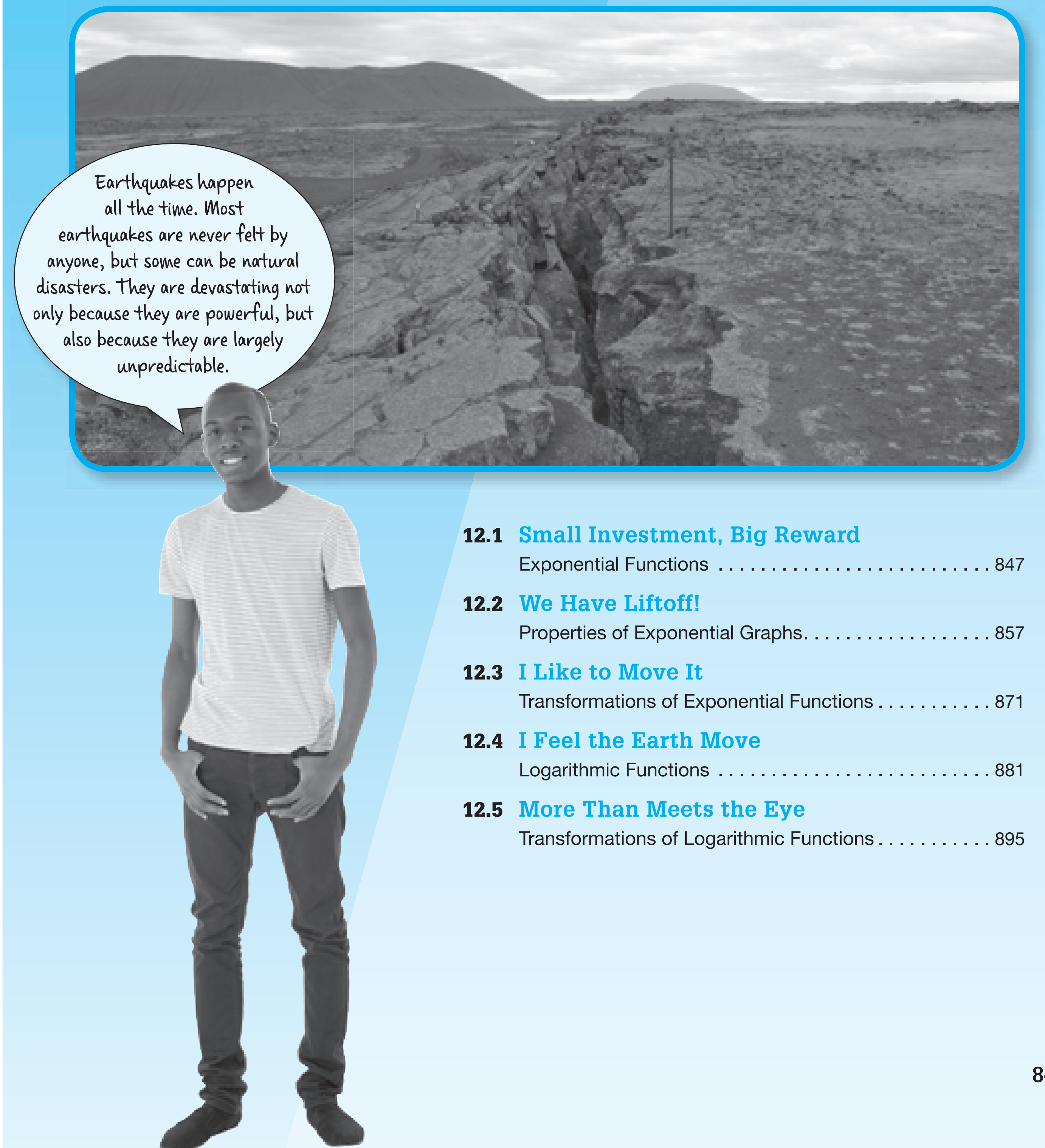
12

Graphing Exponential and Logarithmic Functions



Chapter 12 Overview

This chapter presents opportunities for students to analyze, graph, and transform exponential and logarithmic functions. The chapter begins with an exploration of exponential functions. Students will analyze key characteristics of exponential functions and graphs. Lessons then expand on this knowledge for transformations of exponential functions.

In the later part of the chapter, lessons focus on logarithmic functions. Student will determine key characteristics of logarithmic functions and graphs. Students will also transform logarithmic functions and make generalizations about the effect of a transformation on an inverse function.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
12.1	Exponential Functions	F.IF.4 F.IF.8.b F.LE.5	2	This lesson provides opportunities for students to explore exponential functions. Questions lead students to create exponential graphs using their previous knowledge of geometric sequences. Questions then ask students to analyze the graph of their functions. Questions also use technology to analyze a real-world problem situation using an exponential function.	X	X	X		X
12.2	Properties of Exponential Graphs	F.IF.4 F.IF.7.e F.IF.9	2	This lesson provides opportunities for students to analyze the characteristics of the basic exponential growth and exponential decay functions. Questions introduce students to the irrational number *e* through the development of continuous compound interest.	X	X	X		X
12.3	Transformations of Exponential Functions	F.BF.3	1	This lesson provides opportunities for students to apply their understanding of the transformational function $g(x) = Af(B(x - C)) + D$ to the exponential graph. Questions ask students to identify reflections over the axes and state key characteristics of transformed exponential functions. Questions then ask students to graph horizontal and vertical translations as well as dilations, and to write equations for exponential functions with multiple transformations.	X		X		

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
12.4	Logarithmic Functions	F.IF.4 F.IF.5 F.IF.7.e F.BF.4.a	1	This lesson provides opportunities for students to develop the logarithmic graph as the inverse of the exponential function. Questions ask students to state the key characteristics of the logarithmic function and to state restrictions on any variables. Questions then ask students to explore the graphs of the common logarithm and natural logarithm, and to solve an application of logarithms involving the Richter scale. In the Talk the Talk, students will summarize the key characteristics of the exponential and logarithmic functions.	X	X		X	X
12.5	Transformations of Logarithmic Functions	F.BF.3	2	This lesson provides opportunities for students to apply their understanding of the transformational function $g(x) = Af(B(x - C)) + D$ to the logarithmic graph. Questions ask students to graph reflections, translations, and dilations of logarithmic functions with various bases and examine the effects on the domain and range. Students will then write equations for translated logarithmic functions when provided with graphs. The lesson culminates in an activity where students generalize the effect that a transformation on a function will have on its inverse.	X				

Skills Practice Correlation for Chapter 12

Lesson		Problem Set	Objectives
12.1	Exponential Functions		Vocabulary
		1 – 6	Write explicit formulas for geometric sequences and determine the 10th term
		7 – 12	Write exponential functions to represent geometric sequences and evaluate each function for a given n
		13 – 18	Write exponential functions to represent half-life situations and complete a table of values
12.2	Properties of Exponential Graphs		Vocabulary
		1 – 6	Identify functions as exponential growth or decay functions
		7 – 12	Complete tables and graph exponential functions
		13 – 18	Write exponential functions with given characteristics
		19 – 24	Use the formula for compound interest to answer questions
		25 – 30	Use the formula for population growth to answer questions
12.3	Transformations of Exponential Functions	1 – 6	Complete tables to determine corresponding points on a function given reference points, graph the function, and state the domain, range, and asymptotes
		7 – 12	Describe transformations performed on $f(x)$ to create $g(x)$ and write an equation for $g(x)$ in terms of $f(x)$
		13 – 18	Describe transformations performed on $m(x)$ to produce $t(x)$ and write an equation for $t(x)$
12.4	Logarithmic Functions		Vocabulary
		1 – 6	Write exponential equations as corresponding logarithmic equations
		7 – 12	Write logarithmic equations as corresponding exponential equations
		13 – 18	Graph the inverse of exponential functions and describe the domain, range, asymptotes, and end behavior
		19 – 24	Solve logarithmic equations
12.5	Transformations of Logarithmic Functions	1 – 6	Analyze graphs, describe the transformations performed on $f(x)$ to produce $g(x)$, and write an equation for $g(x)$
		7 – 12	Use the graph of $f(x)$ to sketch the transformed function $m(x)$ and state the domain, range, and asymptotes of $m(x)$
		13 – 18	Write a transformed logarithmic function $c(x)$ in terms of $f(x)$, given characteristics
		19 – 24	Given $f(x)$ and the transformed function $g(x)$, write an equation for $g^{-1}(x)$ in terms of $f^{-1}(x)$
		25 – 30	Complete tables for transformations, write the equation for the transformation function in terms of $f^{-1}(x)$, and identify the transformation on $f(x)$ and its inverse
		31 – 35	Given $f(x)$, write an equation for the inverse function $f^{-1}(x)$

Small Investment, Big Reward

Exponential Functions

12.1

LEARNING GOALS

In this lesson, you will:

- Construct an exponential function from a geometric sequence.
- Classify functions as exponential growth or decay.
- Compare tables, graphs, and equations of exponential functions.

KEY TERM

- half-life

ESSENTIAL IDEAS

- A geometric sequence, when written in function notation, is called an exponential function.
- A half-life refers to the amount of time it takes a substance to decay to half of its original amount.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

Analyze functions using different representations

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

   b. Use the properties of exponents to interpret expressions for exponential functions.

F-LE Linear, Quadratic, and Exponential Models

Interpret expressions for functions in terms of the situation they model

5. Interpret the parameters in a linear or exponential function in terms of a context.

Overview

Students will use a real-world problem situation to explore exponential functions. Students will create exponential graphs using their prior knowledge of geometric sequences and analyze the graphs. Different methods of payment are compared using tables of values, function notation, and graphs. The graphing calculator is used to locate points at which both payment methods result in equal salaries. The term half-life is defined and used within the context of a situation. The table function of the graphing calculator is used to make predictions.

Warm Up

1. What is an arithmetic sequence?

   An arithmetic sequence is a sequence in which the difference between any two consecutive terms is a constant called the common difference.

2. What is an example of an arithmetic sequence?

   Answers will vary.

   The sequence 1, 3, 5, 7 is an arithmetic sequence with a common difference of 2.

3. What is a geometric sequence?

   A geometric sequence is a sequence in which the ratio of any two consecutive terms is a constant called the common ratio.

4. What is an example of a geometric sequence?

   Answers will vary.

   The sequence 2, 4, 8, 16 is a geometric sequence with a common ratio of 2.

5. What is the explicit formula for a geometric sequence?

   The explicit formula for a geometric sequence is $a_n = a_1 \cdot r^{(n-1)}$ where a_1 is the first term of the sequence, r is the common ratio, and n is the term number.

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Exponential Functions

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Have you ever seen a funny picture or video online, and then it suddenly seems like everyone is talking about it? Social media and the internet have made it really easy to pass things along from person to person. You can pin, post, and share anything that you find interesting, thought-provoking, or funny with your friends all over the world. Your friends can in turn share it with their own friends, who share it with their friends, and before you know it, it seems like everyone in the world is exposed to it.

When something becomes extremely popular on the internet in a very short amount of time, it's known as "going viral." Trends can spread across the country or around the world in a matter of days. Some "viral" videos and pictures have produced overnight celebrities and inspired spin-offs in the form of books or TV shows.

What is your favorite "viral" video or picture that you've seen online?

Problem 1

Two different payment strategies are compared. One method requires the use of an explicit formula for a geometric sequence. A worked example shows the explicit formula written in function notation using the properties of powers. Students write functions and use the functions to calculate earnings over a specified period of time. Students calculate when the two methods of payment result in an equal salary and also how long it takes for one salary to surpass the other salary.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- How did you determine Allison's weekly income in the table of values?
- How did you determine Beth's weekly income in the table of values?
- As the week number increases by 1, what happens to the money Allison earns?
- What is a sequence?

PROBLEM 1 Big Things Come to Those Who Wait!

Allison and Beth each receive $10 per week for doing chores for their neighbor. One day, Allison decides to try and increase her income using her knowledge of exponential growth. She proposes that her payment be changed to a penny, and then doubled each week thereafter.

1. Complete the table to represent the amount that Allison and Beth will earn each week.

Week	Allison's Income (dollars)	Beth's Income (dollars)
1	0.01	10.00
2	0.02	10.00
3	0.04	10.00
4	0.08	10.00
5	0.16	10.00
6	0.32	10.00
7	0.64	10.00
8	1.28	10.00

2. How does Allison's income change as the number of weeks increases?

   As the number of weeks increases by 1, the amount that Allison earns is multiplied by 2.

3. Does Allison's income represent an arithmetic or geometric sequence or series? Explain your reasoning and state the general formula.

   Allison's income represents a geometric sequence because the dollar amount is increasing by a common ratio. It is a sequence because the income increases in a pattern, but the amount is not cumulative.

   The general formula for a geometric sequence is $a_n = a_1 \cdot r^{(n-1)}$.

4. Write an equation to represent Allison's income after n weeks.

   $a_n = a_1 \cdot r^{(n-1)}$

   $= 0.01 \cdot 2^{(n-1)}$

- Does Allison's weekly income represent a sequence? Why?
- Does Allison's weekly income in the table of values increase by a common difference or increase by a common ratio?
- What is the general formula for a geometric sequence in terms of the first term and the term number?
- What is the common ratio is Allison's situation?
- What is the first term is Allison's situation?

Guiding Questions for Share Phase, Questions 5 through 8

- What is one-half of 0.01?
- Does Allison start with zero money before she receives her first paycheck?
- What is Allison's weekly income during the 10th week?
- What is Allison's weekly income during the 11th week?
- To calculate Allison's weekly income in the 24th week, did you need to calculate Allison's weekly income in the 23rd week? Why not?
- How did you calculate Allison's weekly income in the 24th week?

5. What is the value of a_n for $n = 0$? Does this value make sense in this problem situation?

For $n = 0$, the value of a_n is 0.005.

Yes this value does make sense. Half of a penny is not real currency and would therefore be zero. Allison starts with no money before receiving her first income.

$$a_0 = 0.01 \cdot 2^{(0-1)}$$
$$= 0.01 \cdot 2^{-1}$$
$$= 0.005$$

6. If the pattern were to continue, how many weeks would it take for Allison to have a larger weekly income than Beth? Complete the table to show your answer.

Allison's weekly income amount would be larger than Beth's after 11 weeks.

Week	Allison's Income (dollars)	Beth's Income (dollars)
9	2.56	10.00
10	5.12	10.00
11	10.24	10.00

You can write the explicit formula for the geometric sequence $a_n = 0.01 \cdot 2^{(n-1)}$ in function notation using the properties of powers.

Statement	Reason
$a_n = 0.01 \cdot 2^{(n-1)}$	Explicit formula for a geometric sequence
$f(n) = 0.01 \cdot 2^{(n-1)}$	Rewrite in function notation
$f(n) = 0.01 \cdot 2^n \cdot 2^{-1}$	Product Rule
$f(n) = 0.01 \cdot 2^n \cdot \frac{1}{2}$	Definition of negative exponents
$f(n) = 0.01 \cdot \frac{1}{2} \cdot 2^n$	Commutative Property of Multiplication
$f(n) = 0.005 \cdot 2^n$	Associative Property of Multiplication

So, $a_n = 0.01 \cdot 2^{(n-1)}$ written in function notation is $f(n) = 0.005 \cdot 2^n$.

Recall that a geometric sequence, when written in function notation, is called an exponential function. The function gets its name from the variable in the exponent.

7. Calculate the income that Allison would earn per week in the:

a. 15th week.

In the 15th week, Allison would earn $163.84.

$f(15) = 0.005 \cdot 2^{15}$

$= 163.84$

b. 20th week.

In the 20th week, Allison would earn $5242.88.

$f(20) = 0.005 \cdot 2^{20}$

$= 5242.88$

c. 24th week.

In the 24th week, Allison would earn $83,886.08.

$f(24) = 0.005 \cdot 2^{24}$

$= 83,886.08$

8. Predict the shape and characteristics of the graph that will model Allison's income as a function of the number of weeks.

Answers will vary.

The graph will be a smooth curve which increases sharply from left to right. The *y*-intercept of the graph will be at (0, 0.005).

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Problem 2

Students once again compare two different methods of payment by completing a table of values, graphing the functions, and writing exponential equations using function notation. The exponential functions are then used to calculate income over different periods of time. The graphing calculator is used to determine when the different functions yield the same salary. Both function models are compared and discussed.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- How did you calculate Quinton's weekly income for the table of values?
- How did you calculate Alisha's weekly income for the table of values?
- Which student's plan increases exponentially?
- Which student's plan is represented by an exponential function?
- Which student's plan increases at a constant rate?
- Which student's plan is represented by a linear function?

PROBLEM 2 The Tortoise and the Hare

1. Beth is amazed at how quickly Allison was able to make a lot of money and decides that she wants in on the action. She asks her two friends, Quinton and Alisha, to help her come up with a plan.

Quinton

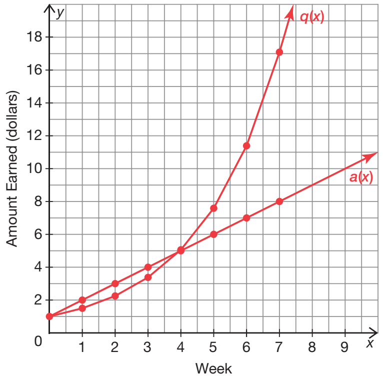
You could start with a dollar and ask for 50% more each week.

Alisha

You could start with a dollar and add another dollar each week.

Whose plan should Beth choose? Complete the table and graph to justify your reasoning. Round to the nearest hundredth.

Week	Quinton's Plan (dollars)	Alisha's Plan (dollars)
0	1.00	1.00
1	1.50	2.00
2	2.25	3.00
3	3.38	4.00
4	5.06	5.00
5	7.59	6.00
6	11.39	7.00
7	17.09	8.00



Beth should choose Quinton's plan.

Quinton's plan represents an exponential function, which means it will grow more quickly over time. Alisha's plan is linear because it is increasing at a constant rate, and will therefore not grow as quickly over time.

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Guiding Questions for Share Phase, Questions 2 through 6

- What general equation is used to represent Quinton's plan?
- What general equation is used to represent Alisha's plan?
- To calculate Beth's weekly income in the 10th week, did you need to calculate Beth's weekly income in the 9th week? Why not?
- How did you determine that Beth earned more than Allison in Week 12?
- Where do the functions for Beth and Allison intersect on the graph of the situation?
- What is the significance of where the graphs of the two functions intersect?
- How many times does Allison's and Beth's income increase each week?
- Does the model of this situation contain continuous functions? Why not?
- Does the decimal such as 18.4 make sense in this situation?
- Should the answers be discrete numbers?
- Whose payment plan earned more money per week in the beginning?
- Whose payment plan earned more money per week during Week 18?
- Whose payment plan earned more money per week during Week 19?
- Which value of the exponential functions helps determine the function that increases faster over a longer time?
- Whose rate of growth is greatest, Allison's or Beth's?
- What number is associated with Beth's *b*-value in the exponential function?
- What number is associated with Allison's *b*-value in the exponential function?

