1 & 2 & -3 & -6 & -30 \end{array}$$

R = −30, so 2 is not a root of the equation.

$$\begin{array}{r|rrrrr} 3 & 1 & 0 & -7 & 0 & -18 \\ & & 3 & 9 & 6 & 18 \\ \hline & 1 & 3 & 2 & 6 & 0 \end{array}$$

R = 0, so 3 is a root of the equation.

c. Rewrite the original polynomial as a product.

$x^4 - 7x^2 - 18 = (x - 3)(x^3 + 3x^2 + 2x + 6)$

d. Determine the possible rational roots of the quotient.

$p = \pm1, \pm2, \pm3, \pm6$

$q = \pm1$

$\frac{p}{q} = \pm1, \pm2, \pm3, \pm6$

e. Use synthetic division to determine one of the roots.

$$\begin{array}{r|rrrr} 1 & 1 & 3 & 2 & 6 \\ & & 1 & 4 & 6 \\ \hline & 1 & 4 & 6 & 12 \end{array} \qquad \begin{array}{r|rrrr} -1 & 1 & 3 & 2 & 6 \\ & & -1 & -2 & 0 \\ \hline & 1 & 2 & 0 & 6 \end{array}$$

$$\begin{array}{r|rrrr} -3 & 1 & 3 & 2 & 6 \\ & & -3 & 0 & -6 \\ \hline & 1 & 0 & 2 & 0 \end{array}$$

R = 0, so −3 is a root of the equation. 1 and −1 are not roots.

f. Rewrite the original polynomial as a product.

$x^4 - 7x^2 - 18 = (x - 3)(x + 3)(x^2 + 2)$

6

Problem 2

Students will determine the roots of three polynomial equations. When the remaining quadratic is not easily factorable, the quadratic formula is used to determine complex roots. Next, they are given the graph of two polynomial functions in which they are able to graphically determine one zero. Using that zero in conjunction with synthetic division and the quadratic formula, students will determine the remaining zeros of the polynomial function.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1, part (a)

- A cubic equation has how many zeros or roots?
- Will -1 satisfy this equation? How do you know?
- If -1 satisfies this equation, is it a zero or root?
- What is the remaining quadratic after the linear factor is factored out of the cubic equation?
- Are the zeros or roots of the quadratic real numbers or imaginary? How do you know?

6

g. Determine the possible rational roots of the quotient.

$p = \pm 1, \pm 2$

$q = \pm 1$

$\frac{p}{q} = \pm 1, \pm 2$

h. Determine the remaining roots.

$x = \pm\sqrt{2}i$

i. Rewrite the original polynomial as a product.

$x^4 - 7x^2 - 18 = (x - 3)(x + 3)(x + \sqrt{2}i)(x - \sqrt{2}i)$

PROBLEM 2 What Bulbs are in Your Garden?

You have learned many different ways to solve higher order polynomials. To determine all the roots or solutions of a polynomial equation:

- Determine the possible rational roots.
- Use synthetic division to determine one of the roots.
- Rewrite the original polynomial as a product.
- Determine the possible rational roots of the quotient.
- Repeat the process until all the rational roots are determined.
- Factor the remaining polynomial to determine any irrational or complex roots.
- Recall that some roots may have a multiplicity.

1. Solve each equation over the set of complex numbers.

   a. $x^3 + 1 = 0$

   $p = \pm 1$

   $q = \pm 1$

   $\frac{p}{q} = \pm 1$

   $(-1)^3 + 1 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

   $x^3 + 1 = (x + 1)(x^2 - x + 1)$

   $x = \dfrac{1 \pm \sqrt{1 - 4(1)(1)}}{2(1)} = \dfrac{1 \pm \sqrt{-3}}{2}$

   $= \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$

   $x = -1, \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$

- How many roots of this cubic equation are real numbers and how many are imaginary numbers?

Guiding Questions for Share Phase, Question 1, parts (b) and (c)

- A quartic equation has how many zeros or roots?
- Will 2 satisfy this equation? How do you know?
- If 2 satisfies this equation, is it a zero or root?
- Will −2 satisfy this equation? How do you know?
- If −2 satisfies this equation, is it a zero or root?
- How many roots of this quartic equation are real numbers and how many are imaginary numbers?

b. $x^4 + 3x^2 - 28 = 0$

$p = \pm 1, \pm 2, \pm 4, \pm 7, \pm 14, \pm 28$

$q = \pm 1$

$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 7, \pm 14, \pm 28$

$(2)^4 + 3(2)^2 - 28 = 16 + 12 - 28 = 0$

$(-2)^4 + 3(-2)^2 - 28 = 16 + 12 - 28 = 0$

$$\begin{array}{r|rrrrr} 2 & 1 & 0 & 3 & 0 & -28 \\ & & 2 & 4 & 14 & 28 \\ \hline & 1 & 2 & 7 & 14 & \boxed{0} \end{array}$$

$x^4 + 3x^2 - 28 = (x - 2)(x^3 + 2x^2 + 7x + 14)$

$$\begin{array}{r|rrrr} -2 & 1 & 2 & 7 & 14 \\ & & -2 & 0 & -14 \\ \hline & 1 & 0 & 7 & \boxed{0} \end{array}$$

$$\begin{aligned} x^4 + 3x^2 - 28 &= (x - 2)(x^3 + 2x^2 + 7x + 14) \\ &= (x - 2)(x + 2)(x^2 + 7) \\ &= (x - 2)(x + 2)(x - \sqrt{7}i)(x + \sqrt{7}i) \end{aligned}$$

$x = \pm 2, \pm\sqrt{7}i$

c. $x^4 - 5x^2 - 6x - 2 = 0$

$p = \pm 1, \pm 2$

$q = \pm 1$

$\frac{p}{q} = \pm 1, \pm 2$

$$\begin{array}{r|rrrrr} 2 & 1 & 0 & -5 & -6 & -2 \\ & & 2 & 4 & -2 & -16 \\ \hline & 1 & 2 & -1 & -8 & \boxed{-18} \end{array} \qquad \begin{array}{r|rrrrr} -2 & 1 & 0 & -5 & -6 & -2 \\ & & -2 & 4 & 2 & 8 \\ \hline & 1 & -2 & -1 & -4 & \boxed{6} \end{array}$$

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & -5 & -6 & -2 \\ & & 1 & 1 & -4 & -10 \\ \hline & 1 & 1 & -4 & -10 & \boxed{-12} \end{array} \qquad \begin{array}{r|rrrrr} -1 & 1 & 0 & -5 & -6 & -2 \\ & & -1 & 1 & 4 & 2 \\ \hline & 1 & -1 & -4 & -2 & \boxed{0} \end{array}$$

$x^4 - 5x^2 - 6x - 2 = (x + 1)(x^3 - x^2 - 4x - 2)$

$p = \pm 1, \pm 2$

$q = \pm 1$

$\frac{p}{q} = \pm 1, \pm 2$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & -4 & -2 \\ & & -1 & 2 & 2 \\ \hline & 1 & -2 & -2 & \boxed{0} \end{array}$$

$x^4 - 5x^2 - 6x - 2 = (x + 1)(x + 1)(x^2 - 2x - 2)$

$x = \frac{2 \pm \sqrt{4 - 4(1)(-2)}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$

$x^4 - 5x^2 - 6x - 2 = (x + 1)(x + 1)(x - 1 + \sqrt{3})(x - 1 - \sqrt{3})$

$x = -1, -1, 1 \pm \sqrt{3}$

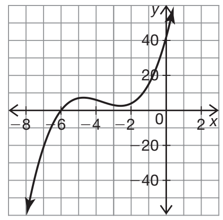
Guiding Questions for Share Phase, Question 2

- Are any of the zeros or roots observable on the graph of the function? Which one(s)?
- What is the remaining quadratic equation after the linear factor is factored out of the cubic function?
- Is the quadratic equation easily factored?
- Are the two remaining roots real or imaginary? How do you know?

6

2. Determine the zeros of each function.

a. $f(x) = x^3 + 11x^2 + 37x + 42$



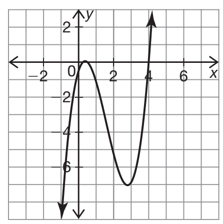
From the graph, I can determine that one of the zeros is −6.

$x^3 + 11x^2 + 37x + 42 = (x + 6)(x^2 + 5x + 7)$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(7)}}{2(1)} = \frac{-5 \pm \sqrt{-3}}{2} = -\frac{5}{2} \pm \frac{\sqrt{3}}{2}i$$

$$x = -6,\ -\frac{5}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{5}{2} - \frac{\sqrt{3}}{2}i$$

b. $f(x) = x^3 - 4.75x^2 + 3.125x - 0.50$



From the graph, I can determine that one of the zeros is 4.

$$\begin{array}{r|rrrr} 4 & 1 & -4.75 & 3.125 & -0.5 \\ & & 4 & -3 & 0.5 \\ \hline & 1 & -0.75 & 0.125 & \boxed{0} \end{array}$$

$x^3 - 4.75x^2 + 3.125x - 0.50 = (x - 4)(x^2 - 0.75x + 0.125)$

$$x = \frac{0.75 \pm \sqrt{(-0.75)^2 - 4(1)(0.125)}}{2(1)} = \frac{0.75 \pm \sqrt{0.0625}}{2} = 0.375 \pm 0.125$$

$x = 4, 0.50, 0.25$

Be prepared to share your solutions and methods.

Check for Students' Understanding

Factor completely and solve over the set of complex numbers.

