Solving Rational Equations

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Chapter 10 Overview

This chapter provides opportunities for students to connect their knowledge of operations with rational numbers to operations with rational expressions. Lessons provide opportunities for students to analyze and compare the process to add, subtract, multiply, and divide rational numbers to the same operations with rational expressions. Students conclude rational expressions are similar to rational numbers and are closed under all the operations. In the later part of the chapter, lessons provide opportunities for students to write and solve rational equations and list restrictions. Student work is presented throughout the chapter to demonstrate efficient ways to operate with rational expressions and efficient ways to solve rational equations based on the structure of the original equation.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
10.1	Adding and Subtracting Rational Expressions	A.SSE.2 A.APR.6 A.APR.7(+)	1	This lesson provides opportunities for students to connect the process for adding and subtracting rational numbers to the process for adding and subtracting rational expressions. Questions first ask students to analyze and compare examples for adding and subtracting rational numbers to performing similar operations involving rational expressions with variables in the numerator. Students conclude that rational expressions are closed for addition and subtraction. Questions then ask students to add and subtract rational expressions with variables in the denominator and identify domain restrictions.		x	x		

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Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
10.2	Multiplying and Dividing Rational Expressions	A.SSE.2 A.APR.6 A.APR.7(+)	1	This lesson provides opportunities for students to connect the process for multiplying and dividing rational numbers to the process for multiplying and dividing rational expressions. Questions ask students to analyze and compare examples for multiplying and dividing rational numbers to performing similar operations involving rational expressions. Students conclude that rational expressions are closed for multiplication and division. Student work is presented to demonstrate efficient ways to operate with rational expressions. Students will multiply and divide rational expressions and identify domain restrictions.		x	x		
10.3	Solving Rational Equations	A.SSE.2 A.REI.2 A.REI.11	2	This lesson provides opportunities for students to solve rational equations. Questions ask students to analyze and compare various methods to solve rational equations and to consider restrictions and extraneous roots. In the last activity, students are presented with twelve rational equations and before solving each, they are asked to sort the equations by considering how the structure of the equation can inform a solution method.	x		x		x
10.4	Using Rational Equations to Solve Real-World Problems	A.CED.1 A.REI.2	2	This lesson provides opportunities for students to use rational equations to solve real-world problems.	x				

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Skills Practice Correlation for Chapter 10

Lesson		Problem Set	Objectives
10.1	Adding and Subtracting Rational Expressions	1 - 8	Calculate the least common denominator for a rational expression and list any domain restrictions
		9 - 16	Determine the sum or difference for rational expressions
		17 - 26	Determine the sum or difference for rational expressions and describe any variable restrictions
10.2	Multiplying and Dividing Rational Expressions	1 - 6	Multiply and divide rational expressions
		7 - 16	Multiply rational expressions and describe any variable restrictions
		17 - 26	Determine quotients and describe any variable restrictions
10.3	Solving Rational Equations		Vocabulary
		1 - 6	Solve rational equations using proportional reasoning and describe any restrictions for the value of x
		7 - 12	Solve rational equations using the least common denominator and describe any restrictions for the value of x or extraneous roots
		13 - 18	Solve rational equations using a graphing calculator and describe any restrictions for the value of x
		19 - 24	Solve rational equations without using a graphing calculator and describe any restrictions for the value of x
10.4	Using Rational Equations to Solve Real-World Problems	1 - 6	Write and solve a rational equation to model each work scenario
		7 - 12	Write and solve a rational equation to model each mixture scenario
		13 - 18	Write and solve a rational equation to model each distance scenario
		19 - 24	Write and solve a rational equation or inequality to model each cost scenario

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There Must Be a Rational Explanation

Adding and Subtracting Rational Expressions

10.1

LEARNING GOALS

In this lesson, you will:

- Add and subtract rational expressions.
- Factor to determine a least common denominator.

ESSENTIAL IDEAS

- The process for adding and subtracting rational expressions is similar to the process for adding and subtracting rational numbers.
- A common denominator must be determined before adding or subtracting rational expressions involving variables.
- When rational expressions contain variables in the denominator, the least common denominator must be determined before adding or subtracting.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions

2. Use the structure of an expression to identify ways to rewrite it.

A-APR Arithmetic with Polynomials and Rational Expressions

Rewrite rational expressions

6. Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form $q(x) + \frac{r(x)}{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

7. Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Overview

The process for adding and subtracting rational expressions is compared to the process for adding and subtracting rational numbers. Students will add and subtract several rational expressions by first determining the common denominator or the least common denominator followed by combining like terms. The set of rational expressions is closed under addition and subtraction. Students then practice adding and subtracting rational expressions with variables and binomials in the denominators. They will use the original form of the rational expression to identify restrictions on the domain of the function. Worked examples and student work is analyzed and different solution methods are shown.

Warm Up

A local artist sells handmade clocks. The average cost to make x clocks is modeled by the equation $C(x) = \frac{1500 + 40x}{x}$. How many clocks should the artist make each week in order to achieve an average cost of $100 per clock?

$$\frac{1500 + 40x}{x} = 100$$

$$100x = 1500 + 40x$$

$$60x = 1500$$

$$x = 25$$

The artist should make 25 clocks per week to achieve an average cost of $100 per clock.

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10.1

There Must Be a Rational Explanation

Adding and Subtracting Rational Expressions

LEARNING GOALS

In this lesson, you will:

- Add and subtract rational expressions.
- Factor to determine a least common denominator.

At some point in most people's lives, the task of putting together a jigsaw puzzle is just one of the things everyone seems to do. In a jigsaw puzzle, small, interlocking pieces with colors or designs fit together to make a larger picture. The pieces can only fit together one way, so various strategies are employed to determine which pieces fit together. Younger children are often encouraged to work on jigsaw puzzles as a way of developing problem-solving skills. Jigsaw puzzles are popular among adults, too, and some of them can get quite complicated. Sometimes, complex designs, a large number of pieces, or even 3-dimensional platforms can make puzzles very challenging. A lot of time, skill, and patience are required to put these puzzles together. One particular puzzle has 24,000 pieces! That certainly isn't child's play!

People who are serious about puzzles can qualify to be on a national puzzling team and compete in international competitions. The most prestigious competition is held every year in Belgium. What types of puzzles or board games do you like? Do you know anybody who is really good at putting puzzles together?

Problem 1

A table shows a comparison between operations performed with rational numbers and operations performed with rational expressions involving variables in the numerator. Students will answer questions related to the examples, and then calculate the sum or difference of several rational expressions by determining a common denominator when necessary. They then explain why the set of rational expressions is closed under addition and subtraction.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- How is the sum of the terms determined if the rational expressions have like denominators?
- How is the difference of the terms determined if the rational expressions have like denominators?
- Why must a common denominator be determined first before adding or subtracting rational expressions?

PROBLEM 1 Oh Snap . . . Look at the Denominator on that Rational

Previously, you learned that dividing polynomials was just like dividing integers. Well, performing operations on rational expressions involving variables is just like performing operations on rational numbers.

	Rational Numbers	Rational Expressions Involving Variables in the Numerator
Example 1	$\frac{1}{6}+\frac{5}{6}-\frac{1}{6}=\frac{5}{6}$	$\frac{1x}{6}+\frac{5x}{6}-\frac{1x}{6}=\frac{5x}{6}$
Example 2	$\frac{3}{2}+\frac{2}{5}-\frac{3}{4}=\frac{3(10)}{2(10)}+\frac{2(4)}{5(4)}-\frac{3(5)}{4(5)}$ $=\frac{30}{20}+\frac{8}{20}-\frac{15}{20}$ $=\frac{23}{20}$	$\frac{3x}{2}+\frac{2y}{5}-\frac{3x}{4}=\frac{3(10)x}{2(10)}+\frac{2(4)y}{5(4)}-\frac{3(5)x}{4(5)}$ $=\frac{30x}{20}+\frac{8y}{20}-\frac{15x}{20}$ $=\frac{15x+8y}{20}$
Example 3	$\frac{3}{5}+\frac{2}{3}-\frac{2}{15}=\frac{3(3)}{5(3)}+\frac{2(5)}{3(5)}-\frac{2}{15}$ $=\frac{9}{15}+\frac{10}{15}-\frac{2}{15}$ $=\frac{17}{15}$	$\frac{3x}{5}+\frac{2y}{3}-\frac{2}{15}-\frac{2x+3y}{5}=\frac{3(3)x}{5(3)}+\frac{2(5)y}{3(5)}-\frac{2}{15}-\frac{(2x+3y)(3)}{5(3)}$ $=\frac{9x}{15}+\frac{10y}{15}-\frac{2}{15}-\frac{6x+9y}{15}$ $=\frac{9x+10y-2-(6x+9y)}{15}$ $=\frac{9x+10y-2-6x-9y}{15}$ $=\frac{3x+y-2}{15}$

1. Analyze the examples.

   a. Explain the process used to add and subtract each expression.

   Example 1 has like denominators, so, you add the numerators and keep the denominator. In Examples 2 and 3, a common denominator must be determined before adding or subtracting.

   b. In Example 2, why is $\frac{3}{2}=\frac{3(10)}{2(10)}$ and why is $\frac{3x}{2}=\frac{3(10)x}{2(10)}$?

   They are equivalent fractions because they are being multiplied by a form of 1, in this case $\frac{10}{10}$. The other terms are being multiplied by different forms of 1, $\frac{2}{5}=\frac{2(4)}{5(4)}$, $\frac{3}{4}=\frac{3(5)}{4(5)}$, $\frac{2y}{5}=\frac{2(4)y}{5(4)}$, $\frac{3x}{4}=\frac{3(5)x}{4(5)}$.

   c. In Example 3, explain how $-\frac{(2x+3y)(3)}{5(3)}=\frac{-6x-9y}{15}$.

   In the numerator, distribute the 3. Then distribute the negative sign.

- When fractions are multiplied by a form of 1, do they remain equivalent?
- What form of 1 was used in each of the examples when the denominators were different?
- How was a common denominator determined in each situation?
- Does the denominator need to be the least common denominator?
- Is it easier to use the least common denominator? Why or why not?

Guiding Question for Share Phase, Question 2

What is the least common denominator in Noelle's situation?

Grouping

Have students complete Questions 3 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 through 5

- What is the least common denominator in each situation?
- What form of 1 was used to change the denominator in each term into a common denominator?
- What is the definition of a rational function?
- Can any expressions that resulted from addition or subtraction be written in the form of a rational function?
- Will the process to determine a common denominator be the same if there were variables in the denominator of the rational expressions?
- Do variables in the denominator always result in restrictions to the domain?

2. Analyze Noelle's work.

Noelle

$$\frac{3x}{3} + \frac{2x}{8} - \frac{1}{2}$$

To determine a common denominator, I multiply all the denominators together: $3 \cdot 8 \cdot 2 = 48$

$$\frac{3x(16)}{3(16)} + \frac{2x(6)}{8(6)} - \frac{1(24)}{2(24)} = \frac{48x}{48} + \frac{12x}{48} - \frac{24}{48}$$
$$= \frac{60x - 24}{48}$$
$$= \frac{5x - 2}{4}$$

Explain how Noelle could have added the rational expressions more efficiently.

Noelle could have rewritten $\frac{3x}{3} + \frac{2x}{8} - \frac{1}{2}$ as $x + \frac{x}{4} - \frac{1}{2}$ and used the least common denominator of 4.

3. Calculate each sum and difference.

a. $\frac{3}{6} + \frac{5x}{4} - \frac{y}{8}$

$$= \frac{1(4)}{2(4)} + \frac{5x(2)}{4(2)} - \frac{y}{8}$$
$$= \frac{4}{8} + \frac{10x}{8} - \frac{y}{8}$$
$$= \frac{10x - y + 4}{8}$$

b. $\frac{x - 2y}{3} + \frac{x}{12} - \frac{z}{4}$

$$= \frac{(x - 2y)4}{3(4)} + \frac{x}{12} - \frac{z(3)}{4(3)}$$
$$= \frac{4x - 8y}{12} + \frac{x}{12} - \frac{3z}{12}$$
$$= \frac{5x - 8y - 3z}{12}$$

c. $\frac{4x}{6} - \frac{2x}{9} - \frac{x}{18}$

$$= \frac{4x(3)}{6(3)} - \frac{2x(2)}{9(2)} - \frac{x}{18}$$
$$= \frac{12x}{18} - \frac{4x}{18} - \frac{x}{18}$$
$$= \frac{7x}{18}$$

4. Is the set of rational expressions closed under addition and subtraction? Explain your reasoning.

Yes. The set of rational expressions is closed under addition and subtraction. The definition of a rational function states that $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions, and $Q(x) \neq 0$. When you add or subtract two or more rational expressions, the sum or difference can be written in this form, as well.

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Problem 2

Two methods are used to add rational expressions that contain variables in the denominator. One method multiplies the denominators together to determine a common denominator and the second method uses the least common denominator to add the rational expressions. Students will calculate the least common denominators for different rational expressions. Student work shows different methods to add rational expressions and the restrictions on the domain are only evident in the original form of the rational expressions. Students then calculate the sum and differences of different rational expressions and list the restriction for the variable.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

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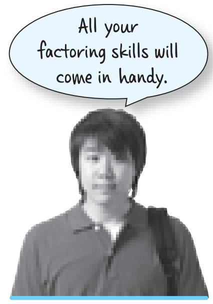
5. Notice that all the variables in the right column of the table are in the numerator. If there were variables in the denominator, do you think the process of adding and subtracting the expressions would change? Explain your reasoning

The process to determine the common denominator would stay the same; however, when there are variables in the denominator, I will need to take into account restrictions. The variable cannot equal a value that will produce a zero in the denominator.

PROBLEM 2 Umm, I Think There Are Some Variables in Your Denominator . . .