2. Write functions to represent Quinton's plan, $q(x)$, and Alisha's plan, $a(x)$.

$q(x) = 1.00 \cdot (1.5)^x$

$a(x) = 1.00 + 1x$

3. Use your choice from Question 1 to determine how much Beth will earn in Week 10.

Beth will earn $57.67 in Week 10.

$$q(10) = 1.00 \cdot (1.5)^{10}$$
$$\approx 57.67$$

4. If Beth and Allison both start using their exponential model to earn income at the same time, who will earn a higher income in Week 12?

Beth will earn a higher income in Week 12.

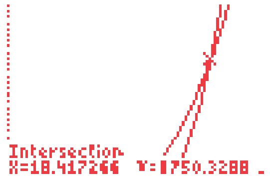
$$\text{Beth: } q(12) = 1.00 \cdot (1.5)^{12}$$
$$\approx 129.75$$

$$\text{Allison: } f(12) = 0.005 \cdot (2)^{12}$$
$$= 20.48$$

5. Use a graphing calculator to determine when Allison's and Beth's incomes will be equal. Does this make sense in this problem situation? Explain your reasoning.

Using my graphing calculator, I graphed $f(x) = 0.005 \cdot 2^x$ and $q(x) = 1.00 \cdot 1.5^x$.

The functions intersect at approximately $x = 18.4$.



This does not make sense in this problem situation.

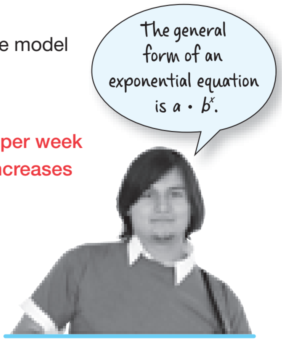
Because x represents the number of weeks, and Allison's and Beth's incomes increase once each week, the models are not continuous functions. Therefore, 18.4 does not make sense because it is not a whole number.

6. Compare Allison's and Beth's function models.

a. As the number of weeks continues to increase, whose model will earn them more per week?

Allison's model is better over time.

Even though Beth's model earned her more money per week in the beginning, after 19 weeks, Allison's income increases at a much faster rate than Beth's.



b. Consider the *a*- and *b*-values of the exponential functions if $y = ab^x$. How do they further support your claim?

Allison's *b*-value, or her income's rate of growth, is larger than Beth's, so eventually Allison's income surpasses Beth's.

Problem 3

The term half-life is defined and used in a situation in which students calculate the amount of time caffeine remains in the bloodstream after ingesting an energy drink. Student work showing different exponential functions are analyzed to determine which one correctly represents the situation. The table function of the graphing calculator is used to predict when the caffeine remaining in the bloodstream will be less than 1 milligram. Students use the properties of exponents to rewrite the function in terms of time. An antibiotic is introduced which changes the half-life of caffeine and students write a new function, complete another table of values, and use the table function of the graphing calculator to determine amounts of caffeine in the bloodstream.

PROBLEM 3 Half-Life of Caffeine

Simeon is studying for a big test and is trying to stay awake. He drank a 12-ounce can of Big Buzz Energy Drink that contains 80 milligrams of caffeine. He is wondering how long the caffeine will stay in his system if the caffeine has a *half-life* of 5 hours.

A **half-life** is the amount of time it takes a substance to decay to half of its original amount.

1. How much caffeine remains in Simeon's system after 5 hours? After 10 hours? Explain your reasoning.

After 5 hours, 40 milligrams of caffeine is left in Simeon's system, because it takes 5 hours to cut the amount in half.

After 10 hours, 20 milligrams of caffeine is left, because after another 5 hours, half of the previous amount is left.

2. Complete the table to determine the amount of caffeine in Simeon's system at each time interval.

Time Elapsed (hours)	0	5	10	15	20
Caffeine in System (mg)	80	40	20	10	5
Number of Half-Life Cycles	0	1	2	3	4

3. What is the initial amount of caffeine in Simeon's system? What is the rate of decay?

The initial amount of caffeine is 80 milligrams.

The rate of decay is $\frac{1}{2}$, or 0.5, for every 5 hours.

Grouping

- Ask a student to read the information and definition. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- How many hours does it take to cut the amount of caffeine in the bloodstream in half?
- How did you calculate the amount of caffeine in the bloodstream at each time interval?
- How did you determine the number of half-life cycles?
- Will there ever be 0 mg of caffeine in Simeon's bloodstream?
- How would you rewrite the base of $\frac{1}{2}$ using an exponent?
- What is $(2^{-1})^{\frac{t}{5}}$ simplified?
- If the half-life cycle is 5 hours, should the exponent be divided into 5 parts or multiplied by 5?

Grouping

Have students complete Questions 5 through 9 with a partner. Then have students share their responses as a class.

4. Emily, Tyler, and Renee were asked to write an exponential function $A(t)$ to represent the amount of caffeine remaining in Simeon's system after t hours.

Emily

$A(t) = 80\left(\frac{1}{2}\right)^{\frac{t}{5}}$

The variable t represents the number of hours, and the half-life occurs in 5 hour cycles, so I divided my exponent by 5.

Tyler

$A(t) = 80\left(\frac{1}{2}\right)^{-\frac{t}{5}}$

The variable t represents the number of hours and since it's a decay function, I made my exponent negative.

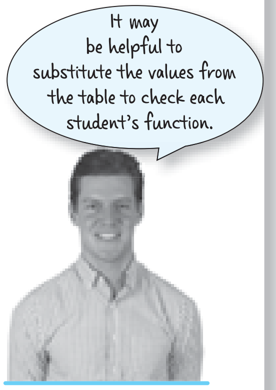
Renee

$A(t) = 80\left(\frac{1}{2}\right)^{5t}$

The variable t represents the number of hours and I multiplied it by 5 to represent the half-life cycle of 5 hours.

a. Why is Tyler's reasoning incorrect?

Even though the function represents exponential decay, the negative exponent is incorrect because the base, $\frac{1}{2}$, is equivalent to 2^{-1}. Therefore, if written this way, the exponent would actually become positive.



b. Why is Renee's reasoning incorrect?

The half-life cycle consists of 5 hours, so the exponent should be divided into 5 parts, not multiplied by 5.

5. How much caffeine remains in Simeon's system after 2 hours?

$A(2) = 80\left(\frac{1}{2}\right)^{\frac{2}{5}}$

$A(2) \approx 60.63$

There is approximately 61 milligrams of caffeine remaining in Simeon's system after 2 hours.

Guiding Questions for Share Phase, Questions 5 through 9

- How was Emily's equation used to calculate the amount of caffeine remaining in Simeon's bloodstream after 2 hours?
- Did Kendra follow the order of operations?
- What did Kendra do before using the exponent?
- If the variable is in the exponent, it is possible to multiply 80 by $\frac{1}{2}$ before using the exponent?
- Will there always be a trace amount of caffeine in Simeon's bloodstream? Why?
- Do exponential functions always approach a horizontal asymptote? Why?
- What is the horizontal asymptote in this situation?
- Will the exponential function ever equal zero?
- Will the amount of caffeine in Simeon's bloodstream be less than 1 mg in 31 hours?
- Will the amount of caffeine in Simeon's bloodstream be less than 1 mg in 32 hours?
- What is the decimal equivalent of $\left(\frac{1}{2}\right)^{\frac{1}{5}}$?
- What is the percentage equivalent of 0.87055?

6. Kendra suggests that she can calculate the amount of caffeine remaining by rewriting the equation as $A(t) = 40^{\frac{t}{5}}$. Is Kendra correct? Explain your reasoning.

Kendra is not correct.

Kendra did not follow the order of operations and multiplied $80 \cdot \left(\frac{1}{2}\right)$ before using the exponent. Because the variable is in the exponent, this is not possible.

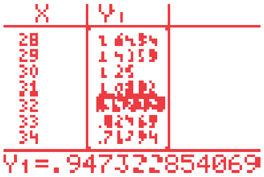
7. Use the table function of a graphing calculator to predict when the caffeine will be completely out of Simeon's system. Does this make sense, given what you know about exponential functions? Explain your reasoning.

The caffeine will never be completely out of Simeon's system because the amount will continue to be divided in half until there is just a trace amount left.

Mathematically, exponential functions always approach a horizontal asymptote. In this case, the horizontal asymptote is 0, and the function will never equal 0.

8. Approximately when will the amount of caffeine remaining in Simeon's system be less than 1 milligram?

The amount of caffeine remaining in Simeon's system will be less than 1 milligram between Hours 31 and 32.



X	Y1
28	[illegible]
29	[illegible]
30	1.25
31	[illegible]
32	[illegible]
33	[illegible]
34	[illegible]

Y1=.947322854069

9. Use the properties of exponents to rewrite your function so that only the variable t is in the exponent. What percentage of caffeine remains after each hour?

$$A(t) = 80\left(\frac{1}{2}\right)^{\frac{1}{5} \cdot t}$$

$$= 80\left(\left(\frac{1}{2}\right)^{\frac{1}{5}}\right)^{t}$$

$$A(t) \approx 80(0.87055)^{t}$$

There is approximately 87% of the previous amount of caffeine remaining after every hour.

Grouping

Have students complete Questions 10 through 14 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 10 through 14

- If the antibiotic extends the half-life to 8 hours, how does the function modeling the situation change?
- How did you calculate the amount of caffeine in the bloodstream at each time interval?
- How did you determine the number of half-life cycles?
- Does 80 mg of caffeine take a longer time or a shorter time to leave Simeon's bloodstream?
- Will the amount of caffeine in Simeon's bloodstream be less than 1 mg in 50 hours?
- Will the amount of caffeine in Simeon's bloodstream be less than 1 mg in 51 hours?
- As the half-life time increases, what happens to the time it will take the substance to decay?
- How is the length of the half-life related to the time it will take the substance to decay?

10. Suppose Simeon is taking an antibiotic that extends the half-life of caffeine to 8 hours. Write a function $B(t)$ that models the amount of caffeine remaining under these new conditions.

$B(t) = 80\left(\frac{1}{2}\right)^{\frac{t}{8}}$

or

$B(t) \approx 80(0.917)^t$

11. Complete the table for the new half-life. Round to the nearest hundredth.

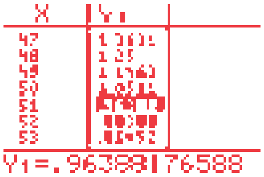
Time Elapsed (hours)	0	5	10	15	20
Caffeine in System (mg)	80	51.87	33.64	21.81	14.14
Number of Half-Life Cycles	0	1	2	3	4

12. How does the medication affect the amount of caffeine remaining in Simeon's system?

The 80 milligrams of caffeine takes longer to leave Simeon's system.

13. Under these new conditions, approximately when will the amount of caffeine remaining in Simeon's system be less than 1 milligram?

Using the table feature of a graphing calculator, I calculated that the amount of caffeine in Simeon's system will be less than 1 milligram between Hours 50 and 51.



```
   X     | Y1
  47     | [illegible]
  48     | [illegible]
  49     | [illegible]
  50     | [illegible]
  51     | [illegible]
  52     | [illegible]
  53     | [illegible]
Y1=.96398176588
```

14. What generalization can you make about the effect of larger or smaller half-lifes on substances?

The larger the half-life time, the longer it will take a substance to decay.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Carbon-14 is a radioactive isotope used to date the age of plants and animals. It has a half-life of about 5730 years. While a plant or animal is alive, the amount of carbon-14 present is constant, but when the plant or animal dies, the amount of carbon-14 begins to decay. To determine the length of time since the plant or animal died, you measure the percent of carbon-14 remaining, and apply the decay formula.

1. Write the decay formula for carbon-14, using the value 100 as the initial amount present. The unit for 100 is not important because you will express the amount of remaining carbon-14 as a percent.

$N = 100(0.5)^{\frac{t}{5730}}$

2. Use your formula to calculate the percentage of carbon-14 remaining after 1000 years.

After 1000 years, 88.6% would be present.

$N = 100(0.5)^{\frac{t}{5730}}$

$N = 100(0.5)^{\frac{1000}{5730}}$

$N \approx 88.60$

3. Use your formula to calculate the percentage of carbon-14 remaining after 10,000 years.

After 10,000 years, 29.8% would be present.

$N = 100(0.5)^{\frac{t}{5730}}$

$N = 100(0.5)^{\frac{10000}{5730}}$

$N \approx 29.83$

4. Use your formula to calculate the percentage of carbon-14 remaining after 1,000,000 years.

After 1,000,000 years, 2.9×10^{-51} % would be present.

$N = 100(0.5)^{\frac{t}{5730}}$

$N = 100(0.5)^{\frac{1000000}{5730}}$

$N \approx 2.91 \text{ x } 10^{-51}$

12.2

We Have Liftoff!

Properties of Exponential Graphs

LEARNING GOALS

In this lesson, you will:

- Identify the domain and range of exponential functions.
- Investigate graphs of exponential functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.
- Explore the irrational number *e*.

ESSENTIAL IDEAS

- For basic exponential growth functions, $f(x) = b^x$, *b* is a value greater than 1.
- For basic exponential decay functions, $f(x) = b^x$, *b* is a fraction or decimal value between 0 and 1.
- The compound interest formula is $A(t) = P\left(1 + \frac{r}{n}\right)^{rt}$ where *A* represents the value after *t* periods, *P* represents the principal amount, *r* represents the interest rate, and *n* represents the number of compound periods per year *t*.
- The natural base $e \approx 2.7182818$ is an irrational number that represents continuous growth and is used to model population changes as well as radioactive decay.

KEY TERM

- natural base *e*

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

  e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Overview

A sorting activity uses the equations, tables, and graphs of exponential growth and exponential decay functions. Students will write exponential growth and decay functions given specified characteristics. They will also summarize the characteristics for the basic exponential growth and decay functions using a table. The irrational number *e*, or natural base *e*, is introduced through the development of continuous compound interest.

Warm Up

Juno is opening her first savings account and is depositing \$100. The bank offers 3% annual interest to be calculated at the end of each year.

1. Write a function $A(t)$ to model the amount of money in Juno's savings account after t years.

   $A(t) = 100(1 + 0.03)^t$

2. Calculate the amount of money in Juno's savings account at the end of one year.

   At the end of one year, the amount of money in Juno's account is \$103.

   $A(t) = 100(1 + 0.03)^t$

   $A(1) = 100(1.03)^1$

   $A(1) = 103$

3. Calculate the amount of money in Juno's savings account at the end of five years.

   At the end of five years, the amount of money in Juno's account is \$115.93.

   $A(t) = 100(1 + 0.03)^t$

   $A(5) = 100(1.03)^5$

   $A(5) \approx 115.927$

4. At this rate, how many years would it take Juno to double her initial investment of \$100?

   It would take Juno more than 23 years to double her initial investment of \$100.

   $A(t) = 100(1 + 0.03)^t$

   $200 = 100(1.03)^t$

   $2 = 1.03^t$

   I can use a graphing calculator to graph $Y_1 = 2$ and $Y_2 = 1.03^t$ to determine their point of intersection at (23.4, 2).

12

12.2

We Have Liftoff!

Properties of Exponential Graphs

LEARNING GOALS

In this lesson, you will:

- Identify the domain and range of exponential functions.
- Investigate graphs of exponential functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.
- Explore the irrational number *e*.

KEY TERM

- natural base *e*

Have you ever tried to remember a long list of things and ended up getting mixed up along the way?

There are lots of tried-and-true ways of memorizing things, and it all depends on what you're trying to memorize. Some people like to make mnemonic devices, where the first letter in each word corresponds to something in the list they're trying to memorize. You may have used one of these when you were learning the order of operations—Please Excuse My Dear Aunt Sally is a great way to help you remember parentheses, exponents, multiplication, division, addition, and subtraction. Some people try doing some sort of movement as they recite their list, so that they can use their muscle memory to help them. Some people like to use rhymes, some people use visualization, and some people rely on good old-fashioned repetition.

But there are some people who are just naturally skilled at remembering things. In fact, there are competitions held around the world to see who can memorize the most digits of pi. In 2005, Chao Lu of China set a world record by memorizing an incredible 67,890 digits of pi! It took him 24 hours and 4 minutes to accurately recite the digits, with no more than 15 seconds between each digit.

Do you have any memory tricks to help you remember things?

Problem 1

Students cut out six graphs and six equations. They sort the cut-outs into "growth" or "decay" functions, tape them onto a graphic organizer, complete a table of values for each function, and analyze the graphs. Students discuss a rule that determines whether a function is an example of exponential growth or exponential decay and explain their reasoning. Students write exponential functions with given characteristics, and complete a table that summarizes the characteristics for the basic exponential growth and exponential decay functions.

Grouping

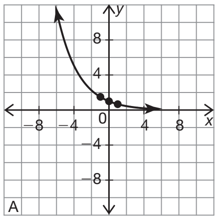
Have students complete all parts of Questions 1 through 4 with a partner. Then have students share their responses as a class.

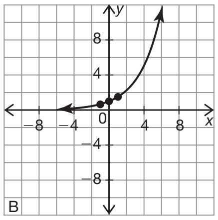
Guiding Questions for Share Phase, Question 1

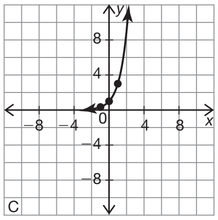
- Is growth associated with increase or decrease?
- Is decay associated with increase or decrease?
- Which graphs describe increasing functions?
- Which equations describe increasing functions?
- Which graphs describe decreasing functions?
- Which equations describe decreasing functions?

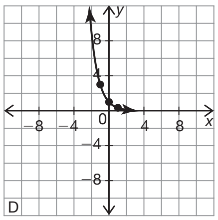
PROBLEM 1 I've Got the Power

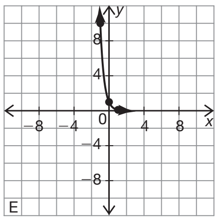
1. Cut out the exponential graphs and equations and match them. Sort them into "growth" or "decay" functions and tape them onto the graphic organizer in this lesson. Finally, complete each table.

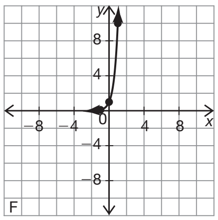




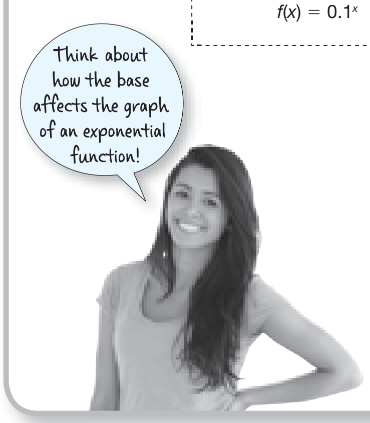








$f(x) = \left(\frac{2}{3}\right)^x$	$f(x) = 10^x$	$f(x) = 1.5^x$
$f(x) = 0.1^x$	$f(x) = 3^x$	$f(x) = \left(\frac{1}{3}\right)^x$



- How did you complete the table of values for each graph/equation?
- Was the graph or the equation used to complete the table of values?