$x^3 + 6x^2 + 13x + 20 = 0$

$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

$$\begin{array}{r|rrrr} -4 & 1 & 6 & 13 & 20 \\ & & -4 & -8 & -20 \\ \hline & 1 & 2 & 5 & \boxed{0} \end{array}$$

$x^3 + 6x^2 + 13x + 20 = (x^2 + 2x + 5)(x + 4)$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$x = \frac{-2 \pm \sqrt{-16}}{2}$$

$$x = \frac{-2 \pm 4i}{2}$$

$x = -1 + 2i, -1 - 2i$

$x^3 + 6x^2 + 13x + 20 = (x + 4)(x + 1 + 2i)(x + 1 - 2i)$

$x = -4, -1 + 2i, -1 - 2i$

6

6

6.6

Identity Theft

Exploring Polynomial Identities

LEARNING GOALS

In this lesson, you will:

- Use polynomial identities to rewrite numeric expressions.
- Use polynomial identities to generate Pythagorean triples.
- Identify patterns in numbers generated from polynomial identities.
- Prove statements involving polynomials.

KEY TERMS

- Euclid's Formula

ESSENTIAL IDEAS

- Polynomial identities such as $(a + b)^2$, $(a - b)^2$, $a^2 - b^2$, $(a + b)^3$, $(a - b)^3$, $a^3 + b^3$, and $a^3 - b^3$ are used to help perform calculations with large numbers.
- Euclid's Formula is used to generate Pythagorean triples given positive integers r and s, where $r > s$: $(r^2 + s^2) = (r^2 - s^2)^2 + (2rs)^2$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-APR Arithmetic with Polynomials and Rational Expressions

Use polynomial identities to solve problems

4. Prove polynomial identities and use them to describe numerical relationships.

Overview

Polynomial identities such as $(a + b)^2$, $(a - b)^2$, $a^2 - b^2$, $(a + b)^3$, $(a - b)^3$, $a^3 + b^3$, and $a^3 - b^3$ are used to help perform calculations involving large numbers without a calculator. Euclid's Formula is stated and used to generate Pythagorean triples. Rules are written that define different sets of numbers and students complete tables listing numbers in each set. Students analyze the sets looking for patterns and create their own rule and set of numbers. In the last activity, students verify algebraic statements by transforming one side of the equation to show that it is equivalent to the other side of the equation.

Warm Up

Expand each polynomial identity.

1. $(a + b)^2$

   $(a + b)^2 = a^2 + 2ab + b^2$

2. $(a - b)^2$

   $(a - b)^2 = a^2 - 2ab + b^2$

3. $a^2 - b^2$

   $a^2 - b^2 = (a + b)(a - b)$

4. $(a + b)^3$

   $(a + b)^3 = (a + b)(a^2 + 2ab + b^2)$

5. $(a - b)^3$

   $(a - b)^3 = (a - b)(a^2 - 2ab + b^2)$

6. $a^3 + b^3$

   $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

7. $a^3 - b^3$

   $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

6

6.6

Identity Theft

Exploring Polynomial Identities

LEARNING GOALS

In this lesson, you will:

- Use polynomial identities to rewrite numeric expressions.
- Use polynomial identities to generate Pythagorean triples.
- Identify patterns in numbers generated from polynomial identities.
- Prove statements involving polynomials.

KEY TERM

- Euclid's Formula

Have you or someone you know ever been the victim of identity theft? With more and more tasks being performed through the use of technology, identity theft is a growing problem throughout the world. Identity theft occurs when someone steals another person's name or social security number in hopes of accessing that person's money or to make fraudulent purchases.

There are many different ways a person can steal another person's identity. Just a few of these methods are:

- rummaging through a person's trash to obtain personal information and bank statements,
- computer hacking to gain access to personal data,
- pickpocketing to acquire credit cards and personal identification, such as passports or drivers' licenses,
- browsing social networking sites to obtain personal details and photographs.

How important is it to you to secure your identity? What actions would you take to ensure that your identity is not stolen?

Problem 1

Polynomial identities are used to perform calculations such as 46^2 and 55^3.

Grouping

- Ask a student to read the information and worked example. Discuss as a class.
- Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Which two integers did you use to represent 46? Any particular reason behind your choice?
- Did your classmates use the same two integers?
- Is the result the same if different integers were used?
- Are some integers easier to use than others? Why?

PROBLEM 1 Check Your Calculator at the Door

You have learned about many different equivalent polynomial relationships. These relationships are also referred to as polynomial identities.

Some of the polynomial identities that you have used so far are shown.

- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a - b)^2 = a^2 - 2ab + b^2$
- $a^2 - b^2 = (a + b)(a - b)$
- $(a + b)^3 = (a + b)(a^2 + 2ab + b^2)$
- $(a - b)^3 = (a - b)(a^2 - 2ab + b^2)$
- $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Polynomial identities can help you perform calculations. For instance, consider the expression 46^2. Most people cannot calculate this value without the use of a calculator. However, you can use a polynomial identity to write an equivalent expression that is less difficult to calculate.

You can use the polynomial identity $(a + b)^2 = a^2 + 2ab + b^2$ to calculate 46^2.

$46^2 = (40 + 6)^2$	Write 46 as the sum of 40 and 6.
$= 40^2 + 2(40)(6) + 6^2$	Apply the polynomial identity $(a + b)^2 = a^2 + 2ab + b^2$.
$= 1600 + 2(40)(6) + 36$	Apply exponents.
$= 1600 + 480 + 36$	Perform multiplication.
$= 2116$	Perform addition.

The value of 46^2 is 2116.

1. Calculate 46^2 in a different way by writing 46 as the difference of two integers squared.

$$46^2 = (50 - 4)^2$$
$$= 50^2 - 2(50)(4) + 4^2$$
$$= 2500 - 2(50)(4) + 16$$
$$= 2500 - 400 + 16$$
$$= 2100 + 16$$
$$= 2116$$

6

Grouping

Have students complete Question 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- Which two integers did you use to represent 112?
- What operation is associated with the two integers?
- Any particular reason behind your choice?
- Which polynomial identity did you use in this situation?
- Did your classmates use the same polynomial identity?
- Is the result the same if different polynomial identities are used?
- Are some polynomial identities easier to use than others for this situation? Why?
- Which two integers did you use to represent 27?
- What operation is associated with the two integers?
- Any particular reason behind your choice?
- Which polynomial identity did you use in this situation?
- Did your classmates use the same polynomial identity?
- Which two integers did you use to represent 55?
- What operation is associated with the two integers?
- Any particular reason behind your choice?

2. Use polynomial identities and number properties to perform each calculation. Show your work.

a. 112^2

Answers will vary.

$$\begin{aligned}112^2 &= (100 + 12)^2\\ &= 100^2 + 2(100)(12) + 12^2\\ &= 10{,}000 + 2(100)(12) + 144\\ &= 10{,}000 + 2400 + 144\\ &= 12{,}400 + 144\\ &= 12{,}544\end{aligned}$$

b. 27^3

Answers will vary.

$$\begin{aligned}27^3 &= (30 - 3)^3\\ &= (30 - 3)(30^2 - 2(30)(3) + 3^2)\\ &= (30 - 3)(900 - 2(30)(3) + 9)\\ &= (30 - 3)(900 - 180 + 9)\\ &= 30(900) - 30(180) + 30(9) - 3(900) + 3(180) - 3(9)\\ &= 27{,}000 - 5400 + 270 - 2700 + 540 - 27\\ &= 19{,}683\end{aligned}$$

c. 55^3

Answers will vary.

$$\begin{aligned}55^3 &= (50 + 5)^3\\ &= (50 + 5)(50^2 + 2(50)(5) + 5^2)\\ &= (50 + 5)(2500 + 2(50)(5) + 25)\\ &= (50 + 5)(2500 + 500 + 25)\\ &= 50(2500) + 50(500) + 50(25) + 5(2500) + 5(500) + 5(25)\\ &= 125{,}000 + 25{,}000 + 1250 + 12{,}500 + 2500 + 125\\ &= 166{,}375\end{aligned}$$

- Which polynomial identity did you use in this situation?
- Did your classmates use the same polynomial identity?

Problem 2

Euclid's Formula uses polynomial identities to generate Pythagorean triples. The formula is stated as are the steps of its proof and students verify the formula by supplying the reasons for the statements using algebraic properties. Students choose any two integers to represent the *r* and *s*-values and using the formula, they will generate a Pythagorean triple. In several problems they are given two integers and generate the Pythagorean triple. Students are also given a Pythagorean triple and will solve for the two integers that generated the triple.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- How do you know which number represents the hypotenuse *c*?
- Is the sum of $4^2 + 5^2$ equal to 9^2?
- Are the three numbers in a Pythagorean triple always integers?
- Is the sum of $39^2 + 80^2$ equal to 89^2?
- Is the sum of any two squared numbers always a perfect square number?
- How do you know when the sum of two squared numbers is also a perfect square?

6

PROBLEM 2 It's Triplets!

Remember that a Pythagorean triple is a set of three positive integers, *a*, *b*, and *c*, such that $a^2 + b^2 = c^2$.

