Searching for Patterns

3

The Aquarium of the Pacific's Watershed Exhibit shows the intricacies of the Los Angeles flood channel system. Originally, many of the channels were small streams, but were converted to concrete flood channels. The impacts of this change help create what the City of Angels is today.

3

Chapter 3 Overview

This chapter begins with opportunities for students to analyze and describe various patterns. Questions ask students to represent algebraic expressions in different forms and use algebra and graphs to determine whether they are equivalent. Lessons provide opportunities for students to identify linear, exponential, and quadratic functions using multiple representations. Lessons introduce students to the concept of building new functions on a coordinate plane by operating on separate functions.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
3.1	Exploring and Analyzing Patterns	A.SSE.1.a A.SSE.1.b A.SSE.2 A.APR.1 F.IF.8.b F.BF.1.b	1	This lesson provides opportunities for students to analyze sequences and describe patterns in various problem situations. Questions ask students to sketch additional terms, or designs, in each sequence and answer questions relevant to each problem situation.	x				
3.2	Using Patterns to Generate Algebraic Expressions	A.SSE.1.a A.SSE.1.b A.SSE.2 A.APR.1 F.IF.8.b F.FB.1.b	1	This lesson revisits the three pattern situations from the previous lesson and asks students to write algebraic expressions for each situation. Peer work is used to present different expressions that represent the same situation. Questions ask students to use algebra and a graphing calculator to prove whether expressions are equivalent or not.	x		x	x	x
3.3	Comparing Multiple Representations of Functions	A.SSE.1.a A.SSE.1.b A.CED.1 A.CED.2 F.IF.4 F.BF.1.b	1	This lesson presents various graphs, tables, equations, and contexts for students to analyze and sort into groups of equivalent relations. The basic concepts of a function and function notation are reviewed. Questions then ask students to analyze a quadratic tile pattern to continue to develop their understanding of equivalent relationships.	x	x	x	x	

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
3.4	Modeling with Functions	A.SSE.1.b A.SSE.2 A.APR.3 A.REI.11	1	This lesson presents a problem situation involving a quadratic for students to model using a table, a function, and a graph. Questions lead students to conclude the quadratic function was created by the product of two linear functions. A second problem situation is presented without scaffolding.	x				
3.5	Analyzing Graphs to Build Functions	A.SSE.1.b A.CED.2 F.IF.5 F.IF.7.a F.IF.7.c	2	This lesson explores operations on functions graphically. Questions ask students to predict and verify graphical behavior of functions. Questions then guide students to consider key points of different graphs of functions and sketch the sum, difference, and product of two functions on a coordinate plane.	x				

3

Skills Practice Correlation for Chapter 3

Lesson		Problem Set	Objectives
3.1	Exploring and Analyzing Patterns	1 – 6	Draw the next three terms of a pattern
		7 – 12	Answer questions about given patterns
		13 – 18	Determine the next number in a given sequence
3.2	Using Patterns to Generate Algebraic Expressions	1 – 6	Write an expression to represent a given pattern
		7 – 12	Determine whether two expressions are equivalent
		13 – 18	Represent a pattern as an expression, a graph, and then identify the function family
3.3	Comparing Multiple Representations of Functions		Vocabulary
		1 – 6	Determine whether two functions are equivalent
		7 – 12	Model a given problem situation using a table, a graph, and a function
		13 – 18	Write expressions and answer questions for given problem situations
		19 – 24	Determine whether two expressions are equivalent
3.4	Modeling with Functions	1 – 6	Complete tables from given situations, and then graph the function
		7 – 12	Define a function to represent a given problem situation, graph the function, and then answer a question
3.5	Analyzing Graphs to Build New Functions		Vocabulary
		1 – 6	Add two functions graphically
		7 – 12	Sketch a new function from a given relationship, and then use a table of values to verify the result
		13 – 18	Use algebra to verify the sum of two functions is equivalent to a third function

3

3.1

Patterns: They're Grrrrrowing!

Exploring and Analyzing Patterns

LEARNING GOALS

In this lesson, you will:

- Identify multiple patterns within a sequence.
- Use patterns to solve problems.

ESSENTIAL IDEAS

- Sequences are used to show observable patterns.
- Patterns are used to solve problems.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity.
2. Use the structure of an expression to identify ways to rewrite it.

A-APR Arithmetic with Polynomials and Rational Expressions

Perform arithmetic operations on polynomials

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

F-IF Interpreting Functions

Analyze functions using different representations

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   b. Use the properties of exponents to interpret expressions for exponential functions.

F-BF Building Functions

Build a function that models a relationship between two quantities

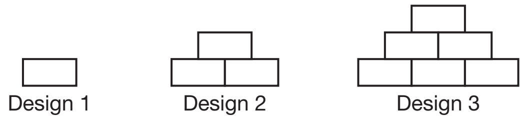
1. Write a function that describes a relationship between two quantities.
   b. Combine standard function types using arithmetic operations.

Overview

Tiling patterns on floors, keeping secrets, and patio designs are used to illustrate sequences described by observable patterns. Students will analyze sequences and describe observable patterns. They sketch other terms or designs in each sequence using their knowledge of the patterns, and then will answer questions relevant to the problem situation. In one situation, a table is used to organize data and help recognize patterns as they emerge.

Warm Up

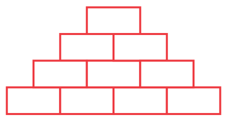
Consider the block designs.



1. Describe Design 4.

   The bottom row of the block design contains four blocks, the second row contains three blocks, the third row contains two blocks, and the top row contains one block.

2. Draw the Design 4.



3. How many blocks are in Design 4?

   The fourth design contains $4 + 3 + 2 + 1$ or 10 blocks.

4. Describe the observable pattern.

   As the design number increases by one, the bottom row of blocks increases by one. Each row above the bottom row of the design has one less block. The top row has one block.

3

3

Patterns: They're Grrrrrowing!

3.1

Exploring and Analyzing Patterns

3

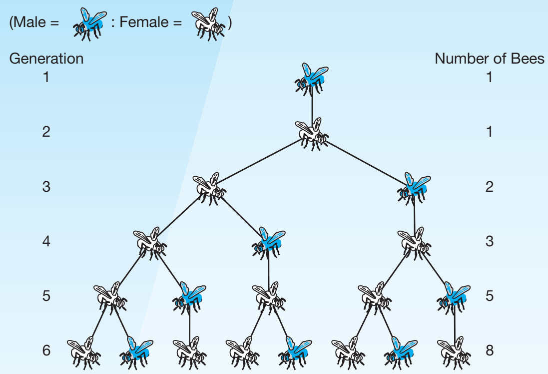
LEARNING GOALS

In this lesson, you will:

- Identify multiple patterns within a sequence.
- Use patterns to solve problems.

You can find patterns everywhere! Sometimes you can describe them in terms of color, shape, size or texture. Other times, a pattern's beauty isn't evident until you describe it using mathematics.

Let's consider a pattern found in nature—the family tree of a male drone bee. Female bees have two parents, a male and a female whereas male bees have just one parent, a female. In this family tree the parents appear below the original male drone bee.



The total number of bees in each generation follows the pattern:

$$1, 1, 2, 3, 5, 8, \ldots$$

What makes this particular pattern fascinating is that it seems to appear everywhere! This pattern is called the Fibonacci Sequence and you can find it in flowers, seashells, pineapples, art, architecture, and even in your DNA!

Do you see the pattern? If so, name the next three terms.

3

Problem 1

The first three terms of a sequence is shown to illustrate a pattern used to tile a square room. Students will analyze the designs and describe observable patterns. They sketch Design 5 in this sequence, and then describe the key features of Design 8. A table is used to organize the observable data for Designs 1 through 8. Finally, given a maximum number of tiles or the size of the floor, students will determine if the outer edge of the tile will be the same color as the center tile or if the outer edge of tile will match the color of the walls.

Grouping

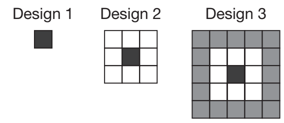
- Ask a student to read the paragraph written at the beginning of Problem 1. Discuss the context.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- How many colors are used in this pattern?
- In what order do the colors alternate in each new square?
- How many additional tiles are used along each edge as the design number increases?
- How many new tiles are added to each design?
- What is the relationship between the total number of squares and the number of tiles along the edge squared?

PROBLEM 1 There's More Than One Way to Tile a Floor

Terrance owns a flooring company. His latest job involves tiling a square room. Terrance's customer, Mr. Rivera, requests a tile pattern of alternating black, white, and gray tiles as shown. Each tile is one square foot.

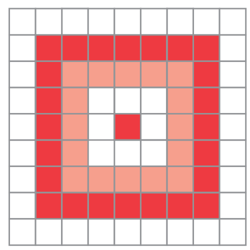


1. Analyze Terrance's design of a tile pattern for a square floor. Describe as many patterns as you can.

   Answers will vary.

   - The colors alternate with each new square: black, white, gray . . .
   - The number of tiles along each edge increases by two in each design: 1, 3, 5, . . .
   - The number of new tiles added to each design is a multiple of 8: 8, 16, . . .
   - The total number of squares is equal to the number of tiles along the edge squared: 1, 9, 25, 49, . . .

2. Sketch the design for a square floor that is 9 feet by 9 feet.



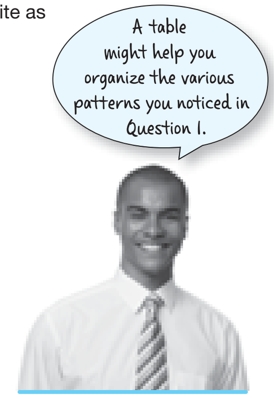
Guiding Questions for Share Phase, Question 3

- What is the total number of tiles in Design 8?
- What color is the outer edge in Design 8?
- What is the length of the sides of the outer most square in Design 8?
- How many black tiles are in Design 8?
- How many gray tiles are in Design 8?
- How many white tiles are in Design 8?
- How many black squares are in Design 8?
- How many white squares are in Design 8?
- How many gray squares are in Design 8?

3. Describe the key features of Design 8 of a square floor. Write as many key features as you can.

Answers will vary.

- There are a total of 225 tiles.
- The outer edge is made of white tiles.
- The square is 15 feet by 15 feet.
- There are 73 black tiles.
- There are 56 gray tiles.
- There are 96 white tiles.
- There are three squares of black tiles, three squares of white tiles, and two squares of gray tiles from the middle square to each edge.



Design	1	2	3	4	5	6	7	8
Square Dimensions	1 × 1	3 × 3	5 × 5	7 × 7	9 × 9	11 × 11	13 × 13	15 × 15
Edge Color	B	W	G	B	W	G	B	W
Number of Black Tiles	1	1	1	25	25	25	73	73
Number of White Tiles	0	8	8	8	40	40	40	96
Number of Gray Tiles	0	0	16	16	16	56	56	56
Total Tiles	1	9	25	49	81	121	169	225

3

Grouping

Have students complete Questions 4 and 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 and 5

- What algebraic expression can be used to represent the dimensions of the square?
- Why does the algebraic expression $2n - 1$ make sense to use to represent the square dimensions?
- What is the relationship between the square dimension and the Design number?
- If $2n - 1 = 45$, what does n equal or in other words, what is the Design number?
- Will Design 23 be a 45×45 square?
- What will be the color of the outer edge of Design 23?
- If the center tile was black, does the match the color of the outer edge of tiles in Design 23?
- If the room is 101 by 101 square, what is its Design number?
- If you start the outer most edge of tiles with white to match the wall color, what color tile will end up as the center tile?
- If the center tile is gray, will the outer edge of the tiles be white to match the walls?

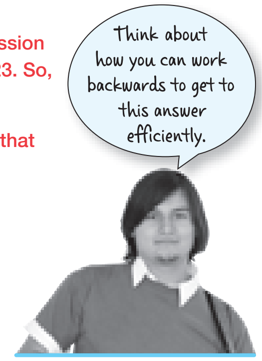
4. A hotel manager wants Terrance to tile their lobby using the same design he created for Mr. Rivera. The lobby measures 45 feet by 45 feet. He wants the outer edge to be the same color as the center tile. Will this occur? Justify your answer.

No. This will not occur.

The square dimensions can be represented by the expression $2n - 1$. Solving the equation $2n - 1 = 45$ results in $n = 23$. So, Design 23 will be 45×45.

Using the pattern of black, white, gray tiles I determined that the outer edge of Design 23 will be white.

The center tile is black so they do not match.



5. Very picky Paula Perkins requests a tile floor from Terrance. She also wants the alternating black, white, and gray tile pattern; however, she wants the outer edge of the tile to match her wall color. The room is 101 feet by 101 feet and the wall color is white. What color must the center tile be to ensure the outer edge is white? Show or explain your work.

To ensure that the outer edge is white, the center tile must be gray.

I first determined that a room that is 101 feet along the outer edge will be Design 51. Then I worked backward starting at white until I reached the first square. The first square must be gray to ensure the last square is white.

Problem 2

Three people are supposed to keep a secret. The next day each person shares the secret with two friends, and the next day those people share the secret with two of their friends, and so on. This pattern continues over a 7 day period of time. Students will create a visual model to represent the problem situation and describe the observable patterns. Next, they determine how many people share the secret on the fourth day. Finally, given the number of people in the senior class, students determine if every senior will know the secret on the 6th day and explain their reasoning.

Grouping

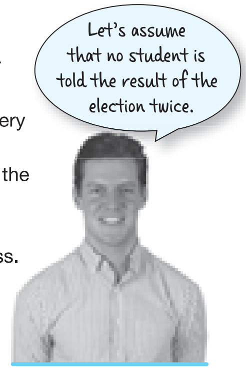
- Ask a student to read the paragraph written at the beginning of Problem 1. Discuss the context.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- One the first day, how many people shared the secret?
- One the second day, how many people shared the secret?
- One the third day, how many people shared the secret?

PROBLEM 2 Can You Keep a Secret?

The class president, vice president, and treasurer of a high school count the ballots for the homecoming king election. The election result is generally kept a secret until the pep rally, when the winner is announced in front of the entire senior class. Unfortunately, this year's ballot counters are not very good at keeping a secret. The very next day, each ballot counter tells two of their friends in the senior class the election result, but makes each friend vow not to spread the result. However, each of the ballot counter's friends cannot keep a secret either. The following day each friend of each ballot counter shares the election result with two of their friends in the senior class. This pattern continues for the entire week leading up to the pep rally.



1. Create a visual model to represent this problem situation. Describe the patterns you observe.

Answers will vary.

- The number of seniors who learn about the secret doubles each day.

Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7
3	6	12	24	48	96	192

- A tree diagram shows the doubling process.

x x x Day 1
x x x x x x Day 2
xx xx xx xx xx xx Day 3

2. How many new seniors will know the winner of the homecoming king election on the fourth day? Explain your reasoning.

The number of new seniors is 24 on the fourth day.

Answers will vary.

- I created a table that extended to day 4.
- I substituted 4 into the expression $3 \cdot 2^{n-1}$.

3. The total number of students in the senior class is 250. If the ballot counters knew the election result on Monday, will every senior already know the winner of the election when the result is announced at the pep rally 6 days later? Explain your reasoning.

No. Not every senior will know the election result.

After 6 days, the number of seniors who know the election result is $3 + 6 + 12 + 24 + 48 + 96 = 189$. On the 7th day, 192 new people learn about the election result and $189 + 192 = 381$, so not all the seniors will know the election result by next Monday. Most likely, though, almost all the seniors will already know the winner when the election result is announced at the pep rally next Monday.

- Does the number of people sharing the secret double each day?
- What algebraic expression can be used to represent the number of people sharing the secret on any given day?
- Why does the algebraic expression $3 \cdot 2^{n-1}$ make sense to represent the number of people sharing the secret on day n?
- How many people are in the senior class?
- How many people know the secret on day 7?

Problem 3

The first three terms of a sequence is shown to illustrate designs for a backyard patio. Students will analyze each design by describing observable patterns and sketch Design 6. Finally, given a maximum number of tiles, they then determine the largest patio design that is doable and explain their reasoning.

3

Grouping

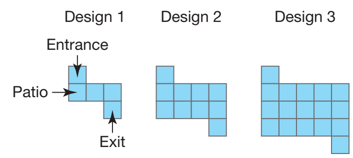
- Ask a student to read the paragraph written at the beginning of Problem 1. Discuss the context.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- As the Design number increases by one, what happens to the number of rows in the patio design?
- How many tiles are used in patio Design 1?
- What are the dimensions of the rectangle between the entrance and the exit in Design 1? In Design 2? In Design 3?
- What happens to the length of the rectangle between the entrance and the exit as the design number increases by 1? To the width?

PROBLEM 3 How Large Is Your Yard?

Maureen and Matthew are designing their backyard patio. There will be an entrance and exit off the front and back of the patio. The sequence shown represents different designs depending on the size of the patio.