When rational expressions contain variables in the denominator, the process remains the same—you still need to determine the least common denominator (LCD) before adding and subtracting.

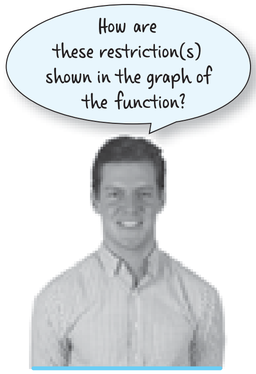
It will save time and effort if you determine the LCD and use it to add and subtract rational expressions.

1. Consider Method A compared to Method B in both columns of the table.

	Rational Numbers	Rational Expressions Involving Variables in the Denominator
Method A	$\frac{1}{3} + \frac{1}{3^2} = \frac{1(3^2)}{3(3^2)} + \frac{1(3)}{3^2(3)}$ $= \frac{3^2}{3^3} + \frac{3}{3^3}$ $= \frac{3^2 + 3}{3^3}$ $= \frac{3(3 + 1)}{3(3^2)}$ $= \frac{3 + 1}{3^2}$	$\frac{1}{x} + \frac{1}{x^2} = \frac{1(x^2)}{x(x^2)} + \frac{1(x)}{x^2(x)}$ $= \frac{x^2}{x^3} + \frac{x}{x^3}$ $= \frac{x^2 + x}{x^3}$ $= \frac{x(x + 1)}{x(x^2)}$ $= \frac{x + 1}{x^2}$ $\frac{1}{x} + \frac{1}{x^2} = \frac{x + 1}{x^2}$ for $x \neq 0$
Method B	$\frac{1}{3} + \frac{1}{3^2} = \frac{1(3)}{3(3)} + \frac{1}{3^2} = \frac{4}{3^2}$	$\frac{1}{x} + \frac{1}{x^2} = \frac{1(x)}{x(x)} + \frac{1}{x^2}$ $= \frac{x + 1}{x^2}$ $\frac{1}{x} + \frac{1}{x^2} = \frac{x + 1}{x^2}$ for $x \neq 0$

Guiding Questions for Share Phase, Questions 1 and 2

- Which method multiplies the denominators together to determine a common denominator?
- Which method uses the least common denominator to add the rational expressions?
- When x has a value of 0, does the rational expression become undefined?
- Did Ruth determine a common denominator or the least common denominator? How do you know?
- Did Samir determine a common denominator or the least common denominator? How do you know?
- How can you identify the restrictions for the value of x?



a. Explain the difference in the methods.

Method A uses the product of the denominators as a common denominator. Method B uses the least common denominator to add the rational expressions.

b. Explain why the statement $\frac{1}{x} + \frac{1}{x^2} = \frac{x + 1}{x^2}$ has the restriction $x \neq 0$.

The restriction for the value of $x \neq 0$ because if $x = 0$, the answer would be undefined in both the original and final forms.

2. Ruth and Samir determine the LCD for the expression: $\frac{1}{x^2 - 1} + \frac{1}{x + 1}$.

Ruth

$\frac{1}{x^2 - 1} + \frac{1}{x + 1}$

$(x^2 - 1)(x + 1)$

LCD: $x^3 + x^2 - x - 1$

Samir

$\frac{1}{x^2 - 1} + \frac{1}{x + 1}$

$\frac{1}{(x - 1)(x + 1)} + \frac{1}{x + 1}$

$(x - 1)(x + 1)$

LCD: $x^2 - 1$

a. Who is correct? Explain your reasoning.

Samir is correct. He factored the denominator on the left and realized that $x + 1$ was already a factor of the denominator, so he only had to multiply $x - 1$ by the denominator on the right.

b. Describe any restriction(s) for the value of x.

$x \neq -1, 1$

Grouping

- Have students complete Question 3 with a partner. Then have students share their responses as a class.
- Then ask a student to read the information. Discuss as a class.

Guiding Questions for Share Phase, Question 3

- Is the least common denominator determined by multiplying the two denominators or is the result just a common denominator?
- Can something in the denominator be factored easily?
- How is factoring helpful when determining the least common denominator?
- Why is it helpful to look at the other denominators in the problem when factoring a binomial or cubic denominator?

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3. Calculate the least common denominator for each pair of rational expressions.

a. $\frac{3}{x+4}, \frac{7x}{x-4}$

LCD: $x^2 - 16;\ x \neq \pm 4$

b. $\frac{-2}{3x-2}, \frac{4x}{3x^2+7x-6}$

LCD: $(3x-2)(x+3);\ x \neq -3, \frac{2}{3}$

c. $\frac{-11}{x}, \frac{7}{x-4}, \frac{x}{x^2-16}$

LCD: $x(x-4)(x+4);\ x \neq \pm 4, 0$

d. $\frac{2x}{x^2-5x+6}, \frac{7x+11}{x^2-6x+9}$

LCD: $(x-2)(x-3)^2;\ x \neq 2, 3$

Notice that even though there are binomials in the denominator, adding two rational expressions is similar to adding two rational numbers.

Rational Expressions Involving Binomials in the Denominator
$\frac{1}{x^2-1} - \frac{1}{x+1} = \frac{1}{(x+1)(x-1)} - \frac{1}{x+1}$
$= \frac{1}{(x+1)(x-1)} - \frac{1(x-1)}{(x+1)(x-1)}$
$= \frac{1}{(x+1)(x-1)} - \frac{x-1}{(x+1)(x-1)}$
$= \frac{1-x+1}{(x+1)(x-1)}$
$= \frac{-x+2}{(x+1)(x-1)}$

Grouping

Have students complete Questions 4 and 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 and 5

- Who determined a common denominator by multiplying the two denominators together?
- Did Marissa use the least common denominator?
- Who multiplied the first expression by $\frac{x}{x}$?
- Who factored $(x - 1)$ out of the problem at the end?
- Who did not have to divide out factors at the end of the problem?
- Why is it important to look at the original form of the rational function when determining any restrictions?

4. Marissa and Salvatore add $\frac{2x + 2}{x + 1} + \frac{1}{x}$.

Marissa

$$\frac{2x + 2}{x + 1} + \frac{1}{x} = \frac{(2x + 2)(x)}{(x + 1)(x)} + \frac{1(x + 1)}{x(x + 1)}$$
$$= \frac{2x^2 + 2x}{(x + 1)(x)} + \frac{x + 1}{x(x + 1)}$$
$$= \frac{2x^2 + 2x + x + 1}{x(x + 1)}$$
$$= \frac{2x^2 + 3x + 1}{x(x + 1)}$$
$$= \frac{(2x + 1)\cancel{(x + 1)}}{x\cancel{(x + 1)}}$$
$$= \frac{2x + 1}{x}$$

Salvatore

$$\frac{2x + 2}{x + 1} + \frac{1}{x} = \frac{2\cancel{(x + 1)}}{\cancel{(x + 1)}} + \frac{1}{x}$$
$$= 2 + \frac{1}{x}$$
$$= \frac{2(x)}{(x)} + \frac{1}{x}$$
$$= \frac{2x + 1}{x}$$

Explain the difference in the methods used.

Marissa determined a common denominator by multiplying the two denominators together. She then had to factor $x + 1$ out of the sum at the end.

Salvatore factored initially and only had to multiply the first expression by $\frac{x}{x}$. He did not have to reduce at the end.

5. Randy says the only restriction on the variable x in Marissa and Salvatore's problem is $x \neq 0$. Cynthia says $x \neq 0, -1$. Who is correct? Explain your reasoning.

Cynthia is correct. When looking at the original form of the problem, $x \neq 0, -1$.

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Grouping

Have students complete Question 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- What is the least common denominator in each situation?
- Can the least common denominator be determined by first factoring or simply multiplying the denominators together?
- How many restrictions are in the domain?
- What form of 1 was used to determine the least common denominator in this problem?
- Are the restrictions of the variable in the denominator easily identified in the sum or difference, or do you need to consider the rational expression in its original form?

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6. Calculate each sum or difference. Make sure to list the restrictions for the variable, and simplify when possible.

a. $\frac{5x - 6}{x^2 - 9} - \frac{4}{x - 3}$

$= \frac{5x - 6}{(x - 3)(x + 3)} - \frac{4(x + 3)}{(x - 3)(x + 3)}$

$= \frac{5x - 6 - 4x - 12}{(x - 3)(x + 3)}$

$= \frac{x - 18}{(x - 3)(x + 3)}; \quad x \neq \pm 3$

b. $\frac{x - 7}{x^2 - 3x + 2} + \frac{4}{x^2 - 7x + 10}$

$= \frac{x - 7}{(x - 2)(x - 1)} + \frac{4}{(x - 2)(x - 5)}$

$= \frac{(x - 7)(x - 5)}{(x - 2)(x - 1)(x - 5)} + \frac{4(x - 1)}{(x - 2)(x - 1)(x - 5)}$

$= \frac{x^2 - 12x + 35 + 4x - 4}{(x - 2)(x - 1)(x - 5)}$

$= \frac{x^2 - 8x + 31}{(x - 2)(x - 1)(x - 5)}; \quad x \neq 1, 2, 5$

c. $\frac{2x - 5}{x} - \frac{4}{5x} - 4$

$= \frac{5(2x - 5)}{5x} - \frac{4}{5x} - \frac{4(5x)}{5x}$

$= \frac{10 - 25 - 4 - 20x}{5x}$

$= \frac{-20x - 19}{5x}; \quad x \neq 0$

d. $\frac{3x - 5}{4x^2 + 12x + 9} + \frac{4}{2x + 3} - \frac{2x}{3}$

$= \frac{3(3x - 5)}{3(2x + 3)^2} + \frac{4(3)(2x + 3)}{3(2x + 3)^2} - \frac{2x(2x + 3)^2}{3(2x + 3)^2}$

$= \frac{9x - 15 + 24x + 36 - 8x^3 - 24x^2 - 18x}{3(2x + 3)^2}$

$= \frac{-8x^3 - 24x^2 + 15x + 21}{3(2x + 3)^2}; \quad x \neq -\frac{3}{2}$

Problem 3

Two worked examples to determine the difference between two rational expressions are given. The two denominators in each example are conjugates of each other. Students will compare the examples noting any patterns in the given expression and the resulting differences. The numerator of the final answer is the sum of both numerators.

Grouping

Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- Are the numerators the same in each example?
- What is the relationship between the denominators in each example?
- What operation is used in both examples?
- In each example, how is the numerator of the final answer related to the sum of both the numerators?
- When determining the difference between $\frac{4}{(x-2)}$ and $\frac{4}{(x+2)}$, why would the numerator be the sum of both the numerators times 2?
- Would the denominators stay the same if both terms were added instead of subtracted?

PROBLEM 3 Can You Spot the Difference?

Consider the worked examples.

Determine the difference.

Example 1

$$\frac{3}{x-1} - \frac{3}{x+1}$$
$$= \frac{3(x+1)}{(x-1)(x+1)} - \frac{3(x-1)}{(x+1)(x-1)}$$
$$= \frac{3x+3-3x+3}{(x-1)(x+1)}$$
$$= \frac{6}{(x-1)(x+1)}$$

Example 2

$$\frac{1}{x-1} - \frac{1}{x+1}$$
$$= \frac{1(x+1)}{(x-1)(x+1)} - \frac{1(x-1)}{(x+1)(x-1)}$$
$$= \frac{x+1-x+1}{(x-1)(x+1)}$$
$$= \frac{2}{(x-1)(x+1)}$$

1. Describe the similarities and the differences in the structure of each example.
   The numerators are the same in both examples. Also, the denominators are conjugates. In both examples, an expression is being subtracted from another expression.

2. Consider the given expression and the resulting difference. What pattern do you notice?
   The numerator of the final answer is the sum of both the numerators.

3. If the numerators in Example 1 of the worked example were doubled, what would be the new answer?
   The answer would be $\frac{12}{(x-1)(x+1)}$.

4. Can you use this pattern to determine $\frac{4}{(x-2)} - \frac{4}{(x+2)}$? Explain your reasoning.
   The pattern would not work exactly. The pattern in the denominator would stay the same but the pattern in the numerator would be the sum of both the numerators times 2.

5. How would the pattern in the worked examples change if you added the terms together?
   If I add the terms together, the denominator stays the same. In the numerator, I have the sum of the numerators multiplied by the *x*-variable.

Be prepared to share your solutions and methods.

- How would the numerators change if both terms were added instead of subtracted?

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Check for Students' Understanding

Calculate the sum and list any restriction(s).

$$\frac{x+2}{x-4} + \frac{2}{x} + \frac{5}{3x-1}$$

$$\frac{x+2}{x-4} + \frac{2}{x} + \frac{5}{3x-1} = \frac{x(3x-1)(x+2)}{x(x-4)(3x-1)} + \frac{2(x-4)(3x-1)}{x(x-4)(3x-1)} + \frac{5x(x-4)}{x(x-4)(3x-1)}$$

$$= \frac{x(3x-1)(x+2) + 2(x-4)(3x-1) + 5x(x-4)}{x(x-4)(3x-1)}$$

$$= \frac{(3x^3 + 5x^2 - 2x) + (6x^2 - 26x + 8) + (5x^2 - 20x)}{x(x-4)(3x-1)}$$

$$= \frac{3x^3 + 16x^2 - 48x + 8}{x(x-4)(3x-1)}$$

$$x \neq 0, 4, \frac{1}{3}$$

10

Different Client, Same Deal

10.2

Multiplying and Dividing Rational Expressions

LEARNING GOALS

In this lesson, you will:

- Multiply rational expressions.
- Divide rational expressions.