1. Determine whether each set of numbers is a Pythagorean triple. Explain your reasoning.

   a. 4, 5, 9

   $4^2 + 5^2 \stackrel{?}{=} 9^2$

   $16 + 25 \stackrel{?}{=} 81$

   $41 \neq 81$

   These numbers are not a Pythagorean triple because $4^2 + 5^2 \neq 9^2$.

   b. 0.4, 0.5, 0.3

   These numbers are not a Pythagorean triple because they are not positive integers.

   c. 89, 80, 39

   $39^2 + 80^2 \stackrel{?}{=} 89^2$

   $1521 + 6400 \stackrel{?}{=} 7921$

   $7921 = 7921$

   These numbers are a Pythagorean triple because $39^2 + 80^2 = 89^2$.

You have just determined whether three positive numbers make up a Pythagorean triple, but suppose that you wanted to *generate* integers that are Pythagorean triples.

2. Describe a process you could use to calculate integers that are Pythagorean triples.

   Answers will vary.

   Student responses could include a process of trial and error, in which they begin with two numbers and square them to see if their sum is also a perfect square. This process would be quite time-consuming.

Grouping

Ask a student to read the information and worked example. Discuss as a class.

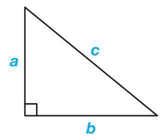
There is an efficient method to generate Pythagorean triples that involves a polynomial identity called *Euclid's Formula*.

Euclid's Formula is a formula used to generate Pythagorean triples given any two positive integers. Given positive integers r and s, where $r > s$, Euclid's Formula is shown.

$$(r^2 + s^2)^2 = (r^2 - s^2)^2 + (2rs)^2$$

The expressions in Euclid's Formula represent the side lengths of a right triangle, a, b, and c, as shown.

$$\begin{array}{ccccc} c^2 & = & a^2 & + & b^2 \\ \downarrow & & \downarrow & & \downarrow \\ (r^2 + s^2)^2 & = & (r^2 - s^2)^2 & + & (2rs)^2 \end{array}$$



You can verify Euclid's Formula by transforming the right side of the equation to show that it is equal to the left side.

Given positive integers a and b, where $a > b$.

$(a^2 + b^2)^2 \stackrel{?}{=} (a^2 - b^2)^2 + (2ab)^2$	Apply Euclid's Formula.
$\stackrel{?}{=} a^4 - 2a^2b^2 + b^4 + (2ab)^2$	Square the binomial.
$\stackrel{?}{=} a^4 - 2a^2b^2 + b^4 + 4a^2b^2$	Apply Product to a Power Rule.
$\stackrel{?}{=} a^4 - 2a^2b^2 + 4a^2b^2 + b^4$	Apply the Commutative Property of Addition.
$\stackrel{?}{=} a^4 + 2a^2b^2 + b^4$	Combine like terms.
$= (a^2 + b^2)^2$	Factor perfect square trinomial.

6

Grouping

Have students complete Question 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

- Which *r*- and *s*-values did you use in this situation?
- Can you use any *r*- and *s*-values in this situation? Why not?
- Any particular reason you chose your *r*- and *s*-values?

Grouping

Have students complete Questions 4 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- Using the integers 4 and 7, is the resulting Pythagorean triple a multiple of a smaller triple? How do you know?
- Using the integers 11 and 5, is the resulting Pythagorean triple a multiple of a smaller triple? How do you know?
- Using the integers 15 and 20, is the resulting Pythagorean triple a multiple of a smaller triple? How do you know?

6

3. Use Euclid's Formula to generate a Pythagorean triple.

a. Choose two integers and use them to generate a Pythagorean triple. Explain your choice in integers.

Answers will vary.

Sample answer: I choose the numbers 2 and 1, where $r = 2$ and $s = 1$.

$(2^2 + 1^2)^2 = (2^2 - 1^2)^2 + (2(2)(1))^2$

$(4 + 1)^2 = (4 - 1)^2 + (2(2))^2$

$5^2 = 3^2 + 4^2$

$25 = 25$

The Pythagorean triple is 3, 4, 5.

I chose the numbers $r = 2$ and $s = 1$ because these numbers are easy to work with.

b. Compare your Pythagorean triple to others in your class. Did everyone get the same triple?

Answers will vary, but there should be a variety of triples.

4. Generate a Pythagorean triple using each pair of given numbers and Euclid's Formula.

a. 4 and 7

$(7^2 + 4^2)^2 = (7^2 - 4^2)^2 + (2(7)(4))^2$

$(49 + 16)^2 = (49 - 16)^2 + (2(28))^2$

$65^2 = 33^2 + 56^2$

$4225 = 4225$

The Pythagorean triple is 33, 56, 65.

b. 11 and 5

$(11^2 + 5^2)^2 = (11^2 - 5^2)^2 + (2(11)(5))^2$

$(121 + 25)^2 = (121 - 25)^2 + (2(55))^2$

$146^2 = 96^2 + 110^2$

$21{,}316 = 21{,}316$

The Pythagorean triple is 96, 110, 146.

c. 15 and 20

$(20^2 + 15^2)^2 = (20^2 - 15^2)^2 + (2(20)(15))^2$

$(400 + 225)^2 = (400 - 225)^2 + (2(300))^2$

$625^2 = 175^2 + 600^2$

$390{,}625 = 390{,}625$

The Pythagorean triple is 175, 600, 625.

Guiding Questions for Share Phase, Questions 5 and 6

- Using any two specific integers, is the resulting Pythagorean triple a unique triple, or could it be a multiple of a smaller triple?
- Under what circumstances is the resulting Pythagorean triple a multiple of a smaller triple?
- If you are given the triple and want to determine the two integers that generated the triple, what is the first step?
- Does 13 represent $r^2 + s^2$, $r^2 - s^2$, or $2rs$? How do you know?

5. Did any of the Pythagorean triples you generated have a common factor? If so, identify them, and explain why you think this happened.

The Pythagorean triple I generated in part (c) has a common factor of 25. This happened because the original integers have a common factor of 5, which is the square root of 25.

6. The integers 5, 12, 13 make up a fairly well-known Pythagorean triple. What two integers generate this triple? Show your work.

$13^2 = 5^2 + 12^2$

So, $r^2 + s^2 = 13$, $r^2 - s^2 = 5$, and $2rs = 12$.

Add $r^2 + s^2 = 13$ and $r^2 - s^2 = 5$ to get $2r^2 = 18$.

$2r^2 = 18$

$r^2 = 9$

$r = \pm 3$

Because r must be a positive integer, $r = 3$.

$2rs = 12$

$2(3)s = 12$

$6s = 12$

$s = 2$

So, the numbers that generate the 5, 12, 13 triple are $r = 3$ and $s = 2$.

Problem 3

Rules are written that define different sets of numbers and students will complete tables listing numbers in each set. They then answer questions related to patterns within each set of numbers and look for relationships between the sets using given numbers.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

6

PROBLEM 3 Is This Your Special Number?

After learning that Euclid's Formula generates numbers that are Pythagorean triples, Danielle and Mike wonder what other formulas they could use to generate interesting patterns. Each came up with their own sets of numbers.

Danielle named her numbers the "Danielle numbers." She defined them as shown.

> The Danielle numbers are any numbers that can be generated using the formula $a^2 + b^2$, where a and b are positive integers and $a > b$.

Following suit, Mike named his numbers the "Mike numbers," and he defined his numbers as shown.

> The Mike numbers are any numbers that can be generated using the formula $a^2 - b^2$, where a and b are positive integers and $a > b$.

1. Complete each table to determine the first few Danielle numbers and the first few Mike numbers. Shade the corresponding cell if a is not greater than b.

Danielle Numbers: $a^2 + b^2$

		b				
		1	2	3	4	5
a	1					
	2	$2^2 + 1^2 = 5$				
	3	$3^2 + 1^2 = 10$	$3^2 + 2^2 = 13$			
	4	$4^2 + 1^2 = 17$	$4^2 + 2^2 = 20$	$4^2 + 3^2 = 25$		
	5	$5^2 + 1^2 = 26$	$5^2 + 2^2 = 29$	$5^2 + 3^2 = 34$	$5^2 + 4^2 = 41$	

Mike Numbers: $a^2 - b^2$

		b				
		1	2	3	4	5
a	1					
	2	$2^2 - 1^2 = 3$				
	3	$3^2 - 1^2 = 8$	$3^2 - 2^2 = 5$			
	4	$4^2 - 1^2 = 15$	$4^2 - 2^2 = 12$	$4^2 - 3^2 = 7$		
	5	$5^2 - 1^2 = 24$	$5^2 - 2^2 = 21$	$5^2 - 3^2 = 16$	$5^2 - 4^2 = 9$	

Guiding Questions for Share Phase, Questions 2 and 3

- Do any numbers appear in both Danielle's table and Mike's table? Which ones?
- Across each row, do Danielle's numbers increase or decrease? By how much?
- Across each row, do Mike's numbers increase or decrease? By how much?
- Across each diagonal, do Mike's numbers increase or decrease? By how much?
- Across each column, do Danielle's numbers and Mike's numbers increase or decrease? By how much?
- Across each row, is there an observable even number and odd number pattern?
- Across each column, is there an observable even number and odd number pattern?
- How can Danielle's rule be used to generate 13?
- How can Mike's rule be used to generate 13?
- How can Mike's rule be used to generate 3?
- What is the smallest number in Danielle's set?

2. Describe any and all patterns you see in each table in Question 1.

Answers will vary.

Sample responses could include (but are not limited to):

- Across each row, the Danielle numbers increase by 3, then 5, then 7, etc.
- Across each row, the Mike numbers decrease by 3, then 5, then 7, etc.
- Across each diagonal, the Mike numbers increase by a constant amount.
- Across each column, the Danielle numbers and the Mike numbers increase by 5 from row 2 to row 3, then they increase by 7 from row 3 to 4, then by 9 from row 4 to 5, etc.
- Across each row, the Danielle numbers and the Mike numbers alternate between odd and even numbers.
- Across each column, the Danielle numbers and the Mike numbers alternate between odd and even numbers.