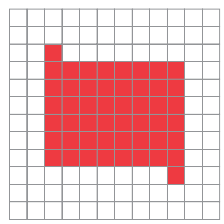


1. Analyze each design in the sequence. Describe as many patterns as you can.

   Answers will vary.

   - There is always a single tile representing the entrance and exit.
   - The number of rows in the patio increases by 1; 1, 2, 3, . . .
   - The number of squares in each row of the patio increases by 1; 3, 4, 5, . . .

2. Sketch Design 6 of the sequence.



3. Matthew has 180 tiles he can use for this project. Identify the largest patio design that he can make. Show or explain your reasoning.

   The largest design Matthew can make is Design 12. Design 12 will use 14 × 12 tiles for the patio plus 2 tiles for the entrance and the exit. That is a total of 170 tiles.

   Design 13 will use 15 × 13 tiles for the patio plus the 2 tiles for the entrance and exit. That is a total of 197 tiles which is too many.

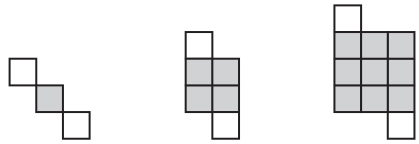
   I can also solve $n^2 + 2n + 2 = 180$ algebraically or graphically.

Be prepared to share your solutions and methods.

- How many tiles are used in Design 6? In Design 12? In Design 13?
- What algebraic expression can be used to determine the total number of tiles used in Design *n*?

Check for Students' Understanding

Consider the block designs.

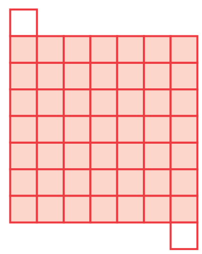


Design 1 Design 2 Design 3

1. Describe Design 7.

   A 7×7 block of squares with one block above the upper left most position of the square and one block below the lower right most position of the square.

2. Draw Design 7.



Design 7

3. How many squares are in Design 7?

   The seventh design contains $49 + 2$ or 51 squares

4. Write an algebraic expression to represent the total number of squares in Design *n*.

   There are $n^2 + 2$ squares in the *n*th Design.

5. Use your algebraic expression to determine the total number of squares in Design 52.

   There are $52^2 + 2$ or 2706 squares in Design 52.

3

3

3.2

Are They Saying the Same Thing?

Using Patterns to Generate Algebraic Expressions

LEARNING GOALS

In this lesson, you will:

- Generate algebraic expressions using geometric patterns.
- Represent algebraic expressions in different forms.
- Determine whether expressions are equivalent.
- Identify patterns as linear, exponential, or quadratic using a visual model, a table of values, or a graph.

ESSENTIAL IDEAS

- Two or more algebraic expressions are equivalent if they produce the same output for all input values.
- Algebra and graphs are used to prove two algebraic expressions are equivalent.
- A visual model, a table of values, and a graph are used to identify patterns as linear, exponential, or quadratic.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.

   a. Interpret parts of an expression, such as terms, factors, and coefficients.

   b. Interpret complicated expressions by viewing one or more of their parts as a single entity.

2. Use the structure of an expression to identify ways to rewrite it.

A-APR Arithmetic with Polynomials and Rational Expressions

Perform arithmetic operations on polynomials

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

F-IF Interpreting Functions

Analyze functions using different representations

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

   b. Use the properties of exponents to interpret expressions for exponential functions.

F-BF Building Functions

Build a function that models a relationship between two quantities

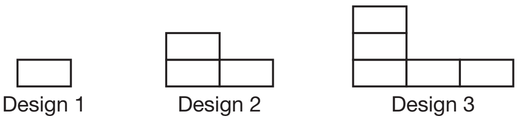
1. Write a function that describes a relationship between two quantities.
   b. Combine standard function types using arithmetic operations.

Overview

This lesson revisits the three scenarios from the previous lesson. Students will write equivalent algebraic expressions for the tile pattern of a square floor to determine the number of new tiles that must be added to create the next square tile design. They then show the expressions are equivalent using the distributive property and combining like terms. In the second activity, equivalent expressions are written to represent the exponential situation for keeping secrets. Students then prove the expressions to be equivalent algebraically and graphically. Next, using the patio design situation, students will determine the number of squares in the next two patio designs, and write equivalent expressions that determine the total number of squares in the patio Design *n*. Again, the expressions are algebraically and graphically proven equivalent. The last activity summarizes the lesson using a geometric pattern. Students will write equivalent expressions to represent the problem situation for Design *n* and show the expressions are equivalent both algebraically and graphically.

Warm Up

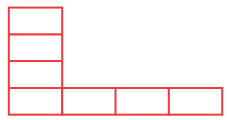
Consider the block designs.



1. Describe Design 4.

The bottom row of the block design contains four blocks, and the left most column contains four blocks.

2. Draw Design 4.



3. How many blocks are in Design 4?

Design 5 contains 7 blocks.

4. Describe the observable pattern.

As the design number increases by one, the bottom row of blocks increases by one and the column on the left increases by one.

3

3.2

Are They Saying the Same Thing?

Using Patterns to Generate Algebraic Expressions

LEARNING GOALS

In this lesson, you will:

- Generate algebraic expressions using geometric patterns.
- Represent algebraic expressions in different forms.
- Determine whether expressions are equivalent.
- Identify patterns as linear, exponential, or quadratic using a visual model, a table of values, or a graph.

3

Are natural habits hard to break? The answer for most grocery stores would be, "Why in the world would we break these habits?" This is the reason why many grocery stores have followed a tried-and-true way for laying out their items in the aisles. Studies have shown that most Americans tend to prefer to shop in a counter-clockwise pattern; thus, most grocery stores have their produce at the front and to the right of the entrance which then leads (in a counter-clockwise manner) toward the bakery. And more cleverly, the bakery is toward the middle or the back of the store. From here, many stores lead you to the meat section and then the dairy section. So, why are the bakery, meat, and dairy sections toward the back of the store? Once again, grocery stores embrace people's natural tendencies. For most families, the most needed items are meats, breads, and milk. So, when these items are toward the back of the store, it provides more chances for customers to make "impulse" purchases along the way—buying things that weren't on the original grocery list!

While scientists don't know what causes this impulse (moving in a counter-clockwise manner, or buying items that aren't necessarily needed), it is extremely strong.

This impulse to move in a counter-clockwise direction can be thought of as a pattern similar to animal migrations. Is this the only way to get to where you are going? Of course not, but for some reason, it seems to be a more comfortable path. When problem solving in mathematics there are often many ways for you to approach a problem, but usually you choose a familiar method. Do you usually find one way to do something and then stick with it, or do you look for different methods?

Problem 1

Using the tile pattern from the previous lesson, students will write an expression to represent the number of new tiles that must be added to an n by n square floor design where n is a positive odd value. Students then analyze four equivalent expressions to determine the number of new tiles added to Design 2 to create Design 3, using the n by n square model. Finally, students show all algebraic expressions are equivalent by combining like terms and using the distributive property.

Grouping

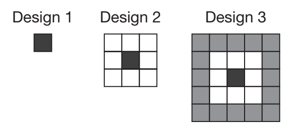
- Ask a student to read the paragraph written at the beginning of Problem 1. Discuss the context.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- How many colors are used in this pattern?
- In what order do the colors alternate in each new square?
- The number of new tiles in each new square floor design increases by how many each time?
- Are all values of n odd values? Why?

PROBLEM 1 Floors by Terrance

Terrance's flooring business from the problem, *There's More Than One Way to Tile a Floor*, was booming! He decides to hire several employees to help lay out his tile designs. It will be necessary for Terrance to describe his tile designs in a clear manner so that all of the employees can create them correctly. Recall that Terrance's square floor design uses alternating black, white, and gray tiles.



1. Describe the pattern in terms of the number of new tiles that must be added to each new square floor design.

   The number of new tiles in each new square floor design increases by 8 each time.

   The number of tiles added to Design 1 to create Design 2 is 8; the number of tiles added to Design 2 to create Design 3 is 16; . . .

2. Write an expression to represent the number of new tiles that must be added to an n by n square floor design. Let n represent the number of tiles along each edge of the square.

   Possible expressions are:

   $4n + 4$

   $2(n + 2) + 2n$

   $(n + 2)^2 - n^2$

   $n + n + n + n + 4$

   $4(n + 2) - 4$

3. Describe which values for n make sense in this problem situation?

   Positive odd values for n make sense in this problem situation.

- Are all values of n positive values? Why?

Grouping

Have students complete Questions 4 and 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 and 5

- In Ramone's expression, what does *n* represent? What is *n* supposed to represent?
- Does the pattern increase by a constant in this situation?
- If a pattern increases by a constant, what does this tell you about the pattern?

4. Ramone determined an expression to represent this pattern. His expression and explanation are shown.

Ramone

Design	1	2	3
New Tiles	0	8	16

The expression $8(n - 1)$ represents Terrance's square floor pattern. I noticed that the number of new tiles is increasing by 8 in each new design.

Explain why Ramone's expression is incorrect.

Ramone's expression is incorrect because he used the term number (1, 2, 3, . . .) for the Designs 1, 2, 3, . . . as *n* instead of the number of squares along each edge.

5. Describe the pattern as new tiles are added as linear, quadratic, exponential, or none of these. Explain your reasoning.

The pattern is linear. I know the pattern is linear because the pattern is increasing by a constant.

3

3

Grouping

Have students complete Questions 6 through 8 with a partner. Then have students share their responses as a class.

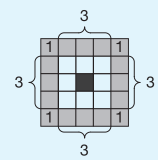
Guiding Questions for Share Phase, Questions 6 through 8

- What does n represent in each expression?
- When writing each algebraic expression, did you replace the value of 3 with the variable n in each situation?
- Whose expression will require use of the distributive property when simplifying?
- Whose expression will require combining like terms when simplifying?
- Whose expression will require binomial multiplication when simplifying?
- How do you simplify the expression $(n + 2)^2$?
- Using Tamara's expression, after multiplying the binomial and combining like terms is there an n^2 term remaining in the expression?

6. Terrance asks his employees to determine the number of new tiles added to Design 2 to create Design 3. Each employee describes a unique method to determine the number of additional tiles needed to create Design 3. Represent each of his employee's explanations with an algebraic expression that describes how many new tiles must be added to an $n \times n$ square to build the next design.

Wilma

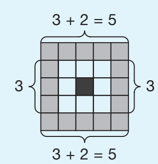
I must add 3 tiles to each of the four sides of the white square, which is 4 · 3 tiles. Then I must add 1 tile at each corner. So the number of additional tiles added to a Design 2 square floor design is 4 · 3 + 4.



Expression: $4n + 4$

Howard

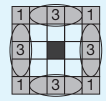
I must add 5 tiles to two of the sides and 3 tiles to the other two sides. The number of additional tiles added to Design 2 square floor design is 2(3 + 2) + 2 · 3.



Expression: $2(n + 2) + 2n$

Tyler

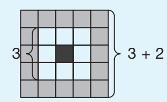
I need to add 3 tiles four times and then add the four corner tiles. The number of additional tiles added to Design 2 square floor design is 3 + 3 + 3 + 3 + 4.



Expression: $n + n + n + n + 4$

Tamara

The way I look at it, I really have two squares. The original square for Design 2 has 3 · 3 tiles. The newly formed Design 3 square has 5 · 5 tiles. So, the number of additional tiles added to the Design 2 square floor design is 5 · 5 − 3 · 3.



Expression: $(n + 2)^2 - n^2$

7. Which expression do you think Terrance should use? Explain your reasoning.

Answers will vary.

Wilma's method requires the least number of calculations.

Grouping

Have students complete Questions 9 and 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 9 and 10

- When 135 is substituted for n in each expression, what is the value of the expression?
- Do all of the expressions produce the same value?
- If all of the expressions result in the same value, what does this imply?
- Is it reasonable to assume all of the expressions are equivalent because you got the same total for $n = 135$? Why not?

3

8. Michael and Louise analyze the expressions they wrote for each student. They both determined that the expression to represent Tamara's method is $(n + 2)^2 - n^2$. Michael claims that this expression is quadratic because of the n^2 term. Louise disagrees and says the expression is linear because the pattern is linear.
Who is correct? Explain your reasoning.
Louise is correct. The expression is linear. Even though the given expression contains two squared terms, these terms add to zero when the expression is simplified which results in a linear expression.

$(n + 2)^2 - n^2 = n^2 + 4n + 4 - n^2$

$= 4n + 4$

9. Use each expression you determined in Question 6 to calculate the number of tiles that must be added to squares with side lengths of 135 tiles to create the next design.

Wilma's expression:

$4n + 4$

$4(135) + 4 = 544$

Tyler's expression:

$n + n + n + n + 4$

$135 + 135 + 135 + 135 + 4 = 544$

Howard's expression:

$2(n + 2) + 2n$

$2(135 + 2) + 2(135) = 544$

Tamara's expression:

$(n + 2)^2 - n^2$

$(135 + 2)^2 - 135^2 = 544$

10. Wilma tells Terrance that since all of the expressions resulted in the same solution, any of the expressions can be used to determine the number of additional tiles needed to make more $n \times n$ designs. Terrance thinks that the employees need to use more values in the expressions than just one to make this conclusion.
Who is correct? Explain your reasoning.
Terrance is correct. The employees cannot state that the methods are all correct based on one additional design. The employees must know that the expressions are equal for all designs.

Grouping

Have students complete Question 11 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 11

- Whose expression is already in simplest terms?
- Combining like terms will show which expressions equivalent?
- Using the distributive property and then combining like terms will show which expressions equivalent?
- Does simplifying each expression result in the same expression? What is it?

Recall that two or more algebraic expressions are equivalent if they produce the same output for all input values. You can verify that two expressions are equivalent by using properties to rewrite the two expressions as the same expression.

11. Show that Wilma, Howard, Tyler, and Tamara's expressions are equivalent. Justify your reasoning.

I can combine like terms to show that Tyler's expression is equivalent to Wilma's expression.

$n + n + n + n + 4$

$4n + 4$

I can use the distributive property, and then combine like terms to show that Howard's expression is equivalent to Wilma's expression.

$2(n + 2) + 2n$

$2n + 4 + 2n$

$4n + 4$

I can square the binomial, and then combine like terms to show that Tamara's expression is equivalent to Wilma's expression.

$(n + 2)^2 - n^2$

$n^2 + 4n + 4 - n^2$

$4n + 4$

3

Problem 2

Using the keeping secrets situation from the previous lesson, students will create a visual model to represent the problem situation and describe the observable patterns. They then determine how many people share the secret on the fourth day. Finally, given the number of people in the senior class, students determine if every senior will know the secret on the 6th day and explain their reasoning.

3

Grouping

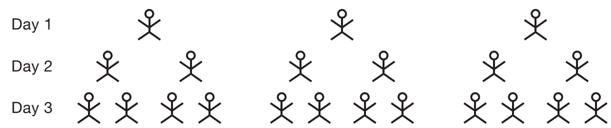
- Ask a student to read the paragraph written at the beginning of Problem 1. Discuss the context.
- Have students complete Question 1 part (a) with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1 part (a)

- Does the number of people sharing the secret double each day?
- What algebraic expression can be used to represent the number of people sharing the secret on any given day?
- Why does the algebraic expression $3 \cdot 2^{n-1}$ make sense to represent the number of people sharing the secret on day *n*?
- How many people are in the senior class?

PROBLEM 2 The Cat's Out of the Bag!

Let's revisit the problem, *Can You Keep a Secret?* about the homecoming king election. The visual model shown represents the number of new seniors who learn about the election result each day that passes.



1. Analyze the pattern.

   a. Complete the table to summarize the number of new seniors who learn about the election result each day. Then write an expression to represent the number of new seniors who learn about the election result on the *n*th day. Finally, describe how each part of your expression relates to the visual model.

Number of Days That Pass	Number of Seniors Who Hear the Results That Day
1	3
2	6
3	12
4	24
5	48
6	96
n	$3 \cdot 2^{n-1}$

The 3 represents the initial number of seniors who counted the ballots.
The base is 2 because the number of seniors who hear the results is doubling each day.

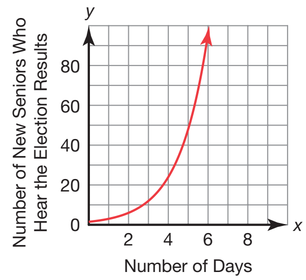
The expression 2^{n-1} represents the additional number of seniors that each senior from the previous day told about the election result.

- How many people know the secret on day 7?
- As the number of days increase by one, what happens to the number of seniors sharing the secret?
- Why is the base of the exponential in the expression 2?
- What does the expression 2^{n-1} represent?