ESSENTIAL IDEAS

- The process for multiplying and dividing rational expressions is similar to the process for multiplying and dividing rational numbers.
- To multiply rational expressions you can simplify and then multiply the numerators and the denominators, or first multiply the numerators and the denominators and then simplify the result.
- When multiplying rational expressions, simplifying earlier saves time and will keep the numbers smaller.
- To divide rational expressions you multiply by the multiplicative inverse.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions.

2. Use the structure of an expression to identify ways to rewrite it.

A-APR Arithmetic with Polynomials and Rational Expressions

Rewrite rational expressions

6. Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form $q(x) + \frac{r(x)}{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

7. Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Overview

The process for multiplying and dividing rational expressions is similar to the process for multiplying and dividing rational numbers. Students will multiply and divide several rational expressions and list restrictions on the variables. The set of rational expressions is closed under multiplication and division. Students will practice multiplying and dividing rational expressions with variables and binomials in the denominators. They then use the original form of the rational expression to identify restrictions on the domain of the function. Worked examples and student work are analyzed and different solution methods are shown.

10

Warm Up

List all restrictions for the value of *x* in each expression.

1. $\frac{-x + 10}{x}$

   $x \neq 0$

2. $\frac{-x - 5}{-5x}$

   $x \neq 0$

3. $\frac{6(2 - x)}{x(4 - x)}$

   $x \neq 0, 4$

4. $\frac{x^2 - 9x + 18}{(x - 3)}$

   $x \neq 3$

10

10

10.2

Different Client, Same Deal

Multiplying and Dividing Rational Expressions

LEARNING GOALS

In this lesson, you will:

- Multiply rational expressions.
- Divide rational expressions.

Imagine that 6 adults order a 12-slice pizza for delivery. Simple division will tell you that each adult gets 2 slices in order to divide the pizza fairly. Consider a situation where a parent brings home a big bag of marbles for her 3 children. If the bag contains 108 marbles, the simple number equation $108 \div 3 = 36$ allows the parent to determine how to fairly divide the marbles up so that each of her children is happy. Have you ever thought that this type of division may not always be the best way to fairly divide resources? Consider the following problem:

Five animals share the resources in a wooded region of land. They all eat fruit, berries, and nuts, and must store an adequate amount of food to get them through the winter. They live on several acres of land that contains exactly 50 pounds of food. How much food should each animal receive? Does your answer change when you learn that the five animals are a squirrel, a mouse, a bird, a deer, and a bear? How would you divide the 50 pounds of food so that each animal gets a "fair" amount? Does it make sense that they all receive the same amount of resources? Is dividing 50 by 5 a fair way to go about this problem?

The mathematical concept "fair division" is an interesting mathematical concept. What criteria do you think should be used to determine whether resources are divided fairly?

Problem 1

A table shows a comparison between multiplying rational numbers and multiplying rational expressions involving variables. In the table, two methods are used to multiply rational expressions, one method involves multiplying first, then simplifying and the other method involves dividing out common factors first, then multiplying. Next, an activity focuses students on determining the restrictions by using the original form of the problem. Students analyze examples of rational expression multiplication with large numbers because they were simplified at the end of the problem, and then they will do the same problem by simplifying first. Students practice multiplying rational expressions and listing their restrictions for the variables. They conclude rational expressions are closed under multiplication.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Which method multiplies first, then simplifies?
- Which method simplifies first, then multiplies?
- Which method could result in working with larger numbers than necessary?
- Which method leaves more room for error? Why?
- Which method is associated with simpler calculations?
- Did Damiere consider the original form of the problem?
- Should Damiere considered the original form of the problem before claiming there are no restrictions?

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PROBLEM 1 Take Advantage of the Clearance on Expressions

Previously, you learned that adding and subtracting rational expressions involved the same process as adding and subtracting rational numbers. Now, you will see that multiplying rational expressions involves the same steps as multiplying rational numbers.

Remember that when you multiply rational numbers, you can simplify at the beginning or the end, and the product is the same; however, simplifying earlier saves time and will keep the numbers smaller.

1. Consider Method A compared to Method B in both columns of the table.

	Rational Numbers	Rational Expressions Involving Variables
Method A	$\frac{2}{15} \cdot \frac{5}{8} = \frac{10}{120}$ $= \frac{1}{12}$	$\frac{2x}{15x^2} \cdot \frac{5x^2}{8} = \frac{10x^3}{120x^2}$ $= \frac{1x}{12}$
Method B	$\frac{\overset{1}{\cancel{2}}}{\underset{3}{\cancel{15}}} \cdot \frac{\overset{1}{\cancel{5}}}{\underset{4}{\cancel{8}}} = \frac{1}{12}$	$\frac{\overset{1}{\cancel{2}}x}{\underset{3}{\cancel{15x^2}}} \cdot \frac{\overset{1}{\cancel{5x^2}}}{\underset{4}{\cancel{8}}} = \frac{1x}{12}$

a. Explain the difference in the methods.

Method A multiplies first, then simplifies. Method B simplifies first, and then multiplies the expressions.

b. Which method do you prefer?

Method A involves more steps and could result in using larger numbers than necessary. I would then have to simplify at the end. This leaves more room for error. Method B is preferable because the calculations are simpler.

2. Brody says $x \neq 0$ for the equation in the table, $\frac{2x}{15x^2} \cdot \frac{5x^2}{8} = \frac{1x}{12}$. Damiere says that there are no restrictions because the answer is $\frac{x}{12}$ and there are no variables in the denominator. Who is correct? Explain your reasoning.

Brody is correct. When looking at the original form of the problem, $x \neq 0$. Although Damiere's calculations are correct, x cannot equal 0 because of the original form of the problem.

Grouping

Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- Did Isha simplify the rational expressions first?
- How could Isha simplify the rational expressions first?
- What is the largest number Shaheen worked with when determining the product of the two rational expressions?
- What is the largest number you worked with when determining the product of the two rational expressions?
- How did you simplify the two rational expressions before determining the product?
- When identifying the restrictions, does the original form of the rational expressions need to be considered?

3. Analyze Isha's work.

Isha

$$\frac{12xyz^2}{11} \cdot \frac{33x}{8z} = \frac{\overset{9}{\cancel{396}}\,x^2y\overset{z}{\cancel{z^2}}}{\underset{2}{\cancel{88}}\,\cancel{z}}$$

$$= \frac{9x^2yz}{2}$$

Explain how Isha could have multiplied the rational expressions more efficiently.

Isha could have reduced first, $\frac{12xyz^2}{11} \cdot \frac{33x}{8z} = \frac{\overset{3}{\cancel{12}}xy\overset{z}{\cancel{z^2}}}{\underset{1}{\cancel{11}}} \cdot \frac{\overset{3}{\cancel{33}}x}{\underset{2}{\cancel{8}}\cancel{z}} = \frac{9x^2yz}{2}$. By simplifying first she does not have to deal with the large numbers.

4. Shaheen multiplies $\frac{5x^2}{3x^2 - 75} \cdot \frac{3x - 15}{4x^2}$ without simplifying first.

Shaheen

$$\frac{5x^2}{3x^2 - 75} \cdot \frac{3x - 15}{4x^2} = \frac{15x^3 - 75x^2}{12x^4 - 300x^2}$$

$$= \frac{15x^2(x - 5)}{3x^2(4x^2 - 100)}$$

$$= \frac{\overset{5}{\cancel{15x^2}}(x - 5)}{\underset{1}{\cancel{3x^2}}(4x^2 - 100)}$$

$$= \frac{5(x - 5)}{4(x^2 - 25)}$$

$$= \frac{5\cancel{(x - 5)}}{4\cancel{(x - 5)}(x + 5)}$$

$$= \frac{5}{4(x + 5)}$$

Complete the same problem as Shaheen, simplifying first, and then list the restrictions.

$$\frac{5x^2}{3x^2 - 75} \cdot \frac{3x - 15}{4x^2} = \frac{5\cancel{x^2}}{\underset{3(x+5)(x-5)}{\cancel{3x^2 - 75}}} \cdot \frac{\overset{3(x-5)}{\cancel{3x - 15}}}{4\cancel{x^2}}$$

$$= \frac{5}{\cancel{3}(x + 5)(x - 5)} \cdot \frac{\cancel{3}(x - 5)}{4}$$

$$= \frac{5}{4(x + 5)},\ x \neq 0, \pm 5$$

Grouping

Have students complete Questions 5 and 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 and 6

- Can the rational expressions be simplified before determining the product?
- How can the rational expressions be simplified before determining the product?
- When identifying the restrictions, does the original form of the rational expressions need to be considered?
- Which restriction(s) are not obvious if only the product of the rational expressions is considered?
- Can the product of any rational expressions be written as another rational expression?

10

5. Multiply each expression. List the restrictions for the variables.

a. $\frac{3ab^2}{4c} \cdot \frac{2c^2}{27ab}$

$\frac{\cancel{3}\cancel{a}b^{\cancel{2}}}{\cancel{4}\cancel{c}} \cdot \frac{\cancel{2}c^{\cancel{2}}}{\cancel{27}\cancel{a}\cancel{b}} = \frac{bc}{18};\quad a, b, c \neq 0$

b. $\frac{3x}{5x - 15} \cdot \frac{x - 3}{9x^2}$

$\frac{\cancel{3x}}{5\cancel{(x - 3)}} \cdot \frac{\cancel{x - 3}}{\cancel{9x^2}}$

$= \frac{1}{15x};\quad x \neq 0, 3$

c. $\frac{x + 5}{x^2 - 4x + 3} \cdot \frac{x - 3}{4x + 20}$

$\frac{\cancel{x + 5}}{(x - 1)\cancel{(x - 3)}} \cdot \frac{\cancel{x - 3}}{4\cancel{(x + 5)}}$

$= \frac{1}{4x - 4};\quad x \neq -5, 1, 3$

d. $\frac{7x - 7}{3x^2} \cdot \frac{x + 5}{9x^2 - 9} \cdot \frac{x^2 - 5x - 6}{x^3 + 6x^2 + 5x}$

$\frac{7\cancel{(x - 1)}}{3x^2} \cdot \frac{\cancel{(x + 5)}}{9\cancel{(x - 1)}\cancel{(x + 1)}} \cdot \frac{\cancel{(x + 1)}(x - 6)}{x(x + 1)\cancel{(x + 5)}}$

$= \frac{7x - 42}{27x^4 + 27x^3};\quad x \neq 0, \pm 1, -5$

6. Is the set of rational expressions closed under multiplication? Explain your reasoning.

Yes. The set of rational expressions is closed under multiplication. The definition of a rational function states that $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions, and $Q(x) \neq 0$. When you multiply two or more rational expressions, the product can be written in this form, as well.

Problem 2

Worked examples to maintain equivalent fractions by multiplying by a form of 1 are provided. One problem has a numerator and denominator that are both fractions. A table shows a comparison between dividing rational numbers and dividing rational expressions involving variables. Students will answer questions related to the examples, and then calculate the quotient of several rational expressions. They then explain why the set of rational expressions is closed under division.

Grouping

Ask a student to read the information and complete Question 1 as a class.

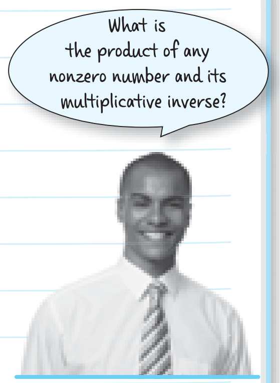
PROBLEM 2 Topsy Turvy World

Dividing rational expressions is similar to the process you use when dividing rational numbers.

Notice that when you multiply $\frac{4}{5}$ by a form of 1, in this case $\frac{3}{3}$, you maintain equivalent fractions:

$$\frac{4}{5} \cdot \frac{3}{3} = \frac{12}{15}$$

$$\frac{4}{5} = \frac{12}{15}$$



This same process works when the numerator and denominator are fractions. Consider $\frac{\frac{4}{5}}{\frac{3}{7}}$.

When you multiply by a form of 1, in this case $\frac{\frac{7}{3}}{\frac{7}{3}}$, you maintain equivalent fractions:

$$\frac{\frac{4}{5} \cdot \frac{7}{3}}{\frac{3}{7} \cdot \frac{7}{3}} = \frac{\frac{28}{15}}{\frac{21}{21}} = \frac{\frac{28}{15}}{1} = \frac{28}{15}$$

$$\frac{\frac{4}{5}}{\frac{3}{7}} = \frac{28}{15}$$

1. What is special about the form of 1 used to multiply $\frac{\frac{4}{5}}{\frac{3}{7}}$?

   The form of 1, $\frac{\frac{7}{3}}{\frac{7}{3}}$, is special because it is the multiplicative inverse of the denominator in the original problem. When you multiply the denominator by its multiplicative inverse, the product is 1.

Grouping

Have students complete Questions 2 and 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 2 and 3

- When a fraction is multiplied by a form of 1, is the result always an equivalent fraction?
- Is the product always 1 when a denominator is multiplied by its multiplicative inverse?
- When dividing rational expressions, is it always necessary to multiply by its multiplicative inverse?
- How do you determine the multiplicative inverse?
- When identifying the restrictions, does the original form of the rational expressions need to be considered?
- How is multiplying rational expressions different than dividing rational expressions?

10

You may recall that, to divide fractions, you multiply the dividend by the multiplicative inverse of the divisor.

	Rational Numbers	Rational Expressions Involving Variables
Example 1	$\frac{1}{5} \div \frac{3}{10} = \frac{1}{\cancel{5}_1} \cdot \frac{\cancel{10}^2}{3}$ $= \frac{2}{3}$	$\frac{xy^2}{5z} \div \frac{3xy}{10z^2} = \frac{\cancel{xy^2}^{y}}{{}_1\cancel{5z}} \cdot \frac{\cancel{10z^2}^{2z}}{3\cancel{xy}}$ $= \frac{2yz}{3};\ x, y, z \neq 0$
Example 2	$\frac{6}{7} \div 4 = \frac{\cancel{6}^3}{7} \cdot \frac{1}{\cancel{4}_2}$ $= \frac{3}{14}$	$\frac{6a^3}{7b} \div 4a = \frac{\cancel{6a^3}^{3a^2}}{7b} \cdot \frac{1}{\cancel{4a}_2}$ $= \frac{3a^2}{14b};\ a, b \neq 0$

2. Analyze the examples shown in the table.

   a. Explain the process for dividing rational expressions.