3. Determine whether each number is a Danielle number, a Mike number, both, or neither. Explain your reasoning.

a. 13

This number is both a Danielle number and a Mike number, because $3^2 + 2^2 = 13$ (Danielle number) and $7^2 - 6^2 = 13$ (Mike number).

b. 3

This number is a Mike number because $2^2 - 1^2 = 3$. The Danielle numbers start at 5 and then always increase, so the number 3 cannot be a Danielle number.

c. 2

This number is neither a Danielle number nor a Mike number. The Danielle numbers start at 5 and then always increase, so the number 2 cannot be a Danielle number. The Mike numbers start at 3 and then always increase, so the number 2 cannot be a Mike number.

6

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 4 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 6

- Are Sandy's numbers similar to Danielle's numbers or Mike's numbers?
- Are Dave's numbers similar to Danielle's numbers or Mike's numbers?
- How is Sandy's rule similar to Danielle's rule?
- How is Dave's rule similar to Mike's rule?
- Across each row, do Dave's numbers increase or decrease? By how much?
- Across each row, do Sandy's numbers increase or decrease? By how much?
- Across each column, do Dave's table and Sandy's numbers increase or decrease? By how much?
- Across each row, is there an observable even number and odd number pattern?
- Across each column, is there an observable even number and odd number pattern?
- How can Dave's rule be used to generate 35?

After hearing about Danielle and Mike's numbers, Dave and Sandy decide to create their own numbers as well. Their definitions are shown.

The Dave numbers are any numbers that can be generated using the formula $a^3 + b^3$, where a and b are positive integers and $a > b$.

The Sandy numbers are any numbers that can be generated using the formula $a^3 - b^3$, where a and b are positive integers and $a > b$.

4. Complete the tables to determine the first few Dave numbers, and the first few Sandy numbers. Shade the corresponding cell if a is not greater than b.

Dave Numbers: $a^3 + b^3$

		b				
		1	2	3	4	5
a	1					
	2	$2^3 + 1^3 = 9$				
	3	$3^3 + 1^3 = 28$	$3^3 + 2^3 = 35$			
	4	$4^3 + 1^3 = 65$	$4^3 + 2^3 = 72$	$4^3 + 3^3 = 91$		
	5	$5^3 + 1^3 = 126$	$5^3 + 2^3 = 133$	$5^3 + 3^3 = 152$	$5^3 + 4^3 = 189$	

Sandy Numbers: $a^3 - b^3$

		b				
		1	2	3	4	5
a	1					
	2	$2^3 - 1^3 = 7$				
	3	$3^3 - 1^3 = 26$	$3^3 - 2^3 = 19$			
	4	$4^3 - 1^3 = 63$	$4^3 - 2^3 = 56$	$4^3 - 3^3 = 37$		
	5	$5^3 - 1^3 = 124$	$5^3 - 2^3 = 117$	$5^3 - 3^3 = 98$	$5^3 - 4^3 = 61$	

6

5. Describe any and all patterns you see in each table in Question 4.

Answers will vary.

Sample responses could include (but are not limited to):

- Across each row, the Dave numbers increase by 7, then 19, then 37, etc., while the Sandy numbers decrease by 7, then 19, then 37, etc.
- Across each column, the Dave numbers and the Sandy numbers increase by 19 from row 2 to row 3, then they increase by 37 from row 3 to 4, then by 61 from row 4 to 5, etc.
- Across each row, the Dave numbers and the Sandy numbers alternate between odd and even numbers.
- Across each column, the Dave numbers and the Sandy numbers alternate between odd and even numbers.

6. Determine whether each number is a Dave number, a Sandy number, both, or neither. Explain your reasoning.

a. 35

This number is a Dave number because $3^3 + 2^3 = 35$. The first few Sandy numbers in numerical order are 7, 19, 26, and 37, and then continue to increase. So, the number 35 cannot be a Sandy number.

b. 5

This number is neither a Dave number nor a Sandy number. The Dave numbers start at 9 and then always increase, so the number 5 cannot be a Dave number. The Sandy numbers start at 7 and then always increase, so the number 5 cannot be a Sandy number.

7. Write a rule that defines your own set of numbers. What interesting patterns do you see with your numbers?

Answers will vary.

Grouping

Have students complete Question 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 7

- What operations are associated with your rule?
- How many terms are used in your rule?
- How many variables are used in your rule?
- Is your rule a polynomial identity?
- How is your rule different than your classmate's rules?

Problem 4

Students will verify algebraic statements by transforming one side of the equation to show that it is equivalent to the other side of the equation.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What operations were performed in this situation?
- What algebraic properties were used in this situation?
- Is there more than one way to solve this problem?
- Did you change the algebraic expression on the left side of the equation or the right side of the equation?
- Is it easier to change the left side to match the right side or the right side to match the left side? Why?
- Which polynomial identities were used to solve this problem?

6

PROBLEM 4 Okay, Now Prove It!

Verify each algebraic statement by transforming one side of the equation to show that it is equivalent to the other side of the equation.

1. $v^6 - w^6 = (v^2 - w^2)(v^2 - vw + w^2)(v^2 + vw + w^2)$

Method 1:

$$v^6 - w^6 \stackrel{?}{=} (v^2 - w^2)(v^2 - vw + w^2)(v^2 + vw + w^2)$$
$$\stackrel{?}{=} (v^4 - v^3w + v^2w^2 - v^2w^2 + vw^3 - w^4)(v^2 + vw + w^2)$$
$$\stackrel{?}{=} v^6 + v^5w + v^4w^2 - v^5w - v^4w^2 - v^3w^3 + v^4w^2 + v^3w^3 + v^2w^4 - v^4w^2 - v^3w^3 - v^2w^4 + v^3w^3 + v^2w^4 + vw^5 - v^2w^4 - vw^5 - w^6$$
$$= v^6 - w^6$$

Method 2:

$$v^6 - w^6 = (v^2 - w^2)(v^2 - vw + w^2)(v^2 + vw + w^2)$$
$$(v^3 + w^3)(v^3 - w^3) \stackrel{?}{=}$$
$$(v + w)(v^2 - vw + w^2)(v - w)(v^2 + vw + w^2) \stackrel{?}{=}$$
$$(v + w)(v - w)(v^2 - vw + w^2)(v^2 + vw + w^2) \stackrel{?}{=}$$
$$(v^2 - w^2)(v^2 - vw + w^2)(v^2 + vw + w^2) =$$

2. $(p^4 + q^4)^2 = (p^4 - q^4)^2 + (2p^2q^2)^2$

Method 1:

$$(p^4 + q^4)^2 \stackrel{?}{=} (p^4 - q^4)^2 + (2p^2q^2)^2$$
$$\stackrel{?}{=} p^8 - 2p^4q^4 + q^8 + 4p^4q^4$$
$$\stackrel{?}{=} p^8 + 2p^4q^4 + q^8$$
$$= (p^4 + q^4)^2$$

Method 2:

$$(p^4 + q^4)^2 \stackrel{?}{=} (p^4 - q^4)^2 + (2p^2q^2)^2$$
$$p^8 + 2p^4q^4 + q^8 \stackrel{?}{=}$$
$$p^8 + 2p^4q^4 + q^8 + 2p^4q^4 - 2p^4q^4 \stackrel{?}{=}$$
$$p^8 - 2p^4q^4 + q^8 + 2p^4q^4 + 2p^4q^4 \stackrel{?}{=}$$
$$(p^4 - q^4)^2 + 4p^4q^4 \stackrel{?}{=}$$
$$(p^4 - q^4)^2 + (2p^2q^2)^2 =$$

3. $m^9 + n^9 = (m + n)(m^2 - mn + n^2)(m^6 - m^3n^3 + n^6)$

Method 1:

$$\begin{aligned} m^9 + n^9 &\stackrel{?}{=} (m + n)(m^2 - mn + n^2)(m^6 - m^3n^3 + n^6) \\ &\stackrel{?}{=} (m^3 - m^2n + mn^2 + m^2n - mn^2 + n^3)(m^6 - m^3n^3 + n^6) \\ &\stackrel{?}{=} (m^3 + n^3)(m^6 - m^3n^3 + n^6) \\ &\stackrel{?}{=} m^9 - m^6n^3 + m^3n^6 + m^6n^3 - m^3n^6 + n^9 \\ &= m^9 + n^9 \end{aligned}$$

Method 2:

$$\begin{aligned} m^9 + n^9 &\stackrel{?}{=} (m + n)(m^2 - mn + n^2)(m^6 - m^3n^3 + n^6) \\ (m^3)^3 + (n^3)^3 &\stackrel{?}{=} \\ (m^3 + n^3)((m^3)^2 - m^3n^3 + (n^3)^2) &\stackrel{?}{=} \\ (m^3 + n^3)(m^6 - m^3n^3 + n^6) &\stackrel{?}{=} \\ (m + n)(m^2 - mn + n^2)(m^6 - m^3n^3 + n^6) &= \end{aligned}$$

Be prepared to share your solutions and methods.

6

Check for Students' Understanding

Create an example for each representation.

1. Difference of Cubes

   Answers will vary.

   $8x^3 - 64 = (2x - 4)(4x^2 + 8x + 16)$

2. Difference of Squares

   Answers will vary.

