Rational Functions

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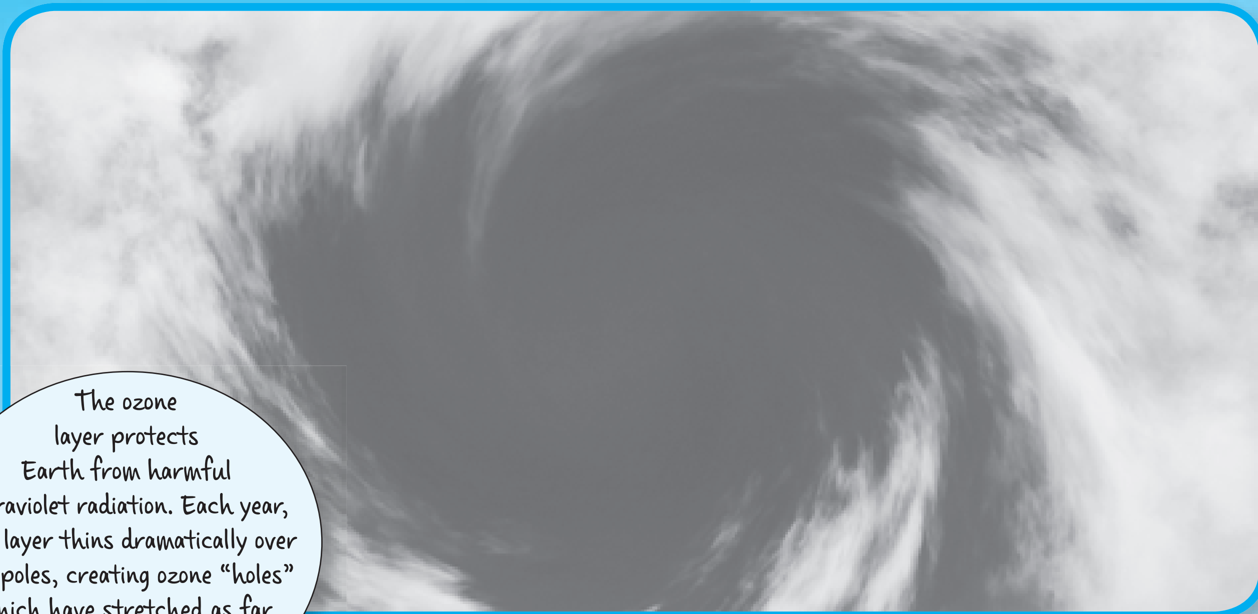

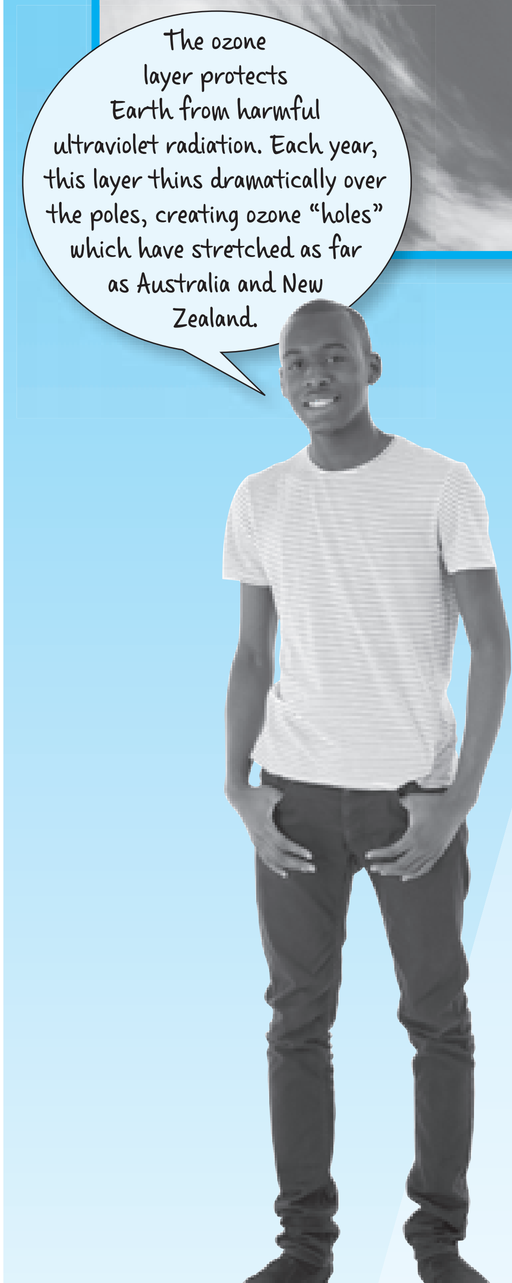

Chapter 9 Overview

This chapter presents opportunities for students to analyze, graph, and transform rational functions. The chapter begins with an analysis of key characteristics of rational functions and graphs. Lessons then expand on this knowledge for transformations of rational functions. Students will determine whether graphs of rational functions have vertical asymptotes, removable discontinuities, both, or neither, and then sketch graphs of rational functions detailing all holes and asymptotes. Finally, students will explore problem situations modeled by rational functions and answer questions related to each scenario.

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Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
9.1	Introduction to Rational Functions	F.IF.7.d (+)	1	This lesson explores the graphs, tables, and values of rational functions including $g(x) = \frac{1}{x}$ and $r(x) = \frac{1}{x^2}$. Questions ask students to compare graphs and analyze key characteristics of rational functions. They then construct a Venn diagram to show the similarities and differences between even and odd reciprocal power functions.	x		x		x
9.2	Translating Rational Functions	F.IF.7.d (+) F.IF.8.a F.BF.3	1	This lesson explores rational functions of the form $g(x) = \frac{1}{x - c}$ for a constant value c. Questions lead students to determine a general formula to identify vertical asymptotes, and that a horizontal translation results when changing the c-value of the reciprocal function. They then determine the vertical asymptotes of different rational functions using algebra.	x		x		
9.3	Exploring Rational Functions Graphically	F.IF.7.d (+) F.IF.8.a F.BF.3	1	This lesson provides opportunities for students to explore transformations of rational functions. Questions ask students to sketch rational functions without a calculator and to identify their key characteristics, as well as to match transformations of rational functions with their graphs.	x		x		

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
9.4	Graphical Discontinuities	A.APR.6 A.APR.7(+) F.IF.7.d (+) F.IF.8.a	2	This lesson explores rational functions with graphical discontinuities. Questions ask students to graph rational functions with holes or asymptotes. They will also develop methods to determine whether graphs have vertical asymptotes, removable discontinuities, both, or neither.	x	x	x	x	
9.5	Using Rational Functions to Solve Problems	A.SSE.2 A.CED.1 A.REI.2 F.IF.5	1	This lesson provides opportunities for students to model problem situations using rational functions. Questions ask students to a create ratios, write rational expressions, describe end behavior, identify the domain and range, and calculate average costs for various problem situations.			x		

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Skills Practice Correlation for Chapter 9

Lesson		Problem Set	Objectives
9.1	Introduction to Rational Functions		Vocabulary
		1 – 8	Determine whether a function is a rational function or not a rational function
		9 – 14	Describe the vertical and horizontal asymptotes from a graph of rational functions
		15 – 20	Determine the domain and range for rational functions in the form $f(x) = \frac{a}{x^n}$, where a is a non-zero real number and n is an integer greater than or equal to 1
		21 – 26	Describe the end behavior of rational functions in the form $f(x) = \frac{a}{x}$, where a is a non-zero real number and n is an integer greater than or equal to 1
		27 – 32	Describe the end behavior of rational functions as x approaches zero from the left and as x approaches zero from the right
		33 – 38	Analyze given key characteristics of rational functions of the form $f(x) = \frac{1}{x^n}$ and identify whether the characteristic is modeled by an odd power of n, an even power of n, or both
9.2	Translating Rational Functions	1 – 6	Determine the vertical and horizontal asymptotes, the domain, and the range from rational functions of the form $\frac{1}{x - c}$
		7 – 12	Determine the domain, range, and vertical and horizontal asymptotes of rational functions without using a graphing calculator
		13 – 18	Write a rational function for the table, graph, or description provided
		19 – 24	Sketch rational functions without using a graphing calculator
		25 – 30	Use algebra to determine the vertical asymptotes of rational functions
		31 – 36	Determine two different rational functions with given characteristics
9.3	Exploring Rational Functions Graphically	1 – 6	Sketch rational functions without using a graphing calculator, and then indicate the domain, range, vertical and horizontal asymptote(s), and y-intercept
		7 – 12	Determine the transformation performed given $f(x) = \frac{1}{x}$
		13 – 18	Sketch a graph of the transformations performed on the function $f(x) = \frac{1}{x}$
		19 – 24	Write a rational function to match a given characteristic

Lesson		Problem Set	Objectives
9.4	Graphical Discontinuities		Vocabulary
		1 – 6	Determine if rational functions have removable discontinuities without using a graphing calculator
		7 – 14	Simplify rational expressions and list any restrictions on the domain
		15 – 22	Determine whether graph of rational functions have a vertical asymptote, a removable discontinuity, both, or neither
		23 – 28	Write an example of a rational function that models given characteristics
		29 – 36	Sketch a graph of a rational function with using a graphing calculator and list any domain restrictions
9.5	Using Rational Functions to Solve Problems	1 – 10	Solve problems using rational functions
		11 – 16	Sketch a graph to solve rational equations

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9.1

A Rational Existence

Introduction to Rational Functions

LEARNING GOALS

In this lesson, you will:

- Graph rational functions.
- Compare rational functions in multiple representations.
- Compare the basic rational function to various basic polynomial functions.
- Analyze the key characteristics of rational functions.

KEY TERMS

- rational function
- vertical asymptote

ESSENTIAL IDEAS

- A rational function is any function that can be written as the ratio of two polynomials. It can be written in the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions, and $Q(x) \neq 0$.
- A horizontal asymptote is a horizontal line that a function gets closer and closer to, but never intersects.
- A vertical asymptote is a vertical line that a function gets closer and closer to, but never intersects.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-IF Interpreting Functions

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

d. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

Overview

Students will explore and compare the graphs, tables, and values of a polynomial function, $f(x) = x$, and its reciprocal function, $g(x) = \frac{1}{x}$. Next, they compare the graphs, tables, and values of a power function, $q(x) = x^2$, and its reciprocal function, $r(x) = \frac{1}{x^2}$. A graphing calculator is used to explore the key characteristics of the reciprocals of all power functions. Students then construct a Venn Diagram to show the similarities and differences between the groups of reciprocal power functions.

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Warm Up

Identify the value(s) of x that make each function undefined.

1. $f(x) = -\frac{3}{4x}$

   The function $f(x)$ is undefined when $x = 0$.

2. $f(x) = \frac{x}{x - 2}$

   The function $f(x)$ is undefined when $x = 2$.

3. $f(x) = \frac{1}{12 - x}$

   The function $f(x)$ is undefined when $x = 12$.

4. $f(x) = \frac{-6}{5 + x}$

   The function $f(x)$ is undefined when $x = -5$.

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9.1

A Rational Existence

Introduction to Rational Functions

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KEY TERMS

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Consider the following mathematical explanation that 1 is equal to 2. Yes, you read that correctly–you will analyze a proof of $1 = 2$.

Let's start by noting that any number multiplied by 0 is equal to 0, correct? Therefore,

$$1 \times 0 = 0$$
$$2 \times 0 = 0.$$

Since the expressions 1×0 and 2×0 both equal zero, then they must be equal to each other by the transitive property. Therefore,

$$1 \times 0 = 2 \times 0.$$

Dividing both sides of an equation by the same value preserves equality. Therefore, you can divide both sides of the equation by 0.

$$\frac{1 \times 0}{0} = \frac{2 \times 0}{0}$$

Anything divided by itself is 1, so $\frac{0}{0} = 1$. This leaves

$$1 \times 1 = 2 \times 1$$
$$1 = 2.$$

There weren't any sneaky tricks or magical sleights of hand in this proof. In fact, all steps were justified according to the rules of algebra . . . so then what's wrong with this proof?

Problem 1

Students will compare the graphs, tables, and values of a polynomial function, $f(x) = x$, and its reciprocal function, $g(x) = \frac{1}{x}$. They then analyze the key characteristics of $g(x) = \frac{1}{x}$, describe the domain and range, and conclude that it is a function. The terms rational function and vertical asymptote are defined. A graphing calculator is used to explore various rational functions of the form $p(x) = \frac{a}{x}$, where a is a constant.

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Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

PROBLEM 1 My World Is Turned Upside Down

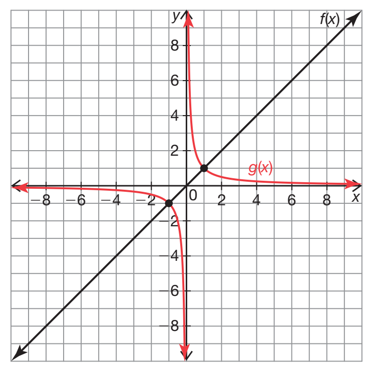
Recall from previous math courses that the reciprocal of any number x is $\frac{1}{x}$. For example, the reciprocal of 5 is $\frac{1}{5}$ and the reciprocal of 0.5 is $\frac{1}{0.5}$, or 2.

Throughout this course you have studied many connections between polynomial functions and real numbers. Does it follow then that polynomial functions also have reciprocals? Is the reciprocal also a polynomial? Is it a function? How would the graph and table of values of $\frac{1}{f(x)}$ compare to the original function $f(x)$?

To begin answering these questions, consider the reciprocal of the basic linear function $f(x) = x$. The reciprocal can be defined as $g(x) = \frac{1}{f(x)}$, or simply $g(x) = \frac{1}{x}$.

1. Consider the graph and table of values for $f(x) = x$. The domain of $f(x)$ is $(-\infty, \infty)$. The points $(-1, -1)$ and $(1, 1)$ are shown and used to create three intervals for analysis.

x	1	2	3	4	5	6
$f(x) = x$	1	2	3	4	5	6
$g(x) = \frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$



-6	-5	-4	-3	-2	-1	x
-6	-5	-4	-3	-2	-1	$f(x) = x$
$-\frac{1}{6}$	$-\frac{1}{5}$	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	$g(x) = \frac{1}{x}$

x	-1	$-\frac{1}{2}$	$-\frac{1}{10}$	$-\frac{1}{100}$	0	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	1
$f(x) = x$	-1	$-\frac{1}{2}$	$-\frac{1}{10}$	$-\frac{1}{100}$	0	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	1
$g(x) = \frac{1}{x}$	-1	-2	-10	-100	und	100	10	2	1

a. Complete the table of values for $g(x) = \frac{1}{x}$. Then plot the points and draw a smooth curve to graph $g(x)$ on the coordinate plane.

See tables in graph.

See graph.

Guiding Questions for Share Phase, Question 1

- Is the graph of $g(x) = \frac{1}{x}$ a continuous graph?
- Does the graph of $g(x) = \frac{1}{x}$ intersect the x-axis?
- Does the graph of $g(x) = \frac{1}{x}$ intersect the y-axis?
- Is $g(x) = \frac{1}{x}$ a function?
- Does the graph of $g(x) = \frac{1}{x}$ increase or decrease?
- Is $g(x) = \frac{1}{x}$ a piecewise function?
- As the x-values approach negative infinity, what happens to the y-values?
- Can the value of x ever be equal to 0? How do you know?
- Is $g(x) = \frac{1}{x}$ increasing or decreasing as x approaches 0 from the left?
- Is $g(x) = \frac{1}{x}$ increasing or decreasing as x approaches 0 from the right?
- When dividing by negative numbers that approach infinity, what do output values approach?
- When dividing by small positive numbers that approach zero, what do output values approach?

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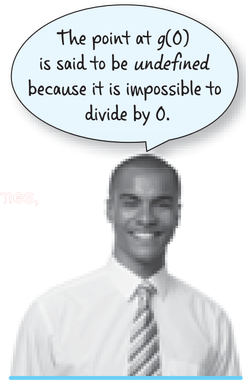
b. Describe the graph of $g(x)$. How is it similar to the graphs of other functions that you've studied? How is it different?

Answers will vary.

Student responses could include:

- The graph is similar to a quadratic in the sense that it curves.
- The graph is different than what I have seen in the past because it looks like 2 pieces.
- The graph is different because it doesn't appear to ever touch the x- or y-axes. This behavior is similar to exponential functions.

c. Describe the end behavior of $g(x)$. Explain your reasoning in terms of the graph, equation, and table of values.



As x approaches negative infinity y approaches 0.

As x approaches positive infinity, y approaches 0.

I can see this in the table because the greater x becomes, the lesser y becomes for all positive values of x. For all negative values of x, the lesser x becomes the closer y approaches zero. This makes sense from the equation because x cannot equal 0.

d. Describe $g(x)$ as x approaches 0 from the left. Explain the output behavior of the function in terms of the equation.

The function $g(x)$ is decreasing. When I divide by negative numbers that approach zero this results in output values that approach negative infinity.

e. Describe $g(x)$ as x approaches 0 from the right. Explain the output behavior of the function in terms of the equation.

The function $g(x)$ is increasing. When I divide by very small positive numbers that approach zero, this results in very large output values that approach infinity.