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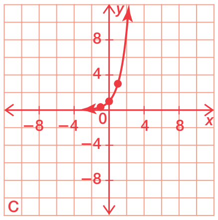
Growth

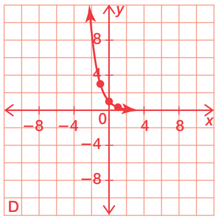
$f(x) = 3^x$



x	$f(x)$
−1	$\frac{1}{3}$
0	1
1	3

Decay

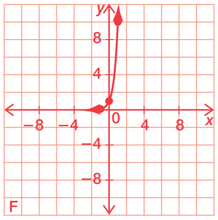
$f(x) = \left(\frac{1}{3}\right)^x$



x	$f(x)$
−1	3
0	1
1	$\frac{1}{3}$

Growth

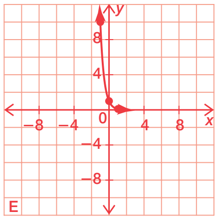
$f(x) = 10^x$



x	$f(x)$
−1	$\frac{1}{10}$
0	1
1	10

Decay

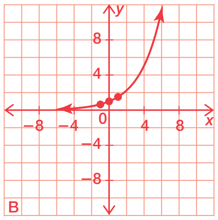
$f(x) = 0.1^x$



x	$f(x)$
−1	10
0	1
1	0.1

Growth

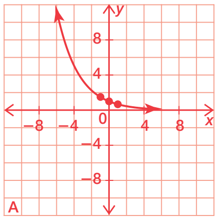
$f(x) = 1.5^x$



x	$f(x)$
−1	$\frac{1}{1.5} = \frac{2}{3}$
0	1
1	$\frac{3}{2}$

Decay

$f(x) = \left(\frac{2}{3}\right)^x$



x	$f(x)$
−1	$\frac{3}{2}$
0	1
1	$\frac{2}{3}$

Guiding Questions for Share Phase, Questions 2 through 4

- Does every basic exponential function have the point (0, 1) in common? Why?
- When any base is raised to the power of 0, what is the result?
- Are the *b*-values of the equations greater than 1 associated with growth or decay situations?
- Are the *b*-values of the equations less than 1 associated with growth or decay situations?
- How would you describe the behavior of the graph $g(x) = (-5)^x$?
- Is the behavior of the graph of $g(x) = (-5)^x$ associated with decay? Why not?
- How would you describe the behavior of the graph $g(x) = 0^x$?
- Is the behavior of the graph of $g(x) = 0^x$ associated with decay? Why not?
- How would you describe the behavior of the graph $g(x) = 1^x$?
- Is the behavior of the graph of $g(x) = 1^x$ associated with decay? Why not?

2. Analyze the exponential growth and decay functions.

a. What point do the graphs have in common? Why?

Every basic exponential function has (0, 1) in common. The *x*-value represents the exponent, and any base raised to the power of 0 will equal 1, no matter what the base is.

b. Compare the equations of the six functions you just sorted. What differentiates an exponential growth from an exponential decay?

In the three exponential growth functions, the *b*-values are greater than 1.

In the three exponential decay functions, the *b*-values are less than 1 but greater than 0.

3. Sarah and Scott's teacher asked them to each write a rule that would determine whether a function was exponential growth or decay, based on its equation.

Sarah

For exponential growth functions, *b* is a value greater than 1, but for exponential decay functions, *b* is a fraction or decimal between 0 and 1.

Scott

For exponential growth functions, *b* is greater than 1. For exponential decay functions, *b* is less than 1.

Why is Scott's reasoning incorrect? Provide a counterexample that would disprove his claim and explain your reasoning.

Answers will vary.

- $g(x) = (-5)^x$ is not exponential decay because the graph oscillates, and is therefore neither growth nor decay.
- $g(x) = 0^x$ is not exponential decay because even though 0 is less than 1, for every *x*-value greater than 0, $g(x)$ is equal to 0. Thus, it would be a constant function for $x > 0$.

4. What *b*-values in exponential functions produce neither growth nor decay? Provide examples to support your answer.

The *b*-values of 0 and 1 produce exponential functions that are neither grow nor decay functions.

$g(x) = 0^x$ is a constant function for $x > 0$. For every *x*-value greater than 0, $g(x) = 0$.

$g(x) = 1^x$ is a constant function. For every *x*-value, $g(x) = 1$.

Grouping

Have students complete all parts of Questions 5 and 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 and 6

- If the graph of the function is increasing over $(-\infty, \infty)$, does the function describe growth or decay?
- If the reference point is (1, 6), what does this tell you about the equation of the function?
- If the graph of the function is decreasing over $(-\infty, \infty)$, does the function describe growth or decay?
- If the reference point is (−1, 4), what does this tell you about the equation of the function?
- Which characteristics are the same for both types of exponential functions?
- Which characteristics are different for both types of exponential functions?

5. Write an exponential function with the given characteristics.

a. Increasing over $(-\infty, \infty)$
Reference point (1, 6)
$f(x) = 6^x$

b. Decreasing over $(-\infty, \infty)$
Reference point (−1, 4)
$f(x) = \left(\frac{1}{4}\right)^x$

c. End behavior $\begin{array}{l} \text{As } x \to -\infty, f(x) \to 0 \\ \text{As } x \to \infty, f(x) \to \infty \end{array}$
Reference point (2, 6.25)
$f(x) = 2.5^x$

6. Summarize the characteristics for the basic exponential growth and exponential decay functions.

	Basic Exponential Growth	Basic Exponential Decay
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$(0, \infty)$	$(0, \infty)$
Asymptote	$y = 0$	$y = 0$
Intercepts	(0, 1)	(0, 1)
End Behavior	As $x \to -\infty$, $f(x) \to 0$ As $x \to \infty$, $f(x) \to \infty$	As $x \to -\infty$, $f(x) \to \infty$ As $x \to \infty$, $f(x) \to 0$
Intervals of Increase or Decrease	Increasing over $(-\infty, \infty)$	Decreasing over $(-\infty, \infty)$

Problem 2

Students write a function modeling the amount of money in a savings account that offers annual interest to be calculated at the end of each year. The function is used to calculate the amount of money in the account over different periods of time. Next, they write a function modeling the amount of money in the savings account if the bank compounded interest *n* times during the year.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- What does $A(t)$ represent in the formula for compound interest?
- What does P represent in the formula for compound interest?
- What does r represent in the formula for compound interest?
- What does t represent in the formula for compound interest?
- If Helen has \$530 in her savings account after one year, how much interest did the account earn?
- If Helen has \$669.11 in her savings account after five years, how much interest did the account earn?
- If the bank offers 3% interest per six month period, what percent do they offer per year?

PROBLEM 2 Let's Compound Some Dough

Helen is opening her first savings account and is depositing \$500. Suppose she decides on a bank that offers 6% annual interest to be calculated at the end of each year.

1. Write a function $A(t)$ to model the amount of money in Helen's savings account after t years.

   $A(t) = 500(1 + 0.06)^t$

2. Calculate the amount of money in Helen's savings account at the end of 1 year?

   After 1 year, Helen will have \$530 in her savings account.

   $A(1) = 500(1 + 0.06)^1$

   $= 530$

3. How much money will be in Helen's savings account at the end of 5 years?

   After five years, Helen will have \$669.11 in her savings account.

   $A(5) = 500(1 + 0.06)^5$

   $A(5) = 669.1127888$

4. Suppose that the bank decides to start compounding interest at the end of every 6 months. If they still want to offer 6% per year, how much interest would they offer per 6-month period?

   If they offer 6% per year, and are compounding it twice a year, that would mean that the bank offers 3% interest per 6-month period.

Guiding Questions for Share Phase, Questions 5 through 8

- Can John and Betty Jo both be correct?
- How is John's solution path different than Betty Jo's solution path?
- If the savings account earns interest that is compounded twice a year, what is the value of *n* in the formula?
- If the savings account earns interest that is compounded monthly, what is the value of *n* in the formula?
- If the savings account earns interest that is compounded daily, what is the value of *n* in the formula?
- What is the relationship between the frequency the account is compounded and the speed at which the money grows?

5. John, Betty Jo, and Lizzie were each asked to calculate the amount of money Helen would have in her savings at the end of the year if interest was compounded twice a year. Who's correct? Explain your reasoning.

John

1st 6 months:

$A(t) = 500(1 + 0.03)$

$A(t) = 515$

2nd 6 months:

$A(t) = 515(1 + 0.03)$

$A(t) = 530.45$

Betty Jo

$A(t) = 500\left(1 + \frac{0.06}{2}\right)^2$

$A(t) = 530.45$

Lizzie

$A(t) = 2(500(1 + 0.03))$

$A(t) = 1030$

John and Betty Jo are both right.

Lizzie calculated the first six months correctly, but then doubled it rather than compounding her interest.

6. Write a function to model the amount of money in Helen's savings account at the end of *t* years, compounded *n* times during the year.

$A(t) = 500\left(1 + \frac{0.06}{n}\right)^{nt}$

7. Determine the amount of money in Helen's account at the end of 3 years if it is compounded:

   a. twice a year.

   Helen will have $597.03 at the end of 3 years if her interest is compounded twice a year.

   Since interest is compounded twice a year, $n = 2$.

   $$A(3) = 500\left(1 + \frac{0.06}{2}\right)^{(2 \cdot 3)}$$
   $$= 500(1.03)^6$$
   $$= 597.0261482645$$

   b. monthly.

   Helen will have $598.34 at the end of 3 years if her interest is compounded monthly.

   Since interest is compounded monthly, $n = 12$.

   $$A(3) = 500\left(1 + \frac{0.06}{12}\right)^{(12 \cdot 3)}$$
   $$= 500(1.005)^{36}$$
   $$= 598.3402624117\ldots$$

   c. daily.

   Helen will have $598.60 at the end of 3 years if her interest is compounded daily.

   Since interest is compounded daily, $n = 365$.

   $$A(3) = 500\left(1 + \frac{0.06}{365}\right)^{(365 \cdot 3)}$$
   $$= 500(1.000164383\ldots)^{1095}$$
   $$= 598.599826468\ldots$$

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8. What effect does the frequency of compounding have on the amount of money in her savings account?

   The more often the account is compounded, the quicker her money grows.

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Problem 3

In this situation, a bank offers 100% interest and the initial principal in the account is \$1. Students complete a chart in which the formulas are listed describing different compound periods: yearly, semi-annually, quarterly, monthly, weekly, daily, hourly, every minute, and every second. As the frequency of the compounded interest increases, the earnings approach the value of the natural base e, which is approximately 2.718281. Students sketch the graph of $m(x) = e^x$, using the graph of the functions $f(x) = 2^x$ and $g(x) = 3^x$. They also approximate the values of the functions on the number line.

Grouping

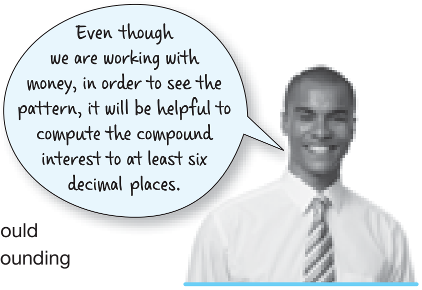
- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- If the interest is compounded monthly, what is the value of n in the formula?
- If the interest is compounded weekly, what is the value of n in the formula?
- If the interest is compounded daily, what is the value of n in the formula?

PROBLEM 3 Easy "e"

Recall that in Problem 2 the variable n represented the number of compound periods per year. Let's examine what happens as the interest becomes compounded more frequently.



1. Imagine that Helen finds a different bank that offers her 100% interest. Complete the table to calculate how much Helen would accrue in 1 year for each period of compounding if she starts with \$1.

Period of Compounding	$n =$	Formula	Amount
Yearly	1	$1\left(1 + \frac{1}{1}\right)^{1 \cdot 1}$	2.00
Semi-Annually	2	$1\left(1 + \frac{1}{2}\right)^{2 \cdot 1}$	2.25
Quarterly	4	$1\left(1 + \frac{1}{4}\right)^{4 \cdot 1}$	2.44140625 . . .
Monthly	12	$1\left(1 + \frac{1}{12}\right)^{12 \cdot 1}$	2.61303529 . . .
Weekly	52	$1\left(1 + \frac{1}{52}\right)^{52 \cdot 1}$	2.692596954 . . .
Daily	365	$1\left(1 + \frac{1}{365}\right)^{365 \cdot 1}$	2.714567482 . . .
Hourly	8760	$1\left(1 + \frac{1}{8760}\right)^{1 \cdot 8760}$	2.718126692 . . .
Every Minute	525600	$1\left(1 + \frac{1}{525600}\right)^{525600 \cdot 1}$	2.718279243 . . .
Every Second	31536000	$1\left(1 + \frac{1}{31536000}\right)^{31536000 \cdot 1}$	2.718281781 . . .

2. Make an observation about the frequency of compounding and the amount that Helen earns. What is it approaching?

   The more frequently that Helen's interest is compounded, the amount becomes closer to approximately \$2.72.

- If the interest is compounded hourly, what is the value of n in the formula?
- If the interest is compounded every minute, what is the value of n in the formula?
- If the interest is compounded every second, what is the value of n in the formula?
- As the interest is compounded more frequently, what does the amount in the account approach?

Grouping

- Ask a student to read the information and definition. Discuss as a class.
- Have students complete Question 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

- How did you decide where to locate the graph of $m(x) = e^x$?
- Is the value of e closer to 2 or closer to 3?
- Should the graph of $m(x) = e^x$ be located closer to $g(x) = 3^x$ or $f(x) = 2^x$?

The amount that Helen's earnings approach is actually an irrational number called e.

$$e \approx 2.718281828459045\ldots$$

It is often referred to as the **natural base e**.

In geometry, you worked with π , an irrational number that was approximated as 3.14159265 . . . and so on. Pi is an incredibly important part of many geometric formulas and occurs so frequently that, rather than write out "3.14159265 . . . " each time, we use the symbol π.

Similarly, the symbol e is used to represent the constant 2.718281 . . . It is often used in models of population changes as well as radioactive decay of substances, and it is vital in physics and calculus.

The symbol for the natural base e was first used by Swiss mathematician Leonhard Euler in 1727 as part of a research manuscript he wrote at age 21. In fact, he used it so much, e became known as Euler's number.

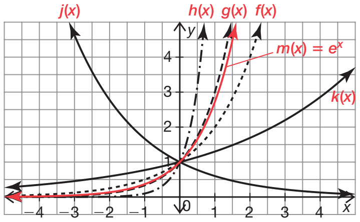
The constant e represents continuous growth and has many other mathematical properties that make it unique, which you will study further in calculus.

The number e goes on forever with no repetition. There are even competitions to see who can memorize the most digits of e. In 2007, Bhaskar Karmakar from India set a World Record for memorizing 5002 digits!

3. The following graphs are sketched on the coordinate plane shown.

   $f(x) = 2^x$, $g(x) = 3^x$, $h(x) = 10^x$, $j(x) = \left(\frac{3}{5}\right)^x$, $k(x) = 1.3^x$.

   a. Label each function.



   b. Consider the function $m(x) = e^x$. Use your knowledge of the approximate value of e to sketch its graph. Explain your reasoning.

   See graph.

   Because $e \approx 2.72$, and 2.72 is closer to 3 than 2, the graph of $m(x) = e^x$ should be closer to $h(x) = 3^x$ than $g(x) = 2^x$.

c. Using the functions $f(x) = 2^x$, $g(x) = 3^x$, $m(x) = e^x$, approximate the values of $f(2)$, $g(2)$, and $m(2)$ on the number line. Explain your reasoning.

$f(2) = 4$

$g(2) = 9$

The base of $m(x)$ is approximately 2.72, which is between 2 and 3 but closer to 3. Therefore, $m(2)$ must fall in between $f(2)$ and $g(2)$, but will be closer to $g(2)$.

PROBLEM 4 It Keeps Growing and Growing and Growing...

1. The formula for population growth is $N(t) = N_0e^{rt}$. Complete the table to identify the contextual meaning of each quantity.

Quantity	Contextual Meaning
N_0	initial amount of population
r	rate of growth
t	time
$N(t)$	population after t years

2. Why is e used as the base?

The constant e is used as the base because the population is continuously growing. The population doesn't increase at set intervals, such as every minute or once a week. It changes continuously.

3. How could this formula be used to represent a decline in population?

I could change the rate of growth to a negative value to represent a decline in population.

Problem 4

A formula using base *e* for population growth is given. Students discuss the components of the formula and relate it to the decline in population or decay. Students use given information and the formula to write a function modeling the growth of a city, and make predictions. The table function of the graphing calculator is also used to estimate population trends.

Grouping

Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What variable in the formula represents the initial amount or population?
- What variable in the formula represents the rate of growth?
- What variable in the formula represents time?
- What variable in the formula represents the population after *t* years?
- Is the population continuously growing?
- Does the population increase at set intervals such as every minute or once a week?
- If the rate of growth were a negative value, how would that change the context of this problem situation?

Guiding Questions for Share Phase, Questions 4 and 5

- What value in this situation is represented by N_0?
- What decimal value in this situation is represented by r?
- In 2013, what is the value of t in the formula?
- In 1980, what is the value of t in the formula?
- How is the table function on the graphing calculator helpful when determining the year Fredericksburg will grow to be 40,000 people?
- In the year 2025, will the population grow to 40,000?
- In the year 2026, will the population grow to 40,000?

4. The population of the city of Fredericksburg, Virginia, was approximately 19,360 in 2000 and has been continuously growing at a rate of 2.9% each year.

a. Use the formula for population growth to write a function to model this growth.

$N(t) = 19{,}360e^{0.029t}$

b. Use your function model to predict the population of Fredericksburg in 2013.

In the model, t represents the number of years since 2000. So, in 2013, $t = 13$.

$N(13) = 19{,}360e^{(0.029 \cdot 13)}$

$= 28225.0274294$

According to the model, in 2013, Fredericksburg would have approximately 28,225 people.

c. What value does your function model give for the population of Fredericksburg in the year 1980?

In the model, t represents the number of years since 2000, so in 1980, $t = -20$.

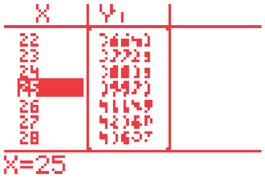
$N(-20) = 19{,}360e^{(0.029 \cdot (-20))}$

$= 10839.6323767$

According to the model, in 1980, Fredericksburg had approximately 10,340 people.

5. Use a graphing calculator to estimate the number of years it would take Fredericksburg to grow to 40,000 people, assuming that the population trend continues.

The population of Fredericksburg would grow to 40,000 between the years 2025 and 2026.



X	Y1
22	[illegible]
23	[illegible]
24	[illegible]
25	[illegible]
26	[illegible]
27	[illegible]
28	[illegible]

X=25

Be prepared to share your solutions and methods.

Check for Students' Understanding

Jan invested $5000 in a savings account for 6 years. The bank pays 2% compounded monthly. At the end of 6 years, how much would Jan's investment be worth?

At the end of 6 years, Jan's investment would be worth $5636.92.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A(6) = 5000\left(1 + \frac{0.02}{12}\right)^{12 \cdot 6}$$

$$A(6) \approx 5000(1.00166)^{72}$$

$$A(6) \approx 5636.921$$

12

12.3

I Like to Move It

Transformations of Exponential Functions

LEARNING GOALS

In this lesson, you will:

- Dilate, reflect, and translate exponential functions using reference points and transformational function form.
- Investigate graphs of exponential functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.
- Describe how transformations of exponential functions affect their key characteristics.