   To divide rational expressions, multiply by the multiplicative inverse of the divisor.

   b. In Example 2, explain why $\frac{1}{4a}$ is the multiplicative inverse of $4a$.

   The product of $\frac{1}{4a}$ and $4a$ is 1.

3. Ranger calculates the quotient of $\frac{4x^2 - 4x}{5x^2} \div \frac{x^2 - 16}{x^2 + 5x - 6} \div \frac{x^2 - 7x + 6}{x^2 + x - 12}$.

Ranger

$$\frac{4x^2 - 4x}{5x^2} \div \frac{x^2 - 16}{x^2 + 5x - 6} \div \frac{x^2 - 7x + 6}{x^2 + x - 12} = \frac{4x(x - 1)}{5x^2} \div \frac{(x + 4)(x - 4)}{(x - 1)(x + 6)} \div \frac{(x - 6)(x - 1)}{(x + 4)(x - 3)}$$

$$= \frac{4\cancel{x}\cancel{(x - 1)}(x + 6)\cancel{(x + 4)}(x - 3)}{5\cancel{x^2}_{x}(x - 4)\cancel{(x + 4)}(x - 6)\cancel{(x - 1)}}$$

$$= \frac{4x^2 + 12x - 72}{5x^3 - 50x^2 + 120x}$$

List the restrictions for the variables.

$x \neq 0, 1, 3, \pm 4, \pm 6$

Grouping

Have students complete Questions 4 and 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 and 5

- What is the multiplicative inverse in this problem?
- When determining the quotient, was it possible to simplify first before multiplying by the multiplicative inverse?
- How were you able to simplify before multiplying by the multiplicative inverse?
- Can the quotient of any rational expression be written as another rational expression?

4. Determine the quotients of each expression.

a. $\frac{9ab^2}{4c} \div \frac{18c^2}{5ab}$

$= \frac{9ab^2}{4c} \cdot \frac{5ab}{18c^2} = \frac{5a^2b^3}{8c^3}$; $a, b, c \neq 0$

b. $\frac{7x^2}{3x^2 - 27} \div \frac{4x^2}{3x - 9}$

$= \frac{7x^2}{3(x - 3)(x + 3)} \cdot \frac{3x - 9}{4x^2}$

$= \frac{7}{4x + 12}$; $x \neq \pm 3, 0$

c. $\frac{3x^2 + 15x}{x^2 - 3x - 40} \div \frac{5x^2}{x^2 - 64}$

$= \frac{3x^2 + 15x}{x^2 - 3x - 40} \cdot \frac{x^2 - 64}{5x^2}$

$= \frac{3x(x + 5)(x - 8)(x + 8)}{5x^2(x + 5)(x - 8)}$

$= \frac{3x + 24}{5x}$; $x \neq 0, \pm 8, -5$

d. $\frac{4x}{x^2y^2 - xy} \div \frac{x^2 - 4}{3x^2 + 19x - 14} \div \frac{x - 2}{xy}$

$= \frac{4x}{xy(xy - 1)} \cdot \frac{(3x - 2)(x + 7)}{(x - 2)(x + 2)} \cdot \frac{xy}{x - 2}$

$= \frac{4x(3x - 2)(x + 7)}{(xy - 1)(x - 2)^2(x + 2)}$;

$x \neq 0, \pm 2,$

$y \neq 0,$

$xy \neq 1$

5. Is the set of rational expressions closed under division? Explain your reasoning.

Yes. The set of rational expressions is closed under division. The definition of a rational function states that $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions, and $Q(x) \neq 0$. When you divide rational expressions, the quotient can be written in this form, as well.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Calculate the quotient. List the restrictions for the variables.

$$\frac{4x - 4}{5x^2} \cdot \frac{x + 6}{x^2 - 16} \div \frac{x^2 - 7x + 6}{x^3 + x^2 - 12x}$$

$$\frac{4x - 4}{5x^2} \cdot \frac{x + 6}{x^2 - 16} \cdot \frac{x^3 + x^2 - 12x}{x^2 - 7x + 6}$$

$$\frac{4x(x - 1)(x + 6)(x + 4)(x - 3)}{5x^2(x - 4)(x + 4)(x - 6)(x - 1)}$$

$$= \frac{4x^2 + 12x - 72}{5x^3 - 50x^2 + 120x}$$

$x \neq 0, \pm 4, 1, 6, 3$

10

10.3

Things Are Not Always as They Appear

Solving Rational Equations

LEARNING GOAL

In this lesson, you will:

- Solve rational equations in one variable.

KEY TERMS

- rational equation
- extraneous solution

ESSENTIAL IDEAS

- A rational equation is an equation that contains one or more rational expressions.
- When both sides of an equation are multiplied by an expression that contains a variable, solutions that were not there before may be introduced.
- Extraneous solutions are solutions that result from the process of solving an equation; but are not valid solutions to the equation.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions.

2. Use the structure of an expression to identify ways to rewrite it.

A-REI Reasoning with Equations and Inequalities

Understand solving equations as a process of reasoning and explain the reasoning

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Represent and solve equations and inequalities graphically

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Overview

The terms rational equation and extraneous solutions are defined. Students will use multiple methods to solve rational equations. Some worked examples are provided and a sorting activity is used to group and solve rational equations by the most efficient method.

10

Warm Up

Solve each proportion.

1. $\frac{x}{12} = \frac{5}{3}$

$\frac{x}{12} = \frac{5}{3}$

$3x = 60$

$x = 20$

2. $\frac{x + 2}{8} = \frac{6}{3}$

$\frac{x + 2}{8} = \frac{6}{3}$

$3x + 6 = 48$

$3x = 42$

$x = 14$

3. $\frac{x + 3}{3} = \frac{0}{x + 5}$

$\frac{x + 3}{3} = \frac{0}{x + 5}$

$(x + 3)(x + 5) = 0$

$x = -3$

$x \neq -5$

10

10

Things Are Not Always as They Appear

10.3

Solving Rational Equations

LEARNING GOAL

In this lesson, you will:

- Solve rational equations in one variable.

KEY TERMS

- rational equation
- extraneous solution

10

A paradox is a statement that leads to a contradiction. Consider the following statements:

"Don't go near the water until you have learned how to swim!"

"Nobody shops at that store anymore, it's always too crowded."

They involve faulty logic. You couldn't actually learn to swim if you never got near water, and obviously a lot of people must still shop at that store if it's always crowded. Some paradoxes use mathematics that lead you to a solution path that does not necessarily make sense in real life. A famous example is Zeno's paradox, which involves traveling a distance approaching a specific value, but never quite getting there. Here is an example of Zeno's paradox:

Suppose you are walking to catch a parked bus that is 20 meters away. You decide to get there by going half the distance every few seconds. This means that you walk 10 meters, then 5 meters, 2.5 meters, 1.25 meters, and so on, until you reach the bus.

The paradox is that if you continually halve the distance between you and the bus, you will never actually reach the bus.

Have you ever followed the correct steps to solve a problem, but then your answer didn't make sense?

Problem 1

The terms rational equation and extraneous solutions are defined. Student work shows several methods for solving rational equations. Some methods involve multiplying both sides of the equation by a common denominator, proportional reasoning, using the Rational Root Theorem, synthetic division, using a graphing calculator, and factoring. Students then use these methods to solve rational equations.

10

Grouping

- Ask a student to read the information and definition. Discuss as a class.
- Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- How many factors are in the denominator in this problem?
- Who multiplied both sides of the equation by a common denominator first?
- How is the least common denominator determined?
- What is the least common denominator in this problem?
- Did Randall and Sully both use the same common denominator?

PROBLEM 1 Method Mayhem

A **rational equation** is an equation that contains one or more rational expressions. You have already solved simple rational equations with a single variable as a denominator and performed simple operations using rational expressions. You will follow the same rules and guidelines when solving more involved rational equations.

There are multiple methods you can use to solve rational equations. Depending on the structure of the equation, some methods will be more efficient than others.

1. Randall and Sully solved the equation $\frac{x+5}{x+2} = \frac{x+1}{x-5}$.

Randall

$$\frac{x+5}{x+2} = \frac{x+1}{x-5}$$

Restrictions: $x \neq -2, 5$

$$\cancel{(x+2)}(x-5)\cdot\frac{x+5}{\cancel{x+2}} = (x+2)\cancel{(x-5)}\cdot\frac{x+1}{\cancel{x-5}}$$

$$(x-5)(x+5) = (x+2)(x+1)$$

$$x^2 - 25 = x^2 + 3x + 2$$

$$-25 = 3x + 2$$

$$-27 = 3x$$

$$x = -9$$

Sully

$$\frac{x+5}{x+2} = \frac{x+1}{x-5}$$

Restrictions: $x \neq -2, 5$

$$(x+5)(x-5) = (x+2)(x+1)$$

$$x^2 - 25 = x^2 + 3x + 2$$

$$-25 = 3x + 2$$

$$-27 = 3x$$

$$x = -9$$

a. Explain Randall's method of solving.

Randall multiplied both sides of the equation by a common denominator, and then solved much the same way Sully did.

b. Sully used proportional reasoning to solve the equation. Explain how he solved the equation.

There is only one factor in the denominator on either side of the equals sign. Sully multiplied the means by the extremes, and then solved for *x*.

c. Which method do you prefer for this problem? Explain your reasoning.

Answers will vary.
Student responses could include a method based on preference.

- Who multiplied the means by the extremes to solve the problem?
- Whose method is shorter?

Grouping

Have students complete Question 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- How many factors are in the denominator in this problem?
- What did Sully multiply both sides of the equation by?
- How did Sully change a quadratic equation into a cubic equation?
- Did Sully's method increase the degree of the equation?
- If the degree of the equation is increased by one, does that create an additional root?
- Is the newly created root actually a solution to the original equation? Why not?
- Using Randall's method, did it increase the degree of the equation like Sully's method?
- Can this problem be solved without creating a cubic equation?

2. Sully was presented with a slightly different equation to solve. Notice that a new factor appears in one of the denominators. Again, he uses proportional reasoning to solve.

Sully

$$\frac{x+5}{(x-5)(x+2)} = \frac{x+1}{x-5}$$

Restrictions: $x \neq 5, -2$

$(x-5)(x+2)(x+1) = (x+5)(x-5)$

$(x^2+2x-5x-10)(x+1) = x^2-25$

$x^3-3x^2-10x+x^2-3x-10 = x^2-25$

$x^3-3x^2-13x+15 = 0$

5	1	−3	−13	15
	↓	5	10	−15
	1	2	−3	0

−3	1	−3	−13	15
	↓	−3	18	−15
	1	−6	5	0

1	1	−3	−13	15
	↓	1	−2	−15
	1	−2	−15	0

$p = \pm 15, \pm 5, \pm 3, \pm 1$

$q = \pm 1$

Possible rational roots $\left(\frac{p}{q}\right)$: $\pm 15, \pm 5, \pm 3, \pm 1$

Using synthetic division, I realize the three roots are 5, −3, and 1. However, from my list of restrictions, I know that $x \neq 5$. So, my solutions to the equation can only be $x = -3$ or $x = 1$. I will check to see if they work.

Check $x = -3$

$$\frac{-3+5}{(-3-5)(-3+2)} \stackrel{?}{=} \frac{-3+1}{-3-5}$$

$$\frac{2}{(-8)(-1)} \stackrel{?}{=} \frac{-2}{-8}$$

$$\frac{2}{8} = \frac{2}{8}$$

Check $x = 1$

$$\frac{1+5}{(1-5)(1+2)} \stackrel{?}{=} \frac{1+1}{1-5}$$

$$\frac{6}{(-4)(3)} \stackrel{?}{=} \frac{2}{-4}$$

$$-\frac{6}{12} \stackrel{?}{=} -\frac{2}{4}$$

$$-\frac{1}{2} = -\frac{1}{2}$$

Thus, $x = -3$ or $x = 1$.

a. What is different about the structure of this equation compared to the equation in Question 1?

The ratio on the left side of the equation has two factors in the denominator.

10

b. Prior to checking his solution, explain why Sully identified three possible roots to the equation.

Sully identified 3 roots because when he cross multiplied and simplified, he ended up with a polynomial of degree 3, which means that the related equation has 3 solutions.

c. Use Randall's method to solve the second equation Sully solved.

$$[\cancel{(x-5)}\cancel{(x+2)}] \cdot \frac{x+5}{\cancel{(x-5)}\cancel{(x+2)}} = [\cancel{(x-5)}(x+2)] \cdot \frac{x+1}{\cancel{x-5}}$$

$x \neq 5, -2$

$x + 5 = (x + 2)(x + 1)$

$x + 5 = x^2 + 3x + 2$

$x^2 + 2x - 3 = 0$

$(x + 3)(x - 1) = 0$

$x = -3, 1$

10

d. Which method is more efficient based on the structure of the original equation? Explain your reasoning.

Answers will vary.

Student responses could include Randall's method because it is shorter and does not involve factoring a cubic. Sully's method creates a cubic function, which is not necessary.

Grouping

Ask a student to read the information and definition. Discuss as a class.