Guiding Questions for Share Phase, Questions 2 and 3

- What is the definition of a function?
- Does every input have to have an output, if it is a function?
- Does the graph pass the vertical line test?
- Does any input have more than one output?
- What happens to the y-values as the x-values increase?
- What happens to the x-values as the y-values increase?
- Is the domain all real numbers?
- Is the range all real numbers?

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2. Henry and Rosie disagree about $g(x) = \frac{1}{x}$.

Henry
The graph and table both clearly show that it is a function.

Rosie
It is not a function. Every input doesn't have an output.

Who is correct? Explain your reasoning.

Henry is correct. The graph passes the vertical line test. Every input does not need an output; it just cannot have more than one output.

3. Analyze the key characteristics of $g(x) = \frac{1}{x}$.

a. Will the graph ever intersect the horizontal line $y = 0$? Explain your reasoning in terms of the graph, table, and equation.

No. The graph will never intersect the line $y = 0$.
I can see this in the graph.
I can tell from the table that as x increases y approaches 0, but will never actually equal zero.
I can tell this from the equation because 0 will never equal one over any number.

b. Will the graph ever intersect the vertical line $x = 0$? Explain your reasoning in terms of the graph, table, and equation.

No. The graph will never intersect the line $x = 0$.
I can see this in the graph.
I can tell from the table because as y increases x approaches 0, but will never actually equal zero.
I can tell this from the equation because 1 divided by 0 is undefined.

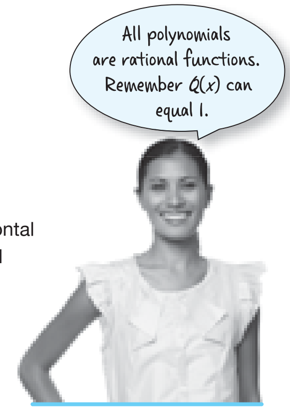
c. Describe the domain and range of $g(x)$.

The domain of $g(x)$ is all real numbers excluding 0.
The range of $g(x)$ is all real numbers excluding 0.

Grouping

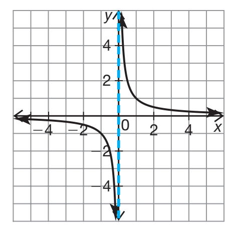
Ask a student to read the information and definitions. Discuss as a class.

The function $g(x) = \frac{1}{x}$ is an example of a *rational function*. A **rational function** is any function that can be written as the ratio of two polynomials. It can be written in the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions, and $Q(x) \neq 0$. You have already seen some specific types of rational functions. Linear, quadratic, cubic, and higher order polynomial functions are types of rational functions.

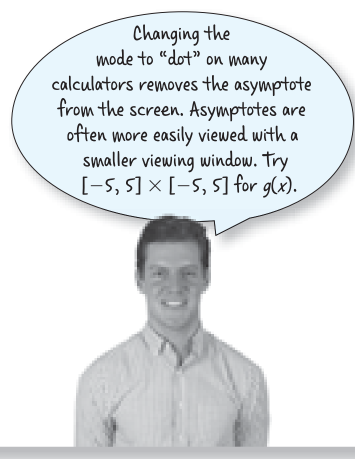


Recall from your study of exponential functions that a horizontal asymptote is a horizontal line that a function gets closer and closer to, but never intersects. In this problem, the function $g(x)$ has a horizontal asymptote at $y = 0$.

The function $g(x) = \frac{1}{x}$ has a *vertical asymptote* at $x = 0$. A **vertical asymptote** is a vertical line that a function gets closer and closer to, but never intersects. The asymptote does not represent points on the graph of the function. It represents the output value that the graph approaches. An asymptote occurs for input values that result in a denominator of 0.



The vertical asymptote is often represented in textbooks and graphing calculators as a dashed or solid line. The convention used in this textbook is to represent asymptotes as dashed lines.



Grouping

Have students complete Question 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

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- Are all linear functions also rational functions?
- Is the square root of x a polynomial?
- If a variable is in the exponent position, is it a polynomial?
- If there is a variable in the denominator of a rational function, will it always have a vertical asymptote?

Grouping

Have students complete Questions 5 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 7

- As the a-value increases beyond 1, what happens to the graph of the function?
- As the graph of the function gets further from the origin, what is happening to the a-values?
- As the a-value moves further from 0 in the negative direction, what happens to the graph of the function?

4. Analyze each function.

$f(x) = x$	$g(x) = \frac{3x}{2}$	$h(x) = \frac{\sqrt{x}}{2x}$
$p(x) = \frac{3}{x} + 2$	$k(x) = 12$	$n(x) = \frac{2^x}{5}$
$j(x) = \frac{4x^2 + 3x + 2}{6x^3 + 10}$	$m(x) = \frac{1}{(x + 2)(x - 3)}$	

a. Circle the rational functions.

See functions in box.

b. Explain why the remaining functions are not rational.

The function $h(x)$ is not a rational function because the square root of x is not a polynomial. The function $n(x)$ is not a polynomial because of the variable in the exponent, which means it cannot be a polynomial.

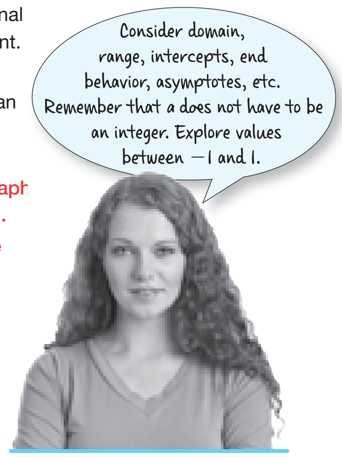
c. Do you think the graphs of all rational functions will have a vertical asymptote? Explain your reasoning.

No. If the rational function does not have a variable in the denominator, there would not be a vertical asymptote. For example, the graph of $k(x)$ will just be a horizontal line.

5. Use a graphing calculator to explore various rational functions of the form $p(x) = \frac{a}{x}$ where a is a constant.

a. Describe changes in the function for various a-values. Make as many conjectures as you can about the key characteristics of functions in this form.

- As the a-value increases beyond 1, the graph of the function gets further from the origin.
- As the a-value moves further from 0 in the negative direction, the graph moves to the second and fourth quadrants, but maintains the same shape as the positive version of the graph.
- When a is between 0 and 1, the graph is closer to the origin.



- As the graph of the function moves to the second and fourth quadrants, what is happening to the a-values?
- When the a-value is between 0 and 1, what happens to the graph of the function?
- As the graph of the function gets closer to the origin, what is happening to the a-values?
- As the x-values in the denominator of the function $p(x)$ approach infinity, what does the function approach?

- As x approaches infinity, will the denominator be greater than the numerator?
- Does $f(x) = 12$ have a horizontal asymptote?
- Is the product of any expression and its reciprocal always equal to 1? Why?
- Is it possible to divide by zero? Why not?

b. Abby and Natasha disagree about functions of the form $p(x) = \frac{a}{x}$ where a is a constant.

Abby
The horizontal asymptote will vary depending on the a-value.

Natasha
All rational functions of this form will have a horizontal asymptote at $y = 0$.

Who is correct? Explain your reasoning.

Natasha is correct. It does not matter how great the a-value is, as x approaches infinity the denominator will be much greater than the numerator, and $p(x)$ will approach zero.

c. List several rational functions that do not have a horizontal asymptote at $y = 0$.

Answers will vary.

$f(x) = 12$

$f(x) = \frac{1}{2}$

$f(x) = \frac{x}{-2}$

6. If $g(x)$ is the reciprocal function of $f(x)$, what is $f(x) \cdot g(x)$ where $g(x) \neq 0$ for any input value? Explain your reasoning.

The value of $f(x) \cdot g(x)$ for any output value will be 1. Anytime I multiply an expression by its reciprocal, the product is 1.

7. The opener to this lesson provides a proof that $1 = 2$. Describe the error in the proof in terms of what you learned in this problem.

I cannot divide by zero. Anytime I divide by zero the expression is undefined and I get an asymptote.

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Problem 2

Students will compare the graphs, tables, and values of a power function $q(x) = x^2$ and its reciprocal function $r(x) = \frac{1}{x^2}$. They then analyze the key characteristics of $r(x) = \frac{1}{x^2}$, describe the domain and range, and conclude that it is a function. A graphing calculator is used to explore the key characteristics of the reciprocals of all power functions. Students conclude all even power functions look relatively similar and all odd power functions look relatively similar. Students will construct a Venn Diagram to show the similarities and differences between the groups of reciprocal power functions.

Grouping

Ask a student to read the information. Discuss and complete Question 1 as a class.

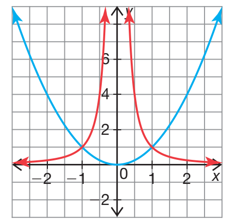
Guiding Questions for Discuss Phase

- Will the graph of the reciprocal of the power function have an x asymptote at zero? How do you know?
- Will the graph of the reciprocal of the power function have a y asymptote at zero? How do you know?
- If y is in fraction form, can it ever equal zero? Why not?
- Can the output value ever be a negative value? Why not?

PROBLEM 2 Power in Rational Behavior

Recall that power functions are any functions of the form $y = x^n$ for $n \geq 1$. In Problem 1, *My World Is Turned Upside Down*, you discovered that the graph of the function $g(x) = \frac{1}{x}$ looks very different than the linear function $f(x) = x$. How will the graphs of the other power functions compare to their reciprocals? Will they all have the same shape? Will they all have asymptotes?

1. Analyze the graph of the quadratic power function $q(x) = x^2$.

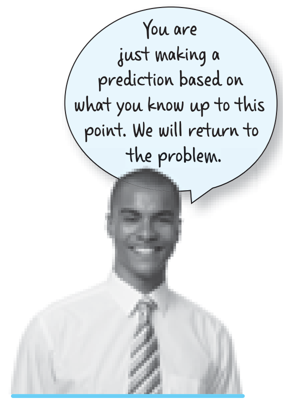


Predict the graph of $r(x) = \frac{1}{x^2}$. Sketch it on the coordinate plane. Explain your reasoning.

See graph.

I know that the graph will have x and y asymptotes at zero because you cannot divide by zero, and y can never equal zero if it is in the form of a fraction.

I also know that the graph cannot be in quadrants III and IV because the output will always be positive because the input values are squared.



- Will the graph of the reciprocal function appear in quadrants III or IV? Why not?

Grouping

Have students complete Questions 2 through 4 with a partner. Then have students share their responses as a class.

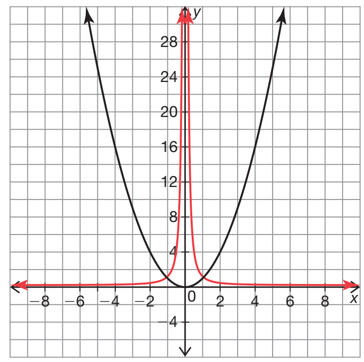
Guiding Questions for Share Phase, Question 2

- Is the reciprocal function in one or two pieces?
- Does $r(x)$ have negative values?
- Does $g(x)$ have negative values?
- Do $r(x)$ and $g(x)$ have the same asymptotes?

2. Consider the graph and table of values for $q(x) = x^2$. The domain of $q(x)$ is $(-\infty, \infty)$. The tables represent three intervals of the domain.

x	-5	-4	-3	-2	-1
$q(x) = x^2$	25	16	9	4	1
$r(x) = \frac{1}{x^2}$	$\frac{1}{25}$	$\frac{1}{16}$	$\frac{1}{9}$	$\frac{1}{4}$	1

x	1	2	3	4	5
$q(x) = x^2$	1	4	9	16	25
$r(x) = \frac{1}{x^2}$	1	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$



x	-1	$-\frac{1}{2}$	$-\frac{1}{10}$	$-\frac{1}{100}$	0	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	1
$q(x) = x^2$	1	$\frac{1}{4}$	$\frac{1}{100}$	$\frac{1}{10000}$	0	$\frac{1}{10000}$	$\frac{1}{100}$	$\frac{1}{4}$	1
$r(x) = \frac{1}{x^2}$	1	4	100	10000	und	10000	100	4	1

a. Complete the table of values for $r(x) = \frac{1}{x^2}$.

See tables.

b. Plot the points and sketch the reciprocal function $r(x)$ on the coordinate plane.

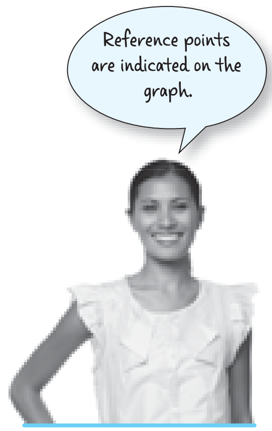
See graph.

c. Describe the shape of the graph of $r(x) = \frac{1}{x^2}$. How is it similar to $g(x) = \frac{1}{x}$? How is it different?

The shape looks like a parabola split in two pieces and turning away from each other.

It is similar to $g(x) = \frac{1}{x}$ because there are asymptotes at $x = 0$ and $y = 0$.

It is different than $g(x)$ because there are no negative y-values.



Guiding Questions for Share Phase, Questions 3 through Question 4, part (b)

- Is the domain of $r(x)$ all real numbers?
- Is the range of $r(x)$ all real numbers?
- As x approaches negative infinity, what does y approach?
- As x approaches positive infinity, what does y approach?
- Using the table of values, as y values increase, what do the x-values approach?
- Why will x never equal zero?
- Will all even power functions look similar? How?
- As the exponent value increases, what effect does it have on the graph?
- Why will the range always be numbers greater than zero?
- Will all odd power functions look similar? How?
- As the exponent value increases, what effect does it have on the graph?
- Why will the range always be all real numbers excluding zero?

9

3. Analyze the key characteristics of $r(x)$.

a. Describe the domain and range of $r(x)$.

The domain of $r(x)$ is all real numbers excluding 0.

The range of $r(x)$ is all positive real numbers.

b. Describe the end behavior of $r(x)$.

As x approaches negative infinity, y approaches 0.

As x approaches positive infinity, y approaches 0.

c. Describe the horizontal and vertical asymptotes of $r(x)$. How can you determine the asymptotes from the graph, table, and equation?

The horizontal and vertical asymptotes of $r(x)$ are $x = 0$ and $y = 0$. I can see this in the graph. I can tell from the table because as y increases, x approaches 0 but will never actually equal zero because it is always a fraction.

I can tell this from the equation because 1 divided by 0 is undefined.

4. Use a graphing calculator to explore the key characteristics of the reciprocals of all power functions. Consider the general shape of the graphs, domain, range, end behavior, horizontal asymptotes, and vertical asymptotes.

a. List your conjectures about the even-powered functions $\left\{\frac{1}{x^2}, \frac{1}{x^4}, \frac{1}{x^6}, \ldots\right\}$. Justify your conjectures.

All even power functions will look relatively similar to each other. The graph becomes steeper as the exponent becomes greater.

The domain will always be all real numbers excluding 0.

The range will always be real numbers greater than 0 because y cannot be zero or negative.

The asymptotes will still be $x = 0$ and $y = 0$.

b. List your conjectures about the odd-powered functions $\left\{\frac{1}{x^3}, \frac{1}{x^5}, \frac{1}{x^7}, \ldots\right\}$. Justify your conjectures.

All odd power functions will look relatively similar to each other. The graph becomes steeper as the exponent becomes greater.

The domain and range will always be all real numbers excluding 0.

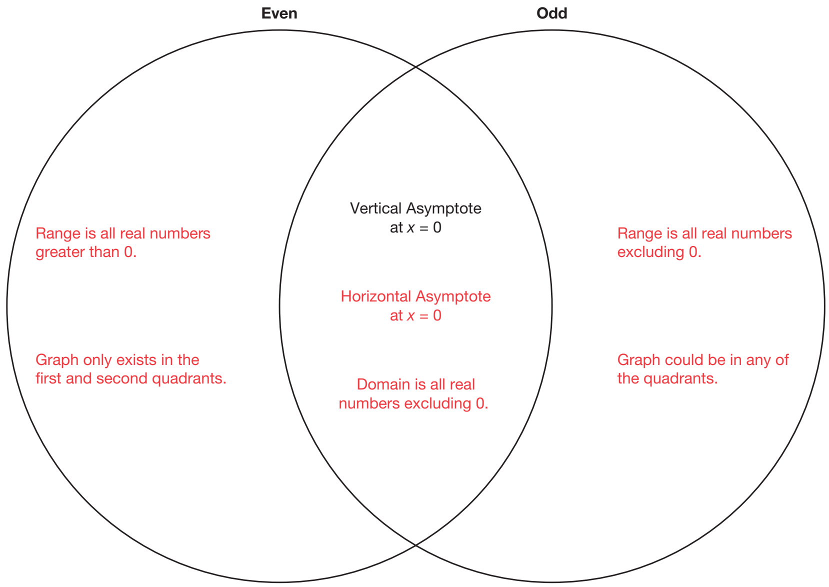
The asymptotes will still be $x = 0$ and $y = 0$.

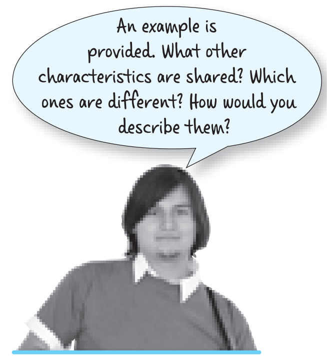
Guiding Questions for Share Phase, Question 4, part (c)

- Where are the key characteristics shared by both groups placed on the Venn Diagram?
- Are the ranges different for even and odd power functions?
- Are the domains different for even and odd power functions?
- Are the quadrants in which the functions lie different for even and odd power functions?
- Are the x asymptotes different for even and odd power functions?
- Are the y asymptotes different for even and odd power functions?