ESSENTIAL IDEAS

- In the general transformational function $g(x) = Af(B(x - C)) + D$, the D-value translates the function $f(x)$ vertically, the C-value translates $f(x)$ horizontally, the A-value vertically stretches or compresses $f(x)$, and the B-value horizontally stretches or compresses $f(x)$.
- Key characteristics of basic exponential functions include a domain of all real numbers, a range of non-negative numbers, and a horizontal asymptote at $y = 0$.
- The domain of exponential functions is not affected by vertical translations, horizontal translations, vertical dilations, and horizontal dilations.
- Vertical translations affect the range and the horizontal asymptote of exponential functions.
- Horizontal translations, vertical dilations, and horizontal dilations do not affect the range and the horizontal asymptote of exponential functions.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-BF Building Functions

Build new functions from existing functions

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

Overview

The general transformational function $g(x) = Af(B(x - C)) + D$ is applied to exponential functions. Students identify key characteristics of transformed exponential functions and analyze graphs of exponential functions by identifying intercepts, asymptotes, intervals of increase and decrease, and end behavior. Dilations and horizontal and vertical translations are graphed. Students use graphs of functions to write equations for transformed exponential functions.

Warm Up

Use a graphing calculator to describe the end behaviors of the graph of each of the following functions.

1. $f(x) = 2^x$

   As $x \to -\infty$, $f(x) \to 0$.

   As $x \to \infty$, $f(x) \to \infty$.

2. $f(x) = 2^{-x}$

   As $x \to -\infty$, $f(x) \to \infty$.

   As $x \to \infty$, $f(x) \to 0$.

3. $f(x) = -2^x$

   As $x \to -\infty$, $f(x) \to 0$.

   As $x \to \infty$, $f(x) \to -\infty$.

4. $f(x) = -2^{-x}$

   As $x \to -\infty$, $f(x) \to -\infty$.

   As $x \to \infty$, $f(x) \to 0$.

12

I Like to Move It

Transformations of Exponential Functions

LEARNING GOALS

In this lesson, you will:

- Dilate, reflect, and translate exponential functions using reference points and transformational function form.
- Investigate graphs of exponential functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.
- Describe how transformations of exponential functions affect their key characteristics.

Andy Warhol was an American pop artist whose work explored the relationship between artistic expression, celebrity culture, and advertisement. A recurring theme throughout Warhol's art is the transformation of the mundane and commonplace into art. His most renowned images are silk-screened reproductions of Campbell's soup cans and publicity photographs of pop culture icons like Marilyn Monroe and Elvis Presley.

Have you ever seen any of Andy Warhol's work?

Problem 1

Students match four exponential functions to their appropriate graphs. They analyze each graph, write equations in terms of $f(x)$, and describe the transformation on $f(x)$. Students complete a table that lists the original functions and their asymptotes, intervals of increase and decrease, and end behavior. Students compare graphs and discuss how the rigid motion performed on each function affected the key characteristics listed in the chart.

Grouping

Have students complete all parts of Questions 1 through 5 with a partner. Then have students share their responses as a class.

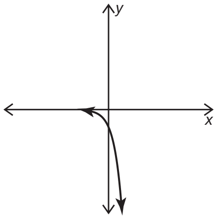
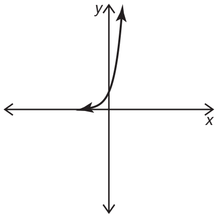
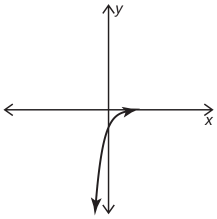
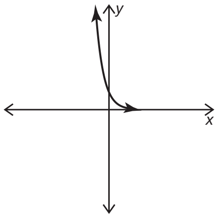
Guiding Questions for Share Phase, Question 1

- Which graph describes exponential growth?
- Which graph describes exponential decay?
- Which graph is a reflection of the graph $f(x)$ across the y-axis?
- Which graph is a reflection of the graph $f(x)$ across the x-axis?
- Which graph is a reflection of the graph $f(x)$ across the x- and y-axis?

PROBLEM 1 It's the Same . . . But Different!

1. The two tables show four exponential functions and four exponential graphs.
   a. Match the exponential function to its corresponding graph, and write the function under the graph it represents.
   b. Explain the method(s) you used to match the functions with their graphs.

Exponential Functions	
$f(x) = 10^x$	$g(x) = 10^{-x}$
$h(x) = -10^x$	$j(x) = -10^{-x}$

Exponential Graphs	
 $h(x) = -10^x$	 $f(x) = 10^x$
 $j(x) = -10^{-x}$	 $g(x) = 10^{-x}$

Answers will vary.

The graph of $f(x) = 10^x$ is an exponential growth function.

The graph of $g(x) = 10^{-x}$ is a reflection of the graph of $f(x)$ across the y-axis.

The graph of $h(x) = -10^x$ is a reflection of the graph of $f(x)$ across the x-axis.

The graph of $j(x) = -10^{-x}$ is a reflection of the graph of $f(x)$ across the x- and y-axes.

Guiding Questions for Share Phase, Questions 2 through 5

- Does the function $h(x) = -f(x)$ describe a function that is reflected across the x-axis or reflected across the y-axis or reflected across both axes?
- Does the function $g(x) = f(-x)$ describe a function that is reflected across the x-axis or reflected across the y-axis or reflected across both axes?
- Does the function $j(x) = -f(-x)$ describe a function that is reflected across the x-axis or reflected across the y-axis or reflected across both axes?
- Which functions have the same interval of increase?
- Which functions have the same interval of decrease?
- Which function(s) are associated with the end behavior of $f(x) \to \infty$ as $x \to \infty$?
- Which function(s) are associated with the end behavior of $f(x) \to 0$ as $x \to \infty$?
- Which function(s) are associated with the end behavior of $f(x) \to -\infty$ as $x \to \infty$?
- Are the equations for the asymptotes for the four functions the same?
- Is the graph of $g(x) = 10^{-x}$ and $k(x) = \left(\frac{1}{10}\right)^x$ the same graph?
- Do the functions $g(x)$ and $k(x)$ represent the same exponential decay function?
- Are both functions $g(x)$ and $k(x)$ the reflection of $f(x) = 10^x$ across the y-axis?

2. Analyze the graphs.

a. Write an equation for $h(x)$ in terms of $f(x)$. Describe the transformation on $f(x)$.

$h(x) = -f(x)$

The transformation on $f(x)$ is a reflection across the x-axis.

b. Write an equation for $g(x)$ in terms of $f(x)$. Describe the transformation on $f(x)$.

$g(x) = f(-x)$

The transformation on $f(x)$ is a reflection across the y-axis.

c. Write an equation for $j(x)$ in terms of $f(x)$. Describe the transformation on $f(x)$.

$j(x) = -f(-x)$

The transformation on $f(x)$ is a reflection of the graph of $f(x)$ across the x- and y-axes.

3. Determine the asymptotes, intervals of increase and decrease, and end behavior for each exponential function.

Function	Asymptotes	Intervals of Increase and Decrease	End Behavior
$f(x) = 10^x$	$y = 0$	Increasing over $(-\infty, \infty)$	As $x \to -\infty$, $f(x) \to 0$ As $x \to \infty$, $f(x) \to \infty$
$g(x) = 10^{-x}$	$y = 0$	Decreasing over $(-\infty, \infty)$	As $x \to -\infty$, $g(x) \to \infty$ As $x \to \infty$, $g(x) \to 0$
$h(x) = -10^x$	$y = 0$	Decreasing over $(-\infty, \infty)$	As $x \to -\infty$, $h(x) \to 0$ As $x \to \infty$, $h(x) \to -\infty$
$j(x) = -10^{-x}$	$y = 0$	Increasing over $(-\infty, \infty)$	As $x \to -\infty$, $j(x) \to -\infty$ As $x \to \infty$, $j(x) \to 0$

4. How would the graph of $k(x) = \left(\frac{1}{10}\right)^x$ compare to the graph of $g(x) = 10^{-x}$?

The graphs of $g(x) = 10^{-x}$ and $k(x) = \left(\frac{1}{10}\right)^x$ would be equivalent. The functions $g(x)$ and $k(x)$ represent the same exponential decay function, which is the reflection of $f(x) = 10^x$ across the y-axis.

$$g(x) = 10^{-x}$$
$$= (10^{-1})^x$$
$$= \left(\frac{1}{10}\right)^x$$

- Do the transformations performed on $f(x)$ affect the horizontal asymptote of the function?
- Does a single reflection across the x- or y-axis affect the intervals of increase or decrease?
- Do reflections over both the x- and y-axis affect the intervals of increase and decrease?

Problem 2

The general transformational function $g(x) = Af(B(x - C)) + D$ is given with a review of the transformations caused by a change in the A-value, B-value, C-value, and D-value. The graph of the function $f(x) = 3^x$ is provided and students use given equations to transform this function on a graph, and complete a table of values listing reference points and their corresponding points. They also identify the domain, range, and asymptotes of each transformed function. Students generalize about the effects of vertical and horizontal translations and dilations on the key characteristics of an exponential function.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

5. How do the transformations on $f(x)$ affect the asymptotes, intervals of increase and decrease, and end behavior?

The transformations on $f(x)$ do not affect the horizontal asymptote of the function.

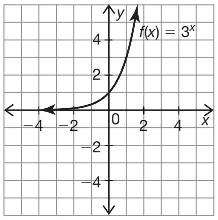
A single reflection over the x- or y-axis affects the intervals of increase and decrease. Reflections across both the x- and y-axes do not affect the intervals of increase and decrease.

The transformations on $f(x)$ all affect the end behavior of the function.

PROBLEM 2 Keep On Moving

Consider the functions $y = f(x)$ and $g(x) = Af(B(x - C)) + D$. Recall that the D-value translates $f(x)$ vertically, the C-value translates $f(x)$ horizontally, the A-value vertically stretches or compresses $f(x)$, and the B-value horizontally stretches or compresses $f(x)$. Exponential functions are transformed in the same manner.

The function $f(x) = 3^x$ is shown. Recall the key characteristics of basic exponential functions, including a domain of all real numbers, a range of positive numbers, and a horizontal asymptote at $y = 0$.



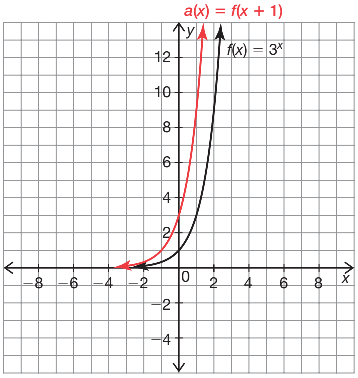
1. Suppose that $a(x) = f(x + 1)$.

a. Describe the transformation on the graph of $f(x)$ that produces $a(x)$.

The graph of $f(x)$ is translated horizontally left 1 unit to produce $a(x)$.

b. Complete the table to determine the corresponding points on $a(x)$, given reference points on $f(x)$. Then, graph and label $a(x)$.

Reference Points on $f(x)$	Corresponding Points on $a(x)$
$\left(-1, \frac{1}{3}\right)$	$\left(-2, \frac{1}{3}\right)$
(0, 1)	(−1, 1)
(1, 3)	(0, 3)



Guiding Questions for Share Phase, Question 1

- Did the transformation performed on $f(x)$ shift the graph horizontally to the left 1 unit or horizontally to the right 1 unit to produce $a(x)$?
- How did you determine the coordinates of the points corresponding to the reference points in this situation?
- Are the domain, range, and asymptotes of $a(x)$ the same as $f(x)$?

Guiding Questions for Share Phase, Questions 2 through 4

- Did the transformation performed on *f*(*x*) shift the graph vertically up 1 unit or vertically down 1 unit to produce *b*(*x*)?
- How did you determine the coordinates of the points corresponding to the reference points in this situation?
- Are the domain, range, and asymptotes of *b*(*x*) the same as *f*(*x*)?
- How is the domain, range, or asymptotes of *b*(*x*) different than those of *f*(*x*)?
- Did the transformation performed on *f*(*x*) shift the graph vertically up 5 units or vertically down 5 units to produce *c*(*x*)?
- How did you determine the coordinates of the points corresponding to the reference points in this situation?
- Are the domain, range, and asymptotes of *c*(*x*) the same as *f*(*x*)?
- How is the domain, range, or asymptotes of *c*(*x*) different than those of *f*(*x*)?
- Did the transformation performed on *f*(*x*) compress the graph horizontally by a factor of 2 or by a factor of $\frac{1}{2}$ to produce *d*(*x*)?
- How did you determine the coordinates of the points corresponding to the reference points in this situation?
- Are the domain, range, and asymptotes of *d*(*x*) the same as *f*(*x*)?

c. Determine the domain, range, and asymptotes of *a*(*x*).

Domain: All real numbers

Range: $y > 0$

Horizontal asymptote: $y = 0$

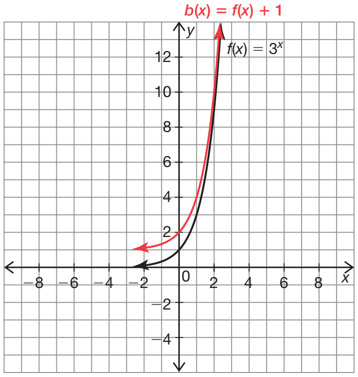
2. Suppose that $b(x) = f(x) + 1$.

a. Describe the transformation on the graph of *f*(*x*) that produces *b*(*x*).

The graph of *f*(*x*) is translated vertically up 1 unit to produce *b*(*x*).

b. Complete the table to determine the corresponding points on *b*(*x*), given reference points on *f*(*x*). Then, graph and label *b*(*x*).

Reference Points on *f*(*x*)	Corresponding Points on *b*(*x*)
$\left(-1, \frac{1}{3}\right)$	$\left(-1, 1\frac{1}{3}\right)$
(0, 1)	(0, 2)
(1, 3)	(1, 4)



c. Determine the domain, range, and asymptotes of *b*(*x*).

Domain: All real numbers

Range: $y > 1$

Horizontal asymptote: $y = 1$

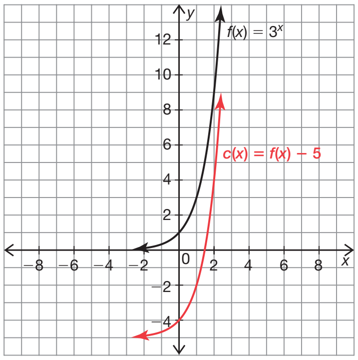
3. Suppose that $c(x) = f(x) - 5$.

a. Describe the transformation on the graph of $f(x)$ that produces $c(x)$.

The graph of $f(x)$ is translated vertically down 5 units to produce $c(x)$.

b. Complete the table to determine the corresponding points on $c(x)$, given reference points on $f(x)$. Then, graph and label $c(x)$.

Reference Points on $f(x)$	Corresponding Points on $c(x)$
$\left(-1, \frac{1}{3}\right)$	$\left(-1, -4\frac{2}{3}\right)$
$(0, 1)$	$(0, -4)$
$(1, 3)$	$(1, -2)$



c. Determine the domain, range, and asymptotes of $c(x)$.

Domain: All real numbers

Range: $y > -5$

Horizontal asymptote: $y = -5$

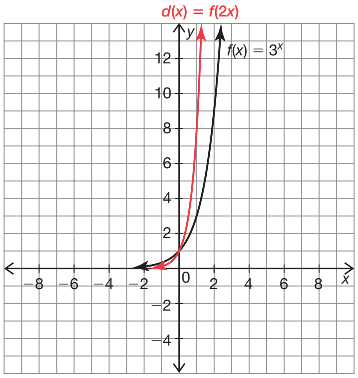
4. Suppose that $d(x) = f(2x)$.

a. Describe the transformation on the graph of $f(x)$ that produces $c(x)$.

The graph of $f(x)$ is compressed horizontally by a factor of $\frac{1}{2}$ to produce $c(x)$.

b. Complete the table to determine the corresponding points on $d(x)$, given reference points on $f(x)$. Then, graph and label $d(x)$.

Reference Points on $f(x)$	Corresponding Points on $d(x)$
$\left(-1, \frac{1}{3}\right)$	$\left(-\frac{1}{2}, \frac{1}{3}\right)$
$(0, 1)$	$(0, 1)$
$(1, 3)$	$\left(\frac{1}{2}, 3\right)$



c. Determine the domain, range, and asymptotes of $d(x)$.

Domain: All real numbers

Range: $y > 0$

Horizontal asymptote: $y = 0$

5. Analyze the transformations performed on $f(x)$ in Questions 1 through 4.

a. Which, if any, of these transformations affected the domain, range, and asymptotes?

Vertical translations of $f(x)$, such as $b(x)$ and $c(x)$, affected the range and the horizontal asymptote.

The domain was not affected.

b. What generalizations can you make about the effects of transformations on the domain, range, and asymptotes of exponential functions?

The domain of exponential functions is not affected by vertical translations, horizontal translations, and horizontal dilations.

Vertical translations affect the range and the horizontal asymptote of exponential functions.

Horizontal translations and horizontal dilations do not affect the range and the horizontal asymptote of exponential functions.

Grouping

Have students complete Questions 5 and 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- Did vertical translations of $f(x)$, such as $b(x)$ and $c(x)$ affect the range?
- Did vertical translations of $f(x)$, such as $b(x)$ and $c(x)$ affect the horizontal asymptote?
- Did vertical translations of $f(x)$, such as $b(x)$ and $c(x)$ affect the domain?
- Is the domain of an exponential function affected by vertical or horizontal translations?
- Is the range of an exponential function affected by vertical or horizontal translations?
- Is the horizontal asymptote of an exponential function affected by vertical or horizontal translations?
- Do horizontal translations affect the range and the horizontal asymptote of exponential functions?
- Do horizontal dilations affect the range and the horizontal asymptote of exponential functions?

Guiding Questions for Share Phase, Question 6

- Which student used his knowledge of transformational function form to describe the graph of $p(x)$ in terms of $f(x)$?
- Which student used his knowledge of properties of exponents to rewrite $p(x)$ in terms of x, then used his knowledge of transformational function form to rewrite $p(x)$ and described its graph in terms of $f(x)$?
- Are the domain, range, and asymptotes of $p(x)$ the same as those of $f(x)$?
- Do vertical dilations affect the domain of exponential functions?
- Do vertical dilations affect the range of exponential functions?
- Do vertical dilations change the location of the horizontal asymptote of exponential functions?

6. Andres and Tomas each described the effects of transforming the graph of $f(x) = 3^x$, such that $p(x) = 3f(x)$.

Andres

$p(X) = 3f(X)$

The A-value is 3 so the graph is stretched vertically by a scale factor of 3.

Tomas

$p(x) = 3f(x)$

$p(x) = 3 \cdot 3^x$

$p(x) = 3^{1+x}$

$p(x) = f(x + 1)$

The C-value is 1 so the graph is horizontally translated 1 unit to the left.

a. Explain Andres' and Thomas' reasoning.

Andres used his knowledge of transformational function form to describe the graph of $p(x)$ in terms of $f(x)$.

Tomas used the properties of exponents to rewrite $p(x)$. Then, he used transformational function form to rewrite $p(x)$, and he described its graph in terms of $f(x)$.

b. Determine the domain, range, and asymptotes of $p(x)$.

Domain: All real numbers

Range: $y > 0$

Horizontal asymptote: $y = 0$

c. What generalizations can you make about the effects of vertical dilations on the domain, range, and asymptotes of exponential functions?