   $25x^2 - 36 = (5x + 6)(5x - 6)$

3. Sum of Cubes

   Answers will vary.

   $8x^3 + 64 = (2x + 4)(4x^2 - 8x + 16)$

6

The Curious Case of Pascal's Triangle

6.7

Pascal's Triangle and the Binomial Theorem

LEARNING GOALS

In this lesson, you will:

- Identify patterns in Pascal's Triangle.
- Use Pascal's Triangle to expand powers of binomials.
- Use the Binomial Theorem to expand powers of binomials.
- Extend the Binomial Theorem to expand binomials of the form $(ax + by)^n$.

KEY TERM

- Binomial Theorem

ESSENTIAL IDEAS

- The formula for a combination of k objects from a set of n objects for $n \geq k$ is:
  $\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!}$
- The Binomial Theorem states that it is possible to extend any power of $(a + b)$ into a sum of the form:
  $(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-APR Arithmetic with Polynomials and Rational Expressions

Use polynomial identities to solve problems

5. Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.

Overview

Students will analyze the patterns in the rows of Pascal's Triangle and create additional rows. They then explore a use of Pascal's Triangle when raising a binomial to a positive integer. Students expand several binomials using Pascal's Triangle. Factorial and the formula for combinations are reviewed. The combination formula is given and the graphing calculator and Pascal's Triangle are used to calculate combinations. The Binomial Theorem is stated and students use it to expand $(a + b)^{15}$. They then expand several binomials with coefficients other than 1.

6

Warm Up

The first six rows of Pascal's Triangle are show. Answer each question.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

1. Calculate the sum of each of the 6 rows.
1, 2, 4, 8, 16, 32

2. Calculate the sum of the 7^{th} row.
64

3. Calculate the sum of the 8^{th} row.
128

4. Calculate the sum of the 9^{th} row.
256

5. Calculate the sum of the 10^{th} row.
512

6

6

6.7

The Curious Case of Pascal's Triangle

Pascal's Triangle and the Binomial Theorem

LEARNING GOALS

In this lesson, you will:

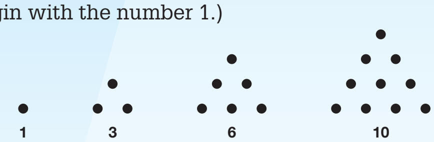
- Identify patterns in Pascal's Triangle.
- Use Pascal's Triangle to expand powers of binomials.
- Use the Binomial Theorem to expand powers of binomials.
- Extend the Binomial Theorem to expand binomials of the form $(ax + by)^n$.

KEY TERM

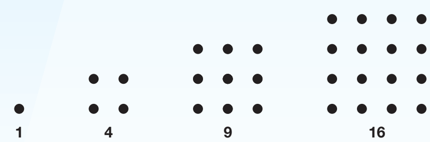
- Binomial Theorem

Some sets of numbers are given special names because of the interesting patterns they create. A *polygonal number* is a number that can be represented as a set of dots that make up a regular polygon. For example, the number 3 is considered a polygonal number because it can be represented as a set of dots that make up an equilateral triangle, as shown.

More specifically, the polygonal numbers that form equilateral triangles are called the *triangular numbers*. The first four triangular numbers are shown. (Note that polygonal numbers always begin with the number 1.)



The *square numbers* are polygonal numbers that form squares. The first four square numbers are shown.



Can you determine the first four pentagonal numbers? How about the first four hexagonal numbers?

6

Problem 1

The first six rows of Pascal's Triangle are shown. Students will analyze the patterns in the rows of Pascal's Triangle and create additional rows using the observable patterns. They then explore a use of Pascal's Triangle when raising a binomial to a positive integer. Students conclude that the coefficients of the terms in each expanded binomial are the same as the numbers in the row of Pascal's Triangle where n is equal to the power of the original binomial, and the sum of the exponents of the a- and b-variables for each term is equal to the power of the original binomial. They expand several binomials using Pascal's Triangle.

Grouping

- Ask a student to read the information and worked example. Discuss as a class.
- Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

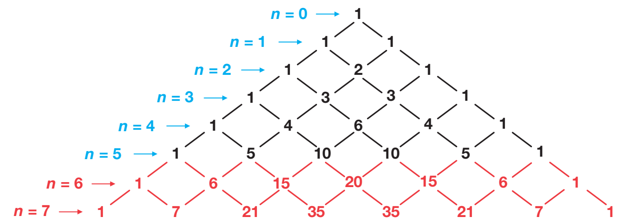
- What do you notice across the first diagonal?
- Which diagonal contains all 1's?
- Which diagonal contains all natural numbers?
- Which diagonal contains all triangular numbers?
- Do you notice any type of symmetry? Where is the axis of symmetry?
- When creating the 6th and 7th row, what number did you start with? Why?
- Which two numbers did you add to get the next number?

6

PROBLEM 1 So Many Patterns, So Little Time

There is an interesting pattern of numbers that makes up what is referred to as Pascal's Triangle.

The first six rows of Pascal's Triangle are shown, where $n = 0$ represents the first row, $n = 1$ represents the second row, and so on.

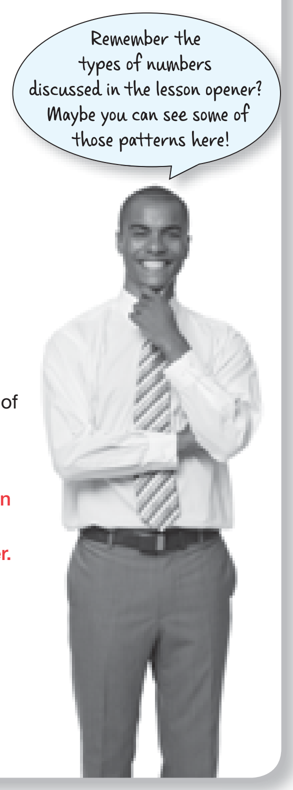


1. Analyze the patterns in Pascal's Triangle.

   a. Describe all the patterns you see in Pascal's Triangle.

   Answers will vary.

   - Across the first diagonal the numbers are all 1s, across the next diagonal the numbers are natural numbers, across the next diagonal the numbers are triangular numbers, etc.
   - All of the numbers are positive.
   - The numbers are symmetric about a vertical line through the apex of the triangle.



   b. Complete the rows for $n = 6$ and $n = 7$ in the diagram of Pascal's Triangle. Describe the pattern you used.

   See diagram above.

   Across each row, I started with the number 1 and then I added the number above and to the left, and the number above and to the right to get the next number.

Grouping

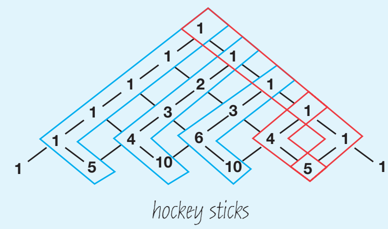
Have students complete Questions 2 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- What do you notice about the sum of the numbers along the longer part of the stick and the number at the end of the shorter part of the stick?
- Is the sum of the numbers along the longer part of the stick equal to the number at the end of the shorter part of the stick?

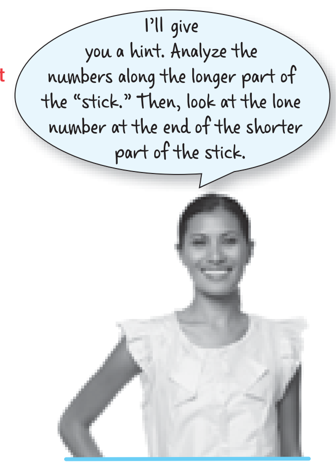
2. Brianna loves hockey. In fact, Brianna is so obsessed with hockey that she drew "hockey sticks" around the numbers in Pascal's Triangle. Lo and behold, she found a pattern! Her work is shown.

Brianna



a. Describe the pattern shown by the numbers inside the hockey sticks that Brianna drew.

The sum of the numbers along the longer part of the stick is equal to the number at the end of the shorter part of the stick.



b. Sketch two more hockey sticks that include numbers that have the same pattern described in part (a).

Sample answers are shown in the drawing above part (a).

6

Guiding Questions for Share Phase, Question 3

- For $n = 0$, does $1 = 2^0$?
- For $n = 1$, does $2 = 2^1$?
- For $n = 2$, does $4 = 2^2$?
- For $n = 3$, does $8 = 2^3$?
- For $n = 4$, does $16 = 2^4$?
- For $n = 5$, does $32 = 2^5$?
- For $n = 1$, does $-1 + 1 = 0$?
- For $n = 2$, does $-1 + 2 - 1 = 0$?
- For $n = 3$, does $-1 + 3 - 3 + 1 = 0$?
- For $n = 4$, does $-1 + 4 - 6 + 4 - 1 = 0$?
- For $n = 5$, does $-1 + 5 - 10 + 10 - 5 + 1 = 0$?

3. Drew and Latasha analyzed Pascal's Triangle, and each described a pattern.

Drew

The sum of the numbers in each row is equal to 2^n, where $n = 0$ represents the first row.

Latasha

If I alternate the signs of the numbers in any row after the first row and then add them together, their sum is 0.

Who's correct? Either verify or disprove each student's work.

Both Drew and Latasha are correct.