9

c. Summarize the similarities and differences between the groups of reciprocal power functions by describing the key characteristics in the Venn diagram. Characteristics that are shared should go in the overlapping space.





Be prepared to share your solutions and methods.

Check for Students' Understanding

9

Suppose you want to purchase a new laptop which will cost $2200.

1. How long would it take to save $2200 if you could save:

   a. $10 per week?

   It would take 220 weeks or just over 4 years.

   b. $20 per week?

   It would take 110 weeks or just over 2 years.

   c. $40 per week?

   It would take 55 weeks or just over 1 year.

   d. $50 per week?

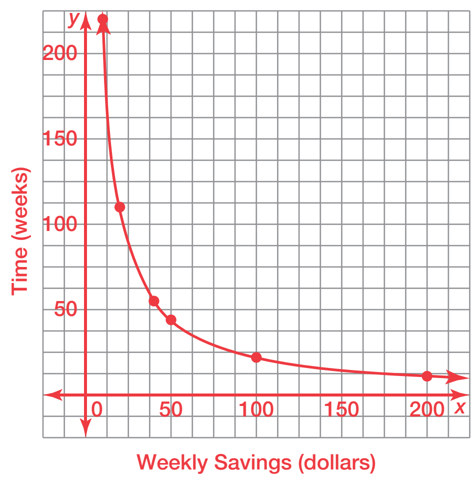
   It would take 44 weeks or about 10 months.

   e. $100 per week?

   It would take 22 weeks or about 5 months.

2. Complete the table. Then use the information in the table to construct a graph of the problem situation.

Weekly Savings (dollars)	Time (weeks)
10	220
20	110
40	55
50	44
100	22
200	11



3. Can this problem situation be modeled by a function? Explain your reasoning.

   Yes. Each dollar amount saved corresponds to exactly one number of weeks to reach the goal of $2200.

4. Write an algebraic equation to model this problem situation.

   $y = \frac{2200}{x}$**, where *x* equals the amount saved per week, and *y* equals the number of weeks to save $2200.**

9

5. Describe the asymptotic behavior of the graph in this situation.

   a. What happens to the graph as *x* approaches zero?

      As *x* approaches zero, *y* approaches infinity.

   b. What happens to the graph as *x* approaches infinity?

      As *x* approaches infinity, *y* approaches zero.

9

9.2

A Rational Shift in Behavior

Translating Rational Functions

LEARNING GOALS

In this lesson, you will:

- Analyze rational functions with a constant added to the denominator.
- Compare rational functions in different forms.
- Identify vertical asymptotes of rational functions.

ESSENTIAL IDEAS

- The general formula to identify the vertical asymptote of a rational function in the form $g(x) = \frac{1}{x - c}$ is $x = c$. The domain is all real numbers except for the c-value. The range is all real numbers except 0.
- Horizontal asymptotes occur at $y = 0$ because the numerator is a constant while a variable is in the denominator.
- The reciprocal of a function of degree n can have at most n vertical asymptotes.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-IF Interpreting Functions

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.
   - **d.** Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   - **a.** Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

F-BF Building Functions

Build new functions from existing functions

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

Overview

Students will use a graphing calculator to explore rational functions of the form $g(x) = \frac{1}{x - c}$ for a constant value c. Students then determine a general formula to identify the vertical asymptote for rational functions in this form, and conclude that changing the c-value has no effect on the function's end behavior. They conclude that changing the c-value on the reciprocal function translates the function to the right or left c units. Students determine the number of vertical asymptotes in functions containing quadratics in their denominator without using a graphing calculator. Algebra is used to determine the vertical asymptotes of different rational functions.

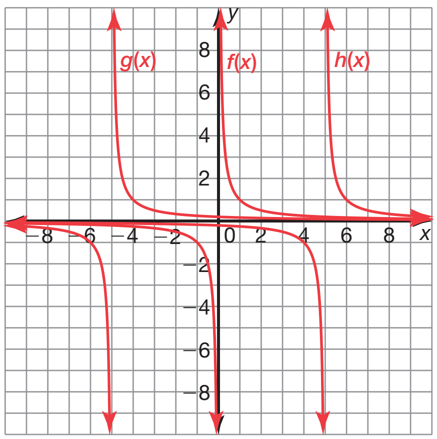
Warm Up

1. Sketch the graph of each rational function on the same coordinate plane.

$f(x) = \frac{1}{x}$

$g(x) = \frac{1}{x + 5}$

$h(x) = \frac{1}{x - 5}$



2. Describe $g(x)$ in terms of $f(x)$.

The function $g(x)$ is $f(x)$ shifted 5 units to the left.

3. Describe $h(x)$ in terms of $f(x)$.

The function $h(x)$ is $f(x)$ shifted 5 units to the right.

9

9

A Rational Shift in Behavior

9.2

Translating Rational Functions

9

LEARNING GOALS

In this lesson, you will:

- Analyze rational functions with a constant added to the denominator.
- Compare rational functions in different forms.
- Identify vertical asymptotes of rational functions.

When cars were first built, they all had manual transmissions. This means the drivers had to press on a clutch and shift gears when starting the car, accelerating, or going up an incline. In the 1940's automatic transmission cars were introduced. The transmission was designed so that the gears shift automatically for the driver as the car accelerates or decelerates. Today, automatic transmission cars make up more than 90% of the cars on the road. Ten percent is a pretty small number, but that still amounts to millions of drivers who choose to manually shift gears while driving. Why would they choose a manual transmission?

Manual transmissions are in less demand, so the lower price tag often attracts drivers. They generally get better gas mileage than automatics, adding to the savings over time. Repairs are usually cheaper, too, as the transmission is less complicated. Some drivers also prefer the control that the manual transmission cars offer, especially having the ability to choose the gear when in poor weather or road conditions.

Have you driven a car? Was it an automatic or manual transmission?

9

Problem 1

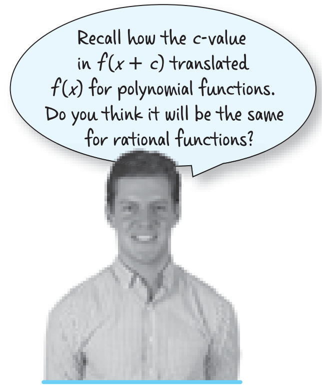
Students will use a graphing calculator to explore rational functions of the form $g(x) = \frac{1}{x - c}$ for a constant value c. They organize information such as vertical and horizontal asymptotes, domain, and range about functions containing different positive and negative c-values in a table. Students then determine a general formula to identify the vertical asymptote for rational functions in this form, and conclude that changing the c-value has no effect on the function's end behavior. They determine the asymptotes, domain, and range of several functions without using the graphing calculator. Given a graph, asymptotes, the domain and range, or a table of values, students will write an equation to fit the information. They conclude that changing the c-value on the reciprocal function translates the function to the right or left c units.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Question 1 with a partner. Then have students share their responses as a class.

PROBLEM 1 Shifty Behavior, Take 1

Recall from *A Rational Existence* that the reciprocal of power functions have a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$. The domain is all real numbers except for 0, because division by 0 is undefined.



In this problem you will use a graphing calculator to explore rational functions of the form

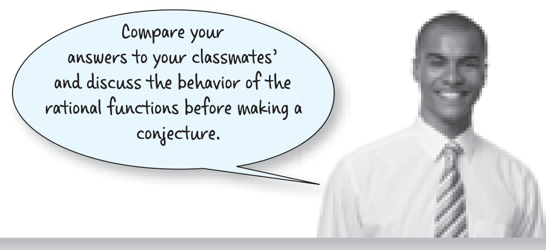
$g(x) = \frac{1}{x - c}$ for a constant value c.

1. Consider the table shown.

a. Identify the vertical asymptote, horizontal asymptote, domain, and range for the given c-values. Then choose different positive and negative c-values to complete the table.

Answers will vary.

c-value	$g(x) = \frac{1}{x - c}$	Vertical Asymptote(s)	Horizontal Asymptote(s)	Domain	Range
1	$g(x) = \frac{1}{x - 1}$	$x = 1$	$y = 0$	All Reals except 1.	All Reals except 0.
-2	$g(x) = \frac{1}{x + 2}$	$x = -2$	$y = 0$	All Reals except -2.	All Reals except 0.



Guiding Questions for Share Phase, Question 1

- What value of x would cause the function to be undefined?
- Why can't y have the value of zero?
- How did you determine the vertical asymptote?
- How did you determine the horizontal asymptote?
- How are the asymptotes related to the domain and range?
- What happens when the value of x equals the value of c?

- Is the value of c in the domain of the function? Why not?
- Does the range include zero? Why not?
- What does the output approach as x approaches negative and positive infinity?

b. Determine the general formula to identify the vertical asymptote of a rational function in the form $g(x) = \frac{1}{x - c}$. Explain your reasoning.

The general formula to determine the vertical asymptote of a function in the form $g(x) = \frac{1}{x - c}$ is $x = c$. The value c, when substituted for x, produces a zero in the denominator.

c. What generalization(s) can you make about the c-value and the domain? The range?

The domain is all real numbers except for the c-value.

The range is all real numbers except for 0.

d. What effect does changing the c-value have on the function's end behavior? Explain your reasoning.

Changing the c-value has no effect on the function's end behavior. As x approaches positive and negative infinity, the output approaches 0.

Grouping

Have students complete Questions 2 through 4 with a partner. Then have students share their responses as a class.

2. Without using a graphing calculator, determine the domain, range, and vertical and horizontal asymptotes of each rational function.

a. $f(x) = \frac{10}{x}$

Domain: All real numbers except 0

Range: All real numbers except 0

Vertical Asymptote: $x = 0$

Horizontal Asymptote: $y = 0$

b. $g(x) = \frac{1}{x + 10}$

Domain: All real numbers except -10

Range: All real numbers except 0

Vertical Asymptote: $x = -10$

Horizontal Asymptote: $y = 0$

c. $j(x) = 10x$

Domain: All real numbers

Range: All real numbers

Vertical Asymptote: None

Horizontal Asymptote: None

d. $g(x) = \frac{1}{x - 10}$

Domain: All real numbers except $x = 10$

Range: All real numbers except 0

Vertical Asymptote: $x = 10$

Horizontal Asymptote: $y = 0$

Guiding Questions for Share Phase, Question 2

- Under what circumstance does a function in this form have no asymptotes?
- Under what circumstance does a function in this form have a horizontal asymptote of $y = 0$?
- Under what circumstance does a function in this form have a vertical asymptote of $x = 0$?
- If a function in this form has no c-value, what does this tell you about its asymptotes?
- What is the basic function in this situation?

9

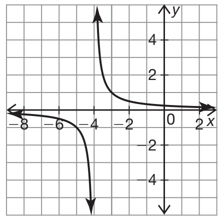
Guiding Questions for Share Phase, Question 3

- How does the graph of the function help identify the asymptotes?
- How does knowing the vertical and horizontal asymptote help identify the function?
- How are the domain and range related to the asymptotes of the function?
- Does the table of values help to identify the asymptotes?

9

3. Write the rational function(s) from the graph, table, or description provided. Explain your reasoning.

a.



Function: $y = \frac{1}{x + 4}$

Explanation: The function is $y = \frac{1}{x}$ translated 4 units to the left.

b. Vertical asymptote at $x = 3$ and a horizontal asymptote at $y = 0$.

Answers will vary.

Function 1: $y = \frac{1}{x - 3}$

Function 2: $y = \frac{3}{2x - 6}$

Explanation: The denominator cannot be 3 in either function, so a vertical asymptote will be at $x = 3$. The 3 in the numerator does not affect the asymptotes. Both functions have a constant in the numerator and a variable in the denominator, so the output will approach 0 as x increases or decreases, creating a horizontal asymptote at $y = 0$.

c. Domain: All Real Numbers except $x = 7$
Range: All Real Numbers except $y = 0$

Function 1: $y = \frac{1}{x - 7}$

Function 2: $y = \frac{10}{x - 7}$

Explanation: The denominator cannot be 7 in either function, so a vertical asymptote will be at $x = 7$. The 10 in the numerator does not affect the asymptotes. Both functions have a constant in the numerator and a variable in the denominator, so the output will approach 0 as x increases or decreases, creating a horizontal asymptote at $y = 0$.

d.

x	y
−2	−0.5
−1	$-\frac{2}{3}$
0	−1.0
1	−2.0
2	undefined
3	2.0
4	1.0

Function: $y = \frac{2}{x - 2}$

Explanation: I know the vertical asymptote is at $x = 2$. I chose $x = 0$ and realized that a 2 must be in the numerator to create an output of −1.0.

e.

x	y
−2	−3.0
−1	undefined
0	3.0
1	1.5
2	1
3	0.75
4	0.60

Function: $y = \frac{3}{x + 1}$

Explanation: I know the vertical asymptote is at $x = -1$. I chose $x = 0$ and realized that a 3 must be in the numerator to create an output of 3.0.

4. Compare the effect that changing the *c*-value has on the reciprocal function to the effect that changing the *C*-value has on any polynomial function $f(x)$ to form $g(x) = f(x - C)$.

The translation is exactly the same. It shifts the function to the right or left *c* units.

Guiding Questions for Share Phase, Question 4

- Does the *c*-value of the reciprocal function translates the graph of the function?
- How do you know which direction the *c*-value of the reciprocal function translates the graph?

9

Problem 2

Students explain why all horizontal asymptotes occurred at $y = 0$ thus far, and what elements in the equation of rational functions determines vertical asymptotes. Without using a graphing calculator, they will sketch different functions given their equations. The vertical asymptotes in the graphs translate both left and right depending on the denominator of the function.

Grouping

- Ask a student to read the information and complete Questions 1 and 2 as a class.
- Have students complete Questions 3 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

- Does $x - 2$ in the denominator of the rational function shift the basic graph $y = \frac{1}{x}$, 2 units to the left or 2 units to the right?
- How does the quadratic expression in the denominator of the rational function affect the graph of the basic function?
- How does a negative numerator affect the graph of the basic function $f(x) = \frac{1}{x}$?

9

PROBLEM 2 Shifty Behavior, Take 2

Recall that rational functions are any functions of the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions, and $Q(x) \neq 0$. So far you have only studied a small subset of all rational functions. Let's consider the structure of the rational functions that you have explored so far.

1. Why have the horizontal asymptotes occurred at $y = 0$? Do you think all rational functions will have a horizontal asymptote at $y = 0$? Explain your reasoning.

   Horizontal asymptotes have occurred at $y = 0$ because the numerator has been a constant while a variable is in the denominator.

   So far, students have only explored rational functions with constants in the numerator. When the numerator contains an expression with a variable, they will see that horizontal asymptotes can occur at places other than $y = 0$.

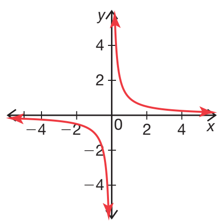
2. What determines a vertical asymptote? Do you think that a rational function could have more than one vertical asymptote?

   Vertical asymptotes are determined by input values for which the denominator is 0.

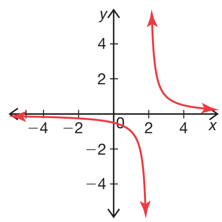
   Students have so far explored functions that have only had one value for which the denominator is 0, and therefore only one vertical asymptote. If a rational function has more than one value that makes the denominator zero, then the rational function will have more than one vertical asymptote.

3. Without using a graphing calculator, sketch each function.

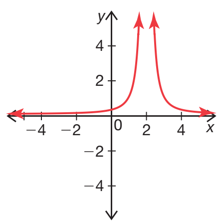
   a. $y = \frac{1}{x}$



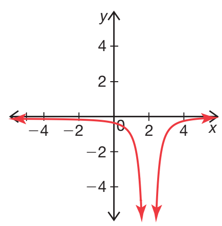
   b. $y = \frac{1}{x - 2}$



   c. $y = \frac{1}{(x - 2)^2}$



   d. $y = \frac{-1}{(x - 2)^2}$



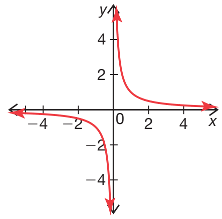
- If part (a) is the basic function, how would you describe the graphic changes from part (a) to part (b)?
- How would you describe the graphic changes from part (b) to part (c)?
- How would you describe the graphic changes from part (c) to part (d)?

Guiding Questions for Share Phase, Question 4

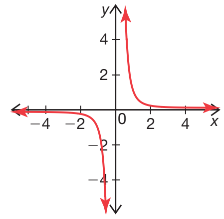
- How would you describe the graphic changes from part (b) to part (c)?
- How would you describe the graphic changes from part (c) to part (d)?

4. Without using a graphing calculator, sketch each function.

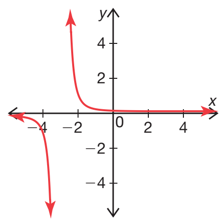
a. $y = \frac{1}{x}$



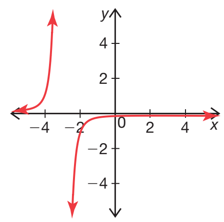
b. $y = \frac{1}{x^3}$



c. $y = \frac{1}{(x + 3)^3}$



d. $y = \frac{-1}{(x + 3)^3}$



5. How would the graphs in Question 3 change for a greater even power? Explain your reasoning.

The general shape would be the same, but the graphs would have a greater steepness.