There is a mathematical reason why Sully identified an extra solution to his equation. Recall that one of the basic principles of algebra is that you can multiply both sides of an equation by a non-zero real number or expression, as long as you maintain balance to the equation. When you multiply both sides of the equation by an expression that contains a variable, you may introduce solutions that are not solutions of the original equation. Notice Sully multiplied by $x - 5$. By doing so, he introduced an additional solution. These extra solutions are called *extraneous solutions*. **Extraneous solutions** are solutions that result from the process of solving an equation, but are not valid solutions to the original equation.

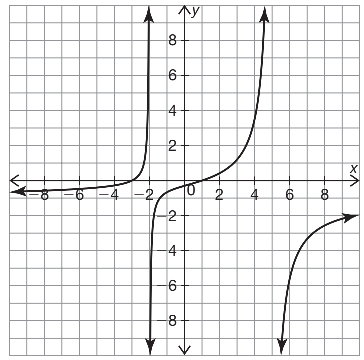
Grouping

Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 and 4

- Does the graph have a vertical asymptote?
- What is the equation of the vertical asymptote?
- Does the graph have a horizontal asymptote?
- Where does the graph of the function cross the *x*-axis?
- When $y = 0$, what are the *x* values?
- What is the significance of the points at which the two graphs intersect?
- Do Mike's solutions agree with Sully's solutions?
- Whose method is easier to use? Why?

3. Sully wanted to graph the equation using a graphing calculator. He rewrote the equation so that one side of the equation equaled zero. Then, he graphed: $y_1 = \frac{x+1}{x-5} - \frac{x+5}{(x-5)(x+2)}$. Consider the graph:



Use the graph to explain why $x \neq 5$ and why $x = -3$ and $x = 1$.

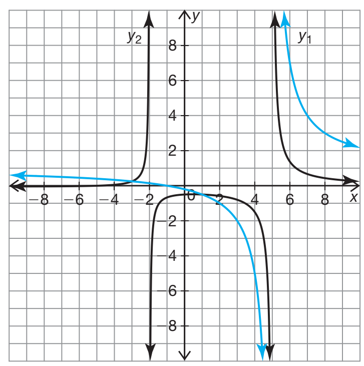
You can use the graph to determine *x* does not equal 5 because there is a vertical asymptote at $x = 5$.

You can use the graph to determine that $x = -3$ and $x = 1$ by noticing that those are the *x*-intercepts of the graph.

4. Mike used a graphing calculator to solve the same equation:

$y_1 = \frac{x+1}{x-5}$

$y_2 = \frac{x+5}{(x-5)(x+2)}$.



How does Mike's graph represent the solutions to the equation?

The points where the two graphs intersect are the solutions to the equation; $x = -3$ and $x = 1$.

Grouping

Have students complete Question 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- Did Sasha or Jake factor the quadratic in the denominator?
- Did Sasha or Jake factor the numerator and divide out the second factor in the denominator?
- Did Sasha or Jake work with a quadratic function?
- How would the use of proportional reasoning create extraneous solutions in this problem?

10

5. Analyze the methods Jake and Sasha used to solve $\frac{2x+4}{x^2-2x-8} = \frac{x+1}{x-4}$.

Sasha

$$\frac{2x+4}{(x-4)(x+2)} = \frac{x+1}{x-4} \quad x \neq 4, -2$$

$$\frac{2x+4}{(x-4)(x+2)} = \frac{x+1}{x-4} \cdot \frac{(x+2)}{(x+2)}$$

$$\frac{2x+4}{(x-4)(x+2)} = \frac{x^2+3x+2}{(x-4)(x+2)}$$

$$\cancel{(x-4)(x+2)}\left[\frac{2x+4}{\cancel{(x-4)(x+2)}} = \frac{x^2+3x+2}{\cancel{(x-4)(x+2)}}\right]$$

$$2x+4 = x^2+3x+2$$

$$x^2+x-2 = 0$$

$$(x+2)(x-1) = 0$$

$$x = -2, 1$$

I know that $x \neq -2$, so I just need to check $x = 1$.

Check $x = 1$

$$\frac{2(1)+4}{(1-4)(1+2)} \stackrel{?}{=} \frac{1+1}{1-4}$$

$$\frac{6}{(-3)(3)} \stackrel{?}{=} \frac{2}{-3}$$

$$-\frac{2}{3} = -\frac{2}{3}$$

Jake

$$\frac{2x+4}{x^2-2x-8} = \frac{x+1}{x-4}$$

$$\frac{2(x+2)}{(x-4)(x+2)} = \frac{x+1}{x-4}$$

$$x \neq 4, -2$$

$$\frac{2\cancel{(x+2)}}{(x-4)\cancel{(x+2)}} = \frac{x+1}{x-4}$$

$$\cancel{(x-4)}\left[\frac{2}{\cancel{x-4}} = \frac{x+1}{\cancel{x-4}}\right]$$

$$2 = x+1$$

$$x = 1$$

a. Describe how Jake and Sasha's methods are similar and different.

Jake and Sasha's methods are similar because they both factor the quadratic in the denominator. Jake's method is different because he also factors the numerator and he is able to divide out the second factor in the denominator. As a result, he has common denominators, and never has to deal with a quadratic function.

b. What would happen if you tried to solve this problem using proportional reasoning?

If I use proportional reasoning to solve this problem, I would end up with a quadratic multiplied by a linear which results in a cubic. I would then have 2 extraneous solutions and I would have to use the Rational Root Theorem and synthetic division to factor and solve.

6. Analyze the methods Seth, Damiere, and Sidonie each used to solve the equation $\frac{6}{x^2 - 4x} + \frac{4}{x} = \frac{2}{x - 4}$.

Seth

$$\frac{6}{x^2 - 4x} + \frac{4}{x} = \frac{2}{x - 4}$$

$$\frac{6}{x(x - 4)} + \frac{4}{x} \cdot \frac{(x - 4)}{(x - 4)} = \frac{2}{x - 4} \cdot \frac{x}{x}$$

$x \neq 0, 4$

$$\frac{6}{x(x - 4)} + \frac{4x - 16}{x(x - 4)} = \frac{2x}{x(x - 4)}$$

$$\frac{6 + 4x - 16}{x(x - 4)} = \frac{2x}{x(x - 4)}$$

$$\cancel{x(x - 4)}\left[\frac{6 + 4x - 16}{\cancel{x(x - 4)}} = \frac{2x}{\cancel{x(x - 4)}}\right]$$

$$6 + 4x - 16 = 2x$$

$$4x - 10 = 2x$$

$$2x = 10$$

$$x = 5$$

Sidonie

$$\frac{6}{x^2 - 4x} + \frac{4}{x} = \frac{2}{x - 4}$$

$$\frac{6}{x(x - 4)} + \frac{4}{x} = \frac{2}{x - 4}$$

$x \neq 0, 4$

$$(x(x - 4)) \cdot \left[\frac{6}{x(x - 4)} + \frac{4}{x} = \frac{2}{x - 4}\right]$$

$$6 + 4(x - 4) = 2x$$

$$6 + 4x - 16 = 2x$$

$$4x - 10 = 2x$$

$$2x = 10$$

$$x = 5$$

10

Grouping

Have students complete Questions 6 and 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- Did Seth, Sidonie, or Damiere factor first then determine the least common denominator?
- Did Seth, Sidonie, or Damiere multiply both sides of the equation to eliminate the denominator?
- Did Seth, Sidonie, or Damiere multiply all the denominators together to determine a common denominator?
- Did Seth, Sidonie, or Damiere create extraneous solutions to this problem?

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Damiere

$$\frac{6}{x(x-4)} + \frac{4}{x} = \frac{2}{x-4}$$

$x \neq 0, 4$

$$\frac{6}{x^2-4x} + \frac{4}{x} = \frac{2}{x-4}$$

$$\frac{(x)(x-4)}{(x)(x-4)} \cdot \frac{6}{x^2-4x} + \frac{(x^2-4x)(x-4)}{(x^2-4x)(x-4)} \cdot \frac{4}{x} = \frac{2}{x-4} \cdot \frac{(x^2-4x)(x)}{(x^2-4x)(x)}$$

$$\frac{6x^2-24x}{x^4-8x^3+16x^2} + \frac{4x^3-16x^2-16x^2+64x}{x^4-8x^3+16x^2} = \frac{2x^3-8x^2}{x^4-8x^3+16x^2}$$

$$\cancel{x^4-8x^3+16x^2} \cdot \frac{6x^2-24x+4x^3-16x^2-16x^2+64x}{\cancel{x^4-8x^3+16x^2}} = \cancel{x^4-8x^3+16x^2} \cdot \frac{2x^3-8x^2}{\cancel{x^4-8x^3+16x^2}}$$

$6x^2 - 24x + 4x^3 - 16x^2 - 16x^2 + 64x = 2x^3 - 8x^2$

$4x^3 - 26x^2 + 40x = 2x^3 - 8x^2$

$2x^3 - 18x^2 + 40x = 0$

$2x(x^2 - 9x + 20) = 0$

$2x(x-4)(x-5) = 0$

$x = 0, 4, 5$

I know that $x \neq 0, 4$, so I will just check $x = 5$.

Check:

$x = 5$

$$\frac{6}{5^2-4(5)} + \frac{4}{5} \stackrel{?}{=} \frac{2}{5-4}$$

$$\frac{6}{25-20} + \frac{4}{5} \stackrel{?}{=} \frac{2}{1}$$

$$\frac{6}{5} + \frac{4}{5} \stackrel{?}{=} 2$$

$$\frac{10}{5} \stackrel{?}{=} 2$$

$$2 = 2$$

a. Describe the methods Seth, Sidonie, and Damiere each used to solve the rational equation.

Seth factored, determined the LCD, and then used a common denominator. Sidonie multiplied both sides of the equation to eliminate the denominator. Damiere multiplied all the denominators together to determine a common denominator.

b. Prior to checking his solution, explain why Damiere identified three possible roots to the equation.

Damiere identified 3 solutions because when he solved, he increased the degree of the function, thereby increasing the number of solutions.

Guiding Questions for Share Phase, Question 7

- Did your method create extraneous solutions? How do you know?
- Is there more than one way to solve this equation?

7. Solve the equation $\frac{10}{x^2 - 2x} + \frac{1}{x} = \frac{3}{x - 2}$. Explain why you chose your solution method.

$$\frac{10}{x^2 - 2x} + \frac{1}{x} = \frac{3}{x - 2}$$

$$\frac{10}{x(x - 2)} + \frac{1}{x} \cdot \left(\frac{x - 2}{x - 2}\right) = \left(\frac{3}{x - 2}\right) \cdot \frac{x}{x}$$

$x \neq 0, 2$

$$\frac{10}{x(x - 2)} + \frac{1x - 2}{x(x - 2)} = \frac{3x}{x(x - 2)}$$

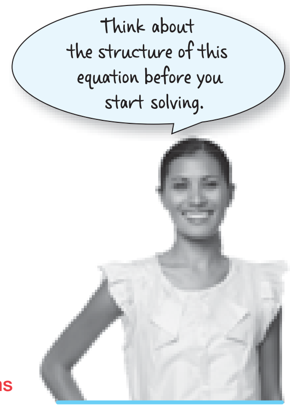
$$\cancel{x(x - 2)} \cdot \frac{10 + x - 2}{\cancel{x(x - 2)}} = \frac{3x}{\cancel{x(x - 2)}} \cdot \cancel{x(x - 2)}$$

$$10 + x - 2 = 3x$$

$$8 = 2x$$

$$x = 4$$

Answers will vary. Student responses could include factoring and multiplying by the LCD to combine terms and simplify.



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Problem 2

Students will cut out 12 equations, sort them into piles with regard to their intended solution method, solve the equations, and paste them onto the pages remaining in the lesson. They also list the domain restrictions for each problem.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- How many groups were used in the sorting process?
- What do the equations in each group have in common?
- How many equations contained extraneous solutions?
- How many equations had no solution?
- How many equations had 2 or more restrictions in the denominator?
- How many equations were solved using proportional reasoning?
- How many equations were solved using the least common denominator?
- How many equations were solved using factoring?

PROBLEM 2 Seeing Structure

1. Cut out each of the equations on the following three pages. Before solving each equation, think about how the structure of the equation informs your solution method. Then, sort the equations based on the solution method you intend to use. Finally, solve each equation. Be sure to list the domain restrictions.

a. $\frac{12}{x+5} = -2$

$x \neq -5$

$\left(\frac{x+5}{1}\right)\frac{12}{x+5} = -2(x+5)$

$12 = -2x - 10$

$2x = -22$

$x = -11$

b. $\frac{x-5}{3} = \frac{x-38}{12} - \frac{x}{4}$

No restrictions

$\left(\frac{4}{4}\right)\cdot\frac{x-5}{3} = \frac{x-38}{12} - \frac{x}{4}\cdot\left(\frac{3}{3}\right)$

$4x - 20 = x - 38 - 3x$

$4x - 20 = -2x - 38$

$6x = -18$

$x = -3$

c. $\frac{x^2-5x}{4} = \frac{8x}{2}$

No restrictions

$\frac{x^2-5x}{4} = \frac{8x}{2}\left(\frac{2}{2}\right)$

$x^2 - 5x = 16x$

$x^2 - 21x = 0$

$x(x - 21) = 0$

$x = 0, 21$

d. $\frac{1}{x-5} = \frac{5}{x^2+2x-35}$

$x \neq 5, -7$

$\frac{1}{x-5} = \frac{5}{(x-5)(x+7)}$

$\frac{(x+7)}{(x+7)}\cdot\frac{1}{x-5} = \frac{5}{(x-5)(x+7)}$

$x + 7 = 5$

$x = -2$

- How many equations were solved using a graphing calculator?
- How many equations were solved using synthetic division?
- How many equations were solved by multiplying all of the denominators to get a common denominator?