Drew:

For $n = 0$: $1 = 1 \rightarrow 2^0$

For $n = 1$: $1 + 1 = 2 \rightarrow 2^1$

For $n = 2$: $1 + 2 + 1 = 4 \rightarrow 2^2$

For $n = 3$: $1 + 3 + 3 + 1 = 8 \rightarrow 2^3$

For $n = 4$: $1 + 4 + 6 + 4 + 1 = 16 \rightarrow 2^4$

For $n = 5$: $1 + 5 + 10 + 10 + 5 + 1 = 36 \rightarrow 2^5$

Latasha:

For $n = 1$: $-1 + 1 = 0$

For $n = 2$: $-1 + 2 - 1 = 0$

For $n = 3$: $-1 + 3 - 3 + 1 = 0$

For $n = 4$: $-1 + 4 - 6 + 4 - 1 = 0$

For $n = 5$: $-1 + 5 - 10 + 10 - 5 + 1 = 0$

6

Guiding Questions for Share Phase, Question 4

- What are Fibonacci numbers?
- Is each number the sum of the previous two numbers?

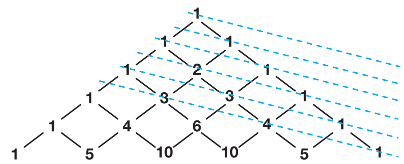
Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 5 and 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- How do the coefficients in each of the terms of the binomial expansion compare to the numbers in the rows of Pascal's Triangle?
- Are the coefficients in each of the terms of the binomial expansion equal to the numbers in the rows of Pascal's Triangle?

4. Consider the numbers along the dashed lines shown.



a. Write the sequence for the sum of numbers along each dashed line.

1, 1, 2, 3, 5, 8

b. Explain how the sums of numbers along the dashed lines in Pascal's Triangle can be linked to a well-known sequence of numbers.

The sums of the numbers along the dashed lines are the Fibonacci numbers. Each of the Fibonacci numbers is the sum of the previous two.

First line: 1

Second line: 1

Third line: $1 + 1 = 2$

Fourth line: $2 + 1 = 3$

Fifth line: $1 + 3 + 1 = 5$

Sixth line: $3 + 4 + 1 = 8$

The patterns shown in Pascal's Triangle have many uses. For instance, you may have used Pascal's Triangle to calculate probabilities. Let's explore how you can use Pascal's Triangle to raise a binomial to a positive integer.

5. Multiply to expand each binomial. Write your final answer so that the powers of *a* are in descending order.

a. $(a + b)^0 = 1$

b. $(a + b)^1 = a + b$

c. $(a + b)^2 = (a + b)(a + b)$
$= a^2 + 2ab + b^2$

d. $(a + b)^3 = (a + b)(a^2 + 2ab + b^2)$
$= a^3 + 3a^2b + 3ab^2 + b^3$

e. $(a + b)^4 = (a + b)(a^3 + 3a^2b + 3ab^2 + b^3)$
$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

- As the exponent of the *a*-variable decreases by 1 in each term, what happens to the exponent of the *b*-variable in each term?
- What is always the first term in the expansion of $(a + b)^n$?
- What is always the last term in the expansion of $(a + b)^n$?

Guiding Questions for Share Phase, Question 6

- Does each successive term in the expansion have an exponent of *a* that is one less than the previous and an exponent of *b* that is one more than the previous?
- How does the sum of the exponents of the *a*- and *b*-variables for each term compare to the power of the original binomial?
- Is the sum of the exponents of the *a*- and *b*-variables for each term equal to the power of the original binomial?

Grouping

Have students complete Question 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 7

- Which row in Pascal's Triangle corresponds with $(a + b)^5$?
- What are the first and last terms associated with $(a + b)^5$?
- What are the coefficients of the terms associated with $(a + b)^5$?
- Which row in Pascal's Triangle corresponds with $(a + b)^6$?
- What are the first and last terms associated with $(a + b)^6$?

6. Analyze your answers to Question 5.

a. Compare the coefficients of each product with the numbers shown in Pascal's Triangle. What do you notice?

The coefficients of the terms in each expanded binomial are the same as the numbers in the row of Pascal's Triangle where *n* is equal to the power of the original binomial.

b. What do you notice about the exponents of the *a*- and *b*-variables in each expansion?

As the exponent of the *a*-variable decreases by 1 in each term, the exponent of the *b*-variable increases by 1 in each term.

Specifically, for $(a + b)^n$, the first term is a^n. Each successive term has an exponent of *a* that is one less than the previous and an exponent of *b* that is one more than the previous, until you reach the last term of b^n.

c. What do you notice about the sum of the exponents of the *a*- and *b*-variables in each expansion?

The sum of the exponents of the *a*- and *b*-variables for each term is equal to the power of the original binomial.

7. Use Pascal's Triangle to expand each binomial.

a. $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

b. $(a + b)^6 =$
$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

c. $(a + b)^7 =$
$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 +$
$7ab^6 + b^7$

- What are the coefficients of the terms associated with $(a + b)^6$?
- Which row in Pascal's Triangle corresponds with $(a + b)^7$?
- What are the first and last terms associated with $(a + b)^7$?
- What are the coefficients of the terms associated with $(a + b)^7$?

6

Problem 2

Factorial and the formula for combinations are reviewed. A worked example using the combination formula is given. Students will use the graphing calculator and Pascal's Triangle to calculate combinations. The Binomial Theorem is stated and students then use it to expand $(a + b)^{15}$. A worked example using the Binomial Theorem with coefficients other than one is provided. Students use the example to expand similar binomials.

Grouping

- Ask a student to read the information. Discuss as a class.
- Complete Question 1 as a class.
- Ask a student to read the information, formula, and worked example. Complete Question 2 as a class.

Guiding Questions for Discuss Phase, Question 1

- Are factorials only defined for whole numbers?
- What does five factorial mean?
- Is 5! the same as 2!3!?
- What is (5)(4)(3)(2)(1)?
- What is (2)(1)(3)(2)(1)?

PROBLEM 2 Binomial Theorem Delirium!

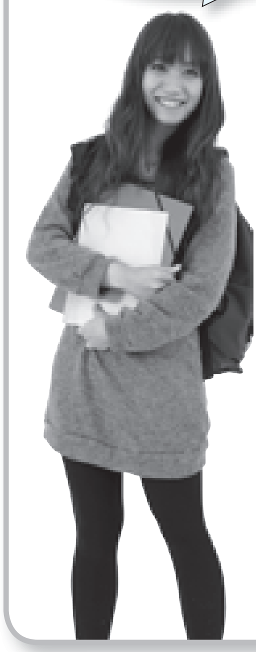
What if you want to expand a binomial such as $(a + b)^{15}$? You could take the time to draw that many rows of Pascal's Triangle, but there is a more efficient way.

Recall that the factorial of a whole number n, represented as $n!$, is the product of all numbers from 1 to n.

You are going to see another method for expanding binomials. But, let's get some notation out of the way first.

1. Perform each calculation and simplify.
   a. $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
   b. $2!3! = (2 \cdot 1)(3 \cdot 2 \cdot 1) = 2(6) = 12$
   c. $\frac{5!}{3!} = \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = \frac{20}{1} = 20$

You may remember that the value of 0! is 1. This is because the product of zero numbers is equal to the multiplicative identity, which is 1.



You may have seen the notation $\binom{n}{k}$ or ${}_nC_k$ when calculating probabilities in another course. Both notations represent the formula for a *combination*. Recall that a combination is a selection of objects from a collection in which order does not matter. The formula for a combination of k objects from a set of n objects for $n \geq k$ is shown.

$$\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!}$$

6

Calculate $\binom{4}{2}$, or ${}_4C_2$.

$\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!}$	Write the formula for a combination.
$n = 4$ and $k = 2$	Identify n and k.
$\binom{4}{2} = \frac{4!}{2!(4-2)!}$	Substitute the values for n and k into the formula.
$= \frac{4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(2 \cdot 1)}$	Write each factorial as a product.
$= \frac{4 \cdot 3 \cdot \cancel{2 \cdot 1}}{(2 \cdot 1)\cancel{(2 \cdot 1)}}$	Divide out common factors.
$= \frac{12}{2} = 6$	Simplify.

2. Explain why n must be greater than or equal to k in the formula for a combination.

In the denominator of the formula, you must calculate $(n - k)!$. Because factorials are defined only for whole numbers, n must be greater than or equal to k.

Grouping

Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

- What is the difference between $\binom{5}{1}$ and ${}_5C_1$?
- Is $\binom{7}{4}$ the same as ${}_7C_4$?

3. Perform each calculation and simplify.

a. $\binom{5}{1} = \frac{5!}{1!(5-1)!}$

$= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(1)(4 \cdot 3 \cdot 2 \cdot 1)}$

$= \frac{5}{1} = 5$

b. ${}_7C_4 = \frac{7!}{4!(7-4)!}$

$= \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{(4 \cdot 3 \cdot 2 \cdot 1)}(3 \cdot 2 \cdot 1)}$

$= \frac{210}{6} = 35$

6

Guiding Questions for Share Phase, Question 4

- If the first row is represented as $n = 0$, and the first number is represented as $k = 0$, is ${}_nC_k$ calculated by looking at the k^{th} or $(k + 1)^{th}$ number in the n^{th} or $(n + 1)^{th}$ row of Pascal's Triangle?
- To calculate ${}_nC_k$, should you look at the kth number in the nth row or the $(k + 1)^{th}$ number in the $(n + 1)^{th}$ row of Pascal's Triangle?

Grouping

Ask a student to read the theorem and complete Question 5 as a class.

4. Sarah and Montel's teacher asks each student to use Pascal's Triangle to calculate ${}_6C_3$. Their answers and explanations are shown.