6. How would the graphs change in Question 4 for a greater odd power? Explain your reasoning.

The general shape would be the same, but the graphs would have a greater steepness.

9

Problem 3

Students will determine the number of vertical asymptotes in functions containing quadratics in their denominator without using a graphing calculator. A worked example factors the quadratic expression in the denominator to determine the equations of the asymptotes. Algebra is used to determine the vertical asymptotes of different rational functions. A graphing calculator is only used to check answers. In the last activity, students are given characteristics, and they determine 2 different rational functional functions fitting the description.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Is it possible for x^2 to equal a negative number?
- How can the expression $x^2 + 4$ be factored?
- How can the expression $x^2 - 4$ be factored?
- What values of x would cause the denominator to equal zero?
- Under what circumstances would this function be undefined?

PROBLEM 3 Shifty Behavior, Take 3

In the previous problems in this chapter, you analyzed rational functions with just 1 vertical asymptote. The vertical asymptote occurred at the value for which the denominator was zero.

1. Without graphing, determine the number of vertical asymptotes for each function. Show all work and explain your reasoning.

a. $f(x) = \dfrac{4}{x^2 + 4}$

There are no vertical asymptotes because $x^2 + 4$ cannot equal 0.

b. $g(x) = \dfrac{4}{x^2 - 4}$

There are 2 vertical asymptotes because the denominator $x^2 - 4$ equals 0 for $x = 2$ and $x = -2$.

c. $h(x) = \dfrac{4}{x^2 + 4x + 4}$

There is one vertical asymptote because the denominator $x^2 + 4x + 4$ equals 0 for $x = 2$ (multiplicity 2).

Recall from the Fundamental Theorem of Algebra that a function of degree n has n zeros. Some of the zeros may be imaginary. Therefore, it follows that the reciprocal of a function of degree n can have at most n vertical asymptotes.

2. Sarah determines the vertical asymptotes for the function $f(x) = \dfrac{1}{2x^2 - 14x - 16}$.

> Sarah
>
> The terms in the denominator have a common factor of 2, so I factored it out first. Then I factored the remaining quadratic.
>
> $f(x) = \dfrac{1}{2(x^2 - 7x - 8)} = \dfrac{1}{2(x - 8)(x + 1)}$
>
> Vertical asymptotes occur when the denominator is zero. So, the asymptotes will occur when $x - 8 = 0$ and when $x + 1 = 0$.
>
> Therefore, the asymptotes occur at $x = 8$ and $x = -1$.

Is Sarah correct? Explain your reasoning.

Yes. Sarah is correct. She factored the expression in the denominator correctly and determined the input values for which the denominator is 0.

- How are the vertical asymptotes related to the values of x?
- Is the expression in the denominator of Sarah's function factored correctly?
- What are the input values for which the denominator is zero?

Grouping

Have students complete Question 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

- What value of x creates an undefined function?
- What operations were used to determine the vertical asymptote(s)?
- What algebraic properties were used to determine the vertical asymptote(s)?
- How can the denominator be factored?
- Is the denominator a perfect square?
- If the denominator cannot be factored, what implications does this have on the vertical asymptotes of the rational function?
- Is the numerator ever used to determine the vertical asymptotes?

3. Analyze each rational function. Use algebra to determine the vertical asymptotes.

a. $f(x) = \frac{5}{7x - 35}$

$7x - 35 = 0$

$7x = 35$

$x = 5$

A vertical asymptote exists at $x = 5$.

b. $g(x) = \frac{1}{x(x - 2)(2x + 3)}$

Vertical asymptotes exist at $x = 0$, $x = 2$, $x = \frac{-3}{2}$.

c. $h(x) = \frac{10}{x^2 - 3x - 10}$

$x^2 - 3x - 10 = 0$

$(x - 5)(x + 2) = 0$

$x = 5, x = -2$

Vertical asymptotes exists at $x = -2, 5$.

d. $h(x) = \frac{x}{2x^2 + 9x + 4}$

$2x^2 + 9x + 4 = 0$

$(2x + 1)(x + 4) = 0$

$x = \frac{-1}{2}, x = -4$

Vertical asymptotes exists at $x = \frac{-1}{2}, -4$.

e. $h(x) = \frac{7}{x^4 - 1}$

$x^4 - 1 = 0$

$(x^2 + 1)(x^2 - 1) = 0$

$x^2 + 1 = 0, x^2 - 1 = 0$

$x = 1, x = -1$

Vertical asymptotes exists at $x = -1, 1$.

f. $f(x) = \frac{2}{x^2 + 2}$

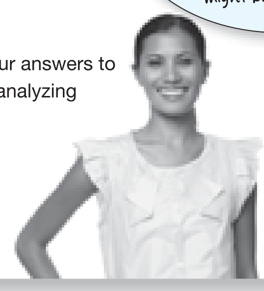
No vertical asymptotes exist. The output is positive for any x-value.

g. $h(x) = \frac{x - 2}{x - 2}$

No vertical asymptotes. Substituting any value in for x results in $h(x) = 1$. The x-value cannot equal 2, but there is no asymptote at this point.

h. $g(x) = \frac{x + 2}{(x + 2)(x - 5)}$

A vertical asymptote exists at $x = 5$.



i. Use a graphing calculator to check your answers to Questions 3 by graphing and then by analyzing the table of values.

9

Grouping

Have students complete Question 4 with a partner. Then have students share their responses as a class.

9

Guiding Questions for Share Phase, Question 4

- How is the vertical asymptote at $x = 3$ rewritten in the form of a factor in the denominator of the rational function?
- How is the vertical asymptote at $x = -1$ rewritten in the form of a factor in the denominator of the rational function?
- How is the vertical asymptote at $x = 0$ rewritten in the form of a factor in the denominator of the rational function?
- How is the vertical asymptote at $x = \frac{1}{2}$ rewritten in the form of a factor in the denominator of the rational function?
- How is the vertical asymptote at $x = 2$ rewritten in the form of a factor in the denominator of the rational function?

4. Determine 2 different rational functions with the characteristics given.

a. vertical asymptotes at $x = 3$, $x = -1$, and $x = 0$

Answers will vary.

$f(x) = \frac{1}{x(x-3)(x+1)}$ or $h(x) = \frac{5}{3x(x-3)(x+1)}$

b. vertical asymptotes at $x = \frac{1}{2}$ and $x = 2$

Answers will vary.

$f(x) = \frac{1}{(2x-1)(x-2)}$ or $g(x) = \frac{18}{(2x-1)(x-2)}$

Be prepared to share your solutions and methods.

Check for Students' Understanding

Determine a rational function with the characteristics given.

1. Vertical asymptotes at $x = -7$, $x = 12$

$f(x) = \dfrac{1}{(x + 7)(x - 12)}$

2. No vertical asymptotes

$f(x) = \dfrac{1}{x^2 + 1}$

3. A vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 0$

$f(x) = \dfrac{1}{x - 5}$

9

9

9.3

A Rational Approach

Exploring Rational Functions Graphically

LEARNING GOALS

In this lesson, you will:

- Graph rational functions.
- Determine graphical behavior of rational functions from the form of the equation.
- Translate rational functions.

ESSENTIAL IDEA

- Rational functions are transformed in the same manner as functions such as $y = f(x)$ and $g(x) = Af(B(x - C)) + D$ where the D-value translates $f(x)$ vertically, the C-value translates $f(x)$ horizontally, the A-value vertically stretches $f(x)$, and the B-value horizontally stretches $f(x)$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-IF Interpreting Functions

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

    d. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

    a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

F-BF Building Functions

Build new functions from existing functions

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

Overview

Students will explore transformations of rational functions. Without using a graphing calculator, students sketch several rational functions and indicate the domain, range, vertical and horizontal asymptotes, and the *y*-intercept. They then match or sketch transformed rational functions with their graphs and vice versa.

9

Warm Up

Explain why the rational function $f(x) = \frac{1}{x}$ has a horizontal asymptote at $y = 0$ and a vertical asymptote at $x = 0$.

The output will approach 0 as x increases or decreases, creating a horizontal asymptote as $y = 0$.

The denominator cannot be 0, so a vertical asymptote is at $x = 0$.

9

9

A Rational Approach

9.3

Exploring Rational Functions Graphically

9

LEARNING GOALS

In this lesson, you will:

- Graph rational functions.
- Determine graphical behavior of rational functions from the form of the equation.
- Translate rational functions.

The word "rational" means to be sensible or reasonable. Humans are said to be rational beings for our ability to use logic to move from a problem to its solution. Most definitions for the word "rational" are subjective. What might seem rational to one person may seem completely irrational to another. A person or group may come up with an idea that seems perfectly reasonable to them, but may seem eccentric to another group. This sometimes makes coming to a decision, or devising a solution path, very difficult for large groups of people.

Have you ever come up with an idea that you thought was perfectly rational, but others didn't quite agree with your rationale? Were you able to use logic to convince them that your idea made sense?

9

Problem 1

Three different methods of graphing a rational function $j(x) = \frac{1}{x^2 - 4}$ are shown. Students will determine which method is most efficient and most accurate. Without using a graphing calculator, students then sketch several rational functions and indicate the domain, range, vertical and horizontal asymptotes, and the y-intercept.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Who graphed the rational function by first separating it into factors?
- Who graphed the rational function by using the reciprocal function?
- Who graphed the rational function using a table of values?
- How many points are necessary to plot the graph of the entire function?
- Why is Jodi's method of plotting points more accurate than Theresa's method?
- Why is Jodi's method of plotting points more accurate than Jin's method?

PROBLEM 1 **Slow Down, Asymptotic Curves Ahead!**

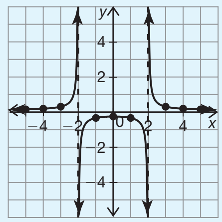
1. Analyze the methods Jodi, Theresa, and Jin each used to graph the rational function $j(x) = \frac{1}{x^2 - 4}$.

Jodi

I created a table and plotted the points.

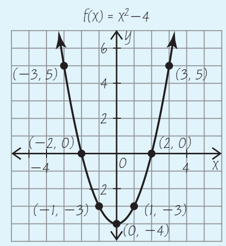
-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$\frac{1}{32}$	$\frac{1}{21}$	$\frac{1}{12}$	$\frac{1}{5}$	undefined	$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{3}$	undefined	$\frac{1}{5}$	$\frac{1}{12}$	$\frac{1}{21}$	$\frac{1}{32}$

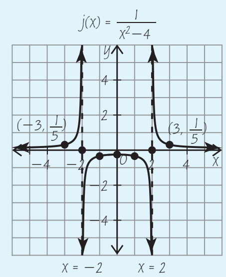
I see that vertical asymptotes occur at $x = -2$ and $x = 2$, where the denominator is 0 and the output is undefined.



Theresa

I graphed the function $f(x) = x^2 - 4$. The function $j(x)$ is the reciprocal of $f(x)$, so I took the reciprocal of several key points and sketched the graph.





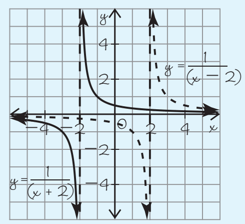
The zeros of the function $f(x) = x^2 - 4$ are $(-2, 0)$ and $(2, 0)$ that become asymptotes at $x = -2$ and $x = 2$ in the function $j(x) = \frac{1}{x^2 - 4}$. The y-intercept shifts from $(0, -4)$ in $f(x)$ to $\left(0, \frac{-1}{4}\right)$ in $j(x)$. By plotting a couple key points and recognizing that a horizontal asymptote is at $y = 0$, I can sketch the function.

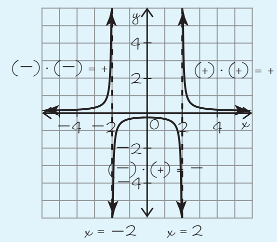
Note that these aren't really "new" methods. Each student is applying previous knowledge to a new situation.

- Are vertical asymptotes included in the domain of a function?
- Are horizontal asymptotes included in the range of a function?

Jin

I used what I know about rational functions and function-building. Since the function $f(x) = \dfrac{1}{x^2 - 4}$ can be rewritten as two separate factors, $f(x) = \left(\dfrac{1}{x + 2}\right)\left(\dfrac{1}{x - 2}\right)$, I graphed each factor separately and multiplied their outputs to determine the graph of their product.





The asymptotes are at $x = -2$ and $x = 2$. Analyzing each function, I saw that the outputs were both negative for the interval $(-\infty, -2)$. Their product will always be positive so $f(x)$ will be above the x-axis for this region. Similarly, a positive and a negative output for the interval $(-2, 2)$ will always be negative. Two positive outputs multiplied together will be positive for the interval $(2, \infty)$.

a. Which method do you think is most efficient? Explain your reasoning.

Answers will vary. Students should realize that Theresa and Jin's methods are more efficient than creating a table and plotting many points.

b. Which method do you think is the most accurate? Explain your reasoning.

Jodi's method of plotting points is the most accurate.

c. How does a vertical asymptote affect the domain of a function?

Vertical asymptotes are values that are not included in the domain of the function.

Grouping

Have students complete Question 2 with a partner. Then have students share their responses as a class.

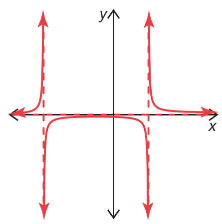
9

Guiding Questions for Share Phase, Question 2

- How is the denominator of the rational function helpful when determining the domain of the function?
- How is the numerator of the rational function helpful when determining the range of the function?
- How did you determine the *y*-intercept of each function?
- Can the denominator of the rational function be factored easily?
- Do all rational functions have a *y*-intercept?
- Do any of the rational functions have zeros?

2. Without using a graphing calculator, sketch each function. Indicate the domain, range, vertical and horizontal asymptotes, and the *y*-intercept for each function.

a. $f(x) = \frac{1}{(x-2)(x+4)}$



Domain: All real numbers except $x = 2$ and $x = -4$

Range: All real numbers except $y = 0$

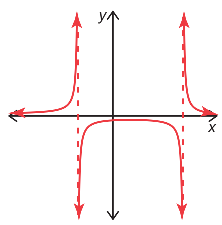
Asymptote(s):

Vertical asymptotes $x = 2$, $x = -4$

Horizontal asymptote $y = 0$

y-intercept: $(0, -\frac{1}{8})$

b. $f(x) = \frac{2}{x^2 - 2x - 8}$



Domain: All real numbers except $x = 4$ and $x = -2$

Range: All real numbers except $y = 0$

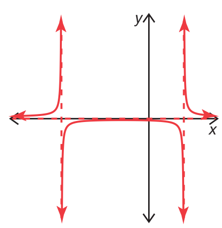
Asymptote(s):

Vertical asymptotes at $x = 4$, $x = -2$

Horizontal asymptote $y = 0$

y-intercept: $(0, -\frac{1}{8})$

c. $h(x) = \frac{1}{x^2 + 3x - 10}$



Domain: All real numbers except $x = -5$ and $x = 2$

Range: All real numbers except $y = 0$

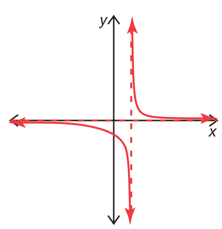
Asymptote(s):

Vertical asymptotes at $x = -5$, $x = 2$

Horizontal asymptote at $y = 0$

y-intercept: $(0, -\frac{1}{10})$

d. $h(x) = \frac{1}{x^3 - 1}$



Domain: All real numbers except $x = 1$

Range: All real numbers except $y = 0$

Asymptote(s):

Vertical asymptote at $x = 1$

Horizontal asymptote at $y = 0$

y-intercept: $(0, -1)$

Problem 2

Students will explore transforming rational functions. The graphs of the function $f(x) = \frac{1}{x}$ after a vertical shift, horizontal shift, or stretch are shown and students identify the function that matches each graph. Next, they are given the transformed function or the characteristics of a function and will sketch the graph.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

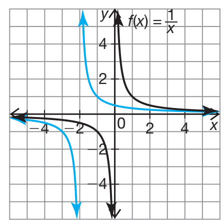
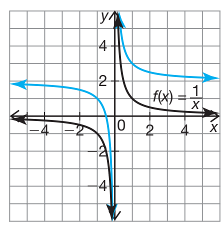
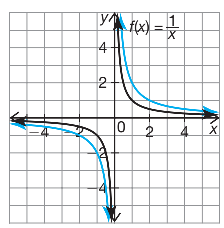
Guiding Questions for Share Phase, Question 1

- Is $j(x)$ a rational function that has been horizontally translated to the left 2 units or translated to the right 2 units? How do you know?
- Is $k(x)$ a rational function that has been vertically translated up 2 units or translated down 2 units? How do you know?
- Is $m(x)$ a rational function that has been vertically stretched as a result of the change in the A-value or a change in the B-value? How do you know?

PROBLEM 2 Ctrl-Alt-Shift

Consider the functions $y = f(x)$ and $g(x) = Af(B(x - C)) + D$. Recall that adding a constant D translates $f(x)$ vertically, while adding a constant C translates $f(x)$ horizontally. Multiplying by the constant A dilates $f(x)$ vertically, while multiplying by the constant B dilates $f(x)$ horizontally. Rational functions are transformed in the same manner.