10

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e. $\frac{3}{x-1} + \frac{2}{5x+5} = \frac{-3}{x^2+1}$

$x \neq -1, 1$

$\left(\frac{5(x+1)}{5(x+1)}\right) \cdot \frac{3}{x-1} + \frac{2}{5(x+1)} \cdot \left(\frac{x-1}{x-1}\right)$

$= \frac{-3}{(x-1)(x+1)} \cdot \left(\frac{5}{5}\right)$

$15x + 15 + 2x - 2 = -15$

$17x + 13 = -15$

$17x = -28$

$x = -\frac{28}{17}$

f. $\frac{x-5}{x-2} = \frac{8}{9}$

$x \neq 2$

$(x-5)(9) = (x-2)(8)$

$9x - 45 = 8x - 16$

$x = 29$

g. $\frac{5}{x} = 25 + \frac{5}{x}$

$x \neq 0$

$0 = 25$

No Solution

h. $\frac{1}{x^2} + \frac{1}{x} = \frac{1}{2x^2}$

$x \neq 0$

$\left(\frac{2}{2}\right) \cdot \frac{1}{x^2} + \left(\frac{2x}{2x}\right) \cdot \frac{1}{x} = \frac{1}{2x^2}$

$2x^2 \cdot \frac{2+2x}{2x^2} = 2x^2 \cdot \frac{1}{2x^2}$

$2 + 2x = 1$

$2x = -1$

$x = -\frac{1}{2}$

i. $\frac{-2}{x+3} + \frac{3}{x-2} = \frac{5}{x^2+x-6}$

$x \neq -3, 2$

$\frac{-2}{x+3} + \frac{3}{x-2} = \frac{5}{(x+3)(x-2)}$

$\frac{(x-2)}{(x-2)} \cdot \frac{-2}{x+3} + \frac{(x+3)}{(x+3)} \cdot \frac{3}{x-2}$

$= \frac{5}{(x+3)(x-2)}$

$-2x + 4 + 3x + 9 = 5$

$x + 13 = 5$

$x = -8$

j. $\frac{7}{x+3} = \frac{8}{x-2}$

$x \neq -3, 2$

$7(x - 2) = 8(x + 3)$

$7x - 14 = 8x + 24$

$-x = 38$

$x = -38$

k. $\frac{x+3}{x^2-1} + \frac{-2x}{x-1} = 1$

$x \neq -1, 1$

$(x^2 - 1)\left[\frac{x+3}{x^2-1} + \frac{-2x}{x-1} = 1\right]$

$x + 3 - 2x(x + 1) = x^2 - 1$

$x + 3 - 2x^2 - 2x = x^2 - 1$

$0 = 3x^2 + x - 4$

$0 = (3x + 4)(x - 1)$

$3x + 4 = 0$ | $x - 1 = 0$

$x = -\frac{4}{3}$ | $x = 1$

$x \neq 1$

$x = \frac{4}{3}$

I know that $x \neq 1$ because that is a restriction.

l. $\frac{3}{x^2+2x} = \frac{6}{x^2}$

$x \neq 0, -2$

$(x^2 + 2x)(6) = (x^2)(3)$

$6x^2 + 12x = 3x^2$

$3x^2 + 12x = 0$

$3x(x + 4) = 0$

$x = 0, -4$

I know that $x \neq 0$ because that is a restriction, so $x = -4$.

Check $x = -4$

$\frac{3}{(-4)^2 + 2(-4)} \stackrel{?}{=} \frac{6}{(-4)^2}$

$\frac{3}{16 - 8} \stackrel{?}{=} \frac{6}{16}$

$\frac{3}{8} \stackrel{?}{=} \frac{6}{16}$

$\frac{3}{8} = \frac{3}{8}$

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Paste your solved rational equations in the space provided.

10

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2. Did you solve the equations using the method you first intended?
Answers will vary. Student responses could include changing methods based on structure.

Be prepared to share your solutions and methods.

10

Check for Students' Understanding

Solve the rational equation and identify any extraneous solutions.

$$\frac{-4}{x-6} + \frac{6}{x+7} = \frac{10}{x^2 + x - 42}$$

$$\frac{-4}{x-6} + \frac{6}{x+7} = \frac{10}{(x+7)(x-6)}$$

$$-4(x+7) + 6(x-6) = 10$$

$$-4x - 28 + 6x - 36 = 10$$

$$2x = 74$$

$$x = 37$$

10

10

Get to Work, Mix It Up, Go the Distance, and Lower the Cost!

10.4

Using Rational Equations to Solve Real-World Problems

LEARNING GOALS

In this lesson, you will:

- Use rational equations to model and solve work problems.
- Use rational equations to model and solve mixture problems.
- Use rational equations to model and solve distance problems.
- Use rational equations to model and solve cost problems.

ESSENTIAL IDEAS

- A work problem is a type of problem that involves the rates of several workers and the time it takes to complete a job.
- A mixture problem is a type of problem that involves the combination of two or more liquids and the concentrations of those liquids.
- A distance problem is a type of problem that involves distance, rate, and time.
- A cost problem is a type of problem that involves the cost of ownership of an item over time.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-CED Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems

A-REI Reasoning with Equations and Inequalities

Understand solving equations as a process of reasoning and explain the reasoning

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Overview

Rational equations are used to model work problems, mixture problems, distance problems, and cost problems.

10

Warm Up

Solve for x.

1. $32 = \frac{x}{2}$

$32 = \frac{x}{2}$

$\frac{32}{1} = \frac{x}{2}$

$x = 64$

2. $\frac{3}{4} + \frac{4}{x} = 1$

$\frac{3}{4} + \frac{4}{x} = 1$

$\frac{3}{4}\left(\frac{x}{x}\right) + \frac{4}{x}\left(\frac{4}{4}\right) = 1\left(\frac{4x}{4x}\right)$

$3x + 16 = 4x$

$x = 16$

3. $\frac{x}{10} + \frac{x}{20} = 1$

$\frac{x}{10} + \frac{x}{20} = 1$

$\frac{x}{10}\left(\frac{2}{2}\right) + \frac{x}{20} = 1\left(\frac{20}{20}\right)$

$2x + x = 20$

$3x = 20$

$x = 6.\overline{66}$

10

10

Get to Work, Mix It Up, Go the Distance, and Lower the Cost!

10.4

Using Rational Equations to Solve Real-World Problems

10

LEARNING GOALS

In this lesson, you will:

- Use rational equations to model and solve work problems.
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- Use rational equations to model and solve distance problems.
- Use rational equations to model and solve cost problems.

If you haven't noticed, learning is not limited to the classroom, but is part of everyday life. In fact, there's the old adage that confirms this: You learn something new everyday.

But learning alone does not equate to success. You need to apply that knowledge and gain experience. In many ways, the classroom sets the foundation for success, but by applying your knowledge, you can shine!

Mathematics is no different. When you gain a deep understanding of the mathematics you use in the classroom and can relate it to real-world situations, you are able to do your own problem-solving and gain success.

How have you used mathematics in your everyday life? How do you think you will use math in the future?

Problem 1

Work problems are modeled using rational equations. When people or teams of people work together a job gets done faster than a single person or team working alone. In each situation, students will create expressions and use them to solve rational equations that determine the amount of time it takes to get a job done working together and working alone.

10

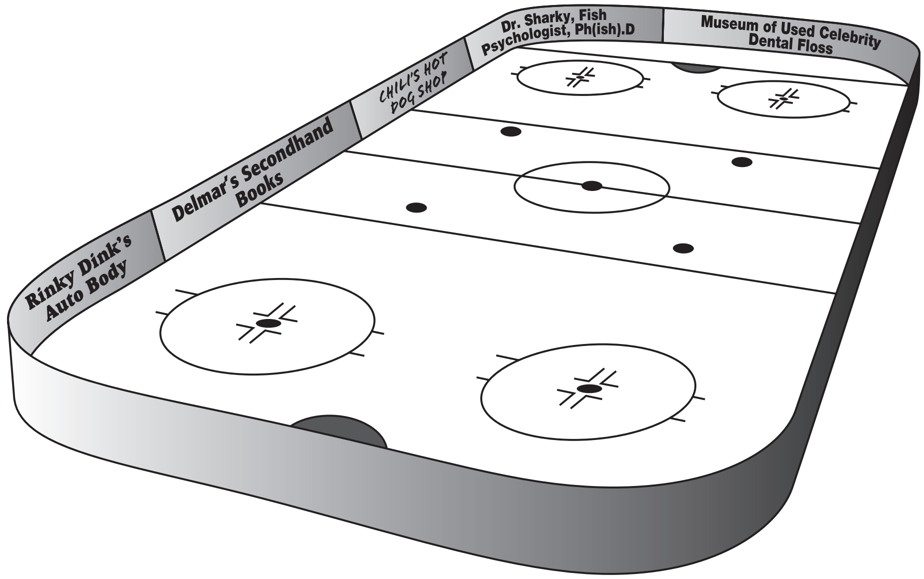
Grouping

Ask a student to read the information. Discuss as a class.

PROBLEM 1 **This Is Quite a Work-Out**

A work problem is a type of problem that involves the rates of several workers and the time it takes to complete a job. For example, the rate at which two painters work and the total time it takes them to paint a house while working together is an example of a work problem.

Anita and Martin are the assistant managers for the marketing department of the Snarky Larks Hockey Team. This hockey season is fast-approaching, and the rink board ads need to be mounted to the rink boards before the season begins.



Each ad is like a giant vinyl sticker that is stuck to each rink board along the inside of the hockey rink. It takes a team of three people to attach each ad: two people hold the ad while a third person carefully presses it to the rink board, being careful that it does not wrinkle.

Up until last year, Anita's team and Martin's team have taken turns doing this job—Anita's team attached the rink boards for the first season, Martin's team attached them for the next season, the following season Anita's team attached them, and so on.

Grouping

Have students complete Question 1 parts (a) through (d) with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1 parts (a) through (d)

- How did you determine the portion of the rink Anita's team completed in 1 hour?
- How did you determine the portion of the rink Martin's team completed in 1 hour?
- How did you determine the portion of the rink Anita's team completed in x hours?
- How did you determine the portion of the rink Martin's team completed in x hours?

1. It takes Anita's team 20 hours to attach all of the rink boards, and it takes Martin's team 30 hours to attach all of the rink boards. This year, however, their boss has asked them to work together to get the job done faster.

a. Determine the portion of the rink each team completes in the given number of hours.

Anita's Team

1 hour: In 1 hour, Anita's team completes $\frac{1}{20}$ of the rink.

5 hours: In 5 hours, Anita's team completes $\frac{5}{20}$, or $\frac{1}{4}$, of the rink.

10 hours: In 10 hours, Anita's team completes $\frac{10}{20}$, or $\frac{1}{2}$, of the rink.

Martin's Team

1 hour: In 1 hour, Martin's team completes $\frac{1}{30}$ of the rink.

5 hours: In 5 hours, Martin's team completes $\frac{5}{30}$, or $\frac{1}{6}$, of the rink.

10 hours: In 10 hours, Martin's team completes $\frac{10}{30}$, or $\frac{1}{3}$, of the rink.

b. Consider the amount of the rink that each team can complete in x hours.

i. Write an expression to represent the portion of the rink that Anita's team can complete in x hours.

Anita's team can complete $\frac{x}{20}$ of the rink in x hours.

ii. Write an expression that represents the portion of the rink that Martin's team can complete in x hours.

Martin's team can complete $\frac{x}{30}$ of the rink in x hours.

10

10

Grouping

Have students complete Question 1 parts (e) through (g) with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Problem 1 Question 1 parts (e) through (g)

- What does the expression $\frac{x}{20} + \frac{x}{30}$ represent with respect to this problem situation?
- Does the expression $\frac{x}{20} + \frac{x}{30}$ represent 1 complete job?
- What is the LCD in the expression $\frac{x}{20} + \frac{x}{30}$?
- Which team works faster?
- If the two teams work together, will that slow Anita's team down or will Martin's team work faster?

c. Each team's rate of work is defined as number of jobs completed per hour. In this case, the rate of work is the number of rinks completed per hour.

i. Determine Anita's team's rate of work.

Anita's team's rate of work is $\frac{1}{20}$ rink per hour.

ii. Determine Martin's team's rate of work.

Martin's team's rate of work is $\frac{1}{30}$ rink per hour.

d. Complete the table.

	Portion of the Rink Completed	Time Spent Working	Rate of Work
	Rinks	Hours	$\frac{\text{Rinks}}{\text{Hour}}$
Anita's Team	$\frac{x}{20}$	x	$\frac{1}{20}$
Martin's Team	$\frac{x}{30}$	x	$\frac{1}{30}$
Entire Job, or 1 Rink	$\frac{x}{20} + \frac{x}{30}$	x	

e. Consider the expression from the table that represents the portion of the rink that Anita's and Martin's teams can complete when working together. If you want to determine the total time it takes the two teams to complete one rink while working together, what should you set this expression equal to?

I will set this expression equal to 1, because this expression represents 1 complete job.

f. Write and solve an equation to determine the total time it takes the two teams to complete the rink.

$$\frac{x}{20} + \frac{x}{30} = 1$$

$$\frac{60}{1}\left(\frac{x}{20} + \frac{x}{30}\right) = 1(60)$$

$$3x + 2x = 60$$

$$5x = 60$$

$$x = 12$$

Working together it will take the two teams 12 hours to complete the rink.

Grouping

Have students complete Question 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- How is this situation similar to the last problem situation?
- How is the situation different than the last problem situation?
- What expression represents Sandra's rate of watering?
- How did you determine the portion of the garden Maureen watered?
- How did you determine the portion of the garden Sandra watered?
- What did you set the expression equal to when solving for the total time it takes for Sandra to water the garden?
- Who waters the garden faster, Sandra or Maureen?

g. Suppose that the two teams work together attaching rink board ads for 4 hours each day. How many days will it take them to complete the job?