Sarah	Montel
I can calculate ${}_nC_k$ by looking at the kth number (from left to right) in the nth row of Pascal's Triangle. So, ${}_6C_3$ is equal to 20.	I can calculate ${}_nC_k$ by looking at the $(k + 1)$th number (from left to right) in the $(n + 1)$th row of Pascal's Triangle. So, ${}_6C_3$ is equal to 35.

Who is correct? Explain your reasoning.

Sarah is correct.

The first row is represented as $n = 0$ and the first number in each row is represented as $k = 0$, so ${}_nC_k$ can be calculated by looking at the kth number (from left to right) in the nth row of Pascal's Triangle.

The **Binomial Theorem** states that it is possible to extend any power of $(a + b)$ into a sum of the form shown.

$$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}a^0b^n$$

5. Use the Binomial Theorem to expand $(a + b)^{15}$. You can use your calculator to determine the coefficients.

$$(a+b)^{15} = \binom{15}{0}a^{15}b^0 + \binom{15}{1}a^{14}b^1 + \binom{15}{2}a^{13}b^2 + \cdots + \binom{15}{14}a^1b^{14} + \binom{15}{15}a^0b^{15}$$

$$= a^{15} + 15a^{14}b^1 + 105a^{13}b^2 + 455a^{12}b^3 + 1365a^{11}b^4 + 3003a^{10}b^5 + 5005a^9b^6 + 6435a^8b^7 + 6435a^7b^8 + 5005a^6b^9 + 3003a^5b^{10} + 1365a^4b^{11} + 455a^3b^{12} + 105a^2b^{13} + 15ab^{14} + b^{15}$$

6

Suppose you have a binomial with coefficients other than one, such as $(2x + 3y)^5$. You can use substitution along with the Binomial Theorem to expand the binomial.

You can use the Binomial Theorem to expand $(a + b)^5$, as shown.

$$(a + b)^5 = \binom{5}{0}a^5b^0 + \binom{5}{1}a^4b^1 + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}a^1b^4 + \binom{5}{5}a^0b^5$$
$$= a^5 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + b^5$$

Now consider $(2x + 3y)^5$.

Let $2x = a$ and let $3y = b$.

You can substitute $2x$ for a and $3y$ for b into the expansion for $(a + b)^5$.

$$(2x + 3y)^5 = (2x)^5 + 5(2x)^4(3y)^1 + 10(2x)^3(3y)^2 + 10(2x)^2(3y)^3 + 5(2x)^1(3y)^4 + (3y)^5$$
$$= 32x^5 + 5(16x^4)(3y) + 10(8x^3)(9y^2) + 10(4x^2)(27y^3) + 5(2x)(81y^4) + 243y^5$$
$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$$

Grouping

Have students complete Question 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- What are the first and last terms associated with $(3x + y)^4$?
- What are the coefficients of the terms associated with $(3x + y)^4$?
- What are the first and last terms associated with $(x - 2y)^6$?
- What are the coefficients of the terms associated with $(x - 2y)^6$?

6. Use the Binomial Theorem and substitution to expand each binomial.

a. $(3x + y)^4$

$$(a + b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Let $a = 3x$ and let $b = y$.

$$(3x + y)^4 = (3x)^4 + 4(3x)^3y + 6(3x)^2y^2 + 4(3x)y^3 + y^4$$
$$= 81x^4 + 4(27x^3)y + 6(9x^2)y^2 + 4(3x)y^3 + y^4$$
$$= 81x^4 + 108x^3y + 54x^2y^2 + 12xy^3 + y^4$$

b. $(x - 2y)^6$

$$(a + b)^6 = \binom{6}{0}a^6b^0 + \binom{6}{1}a^5b^1 + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}a^1b^5 + \binom{6}{6}a^0b^6$$
$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Let $a = x$ and let $b = -2y$.

$$(x - 2y)^6 = x^6 + 6x^5(-2y) + 15x^4(-2y)^2 + 20x^3(-2y)^3 + 15x^2(-2y)^4 + 6x(-2y)^5 + (-2y)^6$$
$$= x^6 + (-12)x^5y + 15x^4(4y^2) + 20x^3(-8y^3) + 15x^2(16y^4) + 6x(-32y^5) + 64y^6$$
$$= x^6 - 12x^5y + 60x^4y^2 - 160x^3y^3 + 240x^2y^4 - 192xy^5 + 64y^6$$

Be prepared to share your solutions and methods.

6

Check for Students' Understanding

Use the Binomial Theorem and substitution to determine each term.

1. The third term of $(x + y)^{20}$.

   $(x + y)^{20} = \binom{20}{2} x^{20-2}y^2 = \frac{20!}{18!2!}x^{18}y^2 = \frac{20 \cdot 19}{2 \cdot 1}x^{18}y^2 = 190x^{18}y^2$

2. The fifth term of $(x + y)^{12}$.

   $(x + y)^{12} = \binom{12}{4} x^{12-4}y^4 = \frac{12!}{8!4!}x^8y^4 = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1}x^8y^4 = 495x^8y^4$

3. The 100th term of $(x - y)^{100}$.

   $(x - y)^{100} = \binom{100}{99} x^{100-99}y^{99} = \frac{100!}{1!99!}xy^{99} = 100xy^{99}$

6

6

Chapter 6 Summary

KEY TERMS

- average rate of change (6.1)
- polynomial long division (6.2)
- synthetic division (6.2)
- Euclid's Formula (6.6)

THEOREMS

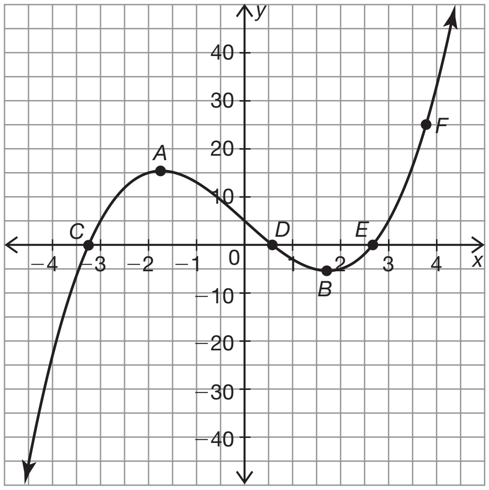
- Remainder Theorem (6.3)
- Factor Theorem (6.3)
- Rational Root Theorem (6.5)
- Binomial Theorem (6.7)

6.1 Analyzing Graphs

A graph can be analyzed over certain intervals or at certain points.

Example

- From point *C* to point *A* the graph is increasing.
- From point *A* to point *B* the graph is decreasing.
- From point *B* to point *F* the graph is increasing.
- Points *C*, *D*, and *E* have a *y*-value of 0.
- Point *A* has a local maximum value of about 15.
- Point *B* has a local minimum value of about −5.



6.1 Determining the Average Rate of Change

The formula for average rate of change is $\frac{f(b) - f(a)}{b - a}$ for an interval (a, b).

Example

The average rate of change over the interval (−1, 4) is

$$f(4) \approx 100,\ f(-1) \approx -200$$

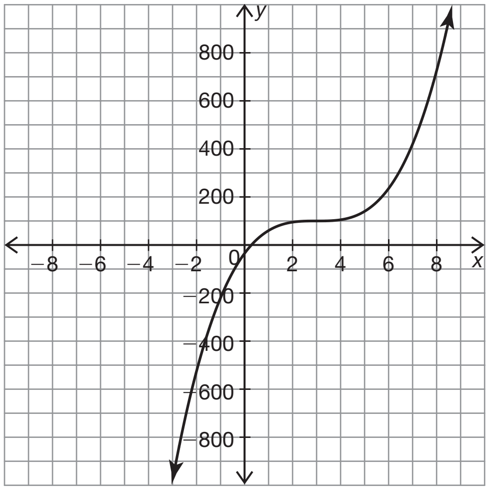
$$= \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(4) - f(-1)}{4 - (-1)}$$

$$= \frac{100 - (-200)}{5}$$

$$= \frac{300}{5}$$

$$= 60$$



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6.2 Using Polynomial Long Division

One polynomial can be divided by another of equal or lesser degree using a process similar to integer division. This process is called polynomial long division. To perform polynomial long division, every power in the dividend must have a placeholder. If there is a gap in the degrees of the dividend, rewrite it so that each power is represented.

Example

The quotient of $3x^3 - 4x^2 + 5x - 3$ divided by $3x + 2$ is $x^2 - 2x + 3$ R -9.

$$
\begin{array}{rrrrr}
 & & x^2 & -\,2x & +\,3 \\ \cline{2-5}
3x + 2\,\big) & 3x^3 & -\,4x^2 & +\,5x & -\,3 \\
 & \underline{3x^3} & \underline{+\,2x^2} & & \\
 & & -6x^2 & +\,5x & \\
 & & \underline{-6x^2} & \underline{-\,4x} & \\
 & & & 9x & -\,3 \\
 & & & \underline{9x} & \underline{+\,6} \\
 & & & & -9
\end{array}
$$

6.2 Determining Factors Using Long Division

When the remainder of polynomial long division is 0, the divisor is a factor of the dividend.