1. The function $f(x) = \frac{1}{x}$ is shown in black on each coordinate plane. Determine whether the second function on each graph is $j(x) = \frac{1}{x + 2}$, $m(x) = \frac{2}{x}$, or $k(x) = \frac{1}{x} + 2$. Explain your reasoning.

	Function: $j(x)$ Explanation: The original function $f(x)$ is translated 2 units to the left. This results from a change in the C-value.
	Function: $k(x)$ Explanation: The original function $f(x)$ is translated 2 units up. This results from a change in the D-value.
	Function: $m(x)$ Explanation: The original function $f(x)$ is vertically stretched. This results from a change in the A-value. I know it is not translated vertically or horizontally because the asymptotes stayed the same.

9

Guiding Questions for Share Phase, Question 2

- What does adding a constant to the basic function do to the graph of the reciprocal function?
- What does adding 5 to the basic function do to the graph of the reciprocal function?
- What does adding a constant to the argument of the function do to the graph of the reciprocal function?
- What does adding 5 to the argument of the function do to the graph of the reciprocal function?

Grouping

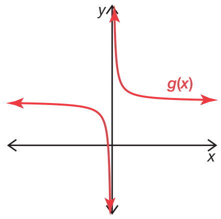
Have students complete Question 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

- If the rational function has a vertical asymptote at $x = 2$, was the graph of the original function translated to the right 2 units, to the left 2 units, up 2 units, or down 2 units? How do you know?
- If the rational function has a horizontal asymptote at $y = 1$, was the graph of the original function translated to the right 2 units, to the left 2 units, up 2 units, or down 2 units? How do you know?

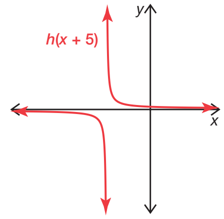
2. Given $f(x) = \frac{1}{x}$.

a. Sketch $g(x) = f(x) + 5$



Explanation: The function $g(x)$ translates the basic reciprocal function up 5 units.

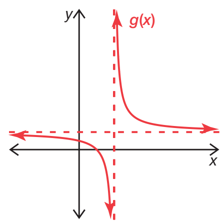
b. Sketch $h(x) = f(x + 5)$.



Explanation: The function $g(x)$ translates the basic function 5 units to the left.

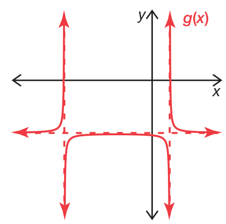
3. Write a rational function $g(x)$ that matches the given characteristics. Sketch the function on the coordinate plane. Answers will vary.

a. Vertical asymptote at $x = 2$
Horizontal asymptote at $y = 1$



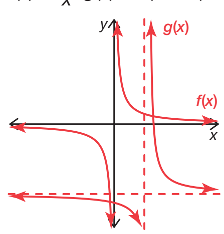
$g(x) = \frac{1}{x - 2} + 1$

b. Vertical asymptote at $x = 1$, $x = -5$
Horizontal asymptote at $y = -3$



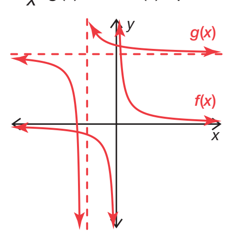
$g(x) = \frac{2}{(3x - 3)(x + 5)} - 3$

c. For $f(x) = \frac{1}{x}$, $g(x) = f(x - 2) - 4$



$g(x) = \frac{1}{x - 2} - 4$

d. For $f(x) = \frac{1}{x}$, $g(x)$ shifts $f(x)$ up and to the left.



$g(x) = \frac{1}{x + 2} + 4$

Be prepared to share your solutions and methods.

- If the rational function has two vertical asymptotes, the function is divided into how many pieces?
- Is there more than one correct graph using these descriptors?
- If $f(x) = \frac{1}{x}$, and $g(x) = f(x - 2) - 4$, what does the 2 do the graph of $f(x)$?
- If $f(x) = \frac{1}{x}$, and $g(x) = f(x - 2) - 4$, what does the -4 do the graph of $f(x)$?
- If $f(x) = \frac{1}{x}$, and $g(x)$ translates $f(x)$ up and to the left, where were the constants added in the reciprocal function?

Check for Students' Understanding

Determine the domain, range, asymptotes, and y-intercept of the rational function.

$$f(x) = \frac{1}{x^2 + 3x - 40}$$

$$\frac{1}{x^2 + 3x - 40} = \frac{1}{(x - 5)(x + 8)}$$

The domain is all real numbers except $x = 5$ and $x = -8$.

The range is all real numbers except $y = 0$.

The function has vertical asymptotes at $x = 5$ and $x = -8$ and a horizontal asymptote at $y = 0$.

The y-intercept is $-\frac{1}{40}$.

9

9

There's a Hole In My Function, Dear Liza

9.4

Graphical Discontinuities

LEARNING GOALS

In this lesson, you will:

- Sketch rational functions with removable discontinuities.
- Rewrite rational expressions.
- Compare removable discontinuities to vertical asymptotes.
- Identify domain restrictions of rational functions.

KEY TERM

- removable discontinuity

ESSENTIAL IDEAS

- A removable discontinuity is a single point at which the graph is not defined.
- The graphs of rational functions have either a removable discontinuity or a vertical asymptote for all domain values that result in division by 0.
- Holes are created in the graphs of rational functions when a common factor divides out of the numerator and denominator of the function.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-APR Arithmetic with Polynomials and Rational Expressions

Rewrite rational expressions

6. Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form $q(x) + \frac{r(x)}{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

7. Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

F-IF Interpreting Functions

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.
   - **d.** Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

   a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

9

Overview

Students will match rational functions with their graphs. The functions that have a common factor in the numerator and denominator have 'holes' whereas the functions that have an undefined value in the denominator have an asymptote. The term removable discontinuity is defined. Students graph several rational functions with holes or asymptotes in the graph. A table shows similarities between rational numbers and rational functions and students list any restrictions in the domain for each example. Given the functions, students will determine whether the graphs of rational functions have vertical asymptotes, removable discontinuity, both, or neither. They then analyze a worked example and explain why a hole and a vertical asymptote are both present in the graph of the function. Finally, students will sketch the graphs of several rational functions detailing all holes and asymptotes.

Warm Up

Describe any restrictions on the domain and range of each rational function.

1. $f(x) = \frac{10}{x}$

   Domain: $x \neq 0$

   Range: $y \neq 0$

2. $g(x) = \frac{10}{x + 1}$

   Domain: $x \neq -1$

   Range: $y \neq 0$

3. $h(x) = \frac{10}{x} + 1$

   Domain: $x \neq 0$

   Range: $y \neq 1$

4. $w(x) = \frac{10}{x} - 1$

   Domain: $x \neq 0$

   Range: $y \neq -1$

9

9

9.4

There's a Hole In My Function, Dear Liza

Graphical Discontinuities

9

LEARNING GOALS

In this lesson, you will:

- Sketch rational functions with removable discontinuities.
- Rewrite rational expressions.
- Compare removable discontinuities to vertical asymptotes.
- Identify domain restrictions of rational functions.

KEY TERM

- removable discontinuity

The ozone layer is a part of the atmosphere that contains high levels of ozone. Ozone absorbs the UV radiation from the sun that might otherwise be harmful to life on Earth. Recently, some experts believe that the ozone levels have been depleting over time. In particular, areas around the North and South poles have developed "ozone holes" which are allowing the harmful rays to enter our atmosphere.

Why is the ozone depleting? What effects will this have on our environment?

Problem 1

Students will match rational functions with their graphs. The functions that have a common factor in the numerator and denominator have 'holes' whereas the functions that have an undefined value in the denominator have a vertical asymptote. The function $y = \frac{x}{x}$ is written as the product of two factors, $y = (x)\left(\frac{1}{x}\right)$, and the reciprocal factors are used to complete a table of values which explains the 'hole' when the value of x equals 0. Multiplying the outputs for each input reveals that $(x)\left(\frac{1}{x}\right) = 1$, graphically it shows why common factors divide to 1 and do not simply just cancel each other out. The term removable discontinuity is defined. Students will graph several rational functions with holes or asymptotes in the graph.

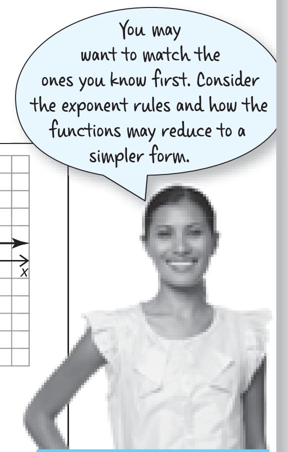
Grouping

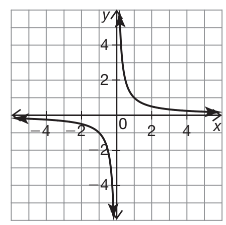
Have students complete Question 1 with a partner. Then have students share their responses as a class.

PROBLEM 1 Mend Your Function, Dear Henry!

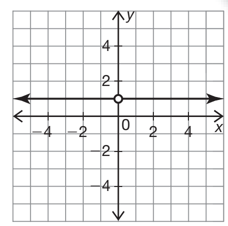
1. Without using a graphing calculator, match each rational function provided with the correct graph. Write the function below the graph.

$y = \frac{1}{x-2}$	$y = \frac{1}{x}$	$y = \frac{x^2}{x}$	$y = \frac{x-2}{x-2}$
$y = \frac{x^3}{x}$	$y = \frac{(x-2)^2}{x-2}$	$y = \frac{x}{x}$	$y = \frac{(x-2)^3}{x-2}$

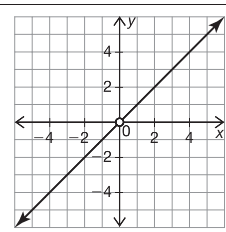




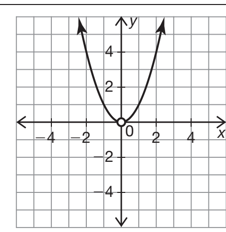
Function: $y = \frac{1}{x}$



Function: $y = \frac{x}{x}$



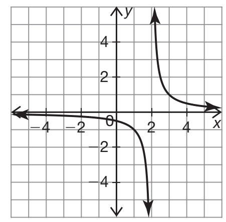
Function: $y = \frac{x^2}{x}$



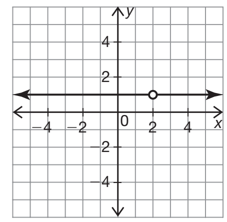
Function: $y = \frac{x^3}{x}$

Guiding Questions for Share Phase, Question 1

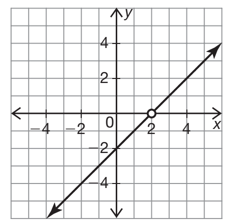
- Which rational functions have holes in their graphs? How do you know?
- Which rational functions have vertical asymptotes in their graphs? How do you know?
- Which rational functions have horizontal asymptotes in their graphs? How do you know?
- Which rational functions are linear functions?
- Does the graph of the rational function that has a common factor in the numerator and denominator have a hole or an asymptote?
- Does the graph of the rational function that has an undefined value in the denominator have a hole or an asymptote?
- Which two rational functions are a horizontal line at $y = 1$?
- Are the domains of the two functions the same or different?
- Are the slopes of both linear functions the same?
- Do both linear functions pass through the origin?
- Which of the two linear functions has a hole at $x = 0$?



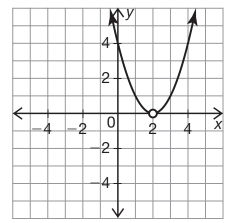
Function: $y = \frac{1}{x-2}$



Function: $y = \frac{x-2}{x-2}$



Function: $y = \frac{(x-2)^2}{x-2}$



Function: $y = \frac{(x-2)^3}{x-2}$

a. Which functions have asymptotes and which functions have "holes" in their graphs? Describe how the structure of the equation determines whether the function will have an asymptote or a "hole."

The functions that have a common factor in the numerator and denominator have "holes." The remaining functions that have an undefined value in the denominator have a vertical asymptote.

b. Compare the graphs of $y = \frac{1}{x-2}$ and $y = \frac{x-2}{x-2}$. How are they the same? How are they different? Describe how the structure of the equation determines these differences.

The function $y = \frac{1}{x-2}$ has an asymptote and two parts to it.

The function $y = \frac{x-2}{x-2}$ is a horizontal line at $y = 1$ with a "hole" at $x = 2$.

The domain for both functions is all real numbers except for $x = 2$.

The range for $y = \frac{1}{x-2}$ is all real numbers except 0.

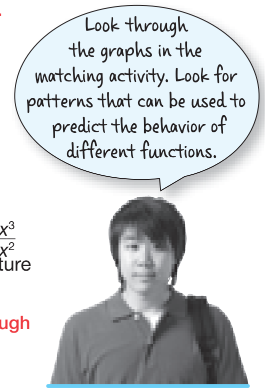
The range for $y = \frac{x-2}{x-2}$ is $y = 1$. Even though this function simplifies to $y = 1$, it still is undefined at $x = 2$ from the original equation.

When comparing the graphs, consider the general shape of the graph, domain, range, asymptotes, end behavior, etc.

c. Compare the graphs of $y = \frac{x}{x}$ and $y = \frac{x-2}{x-2}$. How are they the same? How are they different? Describe the similarities and differences in the domain and range in terms of the structure of their equations.

The graphs of both functions are horizontal lines at $y = 1$.

The domains are different. In the function $y = \frac{x}{x}$, $x \neq 0$, while in the second function, $x \neq 2$. This creates "holes" in different parts of their graphs.



d. Without using a graphing calculator, describe the similarities and differences between the graphs of $y = \frac{x^3}{x^2}$ and $y = x$. Explain your reasoning in terms of the structure of the equations.

The graphs of both functions will be linear, passing through the origin with a slope of 1.

Both functions simplify to $y = x$.

The only difference is that $y = \frac{x^3}{x^2}$ has a "hole" at $x = 0$ because this value is undefined and therefore not a part of the domain.

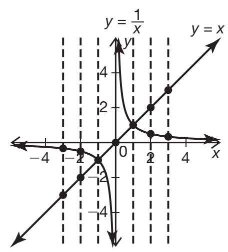
Grouping

Ask a student to read the information and definition. Discuss as a class.

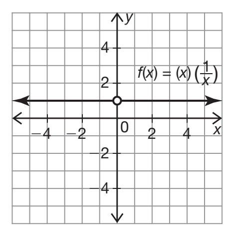
Many rational functions have "holes," or breaks, in the graphs instead of asymptotes. Let's analyze the structure of the function $y = \frac{x}{x}$ to determine why this function has a "hole" in the graph rather than a vertical asymptote at $x = 0$.

The function $y = \frac{x}{x}$ can be rewritten as the product of two factors: $y = (x)\left(\frac{1}{x}\right)$. Looking at these reciprocal factors as separate functions reveals important characteristics.

x	-4	-3	-2	-1	0	1	2	3
$y = (x)\left(\frac{1}{x}\right)$	$(-4)\left(-\frac{1}{4}\right)$ 1	$(-3)\left(-\frac{1}{3}\right)$ 1	$(-2)\left(-\frac{1}{2}\right)$ 1	$(-1)\left(-\frac{1}{1}\right)$ 1	$(0)\left(\frac{1}{0}\right)$ und	$(1)\left(\frac{1}{1}\right)$ 1	$(2)\left(\frac{1}{2}\right)$ 1	$(3)\left(\frac{1}{3}\right)$ 1



Graphical representation of each factor



Graphical representation of the product

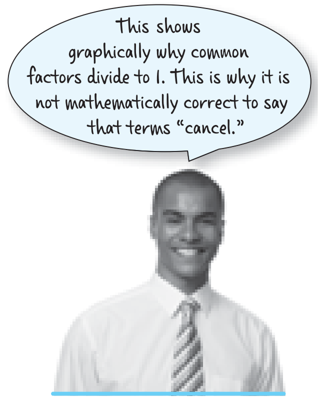
Grouping

Have students complete Question 2 and 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- Who used the exponent rule for division to graph the function?
- Who rewrote the function as a product of two functions to graph the function?
- Did both methods result in the same removable discontinuity?

Multiplying the outputs for each input reveals that $(x)\left(\frac{1}{x}\right) = 1$. This graph is a horizontal line that is undefined at $x = 0$. It is undefined at $x = 0$ because this is the value for which an asymptote exists for the factor $\frac{1}{x}$. Similar reasoning can be used to show that for any function $f(x)$, $f(x) \cdot \frac{1}{f(x)} = 1$, with breaks in the graph for all undefined values where $f(x) = 0$. These breaks, or "holes," in the graph are called *removable discontinuities*. A **removable discontinuity** is a single point at which the graph is not defined. Vertical asymptotes and removable discontinuities must be listed as domain restrictions.



2. Henry and Liza each describe a different way to graph $y = \frac{x^3}{x^2}$.

Henry

I know any function multiplied by its reciprocal is 1.

I can rewrite the function as

$y = \frac{x^3}{x^2} = x \cdot \left(\frac{x^2}{x^2}\right)$.

This means that the output of $y = x$ is multiplied by $y = 1$ with a discontinuity at $x = 0$. The result is the line $y = x$ with a removable discontinuity at $(0, 0)$, so $x \neq 0$.