Vertical dilations do not affect the domain, range, or the horizontal asymptote of exponential functions.

Problem 3

Students analyze pairs of graphs of functions to describe the transformations performed on the first graph to create the second graph. Then they write an equation for the transformed function in terms of the original function given the corresponding points on the transformed graph after the rigid motions were performed. In the last activity, students use the equation of an exponential function and the equation of the function after a transformation to describe in words the transformation performed that produced the second equation. They also express the transformed function as an exponential equation.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

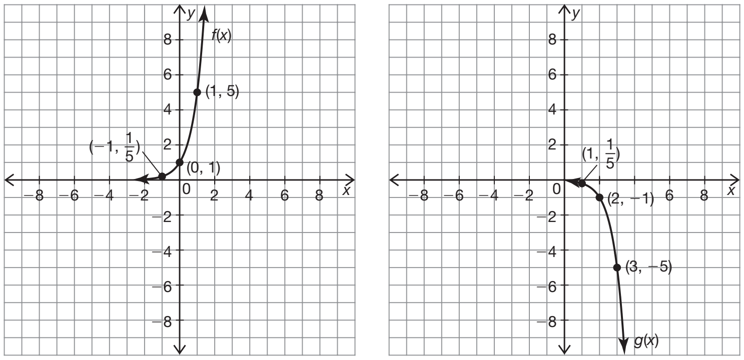
Guiding Questions for Share Phase, Question 1

- Is $g(x)$ written as $g(x) = f(x - 2)$ or written as $g(x) = -f(x - 2)$?
- To create $g(x)$, was the graph of $f(x)$ translated horizontally to the right 2 units or to the left 2 units?
- To create $g(x)$, was the graph of $f(x)$ reflected across the x-axis or reflected across the y-axis?

PROBLEM 3 Multiple Transformations

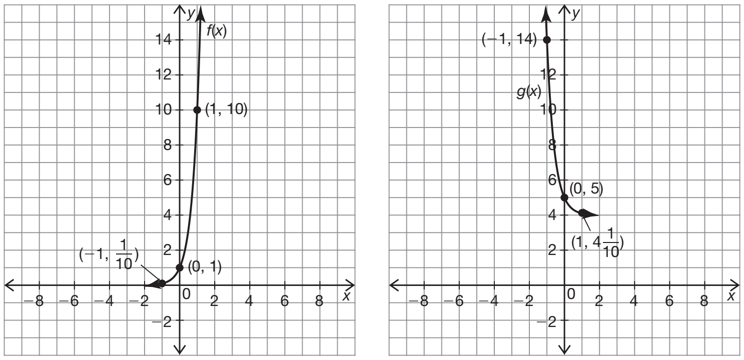
1. Analyze the graphs of $f(x)$ and $g(x)$. Describe the transformations performed on $f(x)$ to create $g(x)$. Then, write an equation for $g(x)$ in terms of $f(x)$. For each set of points shown on $f(x)$, the corresponding points are shown on $g(x)$.

a. $g(x) =$ $-f(x - 2)$



To create $g(x)$, the graph of $f(x)$ is horizontally translated right 2 units and reflected across the x-axis.

b. $g(x) =$ $f(-x) + 4$



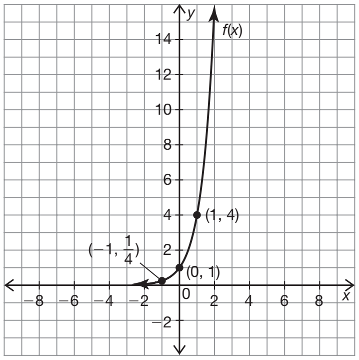
To create $g(x)$, the graph of $f(x)$ is reflected across the y-axis and vertically translated up 4 units.

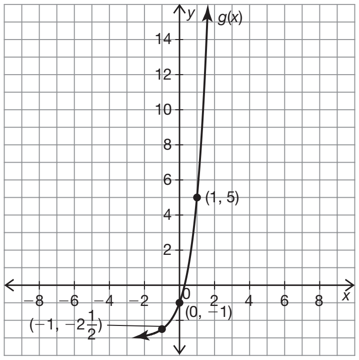
- Is $g(x)$ written as $g(x) = f(-x) + 4$ or written as $g(x) = -f(-x) + 4$?
- To create $g(x)$, was the graph of $f(x)$ reflected across the y-axis, and vertically translated up 4 units or reflected across the x-axis, and vertically translated up 4 units?
- Is $g(x)$ written as $g(x) = f(2x) - 3$ or written as $g(x) = 2f(x) - 3$?
- To create $g(x)$, was the graph of $f(x)$ stretched vertically by a factor of 2 and then vertically translated down 3 units?

Guiding Questions for Share Phase, Question 2

- Was the graph of the function $m(x)$ horizontally translated left 3 units or horizontally translated right 3 units to produce $t(x)$?
- Was the graph of the function $m(x)$ compressed vertically by a factor of 0.5 or compressed horizontally by a factor of 0.5 to produce $t(x)$?
- Was the graph of the function $m(x)$ reflected across the x-axis and vertically translated up 1 unit or down 1 unit to produce $t(x)$?
- Was the graph of the function $m(x)$ reflected across the y-axis and stretched vertically by a factor of 2 or compressed vertically by a factor of 2 to produce $t(x)$?

c. $g(x) =$ $2f(x) - 3$





To create $g(x)$, the graph of $f(x)$ is stretched vertically by a factor of 2, and vertically translated down 3 units.

2. The equation for an exponential function $m(x)$ is given. The equation for the transformed function $t(x)$ in terms of $m(x)$ is also given. Describe the graphical transformation(s) on $m(x)$ that produce(s) $t(x)$. Then, write an exponential equation for $t(x)$.

a. $m(x) = 2^x$

$t(x) = 0.5m(x + 3)$

The graph of the function $m(x)$ is horizontally translated left 3 units and compressed vertically by a factor of 0.5 to produce $t(x)$.

$t(x) = 0.5 \cdot 2^{x+3}$

b. $m(x) = e^x$

$t(x) = -m(x) - 1$

The graph of the function $m(x)$ is reflected across the x-axis and vertically translated down 1 unit to produce $t(x)$.

$t(x) = -e^x - 1$

c. $m(x) = 6^x$

$t(x) = 2m(-x)$

The graph of the function $m(x)$ is reflected across the y-axis and stretched vertically by a factor of 2 to produce $t(x)$.

$t(x) = 2 \cdot 6^{-x}$

Be prepared to share your solutions and methods.

Check for Students' Understanding

Determine if each statement is true or false. If the statement is false, rewrite the statement as true.

1. In the general transformational function $g(x) = Af(B(x - C)) + D$, the D-value translates the function $f(x)$ horizontally, the C-value translates $f(x)$ vertically, the A-value horizontally stretches or compresses $f(x)$, and the B-value vertically stretches or compresses $f(x)$.

   This statement is false.

   In the general transformational function $g(x) = Af(B(x - C)) + D$, the D-value translates the function $f(x)$ vertically, the C-value translates $f(x)$ horizontally, the A-value vertically stretches or compresses $f(x)$, and the B-value horizontally stretches or compresses $f(x)$.

2. Key characteristics of basic exponential functions include a domain of nonnegative numbers, a range of real numbers, and a vertical asymptote at $y = 0$.

   This statement is false.

   Key characteristics of basic exponential functions include a domain of all real numbers, a range of nonnegative numbers, and a horizontal asymptote at $y = 0$.

3. The domain of exponential functions is not affected by translations or dilations.

   This statement is true.

4. Vertical translations do not affect the range and the horizontal asymptote of exponential functions.

   This statement is false.

   Vertical translations affect the range and the horizontal asymptote of exponential functions.

5. Horizontal translations do not affect the range and the horizontal asymptote of exponential functions.

   This statement is true.

6. Horizontal translations do not affect the range and the horizontal asymptote of exponential functions.

   This statement is true.

7. Vertical dilations affect the range and the horizontal asymptote of exponential functions.

   This statement is false.

   Vertical dilations do not affect the range and the horizontal asymptote of exponential functions.

12

12.4

I Feel the Earth Move

Logarithmic Functions

LEARNING GOALS

In this lesson, you will:

- Graph the inverses of exponential functions with bases of 2, 10, and *e*.
- Recognize the inverse of an exponential function as a logarithm.
- Identify the domain and range of logarithmic functions.
- Investigate graphs of logarithmic functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.

ESSENTIAL IDEAS

- The logarithm of a number for a given base is the exponent to which the base must be raised in order to produce the number.
- If $y = b^x$, then x is the logarithm and can be written as $x = \log_b(y)$. The value of the base of a logarithm is the same as the base in the exponential expression, b^x.
- Key characteristics of the inverse of basic exponential functions include a domain of positive numbers, a range of all real numbers, and a vertical asymptote at $x = 0$.
- All exponential functions are invertible.
- A logarithmic function is a function involving a logarithm.
- A common logarithm is a logarithm with base 10, and is usually written as *log* without a base specified.
- A natural logarithm is a logarithm with base *e*, and is usually written as ln.

KEY TERMS

- logarithm
- logarithmic function
- common logarithm
- natural logarithm

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

  e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F-BF Building Functions

Build new functions from existing functions

4. Find inverse functions.

  a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

Overview

The terms logarithm, logarithmic function, common logarithm, natural logarithm are defined. Logarithmic functions are introduced as the inverse of exponential functions. Students will explore the key characteristics of the logarithmic function and state restrictions on the variables for any logarithmic equation. Graphs of logarithmic functions are analyzed and compared to each other. Application problems using logarithmic functions involve the Richter scale and computing the intensity of earthquakes. A graphic organizer summarizes the key characteristics of exponential functions and logarithmic functions.

Warm Up

For each of the following functions, use a graphing calculator to determine the end behaviors of the graph and state the domain and range. Describe the function as basic exponential growth or basic exponential decay.

1. $f(x) = e^x$

As $x \to -\infty$, $f(x) \to 0$.
As $x \to \infty$, $f(x) \to \infty$.

The domain is all real numbers, and the range is $y > 0$.
This function is associated with exponential growth.

2. $f(x) = e^{-x}$

As $x \to -\infty$, $f(x) \to \infty$.
As $x \to \infty$, $f(x) \to 0$.

The domain is all real numbers, and the range is $y > 0$.
This function is associated with exponential decay.

12

I Feel the Earth Move

Logarithmic Functions

12.4

LEARNING GOALS

In this lesson, you will:

- Graph the inverses of exponential functions with bases of 2, 10, and *e*.
- Recognize the inverse of an exponential function as a logarithm.
- Identify the domain and range of logarithmic functions.
- Investigate graphs of logarithmic functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.

KEY TERMS

- logarithm
- logarithmic function
- common logarithm
- natural logarithm

You may have heard about the Richter scale rating. The Richter scale was developed in 1935 by Charles F. Richter of the California Institute of Technology. The Richter scale is used to rate the magnitude of an earthquake—the amount of energy it releases. This is calculated using information gathered by a seismograph.

The Richter scale is logarithmic, meaning that whole-number jumps in the rating indicate a tenfold increase in the wave amplitude of the earthquake. For example, the wave amplitude in a Level 4 earthquake is ten times greater than the amplitude of a Level 5 earthquake, and the amplitude increases 100 times between a Level 6 earthquake and a Level 8 earthquake.

Most earthquakes are extremely small, with a majority registering less than 3 on the Richter scale. These tremors, called microquakes, aren't even felt by humans. Only a tiny portion, 15 or so of the 1.4 million quakes that register above 2 each year, register at 7 or above, which is the threshold for a quake to be considered major.

Problem 1

The basic exponential function $f(x) = 2^x$ is represented using a table of values and a graph. The key characteristics are listed. Student use this information to graph the inverse of the function on the same coordinate plane, complete a table of values, and identify the key characteristics. The term logarithm is introduced as the inverse of an exponential function. Several functions are written in both exponential form and equivalent logarithmic form. Students analyze the exponential equation, $y = b^x$, and its related logarithmic equation, $x = \log_b (y)$.

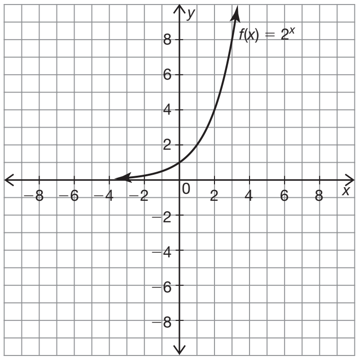
Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

PROBLEM 1 Return of the Inverse

Consider the table and graph for the basic exponential function $f(x) = 2^x$.

x	$f(x) = 2^x$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



You learned that the key characteristics of basic exponential functions are:

- The domain is the set of all real numbers.
- The range is the set of all positive numbers.
- The y-intercept is (0, 1).
- There is no x-intercept.
- There is a horizontal asymptote at $y = 0$.
- The function increases over the entire domain.
- As x approaches negative infinity, $f(x)$ approaches 0.
- As x approaches positive infinity, $f(x)$ approaches positive infinity.

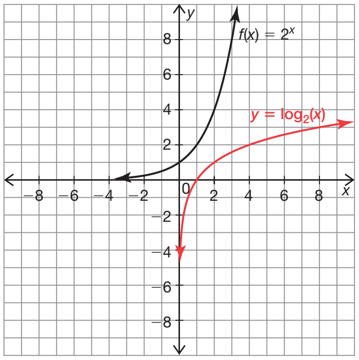
Guiding Questions for Share Phase, Questions 1 and 2, parts (a) and (b)

- How is the table of values for the basic exponential function used to complete the table of values for the inverse of the function?
- Are you able to graph the inverse of the exponential function using the table of values?
- Is the function $f(x) = 2^x$ an invertible function?
- Does the graph of the exponential function pass the Horizontal Line Test?
- Does the graph of the inverse pass the Vertical Line Test?
- Is the domain all positive numbers?
- Is the range all real numbers?
- Is there an x-intercept or a y-intercept?
- Is there a horizontal asymptote or a vertical asymptote?
- Is the function increasing over all real numbers or decreasing over all real numbers?

Recall that for any function f with ordered pairs (x, y), or $(x, f(x))$, the inverse of the function f is the set of all ordered pairs (y, x), or $(f(x), x)$.

1. Graph the inverse of $f(x) = 2^x$ on the same coordinate plane as $f(x)$. Complete the table of values for the inverse of $f(x)$.

x	y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



2. Analyze the key characteristics of the inverse of $f(x) = 2^x$.

a. Is the inverse of $f(x) = 2^x$ a function? Explain your reasoning.

Yes. The function $f(x) = 2^x$ is an invertible function.

The graph of the exponential function passes the Horizontal Line Test. The graph of the inverse passes the Vertical Line Test.

b. Identify the domain, range, intercepts, asymptotes, intervals of increase and decrease, and end behavior of $f^{-1}(x)$.

Domain: $x > 0$

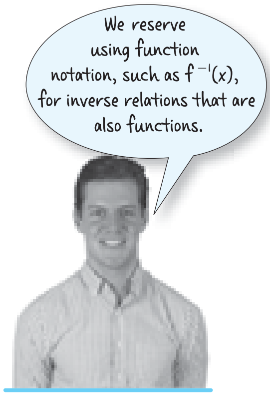
Range: All real numbers

Intercepts: The x-intercept is (1, 0). There is no y-intercept.

Asymptotes: Vertical asymptote at $x = 0$

Intervals of increase and decrease: Increasing over entire domain

End behavior: As x approaches 0 from the right, y approaches negative infinity. As x approaches positive infinity, y approaches positive infinity.



Guiding Questions for Share Phase, Question 2, parts (c) through (g)

- Is the domain of the exponential function the same as the range of the inverse function?
- Is the range of the exponential function the same as the domain of the inverse function?
- Are the equations describing the asymptotes the same for the function and the inverse function?
- Do both the exponential function and its inverse increase over all real numbers?
- How do you write the equation for the inverse of an exponential function?
- Is the inverse written as $x = 2^y$? How do you isolate the dependent variable in this situation?

c. What do you notice about the domain and range of the exponential function and its inverse?

The domain of the exponential function is the same as the range of the inverse function. The range of the exponential function is the same as the domain of the inverse function.

d. What do you notice about the asymptotes of the exponential function and its inverse?

For the exponential function, there is a horizontal asymptote of $y = 0$. For the inverse of the exponential function, there is a vertical asymptote of $x = 0$. The equations of the asymptotes are the same, except the variables are swapped.

e. What do you notice about the intervals of increase and decrease of the exponential function and its inverse?

Both the exponential function and its inverse are increasing over their domains.

f. What do you notice about the end behavior of the exponential function and its inverse?

For the exponential function, as x approaches negative infinity, y approaches 0, whereas for the inverse of the exponential function, as x approaches 0 from the right, y approaches negative infinity. The variables are swapped.

For the exponential function and its inverse, as x approaches positive infinity, y approaches positive infinity.

g. Write the equation for the inverse of $y = 2^x$. Explain your reasoning.

$x = 2^y$

Because the inverse of a function is the relation formed when the independent variable is exchanged with the dependent variable, I can write the inverse as $x = 2^y$.

Grouping

- Ask a student to read the information, definition, and examples. Discuss as a class.
- Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- In this situation, what is the logarithm or exponent?
- In this situation, what is the base?
- In this situation, what is the argument?
- Is the exponent or logarithm any real number?
- Is the base any positive number that is not equal to 1?
- Why can't the base be equal to 1?
- Is the range the set of all positive numbers?

It is necessary to define a new function in order to write the equation for the inverse of an exponential function. The **logarithm** of a number for a given base is the exponent to which the base must be raised in order to produce the number. If $y = b^x$, then x is the logarithm and can be written as $x = \log_b(y)$. The value of the base of a logarithm is the same as the base in the exponential expression b^x.

For example, the number 3 is the logarithm to which base 2 must be raised to produce the argument 8. The base is written as the subscript 2. The logarithm, or exponent, is the output 3. The argument of the logarithm is 8.

argument

logarithm, or exponent $\rightarrow 3 = \log_2(8)$

base

You can write any exponential equation as a logarithmic equation and vice versa.

Example	Exponential Form	⇔	Logarithmic Form
A	$y = b^x$	⇔	$x = \log_b(y)$
B	$16 = 4^2$	⇔	$2 = \log_4(16)$
C	$1000 = 10^3$	⇔	$3 = \log_{10}(1000)$
D	$32 = 16^{1.25}$	⇔	$1.25 = \log_{16}(32)$
E	$a = b^c$	⇔	$c = \log_b(a)$

3. Rewrite your equation in Question 2, part (g), in logarithmic form. Label the graph from Question 1 with your equation.

$x = 2^y \Leftrightarrow y = \log_2(x)$

See graph.

In words, the exponential form of Example B is, "The number 2 is the *exponent* to which the base 4 must be raised to produce 16," whereas the logarithmic form is, "The number 2 is the *logarithm* to which the base 4 must be raised to produce 16."

Think about the key characteristics of the exponential function to make connections to the logarithmic function.