Example

The binomial $2x - 1$ is a factor of $2x^4 + 5x^3 - x^2 + x - 1$ since the remainder is 0.

$$
\begin{array}{rrrrrr}
 & & x^3 & +\,3x^2 & +\,x & +\,1 \\ \cline{2-6}
2x - 1\,\big) & 2x^4 & +\,5x^3 & -\,x^2 & +\,x & -\,1 \\
 & \underline{2x^4} & \underline{-\,x^3} & & & \\
 & & 6x^3 & -\,x^2 & & \\
 & & \underline{6x^3} & \underline{-\,3x^2} & & \\
 & & & 2x^2 & +\,x & \\
 & & & \underline{2x^2} & \underline{-\,x} & \\
 & & & & 2x & -\,1 \\
 & & & & \underline{2x} & \underline{-\,1} \\
 & & & & & 0
\end{array}
$$

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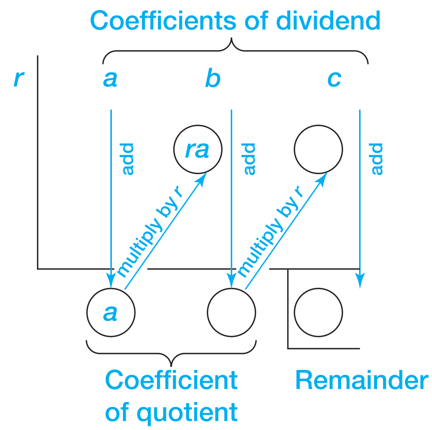
6.3 Using Synthetic Division

Synthetic division is a shortcut method for dividing a polynomial by a binomial $x - r$. To use synthetic division, follow the pattern:



Example

$4x^3 + 3x^2 - 2x + 1$ divided by $x + 3$ is $4x^2 - 9x + 25 \text{ R } \dfrac{-74}{x + 3}$.

-3	4	3	-2	1
		-12	27	-75
	4	-9	25	-74

6.3 Using the Remainder Theorem

The Remainder Theorem states that when any polynomial equation or function, $f(x)$, is divided by a linear factor $(x - r)$, the remainder is $R = f(r)$, or the value of the equation or function when $x = r$.

Example

Let $f(x) = 6x^3 - 2x^2 - x + 1$. When $f(x)$ is divided by $x - 3$, the remainder is 142 since $f(3) = 142$.

$$\begin{aligned} f(3) &= 6(3)^3 - 2(3)^2 - 3 + 1 \\ &= 162 - 18 - 3 + 1 \\ &= 142 \end{aligned}$$

6

6.3

Using the Factor Theorem

The Factor Theorem states that a polynomial has a linear polynomial as a factor if and only if the remainder is 0; $f(x)$ has $(x - r)$ as a factor if and only if $f(r) = 0$.

Examples

Let $f(x) = 2x^3 + 5x^2 - 15x - 12$.

The binomial $(x + 4)$ is a factor of $f(x)$ since $f(-4) = 0$.

$$\begin{aligned} f(-4) &= 2(-4)^3 + 5(-4)^2 - 15(-4) - 12 \\ &= -128 + 80 + 60 - 12 \\ &= 0 \end{aligned}$$

The binomial $(x - 5)$ is not a factor of $f(x)$ since $f(5) \neq 0$.

$$\begin{aligned} f(5) &= 2(5)^3 + 5(5)^2 - 15(5) - 12 \\ &= 250 + 125 - 75 - 12 \\ &= 288 \end{aligned}$$

6

6.4 Factoring Polynomials

There are different methods to factor a polynomial. Depending on the polynomial, some methods of factoring are more efficient than others.

Examples

Factoring out the Greatest Common Factor:
$6x^2 - 36x$
$6x(x - 6)$

Chunking:
$64x^2 + 24x - 10$
$(8x)^2 + 3(8x) - 10$

Let $z = 8x$.
$z^2 + 3z - 10$
$(z - 2)(z + 5)$
$(8x - 2)(8x + 5)$

Factoring by Grouping
$x^3 + 3x^2 + 2x + 6$
$x^2(x + 3) + 2(x + 3)$
$(x^2 + 2)(x + 3)$

Sum or Difference of Cubes
$x^3 - 27$
$(x - 3)(x^2 + 3x + 9)$

$x^3 + 8$
$(x + 2)(x^2 - 2x + 4)$

Difference of Squares
$x^2 - 25$
$(x + 5)(x - 5)$

Perfect Square Trinomials
$9x^2 - 12x + 4$
$(3x - 2)(3x - 2)$

6.5 Using the Rational Root Theorem

The Rational Root Theorem states that a rational root of a polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x^1 + a_0x^0$ with integer coefficients will be of the form $\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

Example

Given the polynomial: $3x^2 + 2x - 6$

$p = \pm1, \pm2, \pm3, \pm6$

$q = \pm1, \pm3$

The possible rational roots are $\pm1, \pm2, \pm3, \pm6, \pm\frac{1}{3}$, and $\pm\frac{2}{3}$.

6

6.5 Solving Polynomials Equations

To determine all roots of a polynomial:

- Determine the possible rational roots.
- Use synthetic division to determine one of the roots.
- Rewrite the original polynomial as a product.
- Determine the possible rational roots of the quotient.
- Repeat the process until all the rational roots are determined.
- Factor the remaining polynomial to determine any irrational or complex roots.
- Recall that some roots may have a multiplicity.

Example

Solve $2x^3 - 9x^2 + 7x + 6 = 0$.

The possible rational roots of $2x^3 - 9x^2 + 7x + 6$ are $\pm\frac{1}{2}$, ±1, $\pm\frac{3}{2}$, ±2, ±3, and ±6 since $p = 6$ and $q = 2$.

Use synthetic division to divide the polynomial by $(x - 3)$.

$$\begin{array}{r|rrrr} 3 & 2 & -9 & 7 & 6 \\ & & 6 & -9 & -6 \\ \hline & 2 & -3 & -2 & \boxed{0} \end{array}$$

$2x^3 - 9x^2 + 7x + 6 = 0$

$(x - 3)(2x^2 - 3x - 2) = 0$

$(x - 3)(2x + 1)(x - 2) = 0$

$x - 3 = 0 \qquad 2x + 1 = 0 \qquad x - 2 = 0$

$x = 3 \qquad x = -\frac{1}{2} \qquad x = 2$

6

6.6 Using Polynomial Identities for Numerical Calculations

Some of the polynomial identities are shown. Polynomial identities can be used to perform calculations.

- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a - b)^2 = a^2 - 2ab + b^2$
- $a^2 - b^2 = (a + b)(a - b)$
- $(a + b)^3 = (a + b)(a^2 + 2ab + b^2)$
- $(a - b)^3 = (a - b)(a^2 - 2ab + b^2)$
- $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Example

To calculate 13^3, use the identity $(a + b)^3 = (a + b)(a^2 + 2ab + b^2)$.

$$\begin{aligned} 13^3 &= (10 + 3)^3 \\ &= (10 + 3)(10^2 + 2(10)(3) + 3^2) \\ &= 13(100 + 60 + 9) \\ &= 13(100) + 13(60) + 13(9) \\ &= 1{,}300 + 780 + 117 \\ &= 2{,}197 \end{aligned}$$

6.6 Using Euclid's Formula to Generate Pythagorean Triples

Euclid's Formula is a formula used to generate Pythagorean triples given any two positive integers.

Given positive integers r and s, where $r > s$, Euclid's Formula is $(r^2 + s^2)^2 = (r^2 - s^2)^2 + (2rs)^2$.

Example

Generate a Pythagorean Triple using the numbers 6 and 13.

Let $r = 13$ and $s = 6$.

$$\begin{aligned} (13^2 + 6^2) &= (13^2 - 6^2) + (2(13)(6))^2 \\ (205)^2 &= (133)^2 + (156)^2 \\ 42{,}025 &= 42{,}025 \end{aligned}$$

So 133, 156, 205 is a Pythagorean triple.

6.7 Using Pascal's Triangle to Expand Binomials

The coefficients for the expansion of $(a + b)^n$ are the same as the numbers in the row of Pascal's Triangle where n is equal to the power of the original binomial.

$n = 0 \longrightarrow$ 1

$n = 1 \longrightarrow$ 1 1

$n = 2 \longrightarrow$ 1 2 1

$n = 3 \longrightarrow$ 1 3 3 1

$n = 4 \longrightarrow$ 1 4 6 4 1

$n = 5 \longrightarrow$ 1 5 10 10 5 1

$n = 6 \longrightarrow$ 1 6 15 20 15 6 1

$n = 7 \longrightarrow$ 1 7 21 35 35 21 7 1

Example

$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

6.7 Using the Binomial Theorem to Expand Binomials

The Binomial Theorem states that it is possible to expand any power of $(a + b)$ into a sum in the following form:

$(a + b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}a^0b^n.$

Example

Expand $(2x - y)^6$.

$$\begin{aligned}(a + b)^6 &= \binom{6}{0}a^6b^0 + \binom{6}{1}a^5b^1 + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}a^1b^5 + \binom{6}{6}a^0b^6 \\ &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6\end{aligned}$$

Let $a = 2x$ and $b = -y$.

$$\begin{aligned}(2x - y)^6 &= (2x)^6 + 6(2x)^5(-y) + 15(2x)^4(-y)^2 + 20(2x)^3(-y)^3 + 15(2x)^2(-y)^4 + 6(2x)(-y)^5 + (-y)^6 \\ &= 64x^6 - 6(32)x^5y + 15(16)x^4y^2 - 20(8)x^3y^3 + 15(4)x^2y^4 - 12xy^5 + y^6 \\ &= 64x^6 - 192x^5y + 240x^4y^2 - 160x^3y^3 + 60x^2y^4 - 12xy^5 + y^6\end{aligned}$$

6