Guiding Questions for Share Phase, Questions 1 part (b) through 4

- Does the graph appear to be linear, quadratic, or exponential? Why?
- Is the data discrete or continuous in this situation?
- Why is the data in this situation considered to be discrete data?
- What are the characteristics of the graph?
- Is the graph increasing or decreasing?
- Does the x-intercept have any relevance in this problem situation?
- If $y = 0$, what does that mean with respect to this problem situation?
- Does the y-intercept have any relevance in this problem situation?
- If $x = 0$, what does that mean with respect to this problem situation?

b. Create a graph of the data from your table on the coordinate plane shown. Then draw a smooth curve to model the relationship between the number of days that pass and the number of seniors who hear the senior election results.



2. Do all the points on the smooth curve make sense in terms of this problem situation? Why or why not?

All the points on the smooth curve do not make sense in terms of this problem situation because the number of days represents discrete data values. Only integers greater than 1 make sense in terms of this problem situation.

3. Describe this pattern as linear, exponential, quadratic, or none of these. Then write the corresponding equation. How does each representation support your answer?

$y = 3 \cdot 2^{n-1}$

This pattern is exponential. The equation can be written in the form $y = ab^x$.

$y = 3 \cdot 2^{(n-1)}$

$= 3 \cdot 2^n \cdot 2^{-1}$

$= 3 \cdot 2^n \cdot \frac{1}{2}$

$= \frac{3}{2} \cdot 2^n$

I can determine from the graph that this pattern is exponential because of the smooth, sharply increasing curve.

I can determine from the table of values that the pattern is exponential because the difference is multiplied by 2 each time.

I can determine that the pattern is exponential from the context because the number of people hearing the secret is doubling each time.

3

4. Describe the key characteristics of your graph. Explain each characteristic algebraically and in terms of this problem situation.

The x-intercept is where the graph crosses the x-axis. There is no x-intercept for this situation because the graph approaches the x-axis, but never crosses it. Contextually, the x-intercept represents the day when no student knew the election results. This doesn't make sense in this problem situation.

The y-intercept is (0, 1.5). Graphically, this represents the point where the graph crosses the y-axis. Contextually, the point represents the number of seniors learning the election result on day 0. This doesn't make sense in this problem situation.

The shape of the graph is a smooth curve that increases. Graphically, this occurs because the values are increasing. The values increase slowly at first, then more quickly. Contextually, this shape occurs because the number of seniors who learn of the election result starts growing slowly then increases more quickly.

3

Guiding Questions for Share Phase, Questions 5 and 6

- How do you use a graphing calculator to determine the number of new seniors that learn about the election results after 8 days?
- After 8 days, how many new seniors learn about the election results?
- After 9 days, how many new seniors learn about the election results?
- Does 6144 seniors seem like a reasonable number of seniors in one high school?

5. After how many days will 500 new seniors learn about the election results?

After 9 days, 500 new seniors will learn about the election results.

Using a graphing calculator, at 8.38 days the number of new seniors learning the election result is 500. This means that the number will be too small at 8 days, so I must round up.

I can also extend my table or use a guess-and-check method.

6. Determine the number of seniors who hear the election results on the twelfth day. Does your answer make sense in the context of this problem? Explain your reasoning.

There will be 6144 new seniors who will hear the secret on this day.

$3 \cdot 2^{11} = 6144$

This does not make sense in the context of this problem. The value is far too big. At some point in this problem the number of new seniors hearing the secret would have to level off.

Problem 3

Using the patio design problem from the previous lesson, students will determine the number of squares in the next two patio designs. They write an expression to determine the total number of squares in the patio Design *n* and explain their reasoning using the visual model. Two alternate expressions are given and students relate each expression to the visual model. They also graph the expressions to determine their equivalence. Next, they identify the parts of the graph that represent the problem situation. Students then determine a design that will contain at least 125 square feet by using the algebraic expression.

Grouping

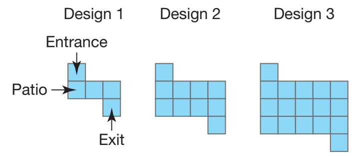
Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- How many squares are in Design 1? How many rows and columns?
- How many squares are in Design 2? How many rows and columns?
- How many squares are in Design 3? How many rows and columns?

PROBLEM 3 Several Spreading Sequences of Squares

Let's revisit the problem, *How Large Is Your Yard?* about backyard patio designs. The model shown represents the first three designs Maureen and Matthew could use. Each square represents 1 square foot.



1. Determine the number of squares in the next two patio designs of the pattern.

   There will be 26 squares in the 4th patio design, and there will be 37 squares in the 5th patio design.

2. Write an expression to determine the total number of squares in patio Design *n*. Describe how each part of your expression relates to the visual model.

   Answers will vary depending on the expression students write.

   $n^2 + 2n + 2$

   $n(n + 2) + 2$

   $2(n + 1) + n^2$

   $(n + 2)^2 - 2(n + 1)$

   $2(n + 2) + n^2 - 2$

- When comparing the total number of squares in each design, is the difference constant?
- Is there more than one way to write this expression correctly?
- How many different ways can this expression be written?
- Why is n^2 a necessary term in this expression?

3

Grouping

Have students complete Questions 3 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3, part (a)

- If Maureen visualized the entire design as a completed square, what expression could represent the top and side of the square pattern?
- What algebraic expression would represent the tiles that needed subtracted when completing the square?
  These are the tiles that were not part of the given pattern, they were only used to complete the square.

3

3. Maureen and Matthew each write different expressions to represent the patio designs.

a.

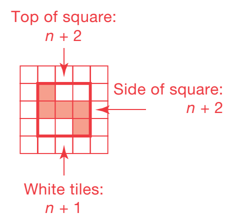
Maureen

$(n + 2)^2 - 2(n + 1)$

Maureen's expression uses subtraction. How can she take away tiles if the number of tiles in each term is increasing?

Describe how each term in Maureen's expression represents the visual model.

Maureen visualized each term of the pattern as a square. The top of the square and the side of the square are both represented as $n + 2$, so $(n + 2)^2$ represents the entire square. She then subtracted the white tiles that completed the square because they are not part of the given pattern. Each set of those tiles is represented as $n + 1$. Because there are two sets, $2(n + 1)$ represents both sets of white tiles that must be removed. When she subtracts these, the difference is the number of tiles in the figure.



Guiding Questions for Share Phase, Questions 3, part (b) and 4

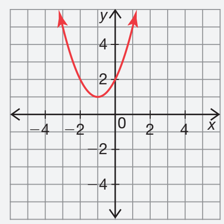
- Is the graph of Matthew's expression the same as the graph of Maureen's expression? What does this imply about the expressions?
- Is the data in this situation discrete or continuous? Why?
- Do negative values make sense in this situation? Why not?
- The graph of this situation lies in which quadrant on the coordinate plane?

b.

Matthew

$n^2 + 2n + 2$

Use a graphing calculator to graph each expression. Is Matthew's expression correct? Explain your reasoning in terms of the graph.



Matthew is also correct. The graphs of the two expressions are the same. Therefore, they are equivalent.

4. Identify the parts of the graph that represent this problem situation.

Only positive integer values make sense in this problem because they are using whole number of tiles.

Only the data values in Quadrant I make sense in terms of the pattern because I cannot have negative tiles.

3

Guiding Questions for Share Phase, Question 5

- What expression can be used to determine if the patio area is at least 125 square feet?
- Why are you able to use only the first two terms of the expression $n^2 + 2n + 2$ to determine the area of the patio?
- For Design 10, what value is substituted into the expression?
- For Design 11, what value is substituted into the expression?

3

Talk the Talk

The first four geometric designs of a sequence are shown. Students will identify two expressions that represent the total number of diamonds used to construct Design *n*. Next, they explain how the expressions relate to the visual model and algebraically prove the two expressions are equivalent. Students also graphically show the expressions are equivalent.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

5. In order to accommodate outdoor furniture, a grill, and a shed, the patio must have an area of at least 125 square feet (not including the walkways). What is the smallest design Matthew can build and still have enough space for these items?

Matthew will have to use the 11th design to ensure he has enough space for all the items.

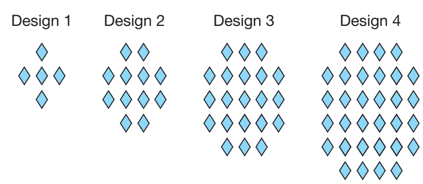
I used the expression $n^2 + 2n + 2$ to determine this answer.

First, I only looked at the $n^2 + 2n$ part of the expression because the +2 represents the walkways.

Then I determined that for Design 11, $11^2 + 2(11) = 143$ which is enough space. I also determined that for Design 10, $10^2 + 2(10) = 120$ which is not enough. So he must use the 11th design.

Talk the Talk

1. Analyze the pattern shown.



a. Identify two expressions that represent the total number of diamonds used to construct Design *n*.

Answers will vary.

Possible expressions are:

$n^2 + 4n$

$(n + 2)^2 - 4$

$n(n + 2) + 2n$

$n(n + 4)$

b. Describe how your expressions relate to the visual model.

Answers will vary depending on the expressions chosen.

c. Algebraically prove your expressions are equivalent.

Answers will vary depending on the expressions chosen.

Possible student responses.

$n(n + 2) + 2n = n^2 + 4n$

$n^2 + 2n + 2n = n^2 + 4n$

$n^2 + 4n = n^2 + 4n$

$(n + 2)^2 - 4 = n^2 + 4n$

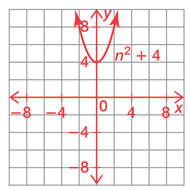
$n^2 + 4n + 4 - 4 = n^2 + 4n$

$n^2 + 4n = n^2 + 4n$

$n(n + 4) = n^2 + 4n$

$n^2 + 4n = n^2 + 4n$

d. Graphically show that your expressions are equivalent.



The graphs for my two expressions are the same.

2. Describe the ways you can prove any two expressions are equivalent.

I can prove two expressions are equivalent by rewriting one or both expressions to show it is the same as the other expression.

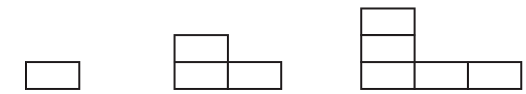
I can graph each expression and show that the graphs are the same.

Be prepared to share your solutions and methods.

3

Check for Students' Understanding

Consider the block designs.



1. Determine the total number of blocks in each design.

Design	1	2	3	4	5	6	10
Blocks	1	3	5	7	9	11	19

2. Write an algebraic expression to represent the total number of blocks in Design n.

There are $2n - 1$ blocks in the nth Design.

3. Use your algebraic expression to determine the total number of blocks in Design 64.

There are 127 blocks in Design 64.

3

Are All Functions Created Equal?

3.3

Comparing Multiple Representations of Functions

LEARNING GOALS

In this lesson, you will:

- Identify equivalent forms of functions in various representations.
- Model situations using tables, graphs, and equations.
- Use functions to make predictions.
- Determine whether two forms of a function are equivalent.

KEY TERMS

- relation
- function
- function notation

ESSENTIAL IDEAS

- A function is a relation such that for each element of the domain there exists exactly one element in the range.
- Function notation is a way to represent functions algebraically. The function $f(x)$ is read as "f of x" and indicated that x is the input and $f(x)$ is the output.
- Tables, graphs, and equations are used to model situations.
- Functions are used to make predictions.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity.

A-CED Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F-IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

F-BF Building Functions

Build a function that models a relationship between two quantities

1. Write a function that describes a relationship between two quantities.
   b. Combine standard function types using arithmetic operations.

Overview

The terms relation, function, and function notation are defined in this lesson. A sorting activity is presented that includes graphs, tables, equations, and contexts. Students will sort the various representations into groups of equivalent relations. The various representations are then identified with respect to their function families.

Students will analyze another tile pattern and use a table to organize data which leads to discovering additional patterns. Next, they create expressions that represent the number of white, gray and total tiles in Design *n*. Within the context of the problem situation, students use algebra to show different functions are equivalent and identify them as quadratic functions.

3

Warm Up

Determine if the table of values represents a function.
If so, determine the family function with which it is associated.

1.

x	y
−5	−13
−4	−12
0	−8
1	−7
5	−3

The table of values describes a linear function.

2.

x	y
−2	−2
0	0
1	4
3	18
5	40

The table of values represents a quadratic function.

3.

x	y
4	1
4	2
4	3
4	4
4	5

The table of values does not represent a function.
Each input value maps to the same output value.

3

3

Are All Functions Created Equal?

3.3

Comparing Multiple Representations of Functions

LEARNING GOALS

In this lesson, you will:

- Identify equivalent forms of functions in various representations.
- Model situations using tables, graphs, and equations.
- Use functions to make predictions.
- Determine whether two forms of a function are equivalent.

KEY TERMS

- relation
- function
- function notation

3

Every year during the first full week in August, the residents of Twinsburg, Ohio literally see double! That's because Twinsburg hosts the annual Twin Day Festival. It is the largest gathering of twins in the world, with thousands of twins, triplets, and multiple-birth families converging on the town for a weekend of games and activities. Although twins develop their own unique personalities, they often stand out in a crowd. It might be an interesting experience for twins and non-twins alike to be in a town completely filled with groups of people who look the same. *Not* having a person who looks just like you might actually make you stand out in the crowd.

Twins only account for about 1% of the pregnancies in the world, but the number of twin births actually varies depending on where you live. For example, the rate of twin births in Massachusetts is much higher than the rate in New Mexico. The highest rates in the world are found in central Africa while the lowest rates are found in Asia.

What do you think might account for differences throughout the world in the rate of twin births? Have you ever known twins? Would you like to have a twin brother or sister?

Problem 1

Students will cut out 24 different representations in the form of graphs, tables, equations, and contexts. Then they analyze them and sort the relations into groups of equivalent representations. Students explain the strategies they used to sort the representation into groups, and how they decided if a relation was or was not a function. They also described each function family associated with the various groupings.

3

Grouping

- Have students cut out the relations at the beginning of Problem 1.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

PROBLEM 1 You Aren't Looking Like Yourself Today

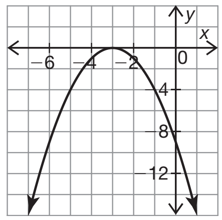
Understanding patterns not only gives insight into the world around you, it provides you with a powerful tool for predicting the future. Pictures, words, graphs, tables, and equations can describe the exact same pattern, but in different ways.

A relation is a mapping between a set of input values and a set of output values. In the problem, *The Cat's Out of the Bag*, you used a visual model, graph, table, and context to describe the relation between the number of ballot counters, and the total number of seniors that learned the result of the homecoming king election. In relations such as this one, there is only one output for each input. This type of relation is called a *function.* A **function** is a relation such that for each element of the domain there exists exactly one element in the range. **Function notation** is a way to represent functions algebraically. The function $f(x)$ is read as "f of x" and indicates that x is the input and $f(x)$ is the output.

Directions: Cut out the relations provided on the following pages. You will encounter graphs, tables, equations, and contexts. Analyze and then sort the relations into groups of equivalent representations. All relations will have at least one match.

Attach your groupings on the blank pages that follow the cut-out pages. Then provide a brief rationale for how you grouped each set of relations.

A



B

$f(x) = x^2 + 2x + 5$

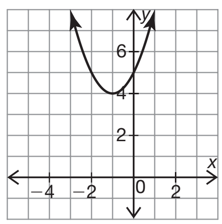
C

x	y
1	2
2	4
3	6
4	8
5	10

D

$f(x) = x^2 + 6x + 5$

E



F

$f(x) = -(x^2 + 6x + 9)$

3

G

$f(x) = 2x$

H

$f(x) = (x + 5)(x + 1)$

I

$f(x) = -(x + 3)(x + 3)$

J

A relation with a line of symmetry at $x = -3$, a vertex that is a maximum value, and a graph that opens down.

K

Louise heard a rumor. She tells the rumor to two people the next day. The two people that she told then tell two more people the following day, who each then go on to tell two more new people the rumor the following day. The relationship between the days that have passed and the number of new people who hear the rumor that day.

L

x	y
−3	8
−2	5
−1	4
0	5
1	8

3

3

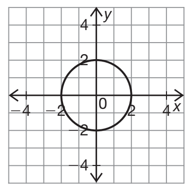
M

x	y
−4	−1
−3	0
−2	−1
−1	−4
0	−9

N

$y = 2^x$

O



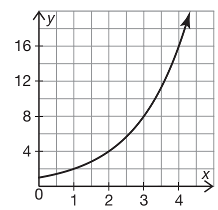
P

$y = (x + 3)^2 - 4$

Q

x	y
0	1
1	2
2	4
3	8
4	16

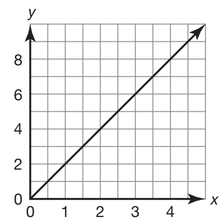
R



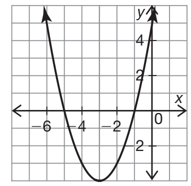
3

3

S



T



U

Erika is worried that her secret got out. On the first day she and her best friend were the only people who knew about the secret. But now, two new people are finding out the secret every day. The relationship between the number of days that have passed and the total number of people who know about her secret.