Liza

A removable discontinuity will exist anywhere that the denominator is 0 for the original function. In this case it is $(0, 0)$. I can use the exponent rules to simplify the function

$$y = \frac{x^3}{x^2}$$
$$= x^{3-2}$$
$$= x^1.$$

Then I can just graph $y = x$ with a hole at $(0, 0)$, so $x \neq 0$.

Which method do you prefer? Explain your reasoning.

Answers will vary.

Guiding Questions for Share Phase, Question 3

- What method did you use to graph the rational function?
- Did you rewrite the function as a product of two functions?
- Did you use an exponent rule to rewrite the function?

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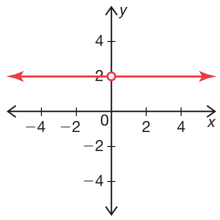
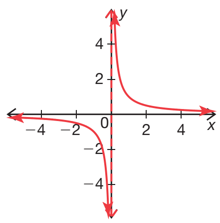
3. Without using a graphing calculator, sketch the graph of each function. Be sure to note any asymptotes or holes in the graph.

a. $y = \frac{2x^2}{x^2}$



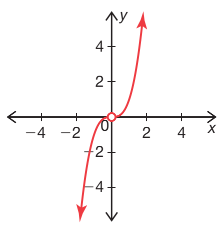
$x \neq 0$

b. $y = \frac{x^2}{x^3}$



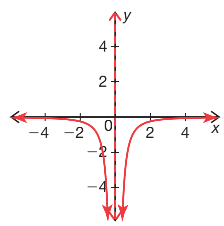
$x \neq 0$

c. $y = \frac{x^4}{x^1}$



$x \neq 0$

d. $y = \frac{-x^2}{x^4}$



$x \neq 0$

Problem 2

A rational function $f(x) = \dfrac{x^2 + x - 6}{x - 2}$ is first factored and then $f(x) = x + 3$ is graphed with a discontinuity at $x = 2$. Students explain why there is a hole in the function at (2, 5) and why $f(x)$ can be written as $f(x) = x + 3$. A table shows similarities between rational numbers and rational functions and students list any restrictions in the domain for each example. Students will simplify different rational expressions and list any restrictions on the domain.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- What does 'cancel each other out' mean?
- Do common factors cancel each other out? Why not?
- How did Henry locate the coordinates of the hole in the graph of the function?
- How is a hole in the graph created?
- What feature of the rational function creates a hole in the graph of the function?
- What feature of the rational function creates an asymptote in the graph of the function?

PROBLEM 2 With What Shall I Mend The Function, Dear Liza?

1. Henry graphed the rational function $f(x) = \dfrac{x^2 + x - 6}{x - 2}$. Analyze his work.

Henry

I know there is a domain restriction, so $x \neq 2$.

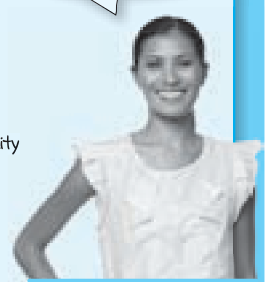
I'm not sure if this is a vertical asymptote or a removable discontinuity, so I'm going to factor the numerator, if possible, to see if a common factor exists.

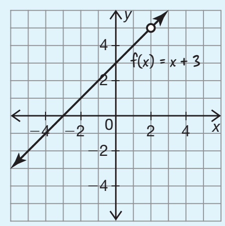
$$f(x) = \frac{x^2 + x - 6}{x - 2} = \frac{(x - 2)(x + 3)}{x - 2}$$

$$= \frac{1(x + 3)}{1} = x + 3$$

I know the output values of $\dfrac{(x - 2)}{(x - 2)} = 1$ with a discontinuity at $x = 2$. Therefore I can simply graph $f(x) = x + 3$. The removable discontinuity is at $(2, 2 + 3)$ and appears as a "hole" in the graph.

Note that the common factors do not "cancel." Many people use this term incorrectly to describe when factors divide to equal 1.





a. Why did Henry include an open circle at (2, 5) and *not* a vertical asymptote at $x = 2$?

Common factors divide to 1, leaving a hole, or removable discontinuity, for the x-value that is not in the domain.

b. Explain why $f(x) = \dfrac{x^2 + x - 6}{x - 2}$ can be rewritten as $f(x) = x + 3$.

The numerator factors into two linear factors. The factor $(x - 2)$ is common to the numerator and denominator so it divides to 1. One is the multiplicative identity.

- Can a rational function have both a hole and an asymptote? What could the equation look like?
- Does the numerator factor into two linear factors?
- Which factor is common to both the numerator and denominator?

Grouping

Ask a student to read the information and complete Questions 2 and 3 as a class.

The graphs of rational functions will have either a removable discontinuity or a vertical asymptote for all domain values that result in division by 0. Simplifying rational expressions is similar to simplifying rational numbers: common factors divide to 1.

2. Analyze the table that shows similarities between rational numbers and rational functions.

		Rational Numbers	Rational Expressions
A common numerator and denominator divide to equal 1.	**Examples**	$\frac{5}{5} = 1$	$\frac{x}{x} = 1$ $x \neq 0$
		$\frac{10.7}{10.7} = 1$	$\frac{5x}{5x} = 1$ $x \neq 0$
		$\frac{0.025 + 0.016}{0.025 + 0.016} = 1$	$\frac{x + 5}{x + 5} = 1$ $x \neq -5$
Common monomial factors divide to equal 1.	**Examples**	$\frac{5 \cdot 3}{5} = \frac{1 \cdot 3}{1}$ $= 3$	$\frac{5x}{5} = \frac{1 \cdot x}{1} = x$ no restrictions
		$\frac{4}{4 \cdot 6} = \frac{1}{1 \cdot 6} = \frac{1}{6}$	$\frac{x}{xz} = \frac{1}{1 \cdot z} = \frac{1}{z}$ $x \neq 0, z \neq 0$
Common binomial factors divide to equal 1.	**Examples**	$\frac{(5 + 3)(16 - 7)}{(5 + 3)} = \frac{1 \cdot (16 - 7)}{1}$ $= 16 - 7$	$\frac{(x + 5)(x - 4)}{(x + 5)} = \frac{1(x - 4)}{1}$ $= (x - 4)$ $x \neq -5$
		$\frac{(9 - 4)}{(9 - 4)(9 + 5)} = \frac{1}{(9 + 5)}$	$\frac{x - 4}{(x - 4)(x + 5)} = \frac{1}{(x + 5)}$ $x \neq 4, -5$

a. Describe how simplifying rational numbers is similar to simplifying rational expressions.

Common factors divide to equal 1. When a rational number or rational expression is completely factored, dividing common factors reduces to an equivalent, simplified form.

b. Why is there a 1 in the numerator after simplifying $\frac{x}{xz} = \frac{1}{z}$?

The common factor x divides to 1. The expression can be rewritten as $\frac{x}{x} \cdot \frac{1}{z} = \frac{1}{z}$.

c. For each example in the rational expressions column, list any restrictions on the domain.

See table.

Grouping

Have students complete Question 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- Can the quadratic in the rational expression be factored easily?
- Can the cubic in the rational expression be factored easily?
- Can the numerator of the rational expression be factored?
- Can the denominator of the rational expression be factored?
- Do the numerator and denominator of the rational expression have a common factor?

3. Liza rewrites the rational expression as shown.

> **Liza**
>
> $\frac{x^2 + 4x + 3}{4x + 3} = x^2$
>
> I divided out the common factors. The numerator and denominator each have a $4x$ and a 3, so I am left with the squared term.

Describe the error in Liza's reasoning.

Liza considered $4x$ and 3 to be separate factors. The terms are added and can therefore not divide out. Common factors divide to 1, and the numerator and denominator of this rational expression have no common factors.

4. Simplify each rational expression. List any restrictions on the domain.

a. $\frac{2x^2 - 8}{x - 2}$

$$\frac{2x^2 - 8}{x - 2} = \frac{2\overset{1}{\cancel{(x - 2)}}(x + 2)}{\underset{1}{\cancel{x - 2}}}$$
$$= 2x + 4; \quad x \neq 2$$

b. $\frac{3xy - 3y}{x^2 - 1}$

$$\frac{3xy - 3y}{x^2 - 1} = \frac{3y\overset{1}{\cancel{(x - 1)}}}{\underset{1}{\cancel{(x - 1)}}(x + 1)}$$
$$= \frac{3y}{x + 1}; \quad x \neq \pm 1$$

c. $\frac{x^2 - 5x + 6}{3x - 9}$

$$\frac{x^2 - 5x + 6}{3x - 9} = \frac{\overset{1}{\cancel{(x - 3)}}(x - 2)}{3\underset{1}{\cancel{(x - 3)}}}$$
$$= \frac{x - 2}{3}; \quad x \neq 3$$

d. $\frac{x^3 - 7x^2 - 18x}{3x^2 - 9x}$

$$\frac{x^3 - 7x^2 - 18x}{3x^2 - 9x} = \frac{\overset{1}{\cancel{x}}(x - 9)(x + 2)}{3\underset{1}{\cancel{x}}(x - 3)}$$
$$= \frac{(x - 9)(x + 2)}{3(x - 3)}; \quad x \neq 0, 3$$

e. $\frac{25x^2 - 9}{5x^2 - 12x - 9}$

$$\frac{25x^2 - 9}{5x^2 - 12x - 9} = \frac{(5x - 3)\overset{1}{\cancel{(5x + 3)}}}{\underset{1}{\cancel{(5x + 3)}}(x - 3)}$$
$$= \frac{5x - 3}{x - 3}; \quad x \neq -\frac{3}{5}, 3$$

f. $\frac{x^3 - 5x^2 - x + 5}{x^2 - 6x + 5}$

$$\frac{x^3 - 5x^2 - x + 5}{x^2 - 6x + 5} = \frac{\overset{1}{\cancel{(x - 1)}}(x + 1)\overset{1}{\cancel{(x - 5)}}}{\underset{1}{\cancel{(x - 1)}}\underset{1}{\cancel{(x - 5)}}}$$
$$= x + 1; \quad x \neq 1, 5$$

- If the numerator and denominator of the rational expression have a common factor, does this always result in a restriction on the domain?
- What besides a common factor could result in a restriction on the domain?

Problem 3

Given the functions, students will determine whether the graphs of rational functions have vertical asymptotes, removable discontinuities, both, or neither. Next, they are given a key characteristic and create two examples of rational functions that meet the criteria. Students then analyze a worked example and explain why a hole and a vertical asymptote are both present in the graph of the function. Finally, students sketch the graphs of several rational functions detailing all holes and asymptotes.

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Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- Does the expression $x + 2$ in the function $j(x)$ divide to 1?
- Does the expression $x + 5$ in the function $h(x)$ create a hole or an asymptote in the graph of the function?
- Is the quadratic in the denominator of the rational expression factorable?
- Does the expression $x - 5$ in the function $j(x)$ create a hole or an asymptote in the graph of the function?

PROBLEM 3 Use Your Head, Dear Henry!

1. Determine whether the graph of the rational function has a vertical asymptote, a removable discontinuity, both, or neither. List the discontinuities and justify your reasoning.

   a. $j(x) = \frac{x + 2}{x(x + 2)}$

   A vertical asymptote at $x = 0$ and a removable discontinuity at $x = -2$

   b. $h(x) = \frac{x}{x + 5}$

   A vertical asymptote at $x = -5$

   c. $j(x) = \frac{5}{5(x + 2)}$

   A vertical asymptote at $x = -2$

   d. $j(x) = \frac{x + 2}{x^2 - 2x - 15}$

   $j(x) = \frac{x + 2}{x^2 - 2x - 15}$

   $j(x) = \frac{(x + 2)}{(x + 3)(x - 5)}$

   Vertical asymptotes at $x = -3$ and $x = 5$

2. Write two examples of rational functions with one or more removable discontinuities. Explain your reasoning.

   Answers will vary.

   The function $f(x) = \frac{(x + 7)(x + 2)}{(x + 2)}$ has a removable discontinuity at $x = -2$.

   The function $j(x) = \frac{x^5}{x}$ has a removable discontinuity at $x = 0$.

   Each function has common factors that divide out.

3. Write a unique function that has a vertical asymptote and a removable discontinuity. Explain your reasoning.

   The function $k(x) = \frac{x}{x(x - 3)}$ has a removable discontinuity at $x = 0$ because of the common factor x. It also has a vertical asymptote at $x = 3$ because $x - 3$ does not have a common factor in the numerator and the domain is restricted for $x = 3$.

- Do all rational functions with one or more removable discontinuities always have a common factor that divides out?
- How is a rational function that has an asymptote created?
- How is a rational function that has a hole created?

Guiding Questions for Share Phase, Question 4

- What is the domain in Liza's function?
- Can x have the value of -2, or 1? How do you know?
- What is the common factor in this situation? What does this imply about the graph?
- Which part of the expression in the denominator creates an asymptote? Where?
- How does the degree of the denominator compare to the degree of the numerator?

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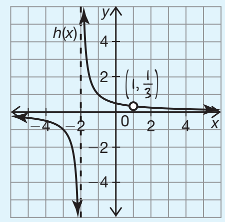
4. Liza graphed the rational function $h(x) = \frac{x - 1}{x^2 + x - 2}$. Analyze her work.

Liza

I'm not sure where the asymptotes are, so I'm going to factor the denominator if possible.

$$h(x) = \frac{x - 1}{x^2 + x - 2} = \frac{x - 1}{(x - 1)(x + 2)}$$
$$= \frac{1}{x + 2}.$$

I know there are domain restrictions at $x = 1$ and $x = -2$. The common factor $(x - 1)$ is in the numerator so $\frac{x - 1}{x - 1} = 1$. Therefore $x = 1$ is a removable discontinuity, while $x = -2$ is a vertical asymptote. I can quickly sketch $h(x) = \frac{1}{x + 2}$ as a horizontal shift of $h(x) = \frac{1}{x}$ two units to the left. I know a discontinuity will exist at $\left(1, \frac{1}{1 + 2}\right)$, or $\left(1, \frac{1}{3}\right)$. A horizontal asymptote is at $y = 0$ and the y-intercept is $\left(0, \frac{1}{2}\right)$.



a. Summarize why $x = -2$ is a vertical asymptote while $x = 1$ appears as a "hole" in the graph.

The domain is all real numbers except $x \neq -2$ and $x \neq 1$. The common factor of $x - 1$ divides out and creates a hole in the graph at $x = 1$. The expression $x + 2$ in the denominator creates an asymptote at $x = -2$.

b. Explain why the graph has a horizontal asymptote at $y = 0$.

The function $h(x)$ approaches 0 as x approaches negative and positive infinity, creating a horizontal asymptote at $y = 0$.

Grouping

Have students complete Question 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5

- Is the expression in the numerator or the denominator of the rational function factorable?
- Is there a common factor in this situation?
- Is the domain of the function restricted? How do you know?

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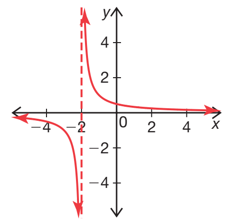
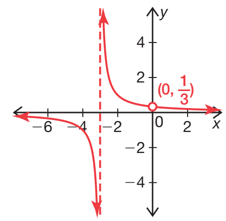
5. Sketch each function without the use of a graphing calculator. Identify any restrictions.

a. $f(x) = \dfrac{x + 2}{x^2 + 4x + 4}$



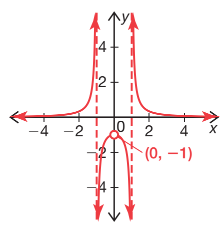
$f(x) = \dfrac{1}{x + 2}$; $x \neq -2$

b. $g(x) = \dfrac{x}{x^2 + 3x}$



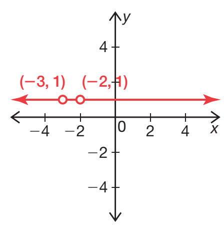
$g(x) = \dfrac{1}{x + 3}$; $x \neq 0, -3$

c. $h(x) = \dfrac{x}{x^3 - x}$



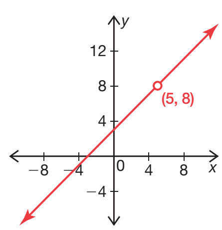
$h(x) = \dfrac{1}{(x - 1)(x + 1)}$; $x \neq -1, 0, 1$

d. $m(x) = \dfrac{x^2 + 5x + 6}{x^2 + 5x + 6}$



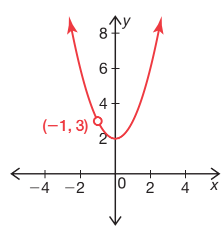
$j(x) = 1$; $x \neq -3, -2$

e. $k(x) = \dfrac{x^2 - 2x - 15}{x - 5}$



$k(x) = x + 3$, $x \neq 5$

f. $k(x) = \dfrac{x^3 + x^2 + 2x + 2}{x + 1}$



$m(x) = x^2 + 2$; $x \neq -1$

Talk the Talk

Students will explain the similarities and differences between rational numbers and rational expressions, and vertical asymptotes and removable discontinuities. They also explain why the phrase 'canceling out' does not properly describe division by a common factor.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Talk the Talk

9

1. Describe the similarities and differences between rational numbers and rational expressions.

   Answers will vary.

   Rational numbers and rational expressions follow the same rules when simplifying. Common factors divide to equal 1.