Problem 2

The terms logarithmic function, common logarithm, and natural logarithm are defined. Students use the graphs of the functions $p(x) = \log_2(x)$ and $q(x) = \log_3(x)$ to sketch the graph of the functions $c(x) = \log x$ and $n(x) = \ln x$. Then they analyze and compare the key characteristics of the four functions and write equations for the inverse of the logarithmic functions.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

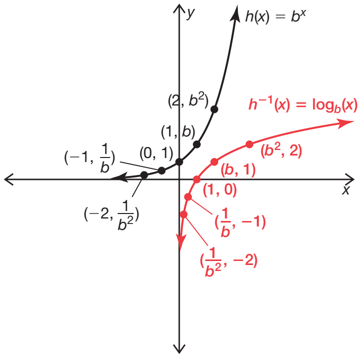
- How did you determine the coordinates of the points on the inverse of $h(x)$?

4. Analyze the exponential equation $y = b^x$ and its related logarithmic equation, $x = \log_b(y)$. State the restrictions, if any, on the variables. Explain your reasoning.

$y = b^x \Leftrightarrow x = \log_b(y)$		
Variable	**Restrictions**	**Explanation**
x	No restrictions	The exponent, or logarithm, can be any real number.
b	$b > 0$ and $b \neq 1$	For an exponential function, the base must be a positive number not equal to 1.
y	$y > 0$	The range of an exponential function is the set of all positive numbers.

PROBLEM 2 A Logarithm by Any Other Name . . .

1. The graph of $h(x) = b^x$ is shown. Sketch the graph of the inverse of $h(x)$ on the same coordinate plane. Label coordinates of points on the inverse of $h(x)$.



2. Write the equation for the inverse of $h(x) = b^x$. Label the graph.

$h^{-1}(x) = \log_b(x)$

See graph.

3. Do you think all exponential functions are invertible? If so, explain your reasoning. If not, provide a counterexample.

Yes. All exponential functions are invertible.

All exponential functions will pass the Horizontal Line Test.

- How were the coordinates of the points on the graph of the exponential function helpful in determining the coordinates of the points on the graph of the inverse?
- Are all exponential functions invertible?
- Do all exponential functions pass the Horizontal Line Test?

Grouping

- Ask a student to read the information and definitions. Discuss as a class.
- Have students complete Question 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- What is the difference between a common logarithm and a natural logarithm?
- Which logarithmic function is graphed between the two given functions? How do you know?
- Which logarithmic function is graphed below the two given functions? How do you know?
- What is the base of the logarithmic equation $n(x) = \ln x$?
- Is the value of *e* closer to 2 or closer to 3?
- What is the base of the logarithmic equation $c(x) = \log x$?
- Is the domain the set of all positive numbers for all four logarithmic functions?
- Is the range the set of all real numbers for all four logarithmic functions?
- Is the *x*-intercept (1, 0) for all four logarithmic functions?
- Do any of the functions have a *y*-intercept?

Recall that the logarithm of a number for a given base is the exponent to which the base must be raised in order to produce the number. If $y = b^x$, then the logarithm is written as $x = \log_b y$, where $b > 0, b \neq 1$, and $y > 0$. A **logarithmic function** is a function involving a logarithm.

Logarithms were first conceived by a Swiss clockmaker and amateur mathematician Joost Bürgi but became more widely known and used after the publication of a book by Scottish mathematician John Napier in 1614. Tables of logarithms were originally used to make complex computations in astronomy, surveying, and other sciences easier and more accurate. With the invention of calculators and computers, the use of logarithm tables as a tool for calculation has decreased. However, many real-world situations can be modeled using logarithmic functions.

Two frequently used logarithms are logarithms with base 10 and base *e*. A **common logarithm** is a logarithm with base 10 and is usually written *log* without a base specified.

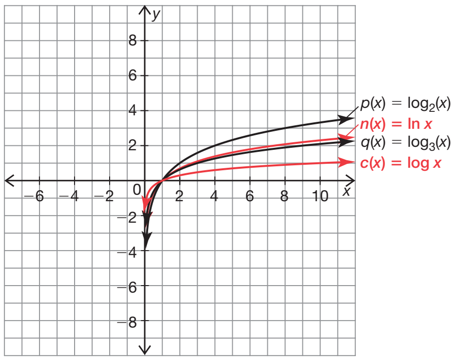
$$c(x) = \log_{10}(x) \quad \Leftrightarrow \quad c(x) = \log x$$

A **natural logarithm** is a logarithm with base *e*, and is usually written as *ln*.

$$n(x) = \log_e(x) \quad \Leftrightarrow \quad n(x) = \ln x$$

4. The functions $p(x) = \log_2(x)$ and $q(x) = \log_3(x)$ have been graphed for you.

   a. Sketch and label the functions $c(x) = \log x$ and $n(x) = \ln x$.



   b. Explain how you determined the graphs of $c(x)$ and $q(x)$.

   I know that the base of the logarithmic equation $n(x) = \ln x$ is the number $e \approx 2.72$. Because *e* is closer to 3 than 2, I sketched the graph of $n(x)$ between $p(x)$ and $q(x)$, but closer to $q(x)$.

   I know that the base of the logarithmic equation $c(x) = \log x$ is 10. Because 10 is greater than 3, I sketched the graph of $c(x)$ between $q(x)$ and the *x*-axis.

- Do all four logarithmic functions have a vertical asymptote at $x = 0$?
- Do all four logarithmic functions increase over the entire domain?
- As *x* approaches 0 from the right, what happens to the *y*-values for all four logarithmic functions?
- As *x* approaches positive infinity, what happens to the *y*-values for all four logarithmic functions?
- Which graph increases more quickly than the other graphs for $x > 1$?

continued on the next page

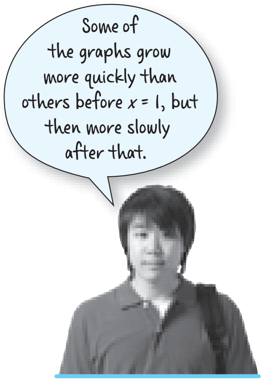
- Which graph increases more slowly than the other graphs for $x > 1$?
- Which graph increases more quickly than the other graphs for $0 < x < 1$?
- Which graph increases more slowly than the other graphs for $0 < x < 1$?

c. Analyze the key characteristics of $p(x)$, $q(x)$, $c(x)$, and $n(x)$. Describe the similarities and differences.

Answers will vary.

Similarities include:

- The domain is the set of all positive numbers.
- The range is the set of all real numbers.
- The x-intercept is (1, 0).
- There is no y-intercept.
- There is a vertical asymptote at $x = 0$.
- The function increases over the entire domain.
- As x approaches 0 from the right, y approaches negative infinity.
- As x approaches positive infinity, y approaches positive infinity.
- They are invertible functions.



Differences include:

- The graph of $p(x)$ increases more quickly than the other graphs for $x > 1$.
- The graph of $c(x)$ increases more slowly than the other graphs for $x > 1$.
- The graph of $c(x)$ increases more quickly than the other graphs for $0 < x < 1$.
- The graph of $p(x)$ increases more slowly than the other graphs for $0 < x < 1$.

d. What is the inverse of the logarithmic function $c(x) = \log x$?

The inverse of the logarithmic function $c(x) = \log x$ is the exponential function $c^{-1}(x) = 10^x$.

e. What is the inverse of the logarithmic function $n(x) = \ln x$?

The inverse of the logarithmic function $n(x) = \ln x$ is the exponential function $n^{-1}(x) = e^x$.

Problem 3

The logarithmic equation to determine the magnitude of an earthquake is given. Students use the equation to determine the magnitude of a standard earthquake, compare the intensity of an earthquake in California to an earthquake in Japan, and compare the intensity of an earthquake in Indonesia to an earthquake in Chile.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What is $\frac{I_0}{I_0}$ simplified?
- What is the value of log 1?
- How was the graphing calculator used in this situation?
- What equation was used in the graphing calculator to represent the Japanese earthquake?
- What equation was used in the graphing calculator to represent the Californian earthquake?
- How did you compare the intensities of the Japanese and Californian earthquakes?
- What equation was used in the graphing calculator to represent the Indonesian earthquake?
- What equation was used in the graphing calculator to represent the Chilean earthquake?
- What number did you use to multiply by the intensity of the Indonesian earthquake to determine the magnitude of the Chilean earthquake?

PROBLEM 3 Ground Shaking

The Richter scale is used to rate the magnitude of an earthquake, or the amount of energy released. An earthquake's magnitude, M, is determined using the equation, $M = \log\left(\frac{I}{I_0}\right)$, where I is the intensity of the earthquake being measured (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake), and I_0 is the intensity of a standard earthquake or "threshold quake" whose seismograph amplitude is 10^{-4} cm.

1. Determine the magnitude of a standard earthquake.

For a standard earthquake, $I = I_0$, so $M = 0$.

If $I = I_0$,

$$M = \log\left(\frac{I_0}{I_0}\right)$$
$$= \log 1$$
$$= 0$$

2. An earthquake in California measured 6.8 on the Richter scale, while an earthquake in Japan measured 7.2. How many times more intense was the Japanese earthquake?

The Japanese earthquake was approximately 2.512 times more intense than the Californian earthquake.

I used a graphing calculator to graph $Y_1 = \log\left(\frac{x}{10^{-4}}\right)$, where x is intensity, $Y_2 = 6.8$, and $Y_3 = 7.2$.

To determine the intensity of the Californian earthquake, I solved $6.8 = \log\left(\frac{x}{10^{-4}}\right)$ graphically.

$$I_{\text{California}} = 630.96$$

To determine the intensity of the Japanese earthquake, I solved $7.2 = \log\left(\frac{x}{10^{-4}}\right)$ graphically.

$$I_{\text{Japan}} = 1584.89$$

To compare the intensities of the Japanese and Californian earthquake, I simplified $\frac{I_{\text{Japan}}}{I_{\text{California}}}$.

$$\frac{I_{\text{Japan}}}{I_{\text{California}}} = \frac{1584.89}{630.96}$$
$$\frac{I_{\text{Japan}}}{I_{\text{California}}} \approx 2.512$$

3. Early in the century an earthquake in Indonesia registered 8.3 on the Richter scale. In the same year, another earthquake was recorded in Chile that was 4 times stronger. What was the magnitude of the Chilean earthquake?

The magnitude of the Chilean earthquake was 8.90.

I used a graphing calculator to graph $Y_1 = \log\left(\frac{x}{10^{-4}}\right)$, where x is intensity, and $Y_2 = 8.3$.

To determine the intensity of the Indonesian earthquake, I solved $8.3 = \log\left(\frac{x}{10^{-4}}\right)$ graphically.

$$I_{\text{Indonesia}} = 19{,}952.62$$

I know the intensity of the Chilean earthquake is 4 times stronger than the intensity of the Indonesian earthquake.

$$\begin{aligned} I_{\text{Chile}} &= 4 \cdot I_{\text{Indonesia}} \\ &= 4(19{,}952.62) \\ &= 79{,}810.48 \end{aligned}$$

To determine the magnitude of the Chilean earthquake, I determined y when $x = 79{,}810.48$. So $y \approx 8.90$.

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Talk the Talk

Students complete a table by cutting out and taping key characteristics next to its corresponding exponential function and logarithmic function.

Guiding Questions for Share Phase, Talk the Talk

- Are the domain and range switched?
- Are the intercepts switched?
- How do the asymptotes relate to each other?
- Are the end behaviors switched?

Talk the Talk

1. Complete the graphic organizer for the exponential function and the logarithmic function at the end of this lesson.

a. Cut out each key characteristic on this page.

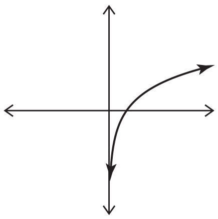
b. Tape each key characteristic to its corresponding function.

a. (0, 1)

b. As $x \to 0$, $y \to -\infty$.
As $x \to +\infty$, $y \to +\infty$.

c. All real numbers

d.



e. $f(x) = b^x$

f. $x > 0$

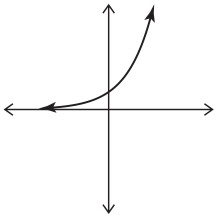
g. $y = 0$

h. All real numbers

i. As $x \to -\infty$, $y \to 0$.
As $x \to +\infty$, $y \to +\infty$.

j. (1, 0)

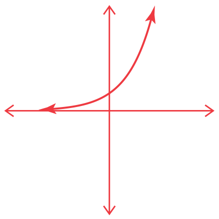
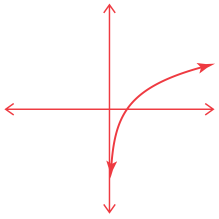
k.



l. $g(x) = \log_b(x)$

m. $x = 0$

n. $y > 0$

	Basic Exponential Function	Basic Logarithmic Function
Function:	**e.** $f(x) = b^x$	**l.** $g(x) = \log_b(x)$
Graph:	**k.** 	**d.** 
Domain:	**c. or h.** All real numbers	**f.** $x > 0$
Range:	**n.** $y > 0$	**c. or h.** All real numbers
Intercepts:	**a.** (0, 1)	**j.** (1, 0)
Asymptotes:	**g.** $y = 0$	**m.** $x = 0$
End behavior:	**i.** As $x \to -\infty$, $y \to 0$. As $x \to +\infty$, $y \to +\infty$.	**b.** As $x \to 0$, $y \to -\infty$. As $x \to +\infty$, $y \to +\infty$.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Write the equation for the inverse of each exponential function.

1. $f(x) = 4^x$

$f^{-1}(x) = \log_4 (x)$

2. $g(x) = e^x$

$g^{-1}(x) = \log_e (x)$

Write the equation for the inverse of each logarithmic function.

3. $h(x) = \ln x$

$h^{-1}(x) = e^x$

4. $j(x) = \log x$

$j^{-1}(x) = 10^x$

More Than Meets the Eye

Transformations of Logarithmic Functions

12.5

LEARNING GOALS

In this lesson, you will:

- Dilate, reflect, and translate logarithmic functions using reference points.
- Investigate graphs of logarithmic functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.

ESSENTIAL IDEAS

- The transformation $g(x) = -f(x)$ is the logarithmic function $f(x)$ reflected across the line $y = 0$.
- The transformation $h(x) = f(-x)$ is the logarithmic function $f(x)$ reflected across the line $x = 0$.
- The transformation $j(x) = -f(-x)$ is the logarithmic function $f(x)$ reflected across the lines $x = 0$ and $y = 0$.
- With respect to reflections, the domain is the set of all positive numbers for the logarithm of x and the set of all negative numbers for the logarithm of the opposite of x.
- The range is the set of all real numbers for any basic logarithmic function regardless of reflection.
- The effects of transformations on $f(x)$ and its inverse $g(x) = f^{-1}(x)$ are as follows:
  - $f(x + C)$ translates $f(x)$ horizontally left C units and translates $g(x)$ vertically down C units.
  - $f(x - C)$ translates $f(x)$ horizontally right C units and translates $g(x)$ vertically up C units.
  - $f(x) + D$ translates $f(x)$ vertically up D units and translates $g(x)$ horizontally right D units.
  - $f(x) - D$ translates $f(x)$ vertically down D units and translates $g(x)$ horizontally left D units.
- A vertical dilation on a function will produce a horizontal dilation the same number of units on its inverse, and a horizontal dilation on a function will produce a vertical dilation the same number of units on its inverse.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-BF Building Functions

Build new functions from existing functions

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

Overview

Students apply their understanding of the transformational function $g(x) = Af(B(x - C)) + D$ to the graph of a logarithmic function. Logarithmic functions are dilated, reflected, and translated using reference points. Key characteristics of logarithmic functions are identified and graphs are explored. Students will write equations for translated logarithmic functions given the graphs. Using a cutout activity, students generalize the effect that a transformation on a function will have on its inverse.

Warm Up

1. Write the inverse of the logarithmic function $h(x) = \log_5 (x)$.

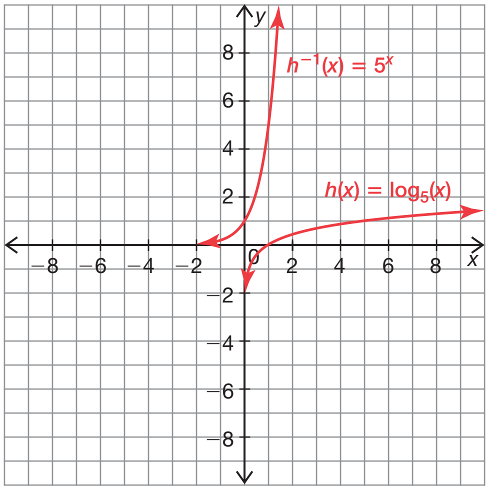
   $h^{-1}(x) = 5^x$

2. Complete the table of values for $h(x)$ and $h^{-1}(x)$.

x	$h(x)$
125	3
25	2
1	0
$\frac{1}{5}$	−1
$\frac{1}{25}$	−2

x	$h^{-1}(x)$
−2	$\frac{1}{25}$
−1	$\frac{1}{5}$
0	1
2	25
3	125

3. Graph $h(x)$ and $h^{-1}(x)$ on the coordinate plane.



4. Is $h^{-1}(x)$ a function? Explain your reasoning.

   Yes. For every value of x, there is one and only one value of $h^{-1}(x) = 5^x$.

12

More Than Meets the Eye

Transformations of Logarithmic Functions

12.5

LEARNING GOALS

In this lesson, you will:

- Dilate, reflect, and translate logarithmic functions using reference points.
- Investigate graphs of logarithmic functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.

'Practice makes perfect'!

When you practice the same motion over and over, you are building up procedural memory in your brain that instructs your muscles to perform a task. The more often your muscles receive those same instructions, the more quickly and efficiently they are able to carry them out until they become like second nature to you. Athletes use this idea of "muscle memory" when conditioning their bodies to perform, and musicians use it to train their fingers to hit the correct keys or strings accurately.

The best way to train your body and mind when learning a new skill is to practice it slowly at first to be sure that your technique is perfect, and then repeat that same quality practice as often as possible. Break the skill or information up into pieces and work on it, a piece at a time, until it is committed to memory. Once you've learned all of the parts, continue to practice, practice, practice until your new skill becomes an ingrained habit.

What skills do you like to practice?

Problem 1

Students match four logarithmic functions with their appropriate graphs. They analyze the graphs of the functions in terms of $f(x)$ and describe each transformation. Key characteristics of each logarithmic function are identified and generalizations are made about the effect of reflections on the domain and range of a logarithmic function.

Grouping

Have a student complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

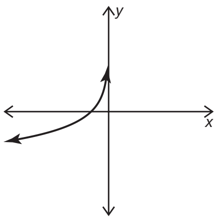
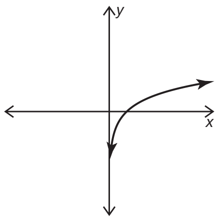
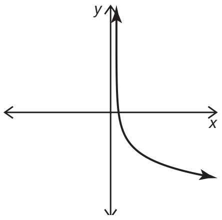
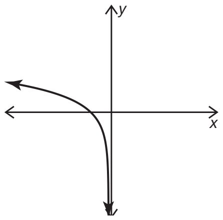
Guiding Questions for Share Phase, Questions 1 through 4

- How did you determine which graph matched each logarithmic equation?
- How would you describe the end behavior of the basic logarithmic function $f(x) = \log_2(x)$?
- Which graph appears to be the graph of the basic logarithmic function reflected across the line $y = 0$?
- Which graph appears to be the graph of the basic logarithmic function reflected across the line $x = 0$?
- Which graph appears to be the graph of the basic logarithmic function reflected across the lines $y = 0$ and $x = 0$?
- Is the range the set of all real numbers for all four logarithmic functions?
- Do all four logarithmic functions have a vertical asymptote at $x = 0$?
- Why will the domain never equal 0?
- Will the range always be the set of all real numbers for any basic logarithmic function, regardless of reflection?