V

$x^2 + y^2 = 4$

W

x	*y*
−5	0
−4	−3
0	5
1	12
2	21

X

x	*y*
−1	1.73
−1	−1.73
0	2
0	−2
1	1.73
1	−1.73

3

3

Equivalent Relations

B, E, L — quadratic

O, V, X — non-function

C, G, S, U — linear

A, F, I, J, M — quadratic

K, N, R, Q — exponential

D, H, P, T, W — quadratic

3

Equivalent Relations

3

Equivalent Relations

3

Equivalent Relations

3

Equivalent Relations

3

Equivalent Relations

3

Guiding Questions for Share Phase, Questions 1 through 3

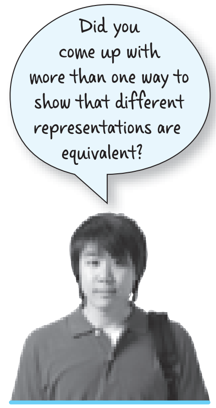
- How many groups did you use to sort the various representations?
- Did any of the representations fit into more than one group? Which ones?
- Did you graph all of the representations?
- Did you use the *y*-intercepts to make any decisions?
- Did you use the function families?
- What are the function families?
- Did all of the graphs pass the Vertical Line Test?
- What is the Vertical Line Test?
- Do the tables show only one *y*-value for each *x*-value?
- Do the equations produce only one output value for each input value?
- How do the relations that are not functions stand apart from the relations that are functions?
- How would you describe the difference between a linear function and a quadratic function?
- How would you describe the difference between a quadratic function and an exponential function?

1. What strategies did you use to sort the representations into your groups?

Answers will vary.

Students should use characteristics of each representation.

- I graphed each representation to determine which ones were the same.
- I looked at the *y*-intercepts and determined which matched the equations and the tables.
- I sorted the representations based on their function family.



2. How do you know which relations are functions and which are not functions? Explain your reasoning in terms of the graph, table, and equation.

Each function shows only one output for each input. This means that the graph passes the Vertical Line Test.

The table shows only one *y*-value for each *x*-value.

The equation produces only one output value for each input value.

The relations that are non-functions have more than one output for a given input.

3

3. Identify the function family associated with each grouping. How can you determine the function family from the graph, table, context, and the equation?

See groupings.

The function families are linear, quadratic, and exponential.

A linear function has a constant rate of change, can be written in the form $y = mx + b$, and its graph is a straight line.

A quadratic function has a common second difference, is shaped like a parabola, and has degree 2.

An exponential function can be written in the form $y = ab^x$, the *y*-values in the table and graph will increase geometrically.

Problem 2

Students will analyze the first three ceramic tile designs by describing all of the observable patterns and organizing their observations in a table format. Through this organization, other patterns emerge. They then answer several questions related to the situation and create algebraic expressions for the number of various colors of tiles in Design *n*. Students show two different expressions equivalent and conclude that the situation is quadratic. Finally, they prove the number of white tiles is always an even number and the total number of tiles is always an odd number.

3

Grouping

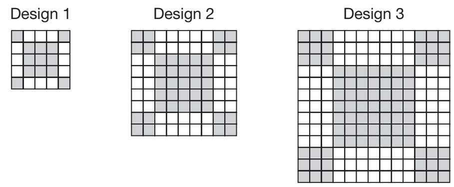
- Ask a student to read the paragraph written at the beginning of Problem 1. Discuss the context.
- Have students complete Question 1 with a partner. Then have students share their responses as a class.
- Have students complete Questions 2 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Is the total number of tiles always a square number?
- How does the number of gray tiles compare to the number of white tiles?
- Is the sum of the white tiles and the gray tiles the same as the total number of tiles?
- Is the number of white tiles always an even number of tiles?
- Is the number of gray tiles always an odd number of tiles?
- Is the total number of tiles always an odd number of tiles?
- How many white rectangles are in each pattern?
- How many gray squares are in each pattern?

PROBLEM 2 Why Are You So Square?

A ceramic tile company creates a new line of decorative kitchen and bathroom tiles. The company will sell larger tiles that are created from combinations of small gray and white square tiles. The designs follow the pattern shown.



1. Analyze the tile designs. Describe all of the various patterns that you notice.

   Answers will vary.

   - The total number of tiles is always a square number.
   - The number of gray tiles is always one more than the number of white tiles.
   - The sum of the white tiles and gray tiles is equal to the total number of tiles.
   - The number of white tiles is always an even number.
   - The number of gray tiles is always an odd number.
   - The total number of tiles is always odd.
   - Each pattern consists of 4 white rectangles and 5 gray squares.

2. Numerically organize the pattern.

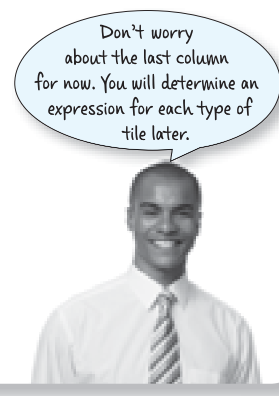
Design Number	1	2	3	4	7	10	*n*
Number of White Tiles, *w*(*n*)	12	40	84	144	420	840	$4n(2n + 1)$
Number of Gray Tiles, *g*(*n*)	13	41	85	145	421	841	$(2n + 1)^2 + 4n^2$
Total Number of Tiles, *t*(*n*)	25	81	169	289	841	1681	$(4n + 1)^2$

3. What new patterns do you notice?

   Answers will vary.

   Students should notice that the number of white tiles is 1 less than the number of gray tiles.

   Students should notice even, odd, and square patterns.



Guiding Questions for Share Phase, Questions 2 through 7

- How did you determine the number of white tiles in Design 4?
- How did you determine the number of gray tiles in Design 4?
- How did you determine the number of white tiles in Design 7?
- How did you determine the number of gray tiles in Design 7?
- How did you determine the number of white tiles in Design 10?
- How did you determine the number of gray tiles in Design 10?
- What is the total number of tiles in Design 11?
- How did you determine the expression for the number of white tiles in the pattern?
- How did you determine the expression for the number of gray tiles in the pattern?
- How did you determine the expression for the total number of tiles in the pattern?

4. How many total tiles are in Design 7? How many of the tiles are white? How many are gray? Explain your reasoning.

The number of total tiles is 841.

The number of white tiles is 420 and the number of gray tiles is 421.

To determine these values, I continued the pattern.

5. A hotel would like to order the largest design possible. They have enough money in their budget to order a design made up of 1700 total gray and white tiles. Which design can they afford? How many tiles in the design will be white? How many will be gray? Explain your reasoning.

The hotel can afford the 10th design. This design has 1681 tiles.

I extended the pattern in the table until I reached 1700 total tiles. I knew it could not be Design 11 because that would be greater than 1700 tiles, which is the most tiles the hotel can purchase with the money in the budget.

The number of white tiles in the design is 840 and the number of gray tiles is 841 since the number of gray tiles is always one more than the number of white tiles.

6. Complete the last column of the table in Question 2 by writing an expression to describe the number of white tiles, gray tiles, and total tiles for Design *n*.

See the table for answers.

Guiding Questions for Share Phase, Question 7

- What is the simplification of Tonya's expression?
- What is the simplification of Alex's expression?
- Do both expressions result in the same graph? What does this imply?

3

7. Tonya and Alex came up with different expressions to represent the number of gray tiles in each pattern. Their expressions are shown.

Tonya	Alex
$4n^2 + (2n + 1)(2n + 1)$	$(4n + 1)^2 - 4n(2n + 1)$

Tonya claims that they are the same expression written different ways. Alex says, "One expression has addition and the other has subtraction. There is no way they are equivalent!"

Who is correct? Justify your reasoning using algebraic and graphical representations.

Tonya is correct.

Tonya's expression	Alex's expression
$4n^2 + (2n + 1)(2n + 1)$	$(4n + 1)^2 - 4n(2n + 1)$
$4n^2 + 4n^2 + 4n + 1$	$16n^2 + 8n + 1 - 4n^2 - 4n$
$8n^2 + 4n + 1$	$8n^2 + 4n + 1$

Both expressions are equivalent to $8n^2 + 4n + 1$.

When I graph both expressions as equations, they produce the same graph which guarantees equivalence.

Grouping

Ask a student to read the information and complete Question 8 as a class.

You may have noticed several patterns in this sequence. An obvious pattern is that the sum of the white tiles and gray tiles is equal to the total number of tiles. This pattern is clear when analyzing the values in the table. However, adding $w(n)$ and $g(n)$ creates a brand new function that looks very different from the function $t(n)$.

In order to prove that the sum of the white tiles and gray tiles is equal to the total number of tiles, you must show that the expressions are equivalent.

$w(n) + g(n)$	$t(n)$
$4n(2n + 1) + (2n + 1)^2 + 4n^2$	$(4n + 1)^2$
$(8n^2 + 4n) + (4n^2 + 4n + 1) + 4n^2$	$(4n + 1)(4n + 1)$
$16n^2 + 8n + 1$	$16n^2 + 8n + 1$

8. Analyze the context, table, and expressions in this problem.

a. Identify the function family that describes the pattern for the number of white tiles. Explain your reasoning.

The function that represents the white tiles is quadratic.

The algebraic representation is in the form $ax^2 + bx + c$.

b. Identify the function family that describes the pattern for the number of gray tiles. Explain your reasoning.

The function family that represents the gray tiles is quadratic.

The algebraic representation is in the form $ax^2 + bx + c$.

c. When you add the functions that represent the number of gray tiles and white tiles, does the new function belong to the same function family? Explain your reasoning.

Yes. Adding the two quadratic functions creates a new quadratic function.

The algebraic representation is also in the form $ax^2 + bx + c$.

Grouping

Have students complete Questions 9 and 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 9 and 10

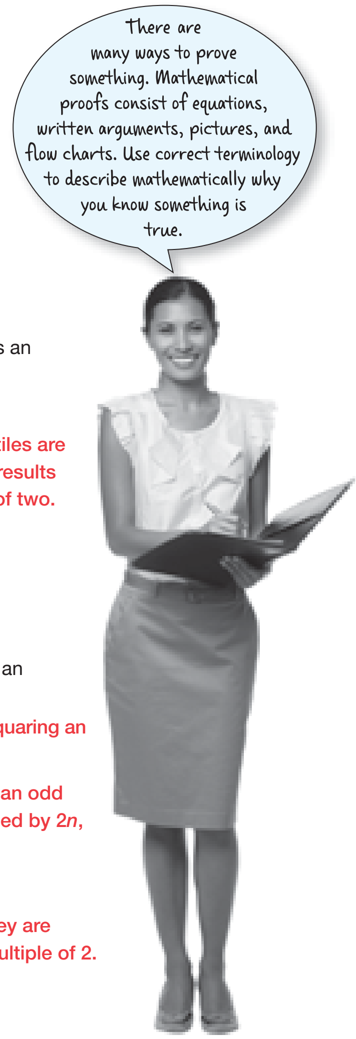
- What expression represents $w(n) + 1$?
- What expression represents $g(n)$?
- Are the white tiles always arranged in rectangles?
- Does multiplying by four always result in an even number?
- Is the product of two odd numbers always an odd number?
- If an even number is represented by $2n$, what expression represents an odd number?
- What is $(2n + 1)(2n + 1)$?
- Why does $(2n + 1)(2n + 1)$ always result in an odd number?

3

9. Describe the relationship between the number of white tiles and gray tiles in each design. Prove that this relationship exists.

The number of white tiles plus one is equal to the number of gray tiles.

$w(n) + 1$	$g(n)$
$4n(2n + 1) + 1$	$(2n + 1)^2 + 4n^2$
$8n^2 + 4n + 1$	$(4n^2 + 4n + 1) + 4n^2$
$8n^2 + 4n + 1$	$8n^2 + 4n + 1$



10. Analyze the tile patterns.

a. Prove that the number of white tiles is always an even number.

Answers will vary.

One possible student response: The white tiles are arranged in rectangles. Multiplying by four results in an even number, since four is a multiple of two.

b. Prove that the total number of tiles is always an odd number.

The total number of tiles is determined by squaring an odd number, such as 3, 5, or 7.

The product of two odd numbers is always an odd number. Since an even number is represented by $2n$, an odd number is $2n + 1$.

$(2n + 1)(2n + 1) = 4n^2 + 4n + 1$

The first two terms will be even because they are multiples of 2, so their sum will also be a multiple of 2. Adding one to the expression results in an odd number.

Talk the Talk

Students will complete statements about the equivalence of functions.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Talk the Talk

Choose a word that makes each statement true. Explain your reasoning.

always	sometimes	never

1. Two functions are **always** equivalent if their algebraic representations are the same.

   The same expression produces the same output for each input.

2. Two functions are **sometimes** equivalent if they produce the same output for a specific input value.

   This may be true, but one value is not enough to be certain.

   For example: $y = 2x + 1$ and $y = x^2 + 1$ produce an output of 1 for an input of 0, but they are not equivalent.

3. Two functions are **always** equivalent if their graphical representations are the same.

   Students should be careful, though, to make sure that their viewing window is large enough to see all graphical behaviors. For example, the functions $y = x$ and $y = |x|$ have the same graphical representation in quadrant 1, but not in quadrants 2 and 3.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Determine if each pair of functions is equivalent.

1. $f(x) = (x + 4)^2 + 6$

$f(x) = x^2 + 8x + 22$

The functions are equivalent.

2. $f(x) = (x + 1)^2 + 2x^2$

$f(x) = (2x + 1)^2 - 2x(x + 1)$

The functions are not equivalent.

3

3.4

Water Under the Bridge

Modeling with Functions

LEARNING GOALS

In this lesson, you will:

- Use multiple representations of functions to model and solve problems.
- Use multiple representations of functions to analyze problems.

ESSENTIAL IDEAS

- Tables, graphs, and equations are used to model situations.
- A function created by the product of two linear factors is a quadratic function.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity.
2. Use the structure of an expression to identify ways to rewrite it.

A-APR Arithmetic with Polynomials and Rational Expressions

Understand the relationship between zeros and factors of polynomials

3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

A-REI Reasoning with Equations and Inequalities

Represent and solve equations and inequalities graphically

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Overview

This lesson presents a storm drain problem situation. Students use this situation to calculate the length of the drain, the width of the drain, and the maximum cross-sectional area of the drain in two different situations. They will create tables of values, equations, and graphs to represent each situation. Students then identify the function that represents the cross-sectional area of the drain as quadratic and the two factors that represent the length and width of the drain as linear. Finally, students analyze the graph by relating the intercepts and axis of symmetry to this problem situation.

Warm Up

1. The table of values shown describes a function.

x	y
−5	24
−4	22
0	14
1	12
5	4

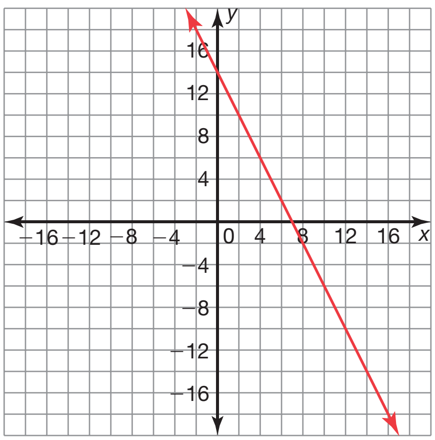
a. Represent the table of values using an equation.

$y = 14 - 2x$

b. Write the equation using function notation.

$f(x) = 14 - 2x$

c. Create a graph on the coordinate plane.



3

3

3.4

Water Under the Bridge

Modeling with Functions

LEARNING GOALS

In this lesson, you will:

- Use multiple representations of functions to model and solve problems.
- Use multiple representations of functions to analyze problems.

3

"It's just water under the bridge" is more than a saying for some hydrologists. To them, it's their career. Some hydrologists specialize in the design of city drainage systems.

So, you might be asking: how important is a city's drainage system? The key function to any drainage system is to channel rain water out of the area at the maximum speed possible. In the early part of the 20th century, the Los Angeles River routinely jumped its banks causing some areas of the city to flood. Outraged citizens demanded a better means of draining water after torrential rains. Hydrologists at the time decided to convert the Los Angeles River from a natural river to a massive storm drain. By pouring concrete and building up the sides of the drain, the city no longer flooded. However, the water that rushes through the drain can reach speeds of 45 miles per hour. These speeds are obviously very dangerous for anyone who might be in the storm drain system at the time of the storm. So while the drain has helped save the city from destruction caused by flooding, many lives have been lost as a result of citizens and rescuers being swept away in the drain system during a storm.

Do you think the city should raise the height of the drain so fewer people fall in? Would that affect how quickly the water flows through the drain?