   Rational expressions have restrictions on the domain to avoid division by 0. Similarly rational numbers are undefined for division by 0.

   The graphs of rational numbers are a horizontal line because they are constants while the graphs of rational expressions vary depending on the function.

2. Describe the similarities and differences between vertical asymptotes and removable discontinuities.

   Both represent restrictions in the domain and breaks in the graph.

   A removable discontinuity is a single point in the graph that is undefined.

   A vertical asymptote also represents a restriction in the domain, but as the input values approach an asymptote the output values get closer and closer to that value but never actually equal that value.

3. Why is it incorrect to describe division by a common factor as "canceling out"?

   Common factors divide to equal 1. "Canceling out" implies that they disappear or reduce to 0, but this is not a true statement.

Be prepared to share your solutions and methods.

Check for Students' Understanding

9

Below are four hints necessary to write a rational function. Read each hint one at a time. Attempt to write the function after each hint before proceeding to the next hint. You may need to readjust your function after each hint.

First hint: It is a rational function.

Answers will vary.

$f(x) = \frac{1}{x}$

Second hint: It is a rational function and $y \neq 0$.

Answers will vary.

$f(x) = \frac{1}{x}$

Third hint: It is a rational function with $y \neq 0$, and it has a removable discontinuity at 4 and -4.

Answers will vary.

$$f(x) = \frac{(x^2 - 16)}{x(x + 4)(x - 4)}$$

Fourth hint: It is a rational function with $y \neq 0$, it has a removable discontinuity at 4 and -4, and it has a vertical dilation factor of 3.

Answers will vary.

$$f(x) = \frac{3(x^2 - 16)}{x(x + 4)(x - 4)}$$

9.5

The Breaking Point

Using Rational Functions to Solve Problems

LEARNING GOALS

In this lesson, you will:

- Model situations with rational functions.
- Use rational expressions to solve real-world problems.

ESSENTIAL IDEA

- Rational expressions are used to solve problems that involve comparing two quantities of the same unit of measure.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A-SSE Seeing Structure in Expressions

Interpret the structure of expressions.

2. Use the structure of an expression to identify ways to rewrite it.

A-CED Creating Equations

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems

A-REI Reasoning with Equations and Inequalities

Understand solving equations as a process of reasoning and explain the reasoning

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

F-IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Overview

A vinegar and oil mixture for salad dressing, the average cost per month for cable television, grams of chocolate in trail mix, a lightning/thunder storm, a quarterback's completion percentage, the average cost of different refrigerators, the sum of an integer and its reciprocal, test scores, average cost of joining a gym, and the Golden Ratio are all situations that are modeled using rational functions. Students will answer questions related to each scenario, create ratios, write rational expressions, describe the behavior of the ratios, identify the domain and range, and calculate average costs.

Warm Up

A 500 ml pitcher of lemonade contains 30% lemon juice plus water.

1. How many milliliters of lemon juice is in a pitcher of lemonade?

   There are 150 ml of lemon juice in a pitcher of lemonade.

   $0.3(500) = 150$

2. How many milliliters of water is in a pitcher of lemonade?

   There are 350 ml of water in a pitcher of lemonade.

   $500 - 150 = 350$

3. Suppose the lemonade was too strong and another 100 ml of water was added to the pitcher. What is the lemon juice concentration now?

   The lemonade now contains 25% lemon juice plus water.

   Let x equal the lemonade juice concentration.

$$\frac{150}{500 + 100} = x\%$$

$$\frac{150}{600} = \frac{x}{100}$$

$$x = 25$$

9

9

The Breaking Point

Using Rational Functions to Solve Problems

9.5

9

LEARNING GOALS

In this lesson, you will:

- Model situations with rational functions.
- Use rational expressions to solve real-world problems.

In Psychology, the term *breaking point* refers to a traumatic moment when a person breaks down. This may happen because of emotional or physical stress, and often ends up being the point at which there must be some sort of resolution to a problem.

Television and movies regularly use this as the moment of climax or resolution to their stories. A criminal may break down because of intense feelings of guilt during a police interrogation. Two characters in a story may try to avoid a conflict over something until their bad feelings reach a breaking point where they must somehow resolve their conflict. During an intense sports competition, often one team struggles to win while the other team hits a breaking point that is too much to overcome.

Can you think of any specific movies where a character reaches a breaking point? How was this moment used in the story?

Problem 1

A vinegar and oil mixture for salad dressing, the average cost per month for cable television, grams of chocolate in trail mix, and a lightning/thunder storm are all situations that are modeled by using rational functions. Students will will answer questions related to each scenario, create ratios, write rational expressions, describe the behavior of the ratios, and identify the domain and range.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- If Delilah adds 6 more teaspoons of olive oil, how many total teaspoons of olive oil are in the salad dressing?
- If Delilah adds 10 more teaspoons of olive oil, how many total teaspoons of olive oil are in the salad dressing?
- Why is 10 in the numerator or denominator of the ratio? How do you know?
- What expression is in the denominator of the ratio?
- Will the ratio ever equal zero?
- Is there an asymptote in this situation? How do you know?

PROBLEM 1 Start Applying Yourself, Rational Function!

Recall that a rational expression is the ratio of two polynomials. Rational expressions can be used to solve problems that involve comparing two quantities of the same unit of measure.

1. Delilah is making her own salad dressing out of red vinegar and olive oil. It's a new recipe so she has to determine the correct proportions. She mixes 10 teaspoons of vinegar and 16 teaspoons of olive oil. After she stirs the mixture, she realizes it's not the consistency she wants, so she adds more olive oil.

   a. What is the ratio of red vinegar to olive oil if she adds 6 teaspoons more of olive oil?

   The ratio of vinegar to olive oil is $\frac{10}{22}$.

   b. What is the ratio of red vinegar to olive oil if she adds 10 teaspoons more of olive oil?

   The ratio of vinegar to olive oil is $\frac{10}{26}$.

   c. Write an expression to represent the ratio of red vinegar to olive oil. Let x represent the number of additional teaspoons of olive oil added to the recipe.

   The ratio is $\frac{10}{16 + x}$.

   d. Describe the behavior of the ratio as the number of additional teaspoons of olive oil increases. Show all of your work and explain your reasoning.

   The ratio approaches zero, but will never be zero because a horizontal asymptote exists at $y = 0$.

   e. The recommended ratio of vinegar to olive oil is 1:7. Determine the amount of olive oil that she must add to the mixture. Show all of your work and explain your reasoning.

   Delilah must add 54 teaspoons of olive oil. I solved the proportion.

   $$\frac{10}{16 + x} = \frac{1}{7}$$

   $$16 + x = 70$$

   $$x = 54$$

   I can also graph the function $f(x) = \frac{10}{16 + x}$ and the horizontal line $g(x) = \frac{1}{7}$ and determine the intersection point.

   f. What are the domain and range of the function? Explain your reasoning.

   The domain is the positive real numbers because x represents the number of teaspoons of olive oil added. The range is $0 < y \leq \frac{10}{16}$ because the ratio can never reach zero, and it can never be greater than the original ratio.

- What equation will help solve for the amount of olive oil Delilah must add to the mixture to achieve the recommended ratio?
- Does the domain include the negative real numbers? Why not?
- Does the range include the negative real numbers? Why not?
- Can the range be greater than the original ratio? Why not?

Grouping

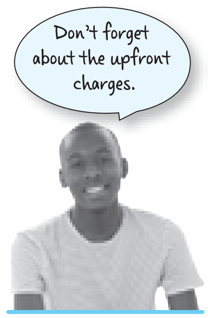
Have students complete Questions 2 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- Does the total cost of cable for the first two months include the cost of installation?
- If \$291 is the total cost of cable for the first two months, how is the average cost for the two months determined?
- Does the total cost of cable for the first year include the cost of installation?
- If \$851.40 is the total cost of cable for the first year, how is the average cost for the first year determined?
- What type of equation best represents the total cost of cable for x months?
- What type of equation best represents the average cost of cable for x months?
- Which equation is in the form $y = mx + b$, the total cost of cable for x months or the average cost of cable for x months?
- Which equation is represented by a rational function, the total cost of cable for x months or the average cost of cable for x months?

2. A door-to-door salesperson for TV Bonanza Cable Company offers cable television for only \$55.95 per month. However, there is a one-time installation cost of \$180.

a. Determine the total cost of cable for the first two months. What is the average cost per month over the first two months?



The total cost for two months is \$291.90.

$$\text{Total cost} = (55.95)(2) + 180$$
$$= 291.90$$

The average cost per month for two months is \$145.95.

$$\text{Average Cost} = \frac{180 + 55.95(2)}{2}$$
$$= 145.95$$

b. Determine the total cost of cable for the first year. What is the average cost per month over the first year?

The total cost for the first year is \$851.40.

$$\text{Total Cost} = (55.95)(12) + 180$$
$$= 851.40$$

The average cost per month for the first year is \$70.95.

$$\text{Average Cost} = \frac{180 + 55.95(12)}{12}$$
$$= 70.95$$

c. Write an equation to represent the total cost of cable for x months.

$y = 55.95x + 180$ where x is the number of months, and y is the cost.

d. Write an equation to represent the average monthly cost of cable for x months.

$y = \frac{55.95x + 180}{x}$ where x is the number of months, and y is the cost.

e. A competitor offers a similar product for \$65 per month and no installation charges. Who is offering the better deal? Show all work and explain your reasoning.

$$55.95x + 180 = 65x$$
$$180 = 9.05x$$
$$19.89 = x$$

The competitor's deal is better for approximately the first 20 months. After 20 months the average cost for TV Bonanza is lower.

- When determining who is offering the better deal, is the amount of time important?
- Does the company that offers the better deal for the first 20 months always offer the best deal?
- Do the two companies ever offer the same deal? What would this look like graphically?

Guiding Questions for Share Phase, Question 3

- How many grams of peanuts are in each package of trail mix?
- How many grams of almonds are in each package of trail mix?
- How many grams of chocolate are in each package of trail mix?
- How many grams would constitute 50% of the mixture of trail mix?
- Can a ratio be created by adding the number of grams to the percentage?
- Why doesn't it make sense to add the number of grams to the percentage?
- Did Tracy or Adrian add the number of grams to the percentage?
- Why must the units be the same when writing ratios?
- What equation is used to determine the grams of chocolate that must be added to get a mixture that is 50% chocolate?
- What unit was used to write the ratio?

9

3. Tracy and Adrian model the following problem with a rational function.

Crunchy College Kid Snack Company manufactures a new brand of trail mix containing peanuts, almonds, and chocolate. Each package contains 400 grams of trail mix, with 50% peanuts, 35% almonds, and 15% chocolate. Herbert loves chocolate. When he gets the bag home, he wants to add enough chocolate so that the mixture is 50% chocolate. How many grams of chocolate should he add?

Tracy

I must first determine the amount of chocolate in the bag.

$(0.15) \cdot (400) = 60g$

The ratio of chocolate to total trail mix must increase to 50%. Adding chocolate increases the total amount of trail mix, so the new ratio is $\frac{60 + x}{400 + x}$. I can set up the proportion $\frac{60 + x}{400 + x} = 0.50$.

The x-value represents the amount of additional chocolate.

Adrian

The current chocolate to trail mix ratio is $\frac{15}{100}$. Adding chocolate to get a mixture with 50% chocolate, add x to the percent chocolate as well as the total, so the rational equation becomes $\frac{15 + x}{100 + x} = 0.50$. The x-value of the intersection point represents the amount of chocolate Herbert must add.

a. Who is correct? Explain your reasoning. If necessary, include the error in the student's reasoning.

Tracy is correct. The number of grams of chocolate is 60 and she correctly adds x to the numerator, representing grams of chocolate, as well as to the denominator, which represents the total amount. Units must be the same when writing ratios.

Adrian is adding the number of grams to the percentage, which will not maintain the correct ratio.

b. Determine the grams of chocolate that Herbert must add to the trail mix to get a mixture that is 50% chocolate.

Herbert must add 280 grams of chocolate.

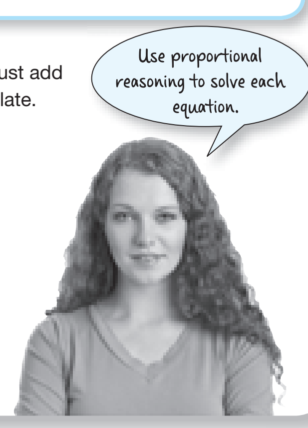
$$0.50 = \frac{60 + x}{400 + x}$$

$$60 + x = (0.50)(400 + x)$$

$$60 + x = 200 + 0.50x$$

$$0.50x = 140$$

$$x = 280$$



Guiding Questions for Share Phase, Question 4

- Is the function that represents the time between seeing lightning and hearing thunder a rational function?
- What equation was used to determine how far away the storm was?
- How was 70 and 3 used in the equation to determine the distance?
- What number was used to represent 'half a second'?
- How was 80 and half a second used in the equation to determine the distance?
- If the storm is overhead, what is the value of d?
- If the numerator is 0, is the output always 0?

9

4. A common misconception is that you can determine how far away a storm is by measuring the time between thunder and lightning. In reality, though, the time between seeing lightning and hearing thunder is a function of both distance and temperature. The time between seeing lightning and hearing thunder is represented by the function $Time = \frac{d}{1.09t + 1050}$, where d is the distance (feet) between the observer and the lightning, and t is the temperature (Fahrenheit).

a. If the temperature outside is 70 degrees and you count 3 seconds between the thunder and the lightning, approximately how far away is the storm? Show all of your work and explain your reasoning.

The storm is approximately 3379 feet away.

$$3 = \frac{d}{1.09(70) + 1050}$$

$$3 = \frac{d}{1126.3}$$

$$d = 3378.9$$

b. If the temperature is 80 degrees and you estimate half a second between thunder and lightning, how far away is the storm? Show all of your work and explain your reasoning.

The storm is approximately 569 feet away.

$$0.5 = \frac{d}{1.09(80) + 1050}$$

$$0.5 = \frac{d}{1137.2}$$

$$d = 568.6$$

c. On a 60-degree day, what is the time between thunder and lightning when the storm is directly overhead? Show all work and explain your reasoning.

If the storm is directly overhead, the distance is 0. Since the numerator is zero, the output will always be 0, so the time must be 0.

$$0 = \frac{d}{1.09(60) + 1050}$$

$$d = 0$$

Problem 2

A quarterback's completion percentage, the average cost of different refrigerators, the sum of an integer and its reciprocal, test scores, average cost of joining a gym, and the Golden Ratio are all situations that are modeled by using rational functions. Students will answer questions related to each scenario, create ratios, write rational expressions, and calculate average costs.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- What rational expression best represents the number of consecutive complete passes the quarterback in second place must throw to break the record?
- What should the rational expression be set equal to, to determine the number of consecutive complete passes the quarterback in second place must throw to break the record?
- Does 58.76 passes make sense in this situation?
- How much does it cost to run the ICY COLD refrigerator for one month?

PROBLEM 2 For The Record

1. In football, a quarterback's completion percentage is the ratio of the number of complete passes to the total number of pass attempts. The current record holder for highest completion percentage is Chad Pennington who completed 66% of his passes over the course of his career in the National Football League. The quarterback in second place completed 3843 passes out of 5853 attempts. Estimate the number of consecutive complete passes the second place quarterback must throw in order to break the record. Show all of your work and explain your reasoning

   The second place quarterback will break the record if he completes the next 59 consecutive passes.

   $\frac{3843 + x}{5853 + x} = 0.66$

   $3843 + x = 3862.98 + 0.66x$

   $0.34x = 19.98$

   $x = 58.76$

2. Josie compares two different refrigerators at the local hardware store. The sales tags are shown.

   ICY COLD
   - Crushed Ice Dispenser
   - Price $699.00
   - Uses 75 Kilowatt hours of electricity per month

   COOL AS A CUCUMBER
   - Gold Star EPA efficiency rating
   - Price: $825
   - Uses 30 kilowatt hours of electricity per month

   Josie does some research online and learns that a kilowatt hour costs approximately $0.06. She also learns that the average refrigerator lasts about 10 years.

   a. Write a function to represent the average cost of each refrigerator per month.

   The average cost per month for Icy Cold is $I(x) = \frac{699 + (0.06)(75)x}{x}$ for x months.

   The average cost per month for Cool As A Cucumber is $C(x) = \frac{825 + (0.06)(30)x}{x}$

   b. Which refrigerator will have a lower average monthly cost over the next ten years? Show all of your work and explain your reasoning.

   The refrigerator with the better efficiency rating ends up being cheaper over time. Substituting 120 months into both equations, I determined that Icy Cold refrigerators will cost $10.33 per month while Cool As A Cumber refrigerators will cost $8.68 per month.

- How much does it cost to run the COOL AS A CUCUMBER refrigerator for one month?
- Which refrigerator is the best buy? Why?
- Which refrigerator is cheaper over time? How much time?
- If the refrigerator doesn't last 10 years, which refrigerator was the best buy?