It will take the two teams $\frac{12}{4}$, or 3 days, to complete the rink.

Maureen is a community volunteer. She volunteers by watering the large vegetable garden in her neighborhood. Sometimes, Maureen's friend Sandra also volunteers.

2. It takes Maureen 90 minutes to water the garden. When Maureen and Sandra are working together, they can complete the job in 40 minutes.

a. Complete the table. Let x represent the time it takes Sandra to water the garden if she works alone.

	Portion of the Garden Watered	Time Spent Watering	Rate of Watering
	Gardens	Minutes	$\frac{\text{Gardens}}{\text{Minute}}$
Maureen	$40\left(\frac{1}{90}\right) = \frac{4}{9}$	40	$\frac{1}{90}$
Sandra	$40\left(\frac{1}{x}\right) = \frac{40}{x}$	40	$\frac{1}{x}$
Entire Job, or 1 Garden	$\frac{4}{9} + \frac{40}{x}$	40	

b. Write and solve an equation to determine the total time it would take Sandra to water the garden if she were working alone.

$$\frac{4}{9} + \frac{40}{x} = 1$$

$$x \neq 0$$

$$\frac{9x}{1}\left(\frac{4}{9} + \frac{40}{x}\right) = 1(9x)$$

$$4x + 360 = 9x$$

$$360 = 5x$$

$$72 = x$$

It would take Sandra 72 minutes to water the garden by herself.

10

Problem 2

Mixture problems are modeled using rational equations. In each situation, students will create expressions and use them to solve rational equations that determine the amounts of liquids that need to be added to reach specified concentrations.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

10

Guiding Questions for Share Phase, Question 1

- How did you determine the amount of salt in 120 mL of 10% salt solution?
- How did you determine the amount of water in 120 mL of 10% salt solution?
- What is the numerator and denominator of the ratio used to determine the concentration of salt solution when Manuel added 80 mL of water?
- How did you change the ratio to a percentage?
- What is the numerator and denominator of the ratio used to determine the concentration of salt solution when Manuel added 180 mL of water?

PROBLEM 2 Shhh! The Mixmaster Needs Complete Concentration

A mixture problem is a type of problem that involves the combination of two or more liquids and the concentrations of those liquids.

1. Manuel is a taking a college chemistry course, and some of his time is spent in the chemistry lab. He is conducting an experiment for which he needs a 2% salt solution. However, all he can find in the lab is 120 milliliters (mL) of 10% salt solution.

   a. How many milliliters of salt and how many milliliters of water are in 120 mL of 10% salt solution?

   There are 0.1(120) = 12 mL of salt and 0.9(120) = 108 mL of water in 120 mL of 10% salt solution.

   b. What would the concentration of the salt solution be if Manuel added 80 mL of water? 180 mL of water?

   $$\frac{12}{120 + 80}$$
   $$= \frac{12}{200}$$
   $$= 0.06 = 6\%$$

   If Manuel added 80 mL of water, the resulting solution would be a 6% salt solution.

   $$\frac{12}{120 + 180}$$
   $$= \frac{12}{300}$$
   $$= 0.04 = 4\%$$

   If Manuel added 180 mL of water, the resulting solution would be a 4% salt solution.

   c. Write and solve an equation to calculate the amount of water Manuel needs to add to the 120 mL of 10% salt solution to make a 2% salt solution. Let x represent the amount of water Manuel needs to add.

   $$\frac{12}{120 + x} = 0.02$$
   $$x \neq -120$$
   $$0.02(120 + x) = 12$$
   $$2.4 + 0.02x = 12$$
   $$0.02x = 9.6$$
   $$x = 480$$

   Manuel needs to add 480 mL of water to make a 2% salt solution.

- What equation was used to calculate the amount of water Manuel needed to add to the 120 mL of 10% salt solution to make a 2% salt solution?
- Are there any restrictions in the denominator?

Grouping

Have students complete Question 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- Can the concentration of the new solution be less than or equal to 5% sulfuric acid? Why not?
- Can the concentration of the new solution be greater than or equal to 20% sulfuric acid? Why not?
- The 20 mL of 20% solution contains how many mL of sulfuric acid?
- The 20 mL of 20% solution contains how many mL of water?
- The 10 mL of 5% solution contains how many mL of sulfuric acid?
- The 10 mL of 5% solution contains how many mL of water?
- How did you determine the concentration of the resulting solution?
- What rational equation represents the amount of 5% sulfuric acid solution Keisha added?

2. Keisha is working on a chemistry experiment. She has 20 mL of a 20% sulfuric acid solution that she is mixing with a 5% sulfuric acid solution.

a. Describe the range of possible concentrations for the new solution.

The new solution will be between 5% and 20% sulfuric acid.

b. Suppose that the 20 mL of 20% sulfuric acid solution is mixed with 10 mL of the 5% sulfuric acid solution. What is the concentration of the resulting solution? Explain your reasoning.

The 20 mL of 20% solution contains 0.2(20) = 4 mL of sulfuric acid and 20 − 4 = 16 mL of water.

The 10 mL of 5% solution contains 0.05(10) = 0.5 mL of sulfuric acid and 10 − 0.5 = 9.5 mL of water.

$$\frac{4 + 0.5}{20 + 10} = \frac{4.5}{30} = 0.15$$

The concentration of the resulting solution will be 15%.

c. Write and solve an equation to calculate the amount of 5% sulfuric acid solution Keisha added if the resulting solution is a 12% sulfuric acid solution. Let x represent the amount of 5% sulfuric acid that Keisha added.

$$\frac{4 + 0.05x}{20 + x} = 0.12$$

$$x \neq -20$$

$$4 + 0.05x = 0.12(20 + x)$$

$$4 + 0.05x = 2.4 + 0.12x$$

$$1.6 = 0.07x$$

$$x \approx 22.9$$

Keisha added approximately 22.9 mL of 5% sulfuric acid solution to create a solution that consists of 12% sulfuric acid.

10

Problem 3

Distance problems are modeled using rational equations. Students will create expressions and use them to solve rational equations that determine the average speed of the current, the average speed of a barge, and the travel times.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

10

Guiding Questions for Share Phase, Question 1

- How did you determine the time traveled with the current?
- How did you determine the time traveled against the current?
- Is the time traveled with the current plus the time traveled against the current equivalent to 20 hours?
- What method was used to solve the rational equation that calculates the average speed of the current?
- What are the restrictions?

PROBLEM 3 Are We There Yet?

A distance problem is a type of problem that involves distance, rate, and time.

1. A river barge travels 140 miles from a loading dock to a warehouse to deliver supplies. Then the barge returns to the loading dock. The barge travels with the current to the warehouse and against the current from the warehouse. The barge's total travel time is 20 hours, and it travels in still water at an average speed of 15 miles per hour.

a. Use the given information to complete the table. Let x represent the average speed of the current.

	Distance Traveled	Time Traveled	Average Speed
	Miles	Hours	$\frac{\text{Miles}}{\text{Hours}}$
With the Current	140	$\frac{140}{15+x}$	$15+x$
Against the Current	140	$\frac{140}{15-x}$	$15-x$
Round Trip	280	20	$\frac{280}{20}=14$

b. You are given that the barge's total travel time is 20 hours. Write an algebraic expression, in terms of the number of hours the barge travels with the current and the number of hours it travels against the current, that is equivalent to 20 hours.

$$\frac{140}{15+x}+\frac{140}{15-x}$$

c. Write and solve an equation to calculate the average speed of the current.

$$\frac{140}{15+x}+\frac{140}{15-x}=20$$

$$x \neq -15, 15$$

$$(15+x)(15-x)\left(\frac{140}{15+x}+\frac{140}{15-x}\right)=20(15+x)(15-x)$$

$$140(15-x)+140(15+x)=20(225-x^2)$$

$$4200=4500-20x^2$$

$$-300=-20x^2$$

$$15=x^2$$

$$3.87\approx x$$

The average speed of the current is approximately 3.87 miles per hour.

Grouping

Have students complete Question 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- What does x represent in this situation?
- Is the barge's trip to the warehouse with the current or against the current?
- Is the barge's trip back to the loading dock with the current or against the current?
- What equations are used to determine the barge's average speed each way?
- What equation is used to determine the time it takes the barge to travel each way?
- What information is needed to calculate the average speed of the barge for the entire trip?
- What information is needed to calculate the barge's total travel time?

2. Calculate each value.

a. What is the barge's average speed during its trip to the warehouse?

$15 + x = 15 + 3.87$

$= 18.87$

The barge's average speed during its trip to the warehouse is about 18.87 miles per hour.

b. What is the barge's average speed during its trip back to the loading dock?

$15 - x = 15 - 3.87$

$= 11.13$

The barge's average speed during its trip back to the loading dock is about 11.13 miles per hour.

c. How long does it take the barge to get from the loading dock to the warehouse?

$\frac{140}{15 + x} = \frac{140}{15 + 3.87}$

$= \frac{140}{18.87} \approx 7.42$

It takes the barge about 7.42 hours to get from the loading dock to the warehouse.

d. How long does it take the barge to return to the loading dock from the warehouse?

$\frac{140}{15 - x} = \frac{140}{15 - 3.87}$

$= \frac{140}{11.13} \approx 12.58$

It takes the barge about 12.58 hours to return to the loading dock from the warehouse.

e. Use your answers to parts (a) and (b) to calculate the average speed of the barge in still water. Verify that your answer matches the given information.

$\frac{18.87 + 11.13}{2} = \frac{30}{2} = 15$

The average speed of the barge in still water is 15 miles per hour, which matches the given information.

f. Use your answers to parts (c) and (d) to calculate the barge's total travel time. Verify that your answer matches the given information.

$7.42 + 12.58 = 20$

The barge's total travel time is approximately 20 hours, which matches the given information.

10

Problem 4

Cost problems are modeled using rational equations. Students will create expressions and use them to solve rational equations that determine the average annual cost of owning a new refrigerator for specified number of years, for an Energy Star refrigerator and a non-certified refrigerator. Comparisons are made to determine the best buy or greatest savings over a period of time.

10

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- How did you determine the average annual cost of owning the new Energy Star refrigerator?
- How did you determine the algebraic expression representing the average annual cost of owning the Energy Star refrigerator for *x* years?
- What inequality is used to determine when Melinda can shop for a new television?

PROBLEM 4 How Much Is It?

A cost problem is a type of problem that involves the cost of ownership of an item over time.

Melinda has decided that it is time to replace her old refrigerator. She purchases a new Energy Star certified refrigerator. Energy Star certified refrigerators use less electricity than those that are not certified. In the long run, the Energy Star refrigerator should cost Melinda less to operate.

1. Melinda purchases a new Energy Star refrigerator for \$2000. The refrigerator costs \$46 per year to operate.

   a. Assume that the refrigerator is reliable and its only costs of ownership are the purchase price and the cost of operation. Determine Melinda's average annual cost of owning the new refrigerator for the given number of years.

   1 year: $\frac{2000 + 46(1)}{1} = \frac{2046}{1}$

   $= 2046$

   Melinda's average annual cost of ownership for 1 year is \$2046.

   5 years: $\frac{2000 + 46(5)}{5} = \frac{2230}{5}$

   $= 446$

   Melinda's average annual cost of ownership for 5 years is \$446.

   10 years: $\frac{2000 + 46(10)}{10} = \frac{2460}{10}$

   $= 246$

   Melinda's average annual cost of ownership for 10 years is \$246.

   b. Write an expression to represent Melinda's average annual cost of owning the new refrigerator for *x* years.

   $\frac{2000 + 46x}{x}$

   c. When Melinda's average annual cost of owning the refrigerator is less than \$400, she plans to shop for a new television. When can Melinda shop for a new television?

   $\frac{2000 + 46x}{x} < 400$

   $2000 + 46x < 400x$

   $2000 < 354x$

   $5.6497 < x$

   Melinda can shop for a new television in about 6 years.

Guiding Questions for Share Phase, Questions 2 and 3

- How did you determine the average annual cost of owning the new non-certified refrigerator?
- How did you determine the algebraic expression representing the average annual cost of owning the non-certified refrigerator for *x* years?
- Can a graphing calculator be used to compare the cost of owning each type of refrigerator? How?
- Graphically, what happens when the cost of owning each type of refrigerator is the same?
- What method was used to solve this problem algebraically?
- Are there any restrictions?

2. Melinda is curious to know how much money the Energy Star certified refrigerator will save her, compared to one that is not certified. A comparable non-certified model costs $1900 to purchase and $60 per year to operate.

a. Assume that this non-certified refrigerator's only costs of ownership are the purchase price and the operation costs. Determine the average annual cost of owning this refrigerator in the given number of years.

1 year: $\frac{1900 + 60(1)}{1} = \frac{1960}{1}$

$= 1960$

The average annual cost of ownership for 1 year is $1960.

5 years: $\frac{1900 + 60(5)}{5} = \frac{2200}{5}$

$= 440$

The average annual cost of ownership for 5 years is $440.

10 years: $\frac{1900 + 60(10)}{10} = \frac{2500}{10}$

$= 250$

The average annual cost of ownership for 10 years is $250.

b. Write an expression to represent the average annual cost of owning the non-certified refrigerator for *x* years.

$\frac{1900 + 60x}{x}$

3. In how many years will the average annual cost of owning the Energy Star certified refrigerator be less than the average annual cost of owning the non-certified refrigerator? Show all of your work.

$\frac{2000 + 46x}{x} = \frac{1900 + 60x}{x}$

$x \neq 0$

$x(2000 + 46x) = x(1900 + 60x)$

$46x^2 + 2000x = 60x^2 + 1900x$

$-14x^2 + 100x = 0$

$x = \frac{-100 \pm \sqrt{(100)^2 - 4(-14)(0)}}{2(-14)}$

$x \approx \frac{-100 \pm 100}{-28}$

$x = 0, x = 7.14$

$x \neq 0$ is a restriction.