PROBLEM 1 Don't Flip Out! It's Just a Reflection

1. The two tables show four logarithmic functions and four logarithmic graphs. Match the logarithmic function to its corresponding graph, and write the function under the graph it represents.

Logarithmic Functions	
$f(x) = \log_2(x)$	$g(x) = -\log_2(x)$
$h(x) = \log_2(-x)$	$j(x) = -\log_2(-x)$

Logarithmic Graphs	
A.  $j(x) = -\log_2(-x)$	**B.**  $f(x) = \log_2(x)$
C.  $g(x) = -\log_2(x)$	**D.**  $h(x) = \log_2(-x)$

2. Analyze the graphs of $f(x)$, $g(x)$, $h(x)$, and $j(x)$. Write an equation for each function $g(x)$, $h(x)$, and $j(x)$ in terms of $f(x)$. Describe each transformation on $f(x)$.

Logarithmic Function	Transformation on $f(x)$
$g(x) = -f(x)$	Reflection across the line $y = 0$
$h(x) = f(-x)$	Reflection across the line $x = 0$
$j(x) = -f(-x)$	Reflection across the lines $x = 0$ and $y = 0$

Recall that all transformations can be written in transformational function form in terms of f(x).

3. Analyze the key characteristics of each logarithmic function. What similarities exist among the four functions?

For the four logarithmic functions, the range is the set of all real numbers. Each function has a vertical asymptote at $x = 0$.

4. What generalizations can you make about the effects of these transformations on the domain and range of a logarithmic function?

The domain is the set of all positive numbers for the logarithm of x. The domain for the logarithm of the opposite of x is the set of all negative numbers. The domain will never equal zero, regardless of how the graph of the logarithmic function is reflected, because there is a vertical asymptote at $x = 0$.

The range will always be the set of all real numbers for any basic logarithmic function, regardless of reflection.

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Problem 2

Students analyze the graphs of $f(x)$ and a transformed function and describe the transformations produced on $f(x)$ to create the transformed function. They write an equation for the transformed function in terms of $f(x)$. Next, the graph of the basic logarithmic function $f(x)$ is shown and students graph the given transformed function of $f(x)$, then state the domain, range, and asymptotes. In the last activity, students write the transformed logarithmic function in terms of $f(x)$ with the given characteristics and graph the transformed function.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

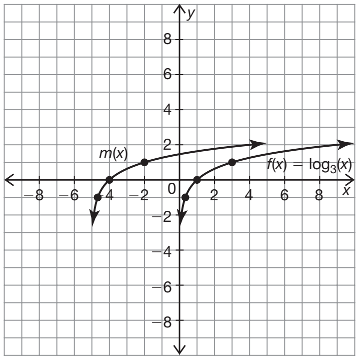
- Was the graph of the logarithmic function translated horizontally to the left 5 units or horizontally to the right 5 units?
- Was the graph of the logarithmic function translated horizontally to the right 1 unit or to the left 1 unit? Was it vertically translated up 2 units, or down 2 units?
- Was the graph of the logarithmic function vertically dilated by a factor of 4 or horizontally dilated by a factor of 4?

PROBLEM 2 You've Got Some Moves!

1. Analyze the graphs of $f(x)$ and the transformed function. Describe the transformations produced on $f(x)$ to create the transformed function. Then, write an equation for the transformed function in terms of $f(x)$. For each set of points shown on $f(x)$, the corresponding points are shown on the transformed function.

   a. Describe the transformation:

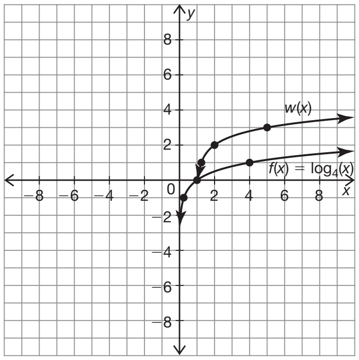
   horizontally translate left 5 units



   $m(x) =$ $f(x + 5)$

   b. Describe the transformation:

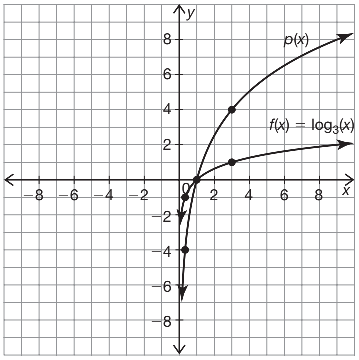
   horizontally translate right 1 unit, vertically translate up 2 units



   $w(x) =$ $f(x - 1) + 2$

   c. Describe the transformation:

   vertical dilation by a factor of 4



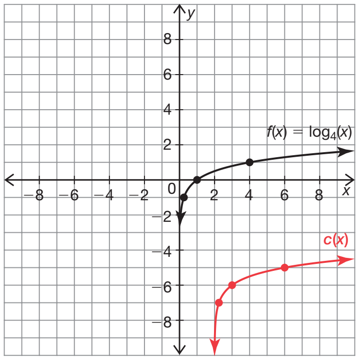
   $p(x) =$ $4f(x)$

Guiding Questions for Share Phase, Questions 2 and 3

- Was the graph of the logarithmic function translated to the left 2 units or to the right 2 units? Was it vertically translated up 6 units or down 6 units?
- Was the graph of the logarithmic function reflected across the x- or y-axis? Was it translated up 3 units or down 3 units?
- Was the graph of the logarithmic function vertically dilated by a factor of $\frac{1}{2}$ or horizontally dilated by a factor of $\frac{1}{2}$?
- Does $f(x - 3)$ or $f(x + 3)$ describe a vertical asymptote at $x = -3$?
- Does $\frac{1}{2}f(x)$ or $f\left(\frac{1}{2}x\right)$ describe the equation that generates the corresponding points?

2. The graph of a basic logarithmic function $f(x)$ is shown. Graph each transformation of $f(x)$. State the domain, range, and asymptotes of your graph.

a. $c(x) = f(x - 2) - 6$

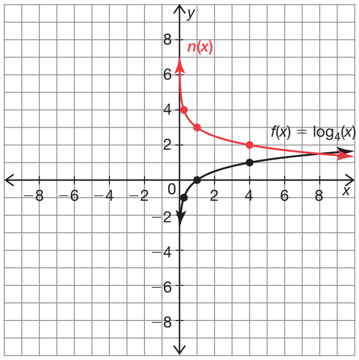


Domain of $c(x)$: $(2, \infty)$

Range of $c(x)$: $(-\infty, \infty)$

Asymptote of $c(x)$: $x = 2$

b. $n(x) = -f(x) + 3$

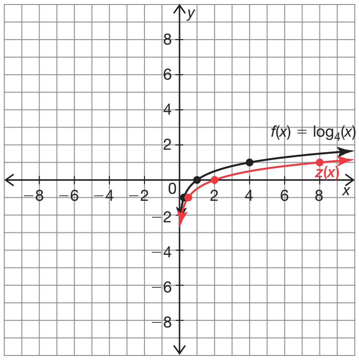


Domain of $n(x)$: $(0, \infty)$

Range of $n(x)$: $(-\infty, \infty)$

Asymptote of $n(x)$: $x = 0$

c. $z(x) = f\left(\frac{x}{2}\right)$



Domain of $z(x)$: $(0, \infty)$

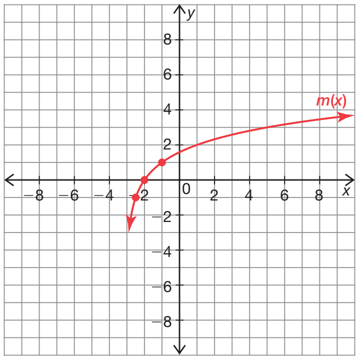
Range of $z(x)$: $(-\infty, \infty)$

Asymptote of $z(x)$: $x = 0$

3. Write a transformed logarithmic function in terms of $f(x)$ with the characteristic(s) given. Then, graph the transformed function.

a. $f(x) = \log_2 (x)$

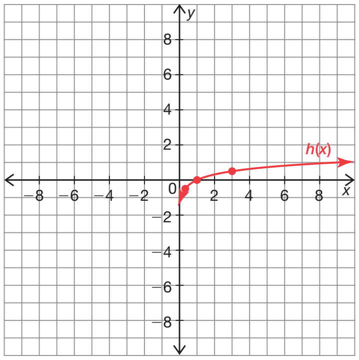
vertical asymptote at $x = -3$



$m(x) =$ $f(x + 3)$

b. $f(x) = \log_3 (x)$

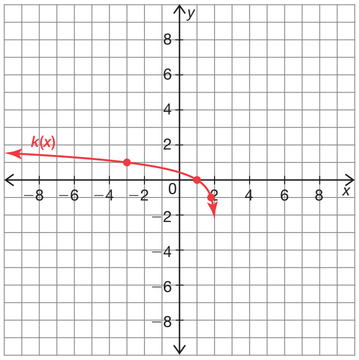
Reference Points on $f(x)$	→	Corresponding Points on $h(x)$
$\left(\frac{1}{3}, -1\right)$	→	$\left(\frac{1}{3}, -\frac{1}{2}\right)$
(1, 0)	→	(1, 0)
(3, 1)	→	$\left(3, \frac{1}{2}\right)$



$h(x) =$ $\frac{1}{2} f(x)$

c. $f(x) = \log_5 (x)$

Domain: $(-\infty, 2)$



Answers will vary.

$k(x) =$ $f(2 - x)$

Problem 3

Using a cutout activity, students organize sets of graphs, graph the transformations, and label each transformation. They also describe the transformations. The given graphs are used to graph the inverse of the transformed functions and describe the relationship between the graph of the inverse of the given transformed function and the original graph. A table is used to summarize and generalize the effects of transformations on $f(x)$ and its inverse function, $g(x) = f^{-1}(x)$. In the last activity, the function $f(x)$ and the transformed function $g(x)$ are used to write an equation for $g^{-1}(x)$ in terms of $f^{-1}(x)$.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

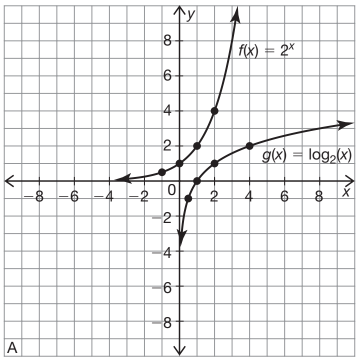
Guiding Questions for Share Phase, Question 1

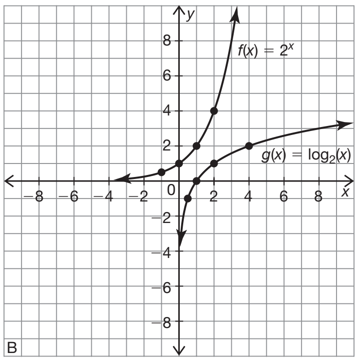
- Was the graph of the logarithmic function translated horizontally to the left 4 units or horizontally to the right 4 units?
- Was the graph of the logarithmic function translated vertically to the up 5 units or vertically down 5 units?

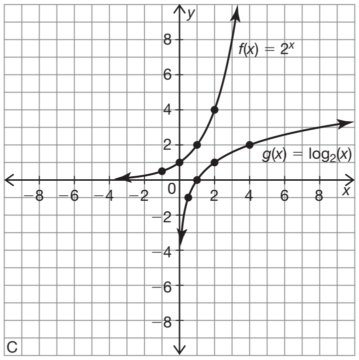
PROBLEM 3 Makin' Moves

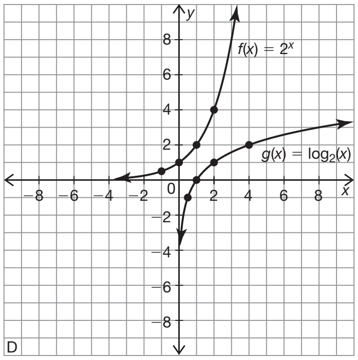
1. Consider the functions $f(x) = 2^x$ and $g(x) = f^{-1}(x)$, or $\log_2(x)$. The graphs of $f(x)$ and $g(x)$ are shown.
   - **a.** Cut out the four grids. Use the graphic organizer in this lesson to organize your information.
   - **b.** On each of your cut outs, graph and label the transformed function A, B, C, or D. Describe the transformation(s) performed on $f(x)$.
   - **c.** Use your cut out to graph the inverse of the transformed function.
   - **d.** Describe and label the graph of the inverse of the transformed function as a transformation on $g(x)$.

Don't forget you can fold the cut outs to determine the inverse graph.





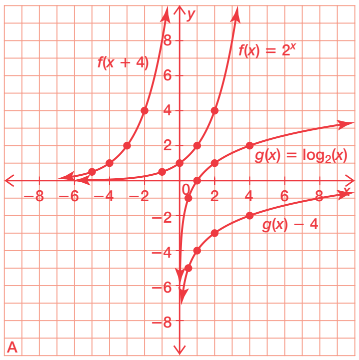




- Was the graph of the logarithmic function translated horizontally to the left 3 units or horizontally to the right 3 units? Was it translated vertically up 6 units or vertically down 6 units?
- Was the graph of the logarithmic function translated horizontally to the left 1 unit or horizontally to the right 1 unit? Was it translated vertically up 1 unit or vertically down 1 unit?

A. $f(x + 4)$

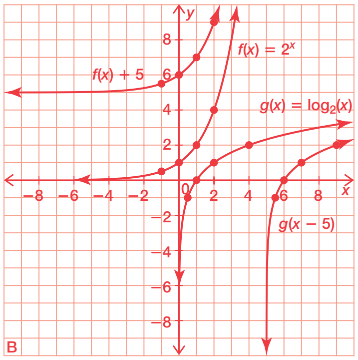
Transformation(s) on $f(x)$: horizontal translation left 4 units



Transformation(s) on $g(x)$: vertical translation down 4 units

B. $f(x) + 5$

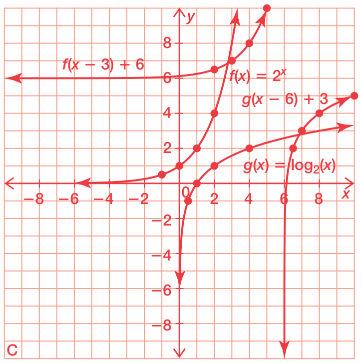
Transformation(s) on $f(x)$: vertical translation up 5 units



Transformation(s) on $g(x)$: horizontal translation right 5 units

C. $f(x - 3) + 6$

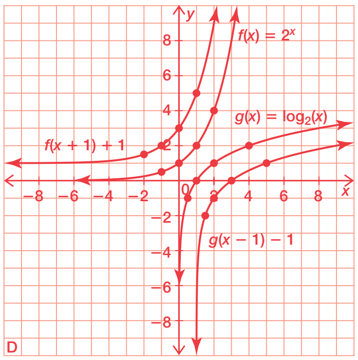
Transformation(s) on $f(x)$: horizontal translation right 3 units and vertical translation up 6 units



Transformation(s) on $g(x)$: horizontal translation left 6 units and vertical translation down 3 units

D. $f(x + 1) + 1$

Transformation(s) on $f(x)$: horizontal translation left 1 unit and vertical translation up 1 unit



Transformation(s) on $g(x)$: horizontal translation right 1 unit and vertical translation down 1 unit

Guiding Questions for Share Phase, Question 2

- Does $f(x + C)$ translate $f(x)$ horizontally left C units or horizontally right C units? Is the inverse associated with a vertical translation up C units or down C units?
- Does $f(x - C)$ translate $f(x)$ horizontally left C units or horizontally right C units? Is the inverse associated with a vertical translation up C units or down C units?
- Does $f(x) + D$ translate $f(x)$ vertically up D units or vertically down D units? Is the inverse associated with a horizontal translation left D units or right D units?
- Does $f(x) - D$ translate $f(x)$ vertically up D units or vertically down D units? Is the inverse associated with a horizontal translation left D units or right D units?

Grouping

Have students complete Question 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

- Is the equation for $g^{-1}(x)$ associated with $f^{-1}(x + 1)$ or $f^{-1}(x) + 1$?
- Is the equation for $g^{-1}(x)$ associated with $f^{-1}(x + 2)$ or $f^{-1}(x) + 2$?

2. Generalize the effects of transformations on $f(x)$ and its inverse function, $f^{-1}(x)$. Complete the table to organize your results.

	Transformation on $f(x)$	**Effect of Transformation on $f^{-1}(x)$**
$f(x + C)$	Translate horizontally left C units	Translate vertically down C units
$f(x - C)$	Translate horizontally right C units	Translate vertically up C units
$f(x) + D$	Translate vertically up D units	Translate horizontally right D units
$f(x) - D$	Translate vertically down D units	Translate horizontally left D units

3. Consider the function $y = f(x)$ and the transformed function $g(x)$. Write an equation for $g^{-1}(x)$ in terms of $f^{-1}(x)$.

a. $y = f(x)$

$g(x) = f(x - 1)$ $g^{-1}(x) =$ $f^{-1}(x) + 1$

b. $y = f(x)$

$g(x) = f(x) - 2$ $g^{-1}(x) =$ $f^{-1}(x + 2)$

c. $y = f(x)$

$g(x) = f(x + 5)$ $g^{-1}(x) =$ $f^{-1}(x) - 5$

d. $y = f(x)$

$g(x) = f(x - 4) + 3$ $g^{-1}(x) =$ $f^{-1}(x - 3) + 4$

- Is the equation for $g^{-1}(x)$ associated with $f^{-1}(x - 5)$ or $f^{-1}(x) - 5$?
- Is the equation for $g^{-1}(x)$ associated with $f^{-1}(x - 3) + 4$ or $f^{-1}(x + 4) - 3$?

Problem 4

Students complete tables of values by writing the transformational function in terms of $f^{-1}(x)$ and identify its effect on the inverse. The affect of vertical and horizontal dilations on the inverse of a function are discussed. Students are given the equation of several functions then write the equation for the inverse function and create the graph.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- How are the reference points on the function $f(x)$ used to determine the coordinates of the corresponding points on the transformed function $f(x)$?
- How are the coordinates of the points on the function $f^{-1}(x)$ used to determine the coordinates of the corresponding points on the transformed function $f^{-1}(x)$?
- Does a vertical dilation on a function produce a horizontal dilation the same number of units on its inverse?
- Does a horizontal dilation on a function produce a vertical dilation the same number of units on its inverse?

PROBLEM 4 Here We Go Again . . .

The function $f(x) = 2^x$ and its inverse function $f^{-1}(x) = \log_2 (x)$ are shown in the table.

$f(x)$	$f^{-1}(x)$
$\left(-1, \frac{1}{2}\right)$	$\left(\frac{1}{2}, -1\right)$
(0, 1)	(1, 0)
(1, 2)	(2, 1)

1. Complete each table. Write the inverse of the transformed function in terms of $f^{-1}(x)$ and identify the effect of the transformation on the inverse.

a.