Problem 1

Rectangular sheets of metal are used to build storm drains. The sheet of metal is bent up on both sides to represent the height of the drain. Students will use a sheet of paper to model a drain and answer questions related to the dimensions of the drain. Next, they are given the width of the sheet of metal used to create the drain and complete a table of values by choosing different heights and calculating the related widths. Students define a function $w(h)$ for the width of the drain given the height of h feet. Next, they define a function $A(h)$ for the cross-sectional area of the drain with a height of h feet, graph the function, and analyze the graph by identifying and relating the intercepts, and the equation of the axis of symmetry to the problem situation. Questions lead students to conclude that the quadratic function that represents the cross-sectional area was created by the product of two linear functions which represent the length and the height of the cross-sectional area of the drain.

Grouping

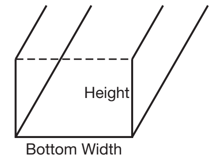
- Ask a student to read aloud the information at the beginning of Problem 1.
- Have students complete Question 1 with a partner. Then have students share their responses as a class.

3

PROBLEM 1 The Water's Five Feet High and Rising

A nearby town hired a civil engineer to rebuild their storm drainage system. The drains in this town are open at the top to allow water to flow directly into them. While designing the drains, the engineer must keep in mind the height and the width of the drain. She needs to consider the height because the water cannot rise above the drain or it will flood the town and cause major destruction. However the drain must also be wide enough that it will not get clogged by debris.

The civil engineer will use rectangular sheets of metal to build the drains. These sheets are bent up on both sides to represent the height of the drain. An end view of the drain is shown.



1. Use a sheet of paper to model a drain.

   a. Compare your model of a drain to your classmate's models. Identify similarities and differences between your models.

   Answers will vary.

   b. How does folding the sides of the drain affect the bottom width of the drain?

   As the height of the side increases, the width of the bottom decreases.

   c. Describe the drain that you think best fits the needs of the town. Explain your reasoning.

   Answers will vary.

   Students should identify that the height needs to be high enough to keep water from overflowing while the width is wide enough to keep the drain from getting clogged.

Guiding Questions for Share Phase, Question 1

- How did you fold your paper to create the model of the drain?
- As the height of the side of the drain increases, what effect does this have on the width of the drain?
- As the width of the drain increases, what effect does this have on the height of the sides of the drain?
- What are the consequences of a drain that is too narrow?
- What are the consequences of a drain that is too short?

Grouping

Have students complete Questions 2 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 2 through 5

- If the height of the drain is 1 foot, what is the width of the drain?
- If the height of the drain is 2 feet, what is the width of the drain?
- What height would cause the width of the drain to be 0?
- What width would cause the height of the drain to be 0?
- If the height of the drain is *h*, what expression would represent the width of the drain?

The sheets of metal being used to create the drain are 8.5 feet wide. The engineer wants to identify possible heights and bottom width measurements she could use to construct the drains.

2. Determine the bottom width for each given height. Then complete the table by choosing different heights and calculating the bottom widths for those heights. If necessary, construct models of each drain.

Height of the Drain (feet)	Bottom Width of the Drain (feet)
0	8.5
0.5	7.5
1.25	6
1.5	5.5
1.75	5
2.5	3.5
3	2.5
3.25	2
4.25	0

3. Describe how to calculate the bottom width for any height.

To calculate the bottom width for any height, multiply the height h by 2, then subtract this from 8.5. If the height is h, then the bottom width is $8.5 - 2h$.

4. Define a function $w(h)$ for the bottom width given a height of h feet.

$w(h) = 8.5 - 2h$

5. The engineer needs to identify the measurements that allow the most water to flow through the drain. What does the engineer need to calculate? What does she need to consider?

The engineer needs to calculate the area so she should consider the height and the bottom width.

3

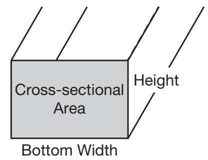
Grouping

- Ask a student to read aloud the information and complete Questions 6 and 7 as a class.
- Have students complete Questions 8 through 11 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 8 through 11

- What shape is the cross-sectional area of the drain?
- How do you determine the area of a rectangle?
- Do you think a square shaped drain with have a greater cross-sectional area than an elongated rectangle? Why or why not?
- When graphing the function *A*(*h*), what unit is measured on the *x*-axis and how does it relate to this problem situation?
- When graphing the function *A*(*h*), what unit is measured on the *y*-axis and how does it relate to this problem situation?

In order to determine the drain dimensions that allows the most water to flow through, the engineer must calculate the cross-sectional area. The cross-sectional area of a drain is shown.



6. Describe how to determine the cross-sectional area of any drain.

To determine the cross-sectional area, I must multiply the height by the bottom width.

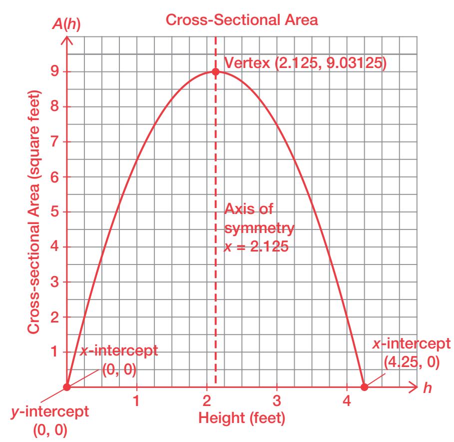
7. Predict and describe the drain with the maximum cross-sectional area.

Answers will vary.

8. Define a function *A*(*h*) for the cross-sectional area of the drain with a height of *h* feet.

$A(h) = h(8.5 - 2h)$

9. Use a graphing calculator to graph the function *A*(*h*). Label your axes.



Guiding Questions for Share Phase, Questions 10 and 11

- What does the maximum point on the parabola represent, with respect to this problem situation?
- Why is the parabola orientation downward?
- What is the height of the drain with the greatest cross-sectional area?
- What is the width of the drain with the greatest cross-sectional area?

10. Analyze your graph.

a. What is the maximum cross-sectional area for the drain pipe? Explain your reasoning.

The maximum point of the graph is (2.125, 9.03125).

The maximum cross-sectional area is 9.03125 square feet, which occurs when the height is 2.125 feet.

b. Identify the intercepts of $A(h)$. What does each mean in terms of this problem situation? Label each intercept on the graph.

The y-intercept is (0, 0).

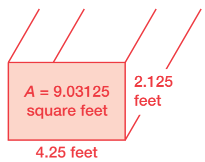
The x-intercepts are (0, 0) and (4.25, 0). This means that the cross-sectional area is zero when the height is 0 or 4.25 feet.

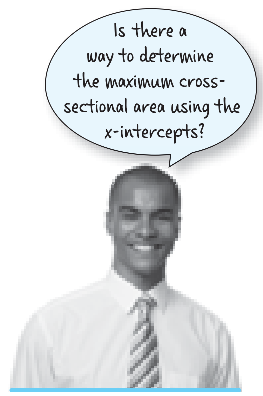
c. Identify the equation of the axis of symmetry. Then label the axis of symmetry on the graph. Finally, describe the relationship between the axis of symmetry and the maximum cross-sectional area.

$x = 2.125$

The axis of symmetry is the x-value of the vertex which represents the cross-sectional area. The axis of symmetry is equal to the height of the drain.

11. Draw and label the drain with the greatest cross-sectional area.





3

Grouping

Have students complete Question 12 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 12

- What does a linear factor/function look like?
- What does a quadratic factor/function look like?
- Is the function $A(h)$ the product of two factors?
- Is the function $A(h)$ a linear function or a quadratic function?
- Are the factors of function $A(h)$ also quadratic factors or linear factors?
- Is the product of two linear factors always a quadratic function? Why?
- How can you prove $h(8.5 - 2h)$ and $8.5h - 2h^2$ are equivalent expressions?

3

12. In this problem you built a new function $A(h) = h(8.5 - 2h)$ using two existing functions.

   a. What is the first factor in this function? What does it represent in terms of this problem situation?

   The first factor is h.

   This represents the height of the drainage system.

   b. What is the second factor in this function? What does it represent in terms of this problem situation?

   The second factor is $8.5 - 2h$.

   This represents the width of the drainage system.

   c. Identify the function families represented by each factor.

   Each factor by itself represents a linear function.

   d. When these factors are multiplied together what type of function is created? Why does this happen?

   When I multiply these two factors together I get a quadratic function.

   This happens because $h(8.5 - 2h)$ is equivalent to $8.5h - 2h^2$ which is quadratic.

Problem 2

This problem is similar to the previous problem without the scaffolding questions.

A sheet of metal that is 15.25 feet wide is folded on both sides to form a storm drain. Students will determine the drain has the maximum cross-sectional area using two different representations. They may represent this quadratic situation using a table of values, a graph, or an equation.

Grouping

- Ask a student to read the paragraph written at the beginning of the problem. Discuss the context.
- Have students complete the problem with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Problem 2

- Is this situation linear, quadratic, or exponential? How do you know?
- What in the description of the problem situation helps to determine the function family?
- Where is the maximum cross-sectional area located on the graph of the function?
- What is the shape of the cross-sectional area of the storm drain?
- What algebraic expression is used to represent the height and length of the cross-sectional area?

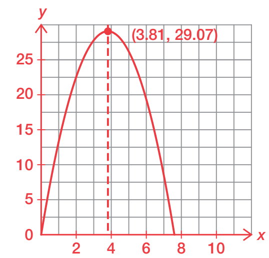
PROBLEM 2 Determine the Best Design

A civil engineering company is hired to design a new drainage system for your town. To construct one of the storm drains, a sheet of metal that is 15.25 feet wide is folded on both sides.

Describe the drain that has the maximum cross-sectional area. Include at least two different representations in your description. Show all work and explain your reasoning.

The maximum cross-sectional area is 29.07 square feet which occurs when the height is 3.81 feet. This is the highest point in the graph. I can also determine this point algebraically using the *x*-intercepts. I can also estimate the vertex from the table of values.

Graphical Solution:



Numeric Solution:

Height	Area
0	0
3	27.75
3.25	28.438
3.5	28.875
3.75	29.063
4	29
4.25	28.688

← Vertex is approximately (4, 29). I know this because the *y*-values increase, reach a peak, and then decrease.

Algebraic Solution:

Solve $x(15.25 - 2x) = 0$ to determine the *x*-intercepts at (0,0) and (7.625,0). The *x*-coordinate of the vertex is the average of the *x*-intercepts, or $x = 3.81$ feet. Substituting this back into the equation gives an output of 29.07 square feet.

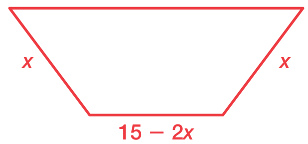
Be prepared to share your solutions and methods.

- What equation is used to graph this function?
- Does the equation that represents this situation contain a maximum or a minimum? How do you know?
- What is the sign of the leading term in the quadratic equation? What does this imply about the orientation of the graph of the equation?
- How did you determine the table of values in the numeric solution?
- What are the *x*-intercepts in this problem situation?
- What do the *x*-intercepts represent in this problem situation?
- Why is the *x*-coordinate of the vertex also the average of the *x*-intercepts?

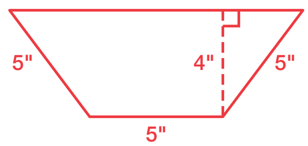
Check for Students' Understanding

A rain gutter is to be constructed with a cross section that is an isosceles trapezoid, a trapezoid where the two non-parallel sides are equal in length. The gutter is to be constructed out of a sheet of metal 15 inches wide.

1. Draw a diagram of this situation. Let x represent the height of the rain gutter.



2. Suppose the height of the isosceles trapezoid is 4 inches and the bottom width of the gutter is 5 inches. Determine the width of the top of the rain gutter.



$$15 - 2x = 5$$
$$-2x = -10$$
$$x = 5$$

$$a^2 + b^2 = c^2$$
$$4^2 + b^2 = 5^2$$
$$16 + b^2 = 2^5$$
$$b^2 = 9$$
$$b = 3$$
$$3 + 3 + 5 = 11$$

The width of the top of the gutter is 11 inches.

3.5

I've Created a Monster, $m(x)$

Analyzing Graphs to Build New Functions

LEARNING GOALS

In this lesson, you will:

- Model operations on functions graphically.
- Sketch the graph of the sum, difference, and product of two functions on a coordinate plane.
- Predict and verify the graphical behavior of functions.
- Build functions graphically.
- Predict and verify the behavior of functions using a table of values.
- Build functions using a table of values.

KEY TERMS

- Zero Product Property
- polynomial
- degree

ESSENTIAL IDEAS

- A graph of a function is a set of an infinite number of points.
- A polynomial is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients.
- The Zero Product Property states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.
- The degree of a polynomial is the greatest variable exponent in the expression.
- When two functions of different degree are added, the resulting function will have at most the highest degree of the functions that were added.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity.

A-CED Creating Equations

Create equations that describe numbers or relationships

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F-IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Overview

Students will use a coordinate plane to add two linear functions to create a new function. The properties for integer operations are reviewed and extended to operations on the graphs of functions. They then use key points such as intercepts, zeros, and intersection points to create new functions. Students will graphically add and subtract a linear function and a constant function, two linear functions, and a linear function and a quadratic function. Next, they multiply two linear functions on a coordinate plane to create a quadratic function and review the Zero Product Property. Finally, students multiply a linear function, to itself to create a quadratic function, and then multiply a quadratic function and a linear function to create a third degree function.

Warm Up

1. The table of values shown describes two functions.

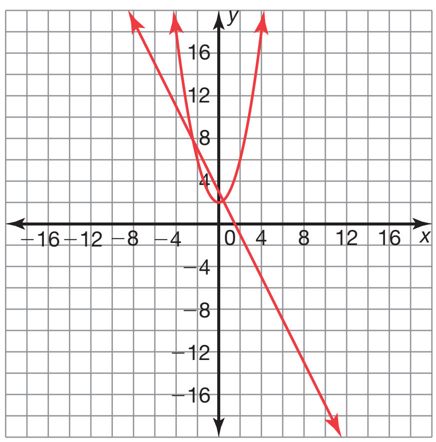
   a. Write each function described by the table of values.

x	$f(x)$	$g(x)$
−3	9	11
−1	5	3
0	3	2
1	1	3
3	−3	11

$f(x) = 3 - 2x$

$g(x) = x^2 + 2$

   b. Sketch a graph of each function on the coordinate plane.



3

3

I've Created a Monster, $m(x)$

3.5

Analyzing Graphs to Build New Functions

3

LEARNING GOALS

In this lesson, you will:

- Model operations on functions graphically.
- Sketch the graph of the sum, difference, and product of two functions on a coordinate plane.
- Predict and verify the graphical behavior of functions.
- Build functions graphically.
- Predict and verify the behavior of functions using a table of values.
- Build functions using a table of values.

KEY TERM

- Zero Product Property
- polynomial
- degree

In 1818 Mary Shelley wrote the science fiction novel *Frankenstein*. It is the tale of Dr. Victor Frankenstein, a scientist who dreams of creating life. He accomplishes this dream by using old body parts and electricity. Unfortunately, he creates a monster! Horrified and filled with regret, Victor decides that he must end the life that he created. His monster has other plans, though. He is lonely and wants Victor to create a woman to keep him company in this cruel world! Crime, drama, and vengeance follow as the creator struggles with his creation.

Frankenstein laid the foundation for many of the horror and science fiction movies that you see today. While Mary Shelley's novel is a literary classic for how it tackles deep issues such as the meaning of life and the ethics of creation, it is also good old-fashioned, scary fun. Do you enjoy scary movies? If so, do you think any of your favorites may have been influenced by this classic tale?

3

Problem 1

Students are given the graph of two linear functions and are asked to predict the function family and the sketch of the new graph if the two functions are added together. An example of adding two linear functions is provided and students answer questions related to adding the two given linear functions. The Commutative Property over Addition, the Additive Inverse, and the Additive Identity Property are reviewed and used to perform operations on the graphs of the two linear functions. When performing operations on two graphs, students will consider key points such as intercepts, zeros, and intersection points.

Grouping

- Ask a student to read aloud the information at the beginning of Problem 1.
- Have students complete Question 1 with a partner. Then have students share their responses as a class.

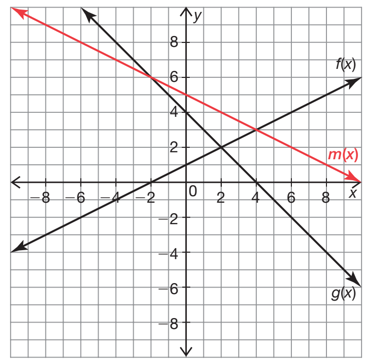
Guiding Questions for Share Phase, Question 1

- What function family is associated with $f(x)$?
- What function family is associated with $g(x)$?
- How did you determine the location of the points on $m(x)$?
- Will $m(x)$, $f(x)$, and $g(x)$ intersect at the same point? Why not?

PROBLEM 1 It's Moving . . . It's Alive!

In the problem, *You're So Square*, you added the functions $w(n)$ and $g(n)$ algebraically to create a new function $t(n)$. Manipulating algebraic representations is a common method for building new functions. However, you can also build new functions graphically. Let's consider two graphs of functions on a coordinate plane and what happens when you add, subtract, or multiply the output values of each.