In 7.14 years, the average annual costs will be equal. So, in 8 years, the average annual cost of owning the certified refrigerator will be less than the average annual cost of owning the non-certified refrigerator.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Describe and correct the error in each problem.

1. $\frac{2}{3} + \frac{4}{x} = 1$

$3x\left(\frac{2}{3} + \frac{4}{x}\right) = 1$

$2x + 12 = 1$

$2x = -11$

$x = -\frac{11}{2}$

The error is on the second line, not multiplying both sides of the equation by 3*x*.

2. $\frac{10}{100 + x} = 5\%$

$\frac{10}{100 + x} = \frac{5}{100}$

$1000 = 500 + x$

$500 = x$

The error is on the third line, not distributing 5 through the *x*-term.

3. $\frac{120 + 80 \cdot x}{x} = 200$

$200 = 200$

The error is that *x* is not a factor in each term on the left side of the equation, so it cannot be divided out.

10

Chapter 10 Summary

KEY TERMS

- rational equation (10.3)
- extraneous solutions (10.3)

10.1 Determining the Least Common Denominator (LCD)

To add or subtract rational numbers, a common denominator is needed. To add or subtract rational expressions, a common denominator is also needed. Determining a common denominator is done in much the same way for both rational numbers and rational expressions.

Example

Rational Numbers	Rational Expressions
$\frac{5}{6}+\frac{3}{4}=\frac{5}{2(3)}+\frac{3}{2(2)}$ $=\frac{5(2)}{2(3)(2)}+\frac{3(3)}{2(2)(3)}$ $=\frac{10}{12}+\frac{9}{12}$ $=\frac{19}{12}$ The least common denominator is 12.	$\frac{7}{2x}+\frac{5}{6x^2}=\frac{7}{2x}+\frac{5}{2(3)x^2}$ $=\frac{7(3)x}{2x(3)x}+\frac{5}{2(3)x^2}$ $=\frac{21x}{6x^2}+\frac{5}{6x^2}$ $=\frac{21x+5}{6x^2}$ The least common denominator is $6x^2$.

10.1 Determining Restrictions for the Value of *x*

When there are variable(s) in the denominator of a rational expression, the value of the variable cannot equal a value that will produce a zero in the denominator.

Example

$\frac{6x+1}{5x^2-10x}=\frac{6x+1}{5x(x-2)}$

$5x(x-2)=0$

$5x=0,\ x-2=0$

$x=0,\ x=2$

The restrictions on the value of x are $x \neq 0$ or $x \neq 2$, since the value of the denominator in the rational expression $\frac{6x+1}{5x^2-10x}$ would equal zero.

10.1 Adding Rational Expressions

To add rational expressions, determine a common denominator and then add. Identify any restrictions on the value of the variable and simplify when possible.

Example

$$\frac{2}{x} + \frac{5}{x+2} = \frac{2(x+2)}{x(x+2)} + \frac{5(x)}{x(x+2)}$$
$$= \frac{2x+4}{x(x+2)} + \frac{5x}{x(x+2)}$$
$$= \frac{7x+4}{x(x+2)};\ x \neq -2, 0$$

10.1 Subtracting Rational Expressions

To subtract rational expressions, determine a common denominator and then subtract. Identify any restrictions on the value of the variable and simplify when possible.

10

Example

$$\frac{4x}{2x+6} - \frac{3}{x+3} = \frac{4x}{2(x+3)} - \frac{3}{x+3}$$
$$= \frac{4x}{2(x+3)} - \frac{3(2)}{2(x+3)}$$
$$= \frac{4x}{2(x+3)} - \frac{6}{2(x+3)}$$
$$= \frac{4x-6}{2(x+3)}$$
$$= \frac{\cancel{2}(2x-3)}{\cancel{2}(x+3)}$$
$$= \frac{2x-3}{x+3};\ x \neq -3$$

10.2 Multiplying Rational Expressions

Multiplying rational expressions is similar to multiplying rational numbers. Factor each numerator and denominator, simplify if possible, and then multiply. Finally, describe any restriction(s) on the variable(s).

Example

$$\frac{4x^3}{9} \cdot \frac{3}{2x} = \frac{\overset{2}{\cancel{4}}\,\overset{x^2}{\cancel{x^3}}}{\underset{3}{\cancel{9}}} \cdot \frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{2}}\,\underset{1}{\cancel{x}}}$$
$$= \frac{2x^2}{3};\ x \neq 0$$

10.2 Dividing Rational Expressions

Dividing rational expressions is similar to dividing rational numbers. Rewrite division as multiplication by using the reciprocal, factor each numerator and denominator, simplify if possible, and then multiply. Finally describe any restriction(s) for the variable(s).

Example

$$\frac{x-2}{x^2-9} \div \frac{2x-4}{x-3} = \frac{x-2}{x^2-9} \cdot \frac{x-3}{2x-4}$$

$$= \frac{\overset{1}{\cancel{x-2}}}{\underset{1}{\cancel{(x-3)}}(x+3)} \cdot \frac{\overset{1}{\cancel{x-3}}}{2\underset{1}{\cancel{(x-2)}}}$$

$$= \frac{1}{2(x+3)};\ x \neq -3, 2, 3$$

10.3 Solving Rational Equations Using Proportional Reasoning

10

A rational equation is an equation containing one or more rational expressions. If a rational equation is in the form of a proportion with only one factor in the denominator on either side of the equation, it can be solved by using proportional reasoning. After using proportional reasoning and solving, describe any restrictions for the value of the variable, check each solution, and identify any extraneous solutions should they occur.

Example

$$\frac{2}{x} = \frac{1}{x-4}$$

Restrictions: $x \neq 0, 4$

$$2(x-4) = x(1)$$
$$2x - 8 = x$$
$$x - 8 = 0$$
$$x = 8$$

Check $x = 8$.

$$\frac{2}{8} \stackrel{?}{=} \frac{1}{8-4}$$

$$\frac{1}{4} = \frac{1}{4}$$

10.3 Solving Rational Equations by Multiplying Both Sides of the Equation by the Least Common Denominator

One method of solving a rational equation is to multiply both sides of the equation by the least common denominator and then solve the resulting equation for the variable. After solving, describe any restrictions for the variable, check each solution, and identify any extraneous solutions should they occur.

Example

$$\frac{7}{x-3} = \frac{3}{x-4} + \frac{1}{2}$$

Restrictions: $x \neq 3, 4$

$$2(x-3)(x-4)\left(\frac{7}{x-3}\right) = 2(x-3)(x-4)\left(\frac{3}{x-4} + \frac{1}{2}\right)$$

$$14x - 56 = (6x - 18) + (x^2 - 7x + 12)$$

$$14x - 56 = x^2 - x - 6$$

$$x^2 - 15x + 50 = 0$$

$$(x-5)(x-10) = 0$$

$$x = 5 \text{ or } x = 10$$

Check $x = 5$.

$$\frac{7}{5-3} \stackrel{?}{=} \frac{3}{5-4} + \frac{1}{2}$$

$$\frac{7}{2} \stackrel{?}{=} 3 + \frac{1}{2}$$

$$3\frac{1}{2} = 3\frac{1}{2}$$

Check $x = 10$.

$$\frac{7}{10-3} \stackrel{?}{=} \frac{3}{10-4} + \frac{1}{2}$$

$$\frac{7}{7} \stackrel{?}{=} \frac{3}{6} + \frac{1}{2}$$

$$1 = 1$$

10.3 Solving Rational Equations Using a Graphing Calculator

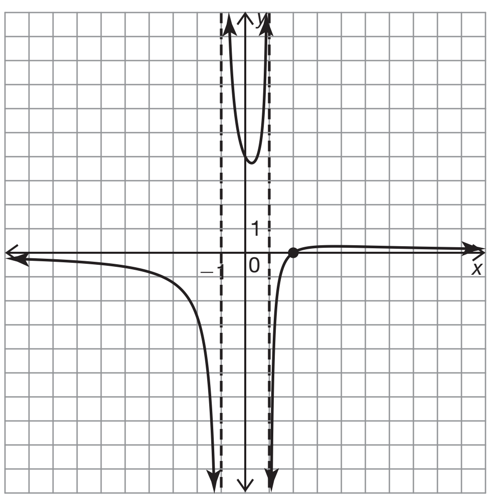
A graph can also be used to solve a rational equation. Rewrite the equation so that one side of the equation equals zero. Then set the non-zero side, $f(x)$, equal to y and graph $y = f(x)$. The vertical asymptotes indicate restrictions on the variable x. The x-intercept(s) of the graph are solution(s) to the original equation.

Example

$$\frac{x}{x-1} + \frac{x-5}{x^2-1} = 1$$

Rewrite the equation so that one side equals 0, then graph $y = \frac{x}{x-1} + \frac{x-5}{x^2-1} - 1$.

The graph shows that $x = -1$ and $x = 1$ are vertical asymptotes and thus represent restrictions on the variable. The graph also shows that $x = 2$ is a possible solution to the original rational equation.



Check $x = 2$.

$$\frac{2}{2-1} + \frac{2-5}{2^2-1} \stackrel{?}{=} 1$$

$$2 + \frac{-3}{3} \stackrel{?}{=} 1$$

$$2 + (-1) \stackrel{?}{=} 1$$

$$1 = 1$$

10.3 Identifying Extraneous Roots

When both sides of a rational equation are multiplied by an expression that contains a variable, additional solutions may be introduced. These additional solutions are called extraneous solutions. Extraneous solutions are solutions that result from the process of solving an equation, but are not valid solutions to the original equation.

Example

$$\frac{5}{x-1} = \frac{10}{x^2-1}$$

Restriction: $x \neq -1, 1$

$$\frac{5}{x-1} = \frac{10}{x^2-1}$$

$$\frac{5}{x-1} = \frac{10}{(x-1)(x+1)}$$

$$(x-1)(x+1)\left[\frac{5}{x-1}\right] = (x-1)(x+1)\left[\frac{10}{(x-1)(x+1)}\right]$$

$$5(x+1) = 10$$

$$x + 1 = 2$$

$$x = 1$$

However $x \neq 1$ since it is a restriction on the variable and thus is an extraneous solution. Therefore, this equation has no solution.

10.4 Writing and Solving an Equation that Best Models a Work Problem

A work problem is a type of problem that involves the rates of several workers and the time it takes to complete a job. To solve a work problem, define an appropriate variable, set up an equation that best models the situation, and solve the equation.

Example

Ramiro can rake a lawn in 2.5 hours; Zelda can rake the same lawn in 2 hours. If they rake the lawn together, how long will it take them to rake the lawn?

Let x represent the number of hours it will take to rake the lawn while working together.

$$\frac{x}{2.5} + \frac{x}{2} = 1$$

$$5\left(\frac{x}{2.5} + \frac{x}{2}\right) = 5(1)$$

$$2x + 2.5x = 5$$

$$4.5x = 5$$

$$x = 1\frac{1}{9} \text{ hours}$$

Working together, it will take Ramiro and Zelda $1\frac{1}{9}$ hours to rake the lawn.

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10.4 Writing and Solving an Equation that Best Models a Mixture Problem

A mixture problem is a type of problem that involves the combination of two or more liquids and the concentration of these liquids. To solve a mixture problem, define an appropriate variable, set up an equation that best models the situation, and solve the equation.

Example

A chemist has 5 liters of a 30% brine solution. She needs to create a solution containing 25% brine by mixing the 5 liters with a second solution containing 20% brine. How much of the second solution should she use?

Let x represent the number of liters of the 20% brine solution needed.

$$\frac{0.3(5) + 0.2x}{5 + x} = 0.25$$

$$1.5 + 0.2x = 1.25 + 0.25x$$

$$0.25 = 0.05x$$

$$x = 5 \text{ liters}$$

The chemist needs to add 5 liters of the 20% brine solution to obtain the desired mixture.

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10.4 Writing and Solving an Equation that Best Models a Distance Problem

A distance problem, is a type of problem that involves distance, rate, and time. To solve a distance problem, define an appropriate variable, set up an equation that best models the situation, and solve the equation.

Example

When there is no wind, Sierra rides her bike at a rate of 15 miles per hour. Last Monday she rode her bike for 6 miles into a headwind and then returned home with the wind at her back. If the entire trip took 1.5 hours, what was the speed of the wind?

Let r represent the speed of the wind.

$$\frac{6}{15 + r} + \frac{6}{15 - r} = 1.5$$

$$(15 + r)(15 - r)\left(\frac{6}{15 + r} + \frac{6}{15 - r}\right) = 1.5(15 + r)(15 - r)$$

$$6(15 - r) + 6(15 + r) = 1.5(225 - r^2)$$

$$180 = 1.5(225 - r^2)$$

$$120 = 225 - r^2$$

$$105 = r^2$$

$$r \approx \pm 10.247; \text{ choose } 10.247 \text{ miles per hour}$$

The speed of the wind was approximately 10.247 miles per hour.

10.4 Writing and Solving an Equation or Inequality that Best Models a Cost Problem

A cost problem is a type of problem that involves the cost of ownership of an item over time. To solve a cost problem, define an appropriate variable, set up an equation or inequality that best models the situation, and solve the equation or inequality.

Example

Tobias purchased a new electric range for $1100 and was told by the salesperson that the yearly average operating cost of the range was approximately $110. In what year of ownership will Tobias's average annual cost of owning the range be $210?

Let x be the year in which Tobias's average annual cost of owning the range is $210.

$$\frac{1100 + 110x}{x} = 210$$

$$1100 + 110x = 210x$$

$$1100 = 100x$$

$$x = 11 \text{ years}$$

In the 11th year of ownership, the average annual cost of owning the range will be $210.

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