$3f(x)$	$f^{-1}\left(\frac{x}{3}\right)$
$\left(-1, \frac{3}{2}\right)$	$\left(\frac{3}{2}, -1\right)$
(0, 3)	(3, 0)
(1, 6)	(6, 1)

Transformation on $f(x)$: vertical dilation of 3

Effect on the inverse: horizontal dilation of 3

b.

$\frac{1}{2}f(x)$	$f^{-1}(2x)$
$\left(-1, \frac{1}{4}\right)$	$\left(\frac{1}{4}, -1\right)$
$\left(0, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 0\right)$
(1, 1)	(1, 1)

Transformation on $f(x)$: vertical dilation of $\frac{1}{2}$

Effect on the inverse: horizontal dilation of $\frac{1}{2}$

c.

$f\left(\frac{x}{4}\right)$	$2f^{-1}(x)$
$\left(-4, \frac{1}{2}\right)$	$\left(\frac{1}{2}, -4\right)$
(0, 1)	(1, 0)
(4, 2)	(2, 4)

Transformation on $f(x)$: horizontal dilation of 4

Effect on the inverse: vertical dilation of 4

d.

$f(2x)$	$\frac{1}{2}f^{-1}(x)$
$\left(-\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, -\frac{1}{2}\right)$
(0, 1)	(1, 0)
$\left(\frac{1}{2}, 2\right)$	$\left(2, \frac{1}{2}\right)$

Transformation on $f(x)$: horizontal dilation of $\frac{1}{2}$

Effect on the inverse: vertical dilation of $\frac{1}{2}$

2. How does a vertical dilation on a function affect its inverse?
A vertical dilation on a function will produce a horizontal dilation the same number of units on its inverse.

3. How does a horizontal dilation on a function affect its inverse?
A horizontal dilation on a function will produce a vertical dilation the same number of units on its inverse.

Grouping

Have students complete Question 4 with a partner. Then have students share their responses as a class.

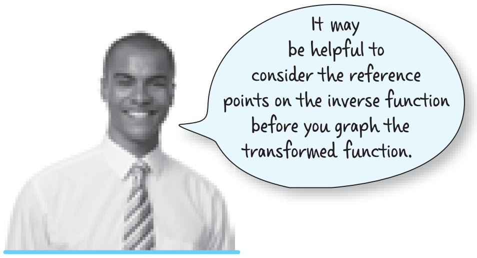
Guiding Questions for Share Phase, Question 4

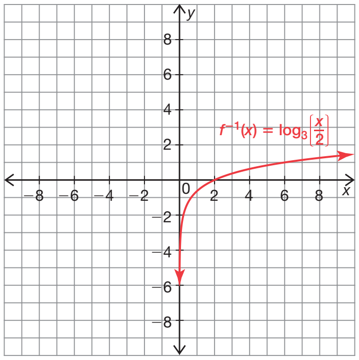
- Which number represents the base of the logarithm in this situation?
- Which number represents the argument in this situation?

4. Given the function $f(x)$, write the equation for the inverse function, $f^{-1}(x)$. Then graph the inverse function.

a. $f(x) = 2 \cdot 3^x$

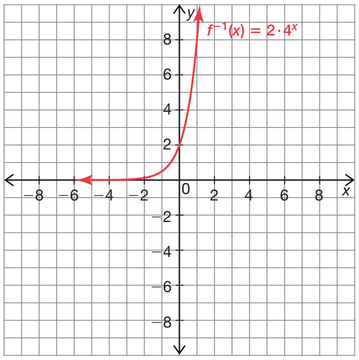
$f^{-1}(x) =$ $\log_3\left(\frac{x}{2}\right)$





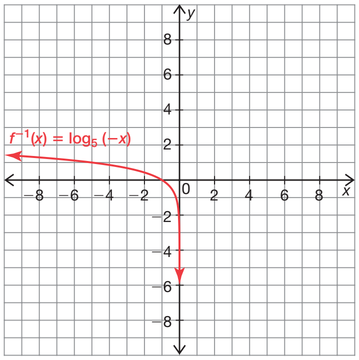
b. $f(x) = \log_4\left(\frac{x}{2}\right)$

$f^{-1}(x) =$ $2 \cdot 4^x$



c. $f(x) = -5^x$

$f^{-1}(x) =$ $\log_5(-x)$



Be prepared to share your solutions and methods.

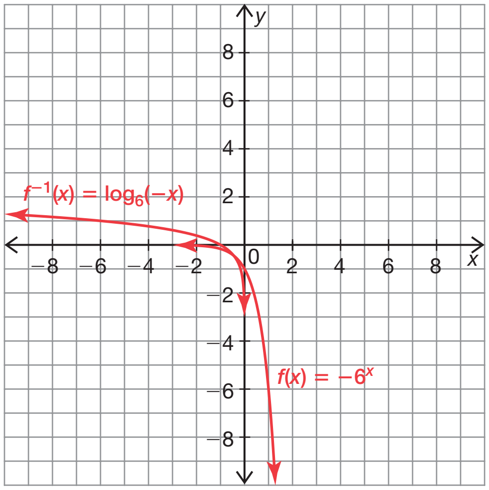
Check for Students' Understanding

Given the function $f(x)$:

- Graph the function.
- Write the equation for the inverse function, $f^{-1}(x)$.
- Graph the inverse function.

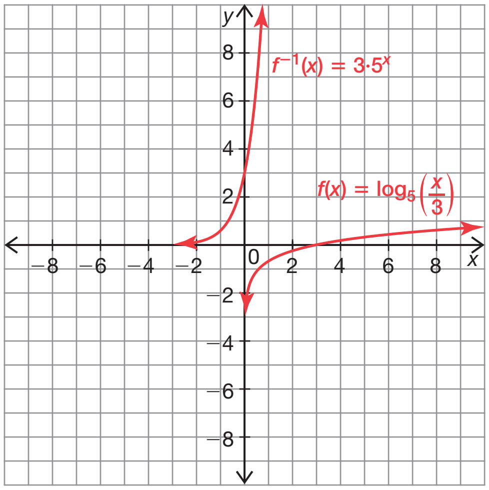
1. $f(x) = -6^x$

$f^{-1}(x) = \log_6(-x)$



2. $f(x) = \log_5\left(\frac{x}{3}\right)$

$f^{-1}(x) = 3 \cdot 5^x$



12

12

Chapter 12 Summary

KEY TERMS

- half-life (12.1)
- natural base *e* (12.2)
- logarithm (12.4)
- logarithmic function (12.4)
- common logarithm (12.4)
- natural logarithm (12.4)

12.1 Constructing an Exponential Function from a Geometric Sequence

The general formula for a geometric sequence is $a_n = a_1 \cdot r^{(n-1)}$. This formula can be written as an exponential function by using properties of exponents and multiplication.

Example

$$a_n = 20 \cdot 5^{(n-1)}$$

$$f(n) = 20 \cdot 5^n \cdot 5^{-1}$$

$$f(n) = 20 \cdot \frac{1}{5} \cdot 5^n$$

$$f(n) = 4 \cdot 5^n$$

12

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12.1 Using an Exponential Function to Solve Half-Life Problems

Half-life refers to the amount of time it takes a substance to decay to half of its original amount. An exponential function can be used to solve problems about half-life.

Example

An exponential function $A(t)$ represents the amount of a drug in a person's system, where t represents elapsed time. If the half-life of the drug occurs in multiple-hour cycles—for example, every 3 hours—divide the exponent t by that amount: $\frac{t}{3}$.

Elapsed Time (Hours)	0	2	4	6	8	20
Drug in System (mg)	160	80	40	20	10	?
Number of Half-Life Cycles	0	1	2	3	4	10

$$A(t) = 160\left(\frac{1}{2}\right)^{\frac{t}{2}}$$

$$A(20) = 160\left(\frac{1}{2}\right)^{\frac{20}{2}}$$

$$A(20) = 160\left(\frac{1}{2}\right)^{10}$$

$$A(20) \approx 160(0.00098)$$

$$A(20) \approx 0.15625$$

After 20 hours, there will be about 0.15625 mg of the drug remaining in the person's system.

12.2 Investigating Exponential Growth and Decay

For exponential growth functions, b is a value greater than 1. For exponential decay functions, b is a fraction or decimal between 0 and 1.

Example

$f(x) = 15^x$	$f(x) = \left(\frac{2}{3}\right)^x$
growth	decay

12.2 Investigating Graphs of Exponential Functions

Every basic exponential function has the point (0, 1) in common. The x-value represents the exponent, and any base raised to the power of 0 will equal 1. Basic functions of exponential growth and decay can be identified by their domain, range, asymptotes, and end behavior as described in the table below.

	Basic Exponential Growth	**Basic Exponential Decay**
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$(0, \infty)$	$(0, \infty)$
Asymptote	$y = 0$	$y = 0$
Intercepts	(0, 1)	(0, 1)
End Behavior	As $x \to -\infty$, $f(x) \to 0$ As $x \to \infty$, $f(x) \to \infty$	As $x \to -\infty$, $f(x) \to \infty$ As $x \to \infty$, $f(x) \to 0$
Intervals of Increase or Decrease	Increasing over $(-\infty, \infty)$	Decreasing over $(-\infty, \infty)$

Examples

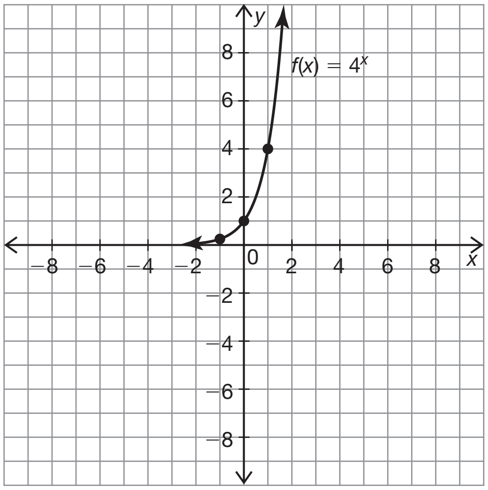
$f(x) = 4^x$

x	$f(x)$
-1	$\frac{1}{4}$
0	1
1	4

Increasing over $(-\infty, \infty)$.

End behavior: As $x \to -\infty$, $f(x) \to 0$

As $x \to \infty$, $f(x) \to \infty$



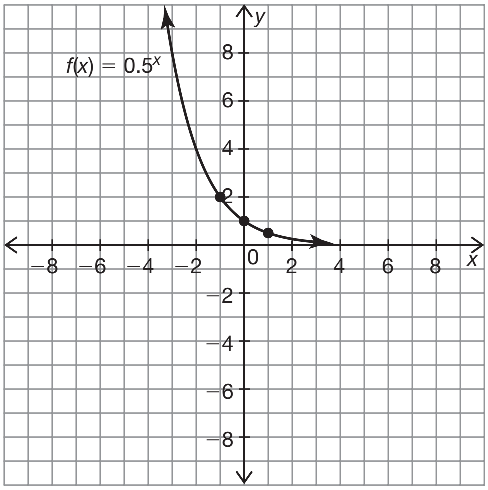
$f(x) = 0.5^x$

x	$f(x)$
-1	2
0	1
1	0.5

Decreasing over $(-\infty, \infty)$.

End behavior: As $x \to -\infty$, $f(x) \to \infty$

As $x \to \infty$, $f(x) \to 0$



12.2 Using Exponential Equations to Solve Compound Interest Problems

The formula for compound interest is $A = P(1 + r)^t$, where A is the amount earned, P is the original amount, or principal, r is the rate, and t is the time in years. If interest is compounded more than once per year, then the formula is: $A(t) = \left(1 + \frac{1}{n}\right)^{n \cdot t}$.

Example

Sarah invests $500 in the bank. Her bank compounds interest 4 times a year at a rate of 4%. How much money will she have in her account after 10 years?

$$\begin{aligned} A(10) &= 500\left(1 + \frac{0.04}{4}\right)^{4 \cdot 10} \\ &= 500(1.01)^{40} \\ &\approx 500(1.4889) \\ &\approx 744.43 \end{aligned}$$

Sarah will have $744.43 in her account after 10 years.

12.2 Using the Natural Base, *e*

The constant e represents continuous growth and is often referred to as the natural base e. The symbol e is used to represent the constant 2.718281. . . and so on. The natural base e is used in the formula for population growth: $N(t) = N_0e^{rt}$.

Example

Miami's population in 2005 was 216,500 people and is growing at a rate of about 3%. According to the population growth model, what would be the approximate population of Miami in 2020?

$$\begin{aligned} N(15) &= 216{,}500e^{(0.03 \cdot 15)} \\ &= 216{,}500e^{0.45} \\ &\approx 339{,}540 \end{aligned}$$

Miami's population in 2020 would be approximately 339,540 people.

12.3 Dilating, Reflecting, and Translating Exponential Functions Using Reference Points

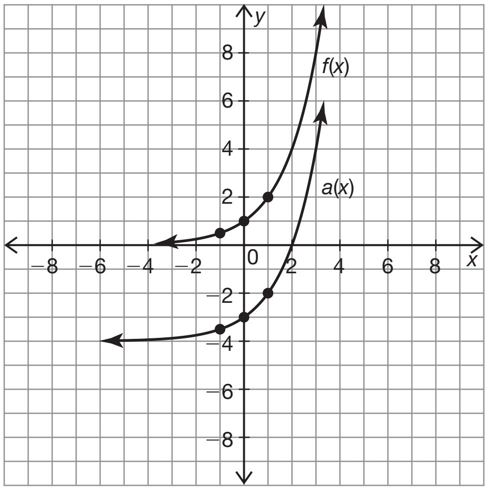
Consider the functions $y = f(x)$ and $g(x) = Af(B(x - C)) + D$. Recall that the D-value translates $f(x)$ vertically, the C-value translates $f(x)$ horizontally, the A-value vertically stretches or compresses $f(x)$, and the B-value horizontally stretches or compresses $f(x)$. Exponential functions are transformed in the same manner.

Example

$f(x) = 2^x$

$a(x) = f(x) - 4$

Reference Points on $f(x)$	Corresponding Points on $a(x)$
$\left(-1, \frac{1}{2}\right)$	$\left(-1, -\frac{7}{2}\right)$
(0, 1)	(0, −3)
(1, 2)	(1, −2)



Domain: All real numbers

Range: $y > -4$

Horizontal asymptote: $y = -4$

12.3 Describing Transformations Performed on Exponential Equations

Using the functions $y = f(x)$ and $g(x) = Af(B(x - C)) + D$, you can describe the transformations to the graph of an exponential function.

Example

$m(x) = 4^x$

$t(x) = -m(x) + 3$

The graph of the function $m(x)$ is translated vertically up 3 units and is reflected across the x-axis to produce $t(x)$.

$t(x) = -4^x + 3$

12.4

Writing Exponential Equations as Logarithmic Equations

The inverse of an exponential equation can be written as a logarithmic equation. The logarithm of a number for a given base is the exponent to which the base must be raised in order to produce the number. If $y = b^x$, then the logarithm is x, and can be written as $x = \log_b y$. The value of the base of a logarithm is the same as the base in the exponential expression b^x.

Examples

$4^3 = 64$	$\log_5\left(\frac{1}{625}\right) = -4$
$\log_4(64) = 3$	$5^{-4} = \frac{1}{625}$

12.4

Graphing the Inverse of an Exponential Function

A logarithmic function is the inverse of an exponential function. It is a function involving a logarithm. Many real-world situations can be modeled using logarithmic functions. Two frequently used logarithms are logarithms with base 10 and base *e*. A logarithm with base 10 is called the common logarithm and is usually written *log* without a base specified. A logarithm with base *e* is called the natural logarithm and is usually written as *ln*.

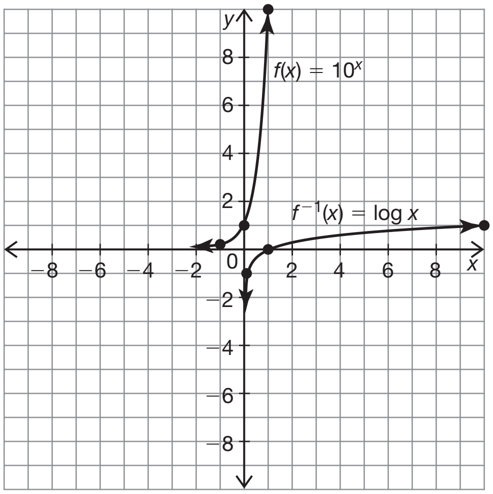
Example

$f(x) = 10^x$

$f^{-1}(x) = \log x$

x	$f(x)$
−1	0.1
0	1
1	10

x	$f^{-1}(x)$
0.1	−1
1	0
10	1



12.5 Describing Transformations Performed on Logarithmic Functions

Using the functions $y = f(x)$ and $g(x) = Af(B(x - C)) + D$, you can describe the transformations to a logarithmic function.

Example

$f(x) = \log_4(x)$

$g(x) = -f(x + 3)$

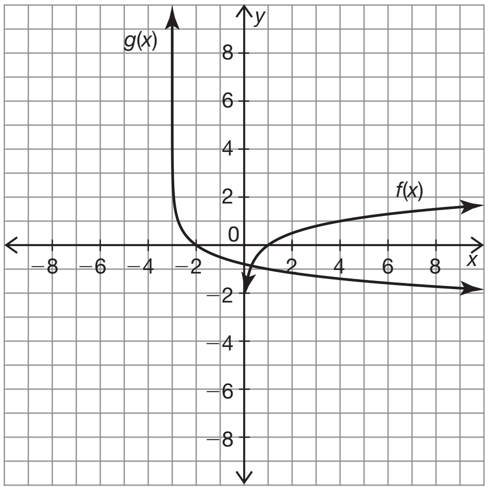
$g(x) = -\log_4(x + 3)$

horizontally translated left 3 units and reflected across the x-axis

Domain of $g(x)$: $(-3, \infty)$

Range of $g(x)$: $(-\infty, \infty)$

Asymptote of $g(x)$: $x = -3$



12.5 Describing the Effects of Transformations on Inverse Functions

Transformations performed on a function and its inverse will have inverse effects.

	Transformation on $f(x)$	**Effect of Transformation on $f^{-1}(x)$**
$f(x + C)$	Translate horizontally left C units	Translate vertically down C units
$f(x - C)$	Translate horizontally right C units	Translate vertically up C units
$f(x) + D$	Translate vertically up D units	Translate horizontally right D units
$f(x) - D$	Translate vertically down D units	Translate horizontally left D units
$Af(x)$	Vertical dilation of 3	Horizontal dilation of 3
$f(Bx)$	Horizontal dilation of $\frac{1}{B}$	Vertical dilation of $\frac{1}{B}$

Example

Consider the transformation on the function $f(x) = 2^x$ and its inverse function $f^{-1}(x) = \log_2(x)$.

$m(x) = \frac{1}{3}f(x)$	$m^{-1}(x) = f^{-1}(3x)$
$\left(-1, \frac{1}{6}\right)$	$\left(\frac{1}{6}, -1\right)$
$\left(0, \frac{1}{3}\right)$	$\left(\frac{1}{3}, 0\right)$
$\left(1, \frac{2}{3}\right)$	$\left(\frac{2}{3}, 1\right)$

Transformation on $f(x)$: vertical dilation of $\frac{1}{3}$

Effect on the inverse: horizontal dilation of $\frac{1}{3}$