1. Analyze the graphs of $f(x)$ and $g(x)$.



a. Predict the function family of $m(x)$ if $m(x) = f(x) + g(x)$. Explain your reasoning.

Predictions will vary.

The function $m(x)$ will belong to the linear function family. I know this because $f(x)$ and $g(x)$ are both linear.

b. Predict and sketch the graph of $m(x)$.

Answers will vary.

c. Explain how you predicted the location of $m(x)$.

Answers will vary.

- At what point will $m(x)$ intersect $f(x)$? How did you determine this point?
- At what point will $m(x)$ intersect $g(x)$? How did you determine this point?

Grouping

- Ask a student to read aloud the information and worked example before Question 2.
- Have students complete Question 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- Are the output values or the input values added when two functions are added? Why?
- Which value is closer to zero, the output value for the function $f(x)$, or the output value for the function $g(x)$ when $x = 6$?
- When two functions are added, what happens to the output value of the new function when one output value of the two functions is zero?

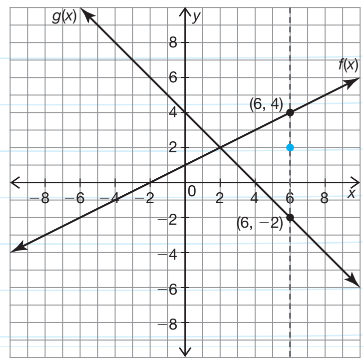
A graph of a function is a set of an infinite number of points. When you add two functions you are adding the output values for each input value. Given two functions, $f(x)$ and $g(x)$, on a coordinate plane, you can graphically add these functions to produce a new function, $m(x)$. To get started, let's consider what happens when you add $f(x)$ and $g(x)$ at a single point.

Let's add the output values for $f(x)$ and $g(x)$ at $x = 6$ to determine $(6, m(6))$.

$m(6) = f(6) + g(6)$
$m(6) = 4 + -2$
$m(6) = 2$

The point on the new function $m(x)$ is (6, 2).

Other points on the graph of $m(x)$ can be determined in a similar way.



2. Analyze the addition of the output values for the input value $x = 6$ in the worked example.

   a. How is this process similar to adding integers on a number line?

   It is the same process except the output values are moving vertically instead of horizontally.

   b. Why is the point $(6, m(6))$ closer to $f(x)$ than $g(x)$?

   The output value for the function $f(x)$ is further from 0 than the output value for the function $g(x)$ at $x = 6$. The sum of 4 and −2 is 2.

   c. Why did the input value of 6 stay the same while the output values changed?

   The output values are added, not the input values. I am determining the vertical shift when the output values are added, not a horizontal shift.

   d. Choose a different input value. Add the output values for $f(x)$ and $g(x)$ to determine a new point on the graph of $m(x)$.

   Answers will vary.

3

Grouping

- Ask a student to read aloud the information and properties before Question 3.
- Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

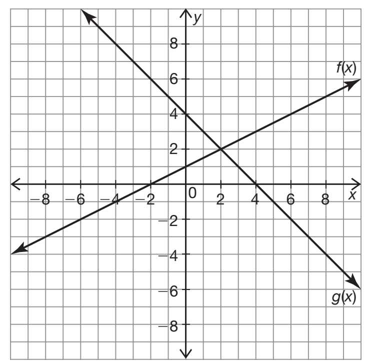
- Are the output values of $f(1) + g(1)$ equal the output values of $g(1) + f(1)$?
- Does the output value of $f(10) + g(10) = 0$? Why?
- Does the output value of $f(4) + g(4) = 0$? Why?

3

Now, let's consider what happens when you add $f(x)$ and $g(x)$ at a few other points. The properties you use in integer operations also extend to operations on the graphs of functions. Recall the integer properties shown in the table.

Property	Definition	Integer Example
Commutative Property over Addition	The commutative property states that the order in which the terms are added does not change the sum. In other words $a + b = b + a$.	$35 + 43 = 43 + 35$
Additive Inverse	The additive inverse of a number is the number such that the sum of the given number and its additive inverse is 0.	The numbers -5 and 5 are additive inverses because $-5 + 5 = 0$.
Additive Identity	The additive identity is 0 because any number added to 0 is equal to itself.	$5 + 0 = 5$

3. Extend the integer properties from the table to operations on the graphs of functions.



a. Use two output values from functions $f(x)$ and $g(x)$ to demonstrate the commutative property over addition for functions.

Answers will vary.

The output values $f(1) + g(1) = g(1) + f(1)$ because $3 + 1.5 = 1.5 + 3$.

On the graph the location of the output is the same regardless of the order in which the functions are added.

b. Determine output values for $f(x)$ and $g(x)$ that demonstrate the Additive Inverse Property. Show that they are additive inverses algebraically and graphically.

The output values $f(10) + g(10) = 0$ because $-6 + 6 = 0$.

The output value of $m(x)$ is 0 for $x = 10$. This falls on the x-axis at (10, 0).

c. Determine output values for $f(x)$ and $g(x)$ that demonstrate the Additive Identity Property. Show that they are additive identities algebraically and graphically.

The output values $f(-2) + g(-2) = g(-2)$ because $0 + 6 = 6$.

The output values $g(4) + f(4) = f(4)$ because $0 + 3 = 3$.

On the graph I can see that the output value stays the same, with no vertical shift.

Guiding Questions for Share Phase, Question 4

- Did Ari add the output values for $g(2)$ and $f(2)$ and keep the input value the same?
- Did Will add the output values for $g(2)$ and $f(2)$ and keep the input value the same?

4. Ari and Will disagree over the location of (2, $m(2)$) when the output values of the functions $f(x)$ and $g(x)$ are added.

Ari	Will
$g(2) + f(2) = (2, m(2))$ $(2, 2) + (2, 2) = (2, 4)$ The location of (2, $m(2)$) is (2, 4). The two points are at the intersection. Adding the output values of the two points equals (2, 2 + 2).	$(2, 2) + 0 = (2, 2)$ The location of (2, $m(2)$) is (2, 2). The lines intersect at one point. A point plus zero is itself.

Who is correct? Explain your reasoning.

Ari is correct. The two points have the same ordered pair so you have to add the output values while keeping the input values the same to determine the new location.

3

Grouping

Have students complete Question 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

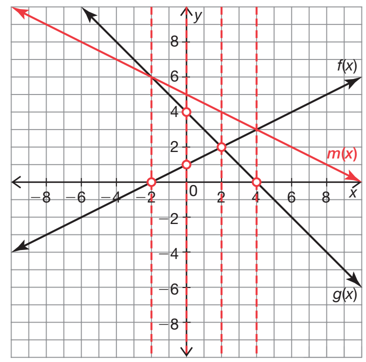
- What key points did you circle on $f(x)$?
- What key points did you circle on $g(x)$?
- Did you circle the x-intercept of $f(x)$ and $g(x)$?
- Did you circle the y-intercept of $f(x)$ and $g(x)$?
- Did you circle the intersection of $f(x)$ and $g(x)$?
- How did you determine the input and output values of $m(x)$?
- How did you verify the graph of $m(x)$?
- Was your earlier prediction of the location of $m(x)$ accurate?

3

When performing operations on two graphs, it isn't practical to consider all sets of ordered pairs. The process is much more efficient if you use key points. Some of the points considered in this problem, such as intercepts, zeros, and intersection points, are good examples of key points.

5. Sketch the graph of $m(x) = f(x) + g(x)$.

a. Circle key points of the graphs of $f(x)$ and $g(x)$.



See graph.

b. Draw dashed vertical lines through your key points.

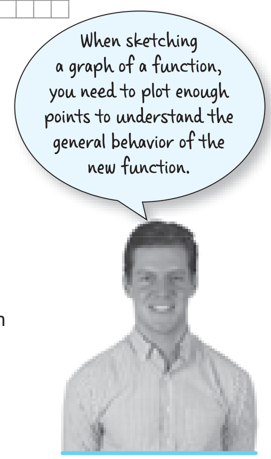
Answers will vary.

$x = -2$; x-intercept of $f(x)$

$x = 0$; y-intercepts of $f(x)$ and $g(x)$

$x = 2$; intersection of $f(x)$ and $g(x)$

$x = 4$; x-intercept of $g(x)$



c. Add the corresponding y-values of $f(x)$ and $g(x)$ on each dashed vertical line to determine points on $m(x)$. Then sketch the graph of $m(x)$. Show or explain your work.

See graph.

For each pair of points, I added the output values of $f(x)$ and $g(x)$ to determine the output values of $m(x)$. The input value of each point stayed the same.

d. Verify your graph of $m(x)$ using one or more pairs of points that are not key points.

Answers will vary.

e. Compare the function you graphed in this question with the prediction you made in Question 1. Describe any errors you may have made in your prediction.

Answers will vary.

Problem 2

Students will build new functions through addition and subtraction. They analyze the graphs of a linear function and a constant function, predict the function family of the new function resulting from the addition or subtraction of the functions, and sketch the function. Next, students add and subtract the outputs of two parallel lines. They conclude that when two parallel line functions are subtracted, the result is always a horizontal line. A table of values is used to organize the values of the functions and students use it to determine algebraic expressions to represent each of the functions. The graph of a linear function and a quadratic function are analyzed next. Again, students will predict the function family of the new function resulting from the addition. They conclude that the result is always a quadratic function. Students are given specific criteria and asked to sketch different functions resulting from the addition of two functions.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

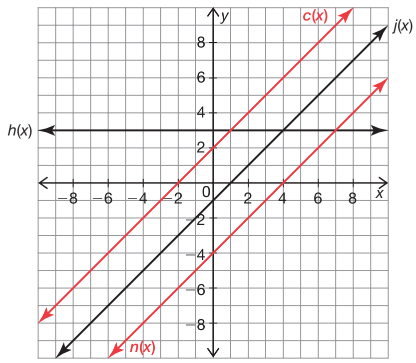
Guiding Questions for Share Phase, Question 1

- What function family is associated with $j(x)$? With $h(x)$?
- How did you determine the location of the points on $c(x)$?
- Will $j(x)$, $h(x)$, and $c(x)$ intersect at the same point? Why not?
- At what point will $c(x)$ intersect $h(x)$? How did you determine this point?
- At what point will $c(x)$ intersect $j(x)$? How did you determine this point?
- Why is $c(x)$ parallel to $j(x)$?
- Did the rate of increase stay the same?
- What effect did adding a constant value of 3 to the output values have on the orientation of the new function on the coordinate plane?

PROBLEM 2 **Keep On Keeping On**

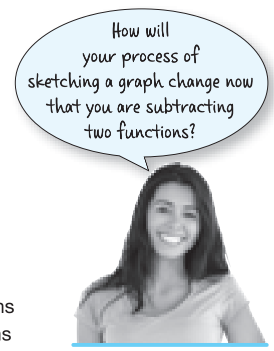
Let's consider operations on different types of graphs. Let's look at a linear function and a constant function.

1. Analyze the graphs of $j(x)$ and $h(x)$.



a. Predict the function family of $c(x)$ if $c(x) = j(x) + h(x)$. Then sketch the graph of $c(x)$.
See graph.

b. Describe the relationship between original functions and $c(x)$. Explain the relationship between the functions in terms of their graphical and algebraic representations.
The new function $c(x)$ is parallel to $j(x)$. The lines are parallel because the rate of increase stayed the same, but adding a constant value of 3 shifted all of the output values up 3 units.



c. Predict the function family of $n(x)$ if $n(x) = j(x) - h(x)$. Then sketch the graph of $n(x)$.
The function $n(x)$ is parallel to $j(x)$ but shifted down 3 units.

d. Describe the relationship between the original functions and $n(x)$. Explain the relationship between the functions in terms of their graphical and algebraic representations.
The new function $n(x)$ is parallel to $j(x)$. It is parallel because the rate of increase stayed the same, but subtracting a constant value of 3 shifted all of the output values down 3 units.

Grouping

Have students complete Question 2 a partner. Then have students share their responses as a class.

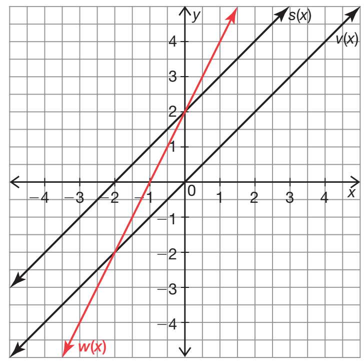
Guiding Questions for Share Phase, Question 2

- Is $w(x)$ a linear function? How do you know?
- What is the y-intercept of $w(x)$?
- How does the slope of $w(x)$ compare to the slope of the original functions?
- What is the slope of each original equation?
- What is the slope when you add the two original equations together?
- What is the y-intercept?
- How is the graph of $s(x) - v(x)$ different than the graph of $s(x) + v(x)$?
- How is the graph of $s(x) - v(x)$ different than the graph of $v(x) - s(x)$?
- Will subtracting the output values of any two parallel lines always result in the graph of a horizontal line?

3

Now let's look at what happens when you add and subtract the outputs of two parallel lines.

2. Analyze the graphs of $s(x)$ and $v(x)$.

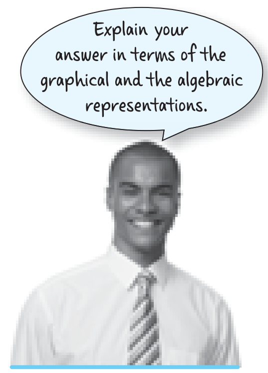


a. Sketch the graph of $w(x) = s(x) + v(x)$.

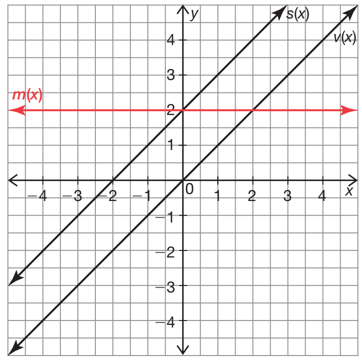
See graph.

b. Describe the shape of $w(x)$ compared to $s(x)$ and $v(x)$. Explain why adding the output values changes the shape of the new graph in this way.

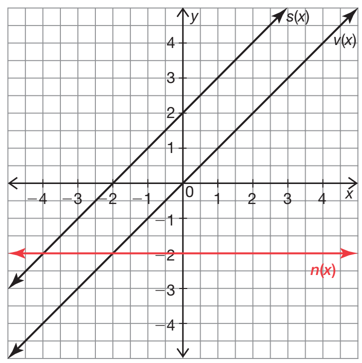
The new function is a linear function with a y-intercept of 2 and a steeper slope than the original functions. This occurs because the slope of each original equation is 1 but when you add them together the slope is 2. The y-intercept is (0, 2)



c. Sketch the graph of $m(x)$ if $m(x) = s(x) - v(x)$.



d. Sketch the graph of $n(x)$ if $n(x) = v(x) - s(x)$.



e. Describe the shape of the graph when you subtract $s(x)$ and $v(x)$. Will subtracting the output values of any two parallel lines have this same result? Explain your reasoning.

The graph will always be a horizontal line. Parallel lines have the same slope, meaning the difference in the output values will be constant. Algebraically, the x terms will have a coefficient of zero when the functions are subtracted, resulting in a function of the form $y = b$, where b is a constant.

Grouping

Have students complete Question 3 with a partner. Then have students share their responses as a class.

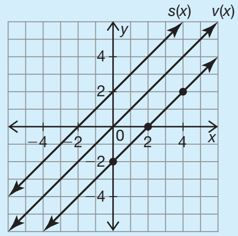
Guiding Questions for Share Phase, Question 3

- Should the graph of the new function be parallel to the original two functions?
- Should the graph of the new function be a horizontal line?
- Did Erik subtract the output values or did he just move the equation down 2 units?
- Did Lily subtract the output values?

3

3. Mrs. Webb asked her students to determine $v(x) - s(x)$. Erik's and Lily's work is shown.

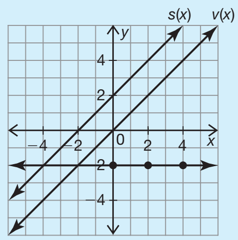
Erik



v(x)	t(x)	Differences
0	−2	−2
2	0	−2
4	2	−2

The new graph is located 2 units below v(x). I know this is correct because each point has a difference of −2

Lily



The new graph is located at $y = -2$.
I know this is correct because I subtracted several points and the y-value was always −2.

Who's correct? Explain why one graph is correct and the error made to create the other graph.

Lily is correct. She correctly subtracted the *y*-values and identified the equation as $y = -2$.

Erik is incorrect because he just moved the graph down 2 units.

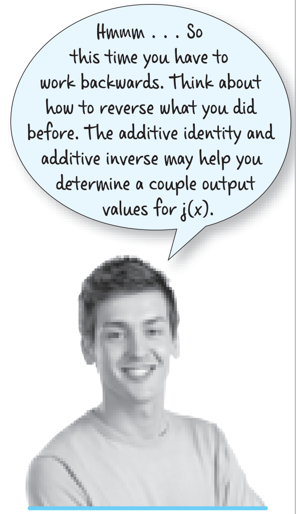
Grouping

Have students complete Question 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

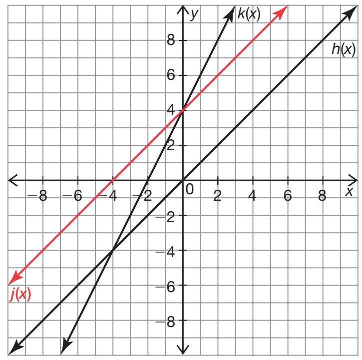
- How is this situation different than the previous problem situations?
- How did you determine the points on the graph of $j(x)$?
- At which input values can the Additive Identity Property be most easily seen?
- At which input values can the Additive Inverse Property be most easily seen?
- If $h(x) = x$, how would you represent $j(x)$?
- If $h(x) = x$, how would you represent $k(x)$?
- Does the table of values show that the first difference is a constant for each graph?
- Are all of the equations in the form $y = mx + b$?

Now, let's work backwards.



4. Analyze the graphs of $h(x)$ and $k(x)$.

a. Draw the function $j(x)$ with outputs such that $h(x) + j(x) = k(x)$.



b. Complete the table of values to verify that $h(x) + j(x) = k(x)$.

x	$h(x)$	$j(x)$	$k(x) = h(x) + j(x)$
−2	−2	2	0
−1	−1	3	2
0	0	4	4
1	1	5	6
2	2	6	8

c. Describe examples of the additive inverse and additive identity properties for output values in this problem.

The Additive Identity Property can be seen at $x = 0$ when $h(0) + j(0) = j(0)$ and $x = 0$. It can also be seen at $x = -4$ when $j(-4) + h(-4) = h(-4)$.

The Additive Inverse Property can be seen at $x = -2$ when $h(-2) + j(-2) = 0$.

d. Use the graph or table of values to determine the algebraic expressions for $h(x)$, $j(x)$, and $k(x)$. Algebraically show that $h(x) + j(x)$ is equivalent to $k(x)$.

$h(x) = x$, $j(x) = x + 4$, $k(x) = 2x + 4$

$x + (x + 4) = 2x + 4$

$2x + 4 = 2x + 4$

3

Grouping

Have students complete Question 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- When the output values were added, was the resulting graph a parabola?
- Does adding the output values preserve the curved shape of the parabola?

3

e. How can you determine from the graph, the table of values, and the algebraic expressions that the functions $h(x)$, $j(x)$, and $k(x)$ are all linear?

graph: The functions are straight lines.

table: The first difference is constant for each graph.

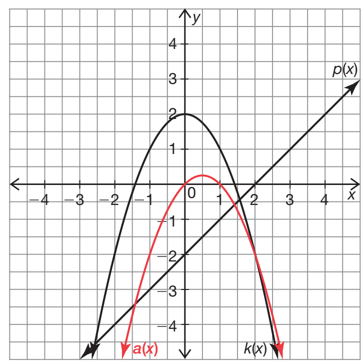
equation: They are all in the form $y = mx + b$.

f. Do you think adding two linear functions will always result in another linear function? Explain your reasoning.

Yes. The first difference will remain constant.

So far, you have only considered two linear functions. Now let's explore a linear function and a quadratic function.

5. Analyze the graphs of $k(x)$ and $p(x)$.



a. Predict the function family of $a(x)$ if $a(x) = k(x) + p(x)$. Explain your prediction.

Answers will vary.

b. Sketch the graph of function $a(x)$. Show or explain your work.

See graph.

I added the output values for each pair of corresponding points which resulted in the parabola.

c. Do you think adding a linear function and a quadratic function will always result in a quadratic function? Explain your reasoning in terms of the algebraic and graphical representations of the functions.

Yes. Adding a linear and a quadratic function will always result in a quadratic function. Adding the output values will preserve the curved shape of the parabola. Algebraically the squared term cannot add to zero when a linear function is added to it.

Grouping

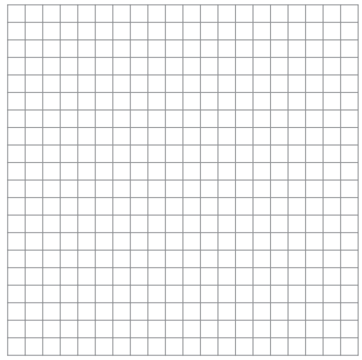
Have students complete Question 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- Does one of the functions have to be a parabola if the sum of the two functions is a parabola?
- Does one of the functions have to be a parabola that opens up if the sum of the two functions is a parabola that opens up?
- Can both functions be parabolas if the sum of the two functions is a parabola?
- Can both functions be parabolas that open up if the sum of the two functions is a parabola that opens up?
- If the sum of two functions is the horizontal line $y = 0$, can the functions both be constant?
- If the sum of two functions is the horizontal line $y = 0$, can the functions both be linear?
- If the sum of two functions is the horizontal line $y = 0$, can the functions both be quadratic?
- If the sum of two functions is the horizontal line $y = 0$, why must they be additive inverses of each other?
- Are functions closed under addition? Why?

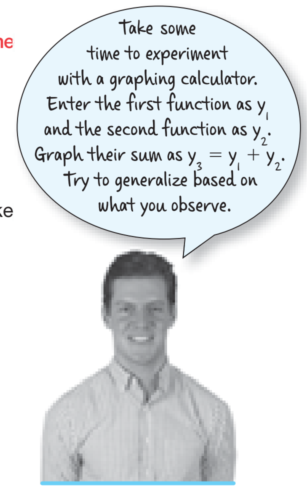
6. Draw the graphs that meet the criteria provided.

a. Sketch the graph of two different functions whose sum is a parabola opening up. What conclusions can you make about the two functions?

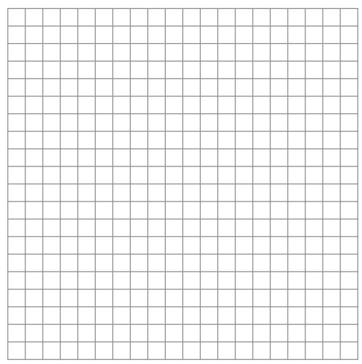


Answers will vary.

One of the graphs must be a parabola opening up. The other graph may be linear or constant. If both graphs are parabolas, either they must both open up, or the parabola opening up must have a greater *a* value.



b. Sketch the graph of two functions whose sum is the horizontal line $y = 0$. What conclusions can you make about the two functions?



Answers will vary.

The functions may be constant, linear, or quadratic. They must be additive inverses of each other. For example, $f(x) = 2x$ and $g(x) = -2x$, $f(x) = 4$ and $g(x) = -4$, or $f(x) = x^2$ and $g(x) = -x^2$.

- Does adding the output values for a given input result in one, unique output for the new function?
- If it has a unique output for each input, is the relation a function by definition?

Problem 3

Students will build new functions through multiplication. They analyze the graphs of two linear functions by predicting the function family of the product of the functions, and sketch the new function. A table of values is used to determine the graphical behavior of the functions. Students then describe the patterns and analyze the first and second differences for each function. The Zero Property is stated and students relate it to the *x*-intercepts of the functions. After analyzing more functions, students conclude when functions are multiplied, the shape of the graph is changed. Students multiply two perpendicular functions, they square a function, and finally multiply a linear and quadratic function.

3

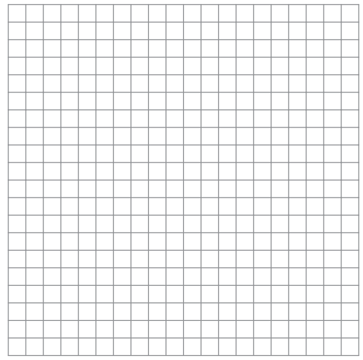
Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- How would you describe the graph of $f(x)$?
- How would you describe the graph of $g(x)$?
- Do you think the graph of $h(x)$ will be linear because the two original functions are linear?

c. Sketch the graph of two functions whose sum is not a function. What conclusions can you make about the two functions?

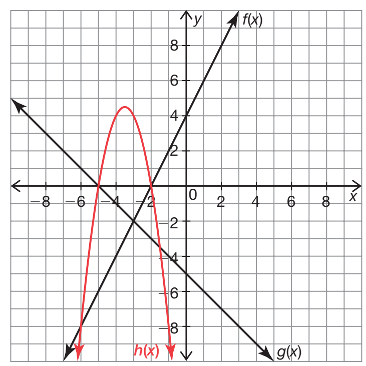


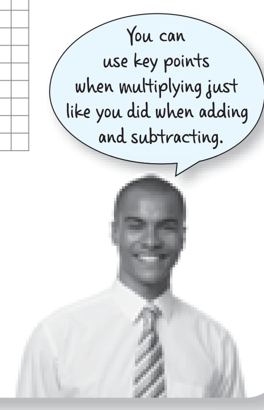
This is not possible. Functions are closed under addition because adding the output values for a given input results in one, unique output for the new function. Having a unique output for each input is, by definition, a function.

PROBLEM 3 They're Multiplying!!

Just as you added and subtracted functions in the previous problems, you can also build functions through multiplication.

1. Analyze the graphs of $f(x)$ and $g(x)$.





a. Predict the function family of $h(x)$ if $h(x) = f(x) \cdot g(x)$. Explain your reasoning.

Answers will vary.

I think the graph will be linear because both of the original graphs are linear.

I think the graph will be quadratic because $x \cdot x = x^2$.

- What operation is used on the output values when determining the product of the functions?
- Did you identify key points to determine the location of the new graph?
- Did you multiply the pairs of output values to determine the location of the new graph?
- How does it make sense that the new graph would be quadratic?

b. Sketch the graph of $h(x)$. Show or explain your work.

See graph.

To determine the location of the graph, I identified key points and multiplied the pairs of output values.

c. Describe the differences between the graphs of $f(x)$ and $g(x)$ and the graph of $h(x)$.

The new function is a parabola that opens down while $f(x)$ is a linear graph with a positive slope and $g(x)$ is a linear graph with a negative slope.

d. Was your prediction in part (a) correct? What was the same/different after you multiplied the output values of key points?

Answers will vary.

2. You can analyze a table of values to determine the graphical behavior of functions.

a. Complete the table of values for $h(x) = f(x) \cdot g(x)$.

x	$f(x)$	$g(x)$	$h(x)$
−7	−10	2	−20
−6	−8	1	−8
−5	−6	0	0
−4	−4	−1	4
−3	−2	−2	4
−2	0	−3	0
−1	2	−4	−8
0	4	−5	−20

b. What patterns do you notice in the table?

The function $f(x)$ is increasing by 2.

The function $g(x)$ is decreasing by 1.

The first difference in the output values of $h(x)$ is not constant.

The output values of $h(x)$ increase and then decrease while the output values for $f(x)$ and $g(x)$ only increase or decrease.

c. Analyze the first and second differences for each function. How do you know $f(x)$ and $g(x)$ are linear but $h(x)$ is not?

The function $f(x)$ is linear because the first difference is a constant of 2.

The function $g(x)$ is linear because the first difference is a constant of −1.

The function $h(x)$ is quadratic because the first difference is not constant, but the second difference is a constant of 4.

Grouping

Have students complete Questions 2 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- How did you determine the $h(x)$ values for the table of values?
- Is the function $f(x)$ increasing or decreasing?
- Is the function $g(x)$ increasing or decreasing?
- Is the first difference in the output values of $h(x)$ constant?
- Do the output values of $h(x)$ increase or decrease?
- What is the first difference of $f(x)$?
- What is the first difference of $g(x)$?
- What is the first difference of $h(x)$?

3

3. Consider the sign of the output values for each function in the table.

a. For which input values are the output values of $h(x)$ negative? For which input values are the output values of $h(x)$ positive?

The output values are negative for the input values $(-\infty, -5)$ and $(-2, \infty)$.

The output values are positive for the input values $(-5, -2)$.

b. How does the sign of the output values of $f(x)$ and $g(x)$ determine the sign of the output values of $h(x)$?

Multiplying two negative output values results in a positive output value.

Multiplying one negative and one positive value results in a negative output value.

4. Consider the x-intercepts for $f(x)$, $g(x)$ and $h(x)$.

a. Identify the x-intercepts for each function.

$f(x)$: $(-2, 0)$

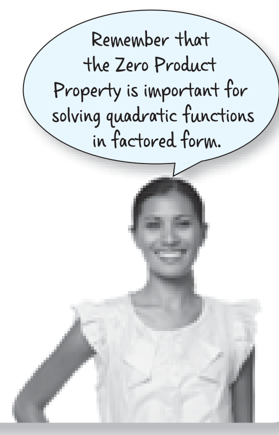
$g(x)$: $(-5, 0)$

$h(x)$: $(-2, 0)$ and $(-5, 0)$

b. What pattern do you notice in the x-intercepts?

The x-intercepts of the functions $f(x)$ and $g(x)$ are the x-intercepts of $h(x)$. The functions $f(x)$ and $g(x)$ have one x-intercept while $h(x)$ has two.

The **Zero Product Property** states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.



c. How does the Zero Product Property relate to the x-intercepts of the three functions?

The functions $f(x)$ and $g(x)$ are the factors of $h(x)$. If their zeros are $(-2, 0)$ and $(-5, 0)$, then the zeros of the function produced from their multiplication must also include these zeros.

Guiding Questions for Share Phase, Question 4

- The function $f(x)$ has how many x-intercepts?
- The function $g(x)$ has how many x-intercepts?
- The function $h(x)$ has how many x-intercepts?
- What is the relationship between the x-intercepts of $f(x)$ and $g(x)$?
- Are the functions $f(x)$ and $g(x)$ considered factors of $h(x)$?
- Should the zeros of the factors also be zeros of the function produced from their multiplication?

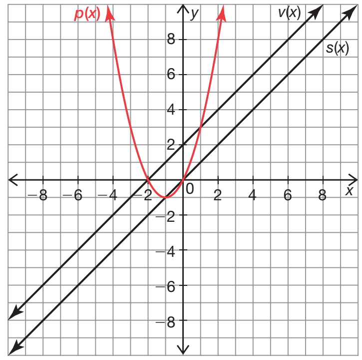
Grouping

Have students complete Questions 5 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- Will the product of the two linear functions be linear or quadratic?
- Why are the *x*-intercepts of *p*(*x*) the same as the *x*-intercepts of *s*(*x*) and *v*(*x*)?
- What are the coordinates of the vertex of *p*(*x*)?
- What do you notice about the signs of the functions *s*(*x*) and *v*(*x*) at the vertex of the parabola?
- Did adding two functions change the shape of the new graph?
- Did multiplying two functions change the shape of the new graph?

5. Analyze the graphs of *s*(*x*) and *v*(*x*).



a. Sketch the graph of *p*(*x*) if $p(x) = s(x) \cdot v(x)$.

See graph.

b. Identify the *x*-intercepts of *p*(*x*). Explain the relationship between the *x*-intercepts of *p*(*x*) and the *x*-intercepts of *s*(*x*) and *v*(*x*).

The *x*-intercepts are (0,0) and (−2,0). These are the *x*-intercepts of *s*(*x*) and *v*(*x*).

c. Identify the vertex of *p*(*x*). What is the relationship between the vertex of *p*(*x*) and the functions *s*(*x*) and *v*(*x*)?

The vertex is at (−1, −1). The vertex occurs when the signs of the functions *s*(*x*) and *v*(*x*) change, causing the new function to change directions.

d. In Problem 2 of this lesson, you added the functions *s*(*x*) and *v*(*x*) to create function *w*(*x*). How is multiplication the same? How is it different?

The process is similar in that the output values are manipulated. The shape of the graph changed. Adding functions changed the slope, but multiplying the functions completely changed the shape of the graph.

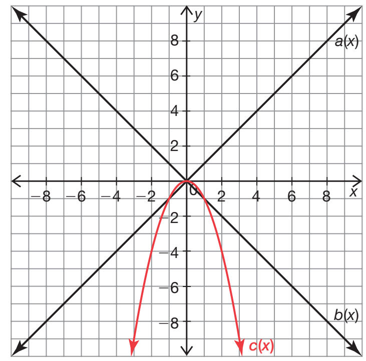
3

Guiding Questions for Share Phase, Questions 6 and 7

- What equation best represents $a(x)$?
- What equation best represents $b(x)$?
- What is the relationship between $a(x)$ and $b(x)$?
- Does the graph of $c(x)$ open upward or downward?
- What is the location of the vertex of $c(x)$?
- What equation best represents $r(x)$?
- Does the graph of $d(x)$ open upward or downward?
- If the equation that represents $r(x)$ was $y = -x$, do you think the parabola would open downward?

3

6. Analyze the graphs of $a(x)$ and $b(x)$.



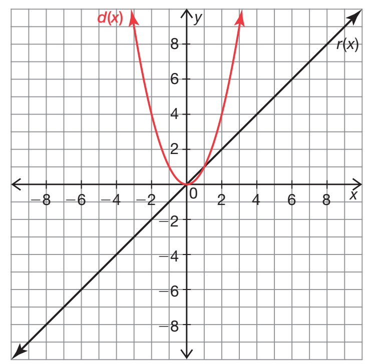
a. Sketch the graph of $c(x)$ if $c(x) = a(x) \cdot b(x)$.

See graph.

b. Describe the shape of $c(x)$.

The graph is a parabola opening down with vertex (0, 0).

7. Analyze the graph of $r(x)$.



a. Sketch the graph of $d(x)$ if $d(x) = r(x) \cdot r(x)$.

See graph.

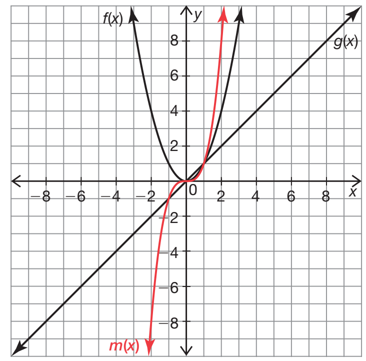
b. Describe the shape of $d(x)$.

The graph is a parabola opening up.

Guiding Question for Share Phase, Question 8

If a linear function is multiplied by a quadratic function, what is the degree of the product of the functions?

8. Analyze the graphs of $f(x)$ and $g(x)$.



a. Sketch the graph of $m(x)$ if $m(x) = f(x) \cdot g(x)$.

See graph.

b. Describe the shape of $m(x)$.

Multiplying a linear function and a quadratic function results in a graph that is S-shaped.

c. Do you think multiplying a quadratic function and a linear function will always result in a graph with this shape? Explain your reasoning.

Answers will vary.

3

Talk the Talk

The terms polynomial and degree of polynomial are defined. Students will determine the function family for the sum of different polynomials used in the lesson.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

3

Talk the Talk

While you may not have realized it, the functions you worked with throughout this lesson are *polynomials*. A **polynomial** is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. The **degree** of a polynomial is the greatest variable exponent in the expression. For example, $4x^3 + 2x^2 + 5x + 1$ is a polynomial expression of degree three, $2x$ is a polynomial of degree 1, and a constant such as 5 has degree zero since it can be written as $5x^0$.

1. Given the functions,
   - $y_1 = ax^2$,
   - $y_2 = bx$, and
   - $y_3 = c$

   generalize the function family of the polynomial when:

   a. $y_1 + y_2$
   The resulting function will be quadratic.

   b. $y_1 + y_3$
   The resulting function will be quadratic.

   c. $y_2 + y_3$
   The resulting function will be linear.

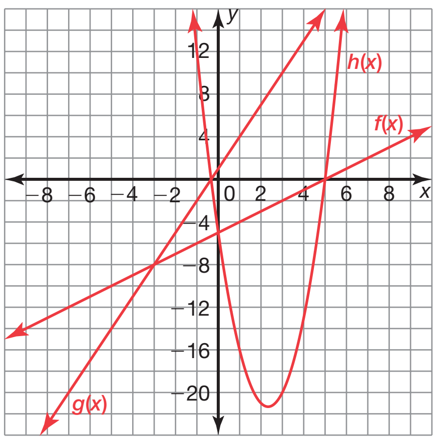
2. When two functions of different degree are added, what can you say about the degree of the resulting function?
   The resulting function will have, at most, the highest degree of the functions that were added.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Multiply $f(x) = x - 5$ and $g(x) = 3x + 1$ to produce $h(x)$. Complete the table of values and then graph each function.

x	$f(x) = x - 5$	$g(x) = 3x + 1$	$h(x)$
−3	−8	−8	64
−2	−7	−5	35
−1	−6	−2	12
0	−5	1	−5
1	−4	4	−16
2	−3	7	−21
3	−2	10	−20



3

3

Chapter 3 Summary

KEY TERMS

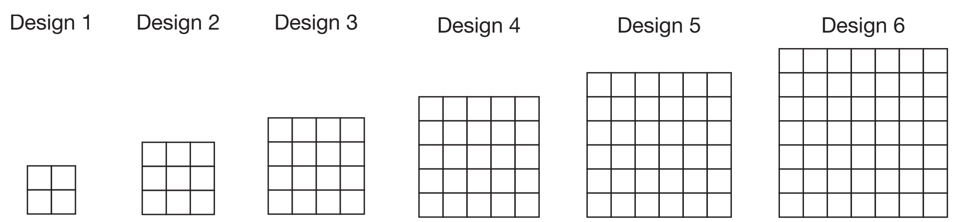
- relation (3.3)
- function (3.3)
- function notation (3.3)
- Zero Product Property (3.5)
- polynomial (3.5)
- degree (3.5)

3.1 Identifying Patterns Within a Sequence

Patterns are found throughout nature and our everyday lives. Some patterns can be described numerically.

Example

Draw the next three terms for the pattern shown.



Design 1 is made of 4 small tiles, Design 2 is made of 9 small tiles, and so on. The total number of tiles in each pattern is a perfect square. So, the terms are $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, and $7^2 = 49$.

3

3.1 Using Patterns to Solve Problems

Once you determine a pattern, you can predict the next term in the sequence.

Example

Ji is the coach for a soccer team. He wants to develop a phone tree for communicating with the team. He will call two people and each of those two people will call two more people. This pattern will continue until all team members and coaches have been contacted.

Rounds of Calls	Number of Calls Made	Cumulative Total Number of Calls Made
First round (Ji)	2	2
Second round (two teammates)	4	6
Third round (four teammates)	8	14
Fourth round (eight teammates)	16	30

The number of calls made each day is doubling. So, during the fifth round, $2 \cdot 16 = 32$ calls are made. During the sixth round, $2 \cdot 32 = 64$ calls are made.

3.2 Writing Algebraic Expressions to Describe Patterns

Algebraic expressions can be used when you want to predict patterns or represent real-life scenarios using mathematics.

Example

A website goes live and receives 83 visits in the first day. During the second day, the site receives 91 visits. The third day, the site receives 107 visits. Use an algebraic expression to represent the number of visits the site receives each day.

Day	Number of Visits to the Website
First day	83
Second day	91
Third day	107
Fourth day	139
Fifth day	203

The difference between each term in the pattern is $4(2^1)$ or 8, then $4(2^2)$ or 16, then $4(2^3)$ or 32, and so on. The first term is 8 more than 75. So, an expression to represent the number of visits the site receives each day is $4(2^x) + 75$.

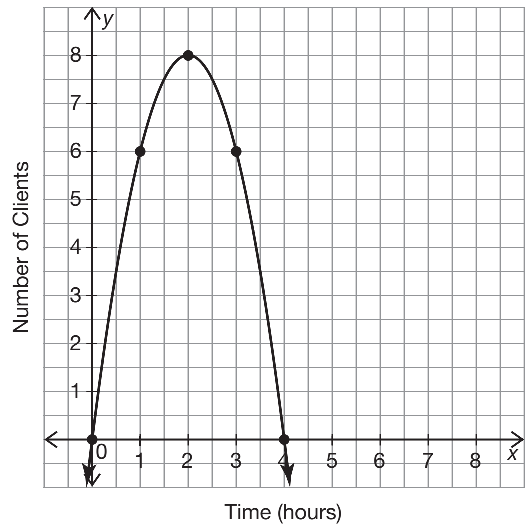
3.2 Representing Patterns

Numeric patterns can be represented as expressions, tables, and graphs. After the pattern is graphed on a coordinate plane, the graph can be identified as linear, exponential, quadratic, or none of these.

Example

A salesperson works for four hours. During the first hour, she has 6 clients. During the second hour, she has 8 clients. During the third hour, she has 6 clients. During the fourth hour, she has 0 clients. The table shown represents the number of clients the salesperson has after each hour and also an expression to represent the number of clients the salesperson sees each hour. The graph represents a quadratic model of the data.

Time (hours)	Number of Clients
1	6
2	8
3	6
4	0
n	$-2n^2 + 8n$



3

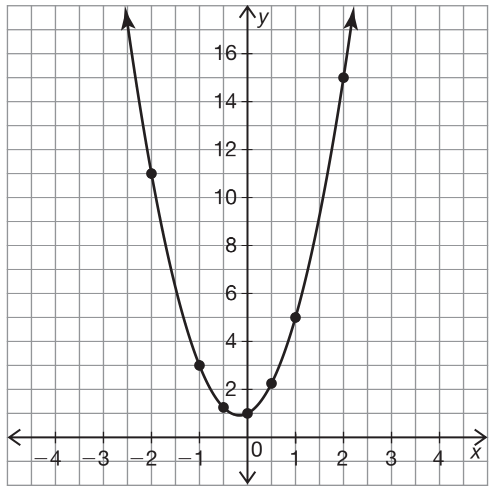
3.3 Identifying Functions

A relation describes how input values are mapped to output values in a pattern. A function is a relation that has only one output for each input. Function notation is a way to represent functions algebraically.

Example

The table and the graph represent relations. Because there is only one output for each input in the relations, they are functions. Because the same outputs are matched to each input, the functions are equivalent.

x	y
−3	25
−2	11
−1	3
0	1
1	5
2	15
3	31



3

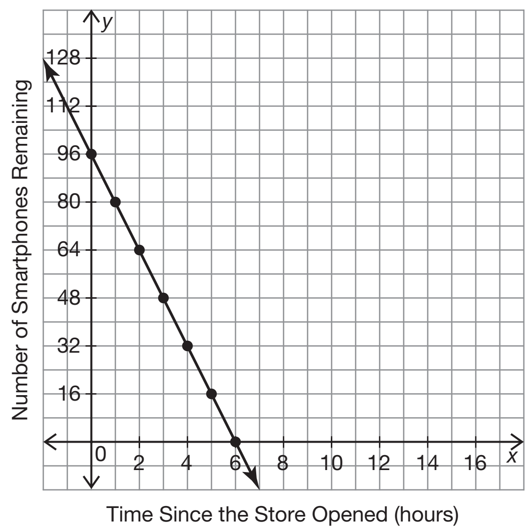
3.3 Modeling Real-World Scenarios

Tables, graphs, and equations can be used to represent real-world scenarios. A table shows the numeric values in columns. A graph shows the relation visually. The graph clearly shows whether the graph is a function or not. An equation uses numbers and variables to model the scenario.

Example

A store is having a sale on smartphones. The store opens at 8:00 AM and has 96 smartphones. After 1 hour, the store has 80 smartphones remaining. After 2 hours, the store has 64 smartphones remaining. If the pattern of sales continues at this rate, at what time will the store run out of smartphones? The number of smartphones in the store can be represented by a table and a graph. Based on the graph, the store will have zero smartphones after 6 hours.

Time	Time Since the Store Opened (hours)	Number of Smartphones Remaining
8:00 AM	0	96
9:00 AM	1	80
10:00 AM	2	64
11:00 AM	3	48
Noon	4	32
1:00 PM	5	16
2:00 PM	6	0



3

3.4

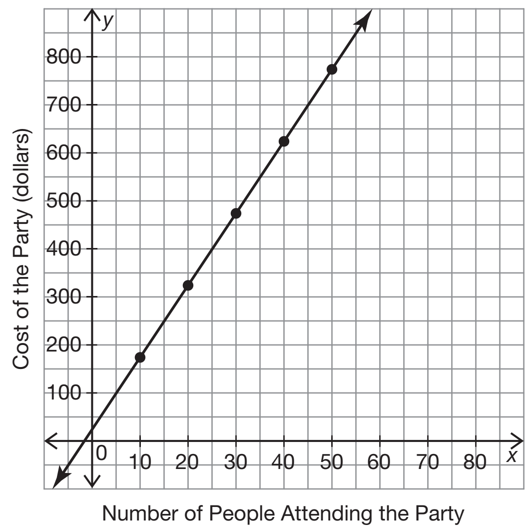
Graphing Functions

A table, graph, and equation can be used to represent real-world situations. Any of these forms can be used to analyze the situation.

Example

The cost of a party can be calculated by multiplying the number of people attending by 15 and then adding 24. An equation to model the situation is $y = 15x + 24$. The table lists the cost for specific numbers of people attending. The graph of the function is also shown. Based on the graph, the cost of the party will keep increasing as the number of people attending the party increases. Based on the table, for every 10 additional people attending the party, the cost of the party increases by $150.

Number of People Attending the Party	Cost of the Party (dollars)
10	174
20	324
30	474
40	624
50	774



3

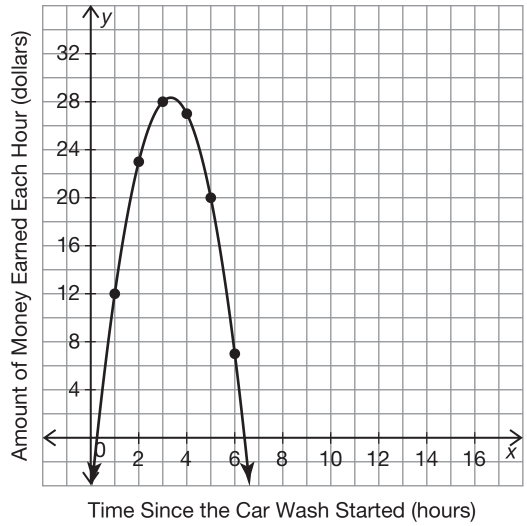
3.4 Using a Graph and Function to Analyze Problems

The table or graph for a situation can provide valuable information about the scenario. The minimum or maximum of a quadratic equation can provide context for the situation or aid in predicting or analyzing the scenario.

Example

Larissa is running a car wash as a fundraiser for her school. In the first hour, the group earns $12 in donations for car washes. The amount of money earned each hour is listed in the table and shown in the graph. The group earns a maximum of $28 in the third hour of the car wash.

Time Since the Car Wash Started (hours)	Amount of Money Earned Each Hour (dollars)
1	12
2	23
3	28
4	27
5	20
6	7



3

3.5 Adding or Subtracting Functions Graphically

To add or subtract two functions graphically, first identify several key points. Some key points include intersection points, *x*-intercepts, and *y*-intercepts. For each input value, add or subtract the output values for each function to calculate the output value of the new function. Then draw the graph of the new function.

Example

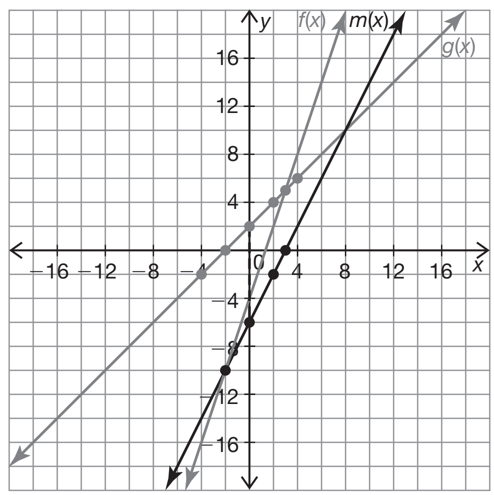
Add $g(x) = x + 2$ and $f(x) = 2x - 6$ graphically.

Use the key points (−2, 0), (−2, −10), (0, 2), (0, −6), (3, 0), and (3, 5).

The new function contains the points (−2, 0 − 10) or (−2, −10), (0, 2 − 6) or (0, −4), and (3, 0 + 5) or (3, 5).

The graph of the new function, $m(x)$ is shown.

The equation for the function $m(x)$ is $m(x) = 3x - 4$.



3

3.5 Multiplying Functions Graphically

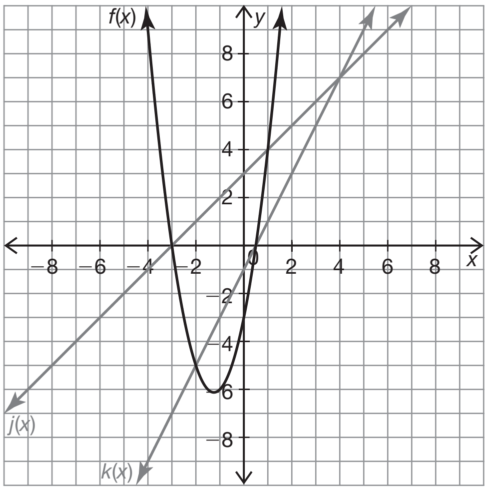
To multiply two functions graphically, first identify several key points. Some key points include intersection points, x-intercepts, and y-intercepts. For each input value, multiply the output values for each function to calculate the output value of the new function. Then draw the graph of the new function.

Example

Multiply $j(x) = x + 3$ and $k(x) = 2x - 1$ to determine $f(x)$.

Use a table of values to graph the functions.

x	$j(x) = x + 3$	$k(x) = 2x - 1$	$f(x)$
−3	0	−7	0
−2	1	−5	−5
−1	2	−3	−6
0	3	−1	−3
1	4	1	4
2	5	3	15
3	6	5	30



3

3