Guiding Questions for Share Phase, Questions 3 through 5

- What is the smallest positive integer?
- What is the reciprocal of the smallest positive integer?
- Does the graph of $y = x + \frac{1}{x}$ have a minimum in the first quadrant? What are the coordinates?
- What rational expression best represents the number of consecutive questions answered correctly to meet Scott's goal?
- What should the rational expression be set equal to, to determine the number of consecutive questions answered correctly to meet Scott's goal?
- If Manuel plays 1 hour of racquetball this month, what was the cost per hour?
- If Manuel plays 2 hours of racquetball this month, what was the cost per hour?
- If Manuel plays 3 hours of racquetball this month, what was the cost per hour?
- If Manuel plays 4 hours of racquetball this month, what was the cost per hour?
- If Manuel plays 5 hours of racquetball this month, what was the cost per hour?
- If Manuel plays 6 hours of racquetball this month, what was the cost per hour?
- What rational expression best represents the number of hours Manuel would have to play racquetball before the average cost is less than $7 per hour?
- What should the rational expression be set equal to, to determine the number of hours Manuel would have to play racquetball before the average cost is less than $7 per hour?

3. What is the least possible positive value for the sum of an integer and its reciprocal? Show all of your work and explain your reasoning.

The least possible value is $x = 1$ with a sum of $1 + 1 = 2$. I graphed the function $y = x + \frac{1}{x}$. The function has a minimum at (1, 2) in Quadrant I. The negative values are excluded for this problem because it specifically asks for the least possible positive value.

4. Scott is taking a test that has two different parts to it. His goal is to get a 90%. He finished Part 1, and a quick scan by the teacher reveals that he got 18 out of the 23 questions correct. He begins Part 2. If he answers each consecutive question correctly, how many must he answer correctly for his grade to be higher than a 90%? Show all of your work and explain your reasoning.

Scott must answer 27 consecutive questions correctly to get a 90%. To get a grade higher than a 90%, he must answer at least 28 consecutive questions correctly.

$$\frac{18 + x}{23 + x} = 0.90$$

$$18 + x = 20.7 + 0.9x$$

$$0.10x = 2.7$$

$$x = 27$$

5. Manuel enjoys racquetball, so he is considering joining a local gym. Joining the gym costs $30 each month, and they charge $2 per hour for using the racquetball courts. They also allow people who are not members of the club to use the courts for $7 per hour. If he joins the gym, how many hours would he have to play before the average cost is less than $7 per hour? Show all work and explain your reasoning.

Manuel would have to play more than 6 hours of racquetball each month.

$$\frac{30 + 2x}{x} = 7$$

$$30 + 2x = 7x$$

$$30 = 5x$$

$$x = 6$$

Guiding Questions for Share Phase, Question 6

- How is the ratio of the sum of the length and width to the length of a rectangle written?
- If the length has a value of 1, what is the width of the rectangle? How was this determined?
- How can you solve this situation graphically?
- What equations are used to solve this situation graphically?

9

6. The ancient Greeks felt as though certain rectangles in art and architecture were much more pleasing to the eye than others. When the ratio of the sum of the length and width to the length is approximately 1.618, they felt the rectangle was perfectly proportionate. This ratio came to be known as the *Golden Ratio*. Determine the length and width of several rectangles with dimensions that are in the *Golden Ratio*.

Answers will vary.

The ratio described is $\frac{\ell + w}{\ell}$. The Golden Ratio is 1.618. I substituted various values for length and solved graphically to determine the width.

For example, for $\ell = 2$ units, $1.618 = \frac{w + 2}{2}$.

Solving graphically I determined that $w = 1.236$ units. Therefore a rectangle with length 2 units and width 1.236 units is in the golden ratio.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Use a rational expression to solve this problem situation.

A saline or salt solution of 120 ml contains 10% salt. How much water would need to be added to the solution for it to contain only 2% salt?

Let x equal the amount of water added to the solution.

$$\frac{12}{120 + x} = 2\%$$

$$\frac{12}{120 + x} = \frac{2}{100}$$

$$1200 = 240 + 2x$$

$$960 = 2x$$

$$480 = x$$

Adding 480 ml of water will make the solution contain only 2% salt.

9

9

Chapter 9 Summary

KEY TERMS

- rational function (9.1)
- vertical asymptote (9.1)
- removable discontinuity (9.4)

9

9.1 Determining Whether a Function Is or Is Not a Rational Function

A rational function is any function that can be written as the ratio of two polynomials.

It can be written in the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions, and $Q(x) \neq 0$.

Example

$f(x) = \frac{x}{x + 7}$

The function $f(x)$ is a rational function because it is the ratio of two polynomials.

$g(x) = \frac{2^x}{x - 3}$

The function $g(x)$ is not a rational function because the numerator of the function has a variable in the exponent. The term 2^x is not a polynomial.

9.1 Determining the Domain and Range of a Rational Function

The domain of a rational function $f(x) = \frac{P(x)}{Q(x)}$ is the set of all real numbers that can be input as that variable x such that $f(x)$ is a real number. The range is the set of all values that the rational function can output when x is a value from the domain.

Example

$f(x) = \frac{-2}{x^3}$

The domain of $f(x)$ is the set of real numbers excluding 0. The range of $f(x)$ is the set of real numbers excluding 0.

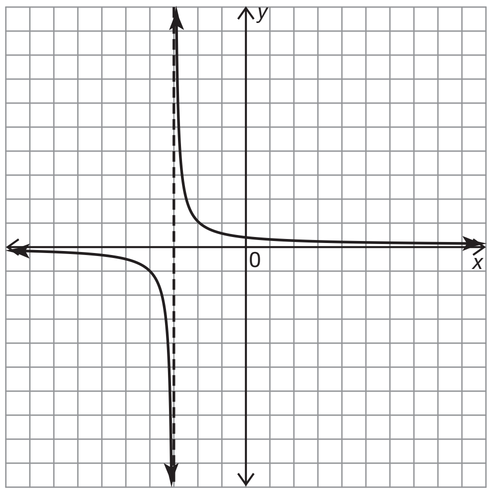
9.1

Describing Vertical and Horizontal Asymptotes of a Rational Function

A vertical asymptote is a vertical line that a function gets closer and closer to, but never intersects. A horizontal asymptote is a horizontal line that a function gets closer and closer to, but never intersects. These asymptotes do not represent points on the graph of the function. They represent the output value that the graph approaches. In particular, a vertical asymptote generally occurs for input values that result in a denominator of 0.

9

Example

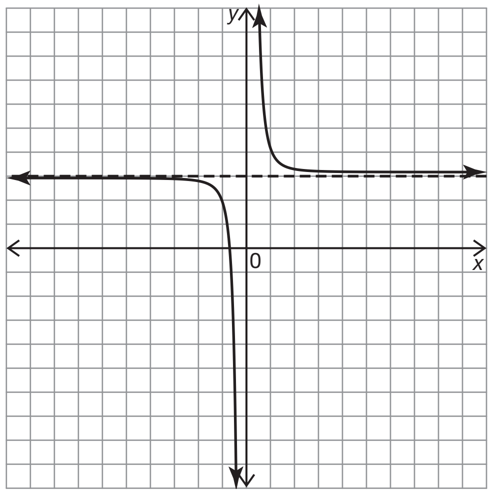


The vertical asymptote is the line $x = -3$. The horizontal asymptote is the x-axis or the line $y = 0$.

9.1 Describing the End Behavior of a Rational Function

The end behavior of a rational function is the characteristics of the function as the values of *x* approach positive and negative infinity.

Example

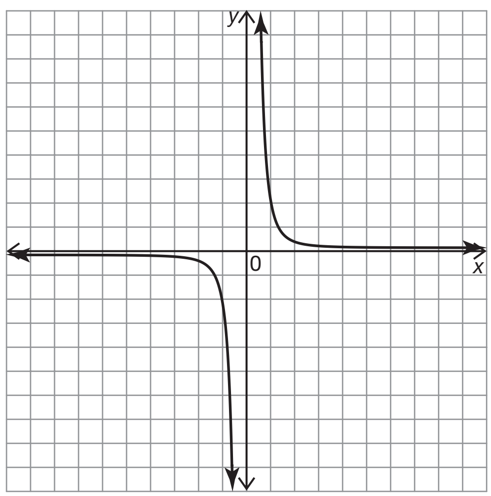


In this graph the end behavior of the rational function can be stated as follows. As *x* approaches negative infinity, *y* approaches 3. As *x* approaches positive infinity, *y* approaches 3.

9.1 Describing the Behavior of a Rational Function as *x* Approaches Zero From the Left and as *x* Approaches Zero From the Right

The behavior of a rational function on the left and right side of zero is the characteristics of the function as the values of *x* approach zero from the left and right.

Example



In this graph the behavior of the rational function as the values of *x* approach zero from the left and right can be stated as follows. As *x* approaches zero from the left, the *y* values approach negative infinity. As *x* approaches zero from the right, the *y* values approach infinity.

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9.2 Determining the Domain, Range, and Vertical and Horizontal Asymptotes for a Rational Function

Just like all of the other functions previously studied, it is important to identify the domain and range of a rational function. Since the graph of rational function often has vertical and horizontal asymptotes it is equally important to identify them also.

Example

$f(x) = \dfrac{3}{x + 5}$

Domain: All real numbers except -5.

Range: All real numbers except 0.

Vertical Asymptote: $x = -5$

Horizontal Asymptote: $y = 0$

9.2 Analyzing Rational Functions Using Algebra to Determine Vertical Asymptotes

Vertical asymptotes for a rational function occur at the value(s) for which the denominator is zero. Sometimes algebra is needed to help identify the vertical asymptote(s).

Example

$f(x) = \dfrac{1}{x^2 + 2x - 8}$

$$x^2 + 2x - 8 = 0$$

$$(x + 4)(x - 2) = 0$$

$$x = -4, x = 2$$

Vertical asymptotes exist at $x = -4$ and $x = 2$.

9.2

Putting the Pieces Together and Using Them to Sketch a Rational Function Given Its Equation

Once the domain, range, and vertical and horizontal asymptotes are known a sketch of the rational function can be made.

Example

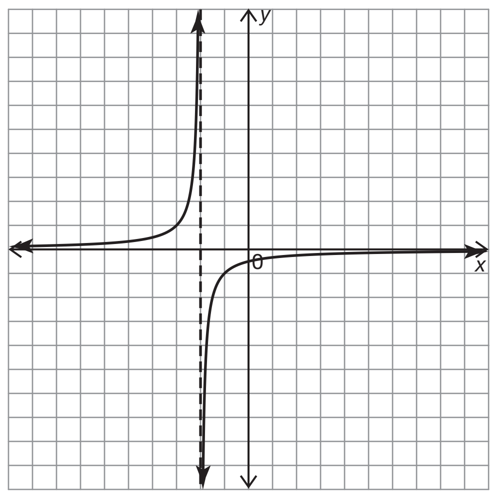
$f(x) = \frac{-1}{x + 2}$

Domain: All real numbers except -2.

Range: All real numbers except 0.

Vertical Asymptote: $x = -2$

Horizontal Asymptote: $y = 0$



9.2

Writing a Rule for a Rational Function Given Its Description

If enough characteristics of a rational function are known, an equation modeling these characteristics can be written. Be aware that there are many correct answers implying different rational functions can share similar characteristics.

Example

Vertical asymptote at $x = 4$. The range is all real numbers except $y = 0$.

Sample answer: $f(x) = \frac{3}{x - 4}$

The denominator cannot be 4 so there will be a vertical asymptote at $x = 4$. The function has a constant in the numerator and a variable in the denominator, so the output will approach 0 as x increases or decreases, creating a horizontal asymptote at $y = 0$.

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9.3

Translating and Dilating a Rational Function

Consider the functions $y = f(x)$ and $g(x) = Af[B(x - C)] + D$. Recall that adding a constant D translates $f(x)$ vertically, while adding a constant C translates $f(x)$ horizontally. Multiplying by the constant A dilates $f(x)$ vertically, while multiplying by the constant B dilates $f(x)$ horizontally. Rational functions are transformed in the same manner.

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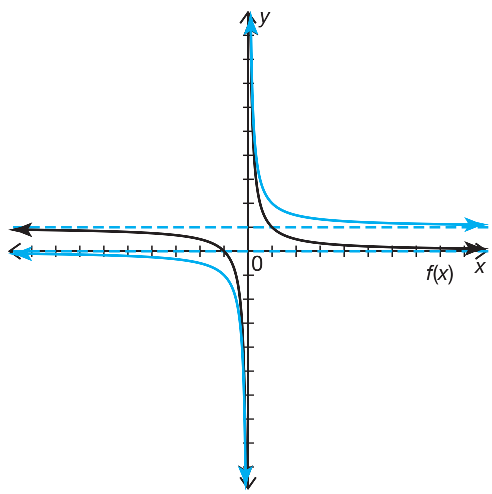
Example

The function $f(x) = \frac{1}{x}$ is shown in black on the coordinate plane. Determine whether the second function on the coordinate plane is the graph of $g(x) = \frac{1}{x - 1}$, $p(x) = \frac{1}{x + 1}$, or $q(x) = \frac{1}{x} + 1$. Explain your reasoning.



Function: $q(x) = \frac{1}{x} + 1$

Explanation: The original function $f(x) = \frac{1}{x}$ has been translated 1 unit up. This results from a change in the D-value.

9.3 Sketching a Rational Function in Detail

Given the equation of a rational function and having developed the tools to analyze its characteristics, a detailed sketch of the function can be made.

Example

$f(x) = \frac{1}{x^2 - x - 6}$

$\frac{1}{x^2 - x - 6} = \frac{1}{(x + 2)(x - 3)}$

$(x + 2)(x - 3) = 0$

$x = -2, x = 3$

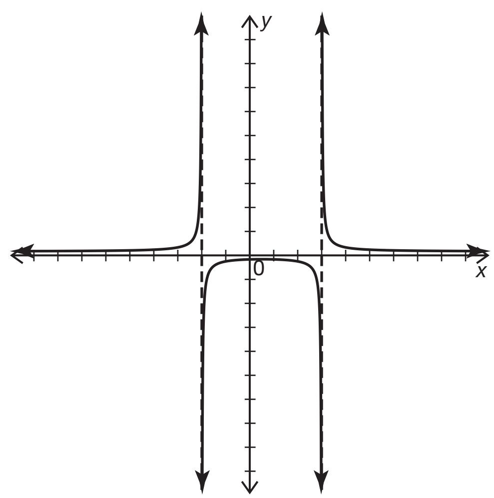
Domain: All real numbers except $x = -2$ and 3.

Range: All real numbers except 0.

Vertical Asymptotes: $x = -2$ and $x = 3$

Horizontal Asymptote: $y = 0$

y-intercept: $\left(0, -\frac{1}{6}\right)$



9.3 Writing the Equation of a Rational Function that Matches the Given Characteristics

The more that is known about the characteristics of a rational function, the easier it is to write an equation modeling the given characteristics. Be aware that there are many correct answers implying different rational functions can share similar characteristics.

Example

Given $f(x) = \frac{1}{x}$ and $g(x) = f(x - 4) + 3$, write a rational equation modeling $g(x)$.

Sample answer: $g(x) = \frac{1}{x - 4} + 3$

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9.4 Comparing Removable Discontinuities to Vertical Asymptotes

A removable discontinuity is a single point at which the graph of a function is not defined. A vertical asymptote is a vertical line that a function gets closer and closer to, but never intersects. When looking for removable discontinuities, look for common factors in the numerator and denominator of the rational function. The zeros of these common factors, if they exist, represent removable discontinuities.

Example

$$f(x) = \frac{x(x + 7)}{(x - 3)(x + 7)}$$

The function $f(x)$ has a removable discontinuity at $x = -7$ and a vertical asymptote at $x = 3$. Notice that $x + 7$ is a common factor in both the numerator and denominator of the rational function.

9.4 Simplifying Rational Expressions

When simplifying a rational expression it is necessary to list all restrictions on the variable along with the answer.

Example

$$\frac{x^2 + 6x - 7}{x - 1}$$

$$\frac{x^2 + 6x - 7}{x - 1} = \frac{\overset{1}{\cancel{(x - 1)}}(x + 7)}{\underset{1}{\cancel{(x - 1)}}}$$

$$= x + 7;\ x \neq 1$$

9.4 Sketching Rational Functions with Removable Discontinuities

A removable discontinuity is a single point at which the graph is not defined.

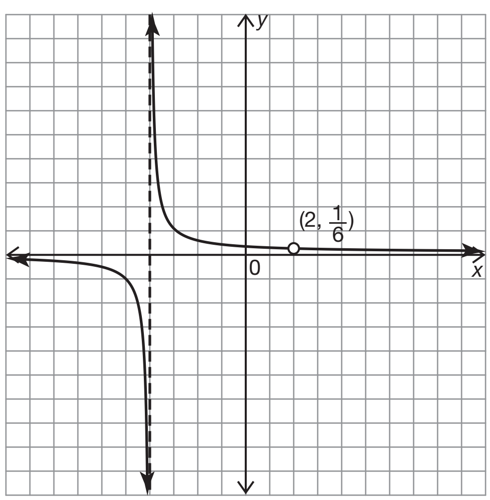
Example

$$f(x) = \frac{(x - 2)}{(x + 4)(x - 2)}$$

$$f(x) = \frac{(x - 2)}{(x + 4)(x - 2)}$$

$$= \frac{\overset{1}{\cancel{(x - 2)}}}{(x + 4)\underset{1}{\cancel{(x - 2)}}}$$

$$= \frac{1}{x + 4};\ x \neq -4, 2$$



9.5 Modeling Situations with Rational Functions

Rational expressions can be used to solve problems that involve comparing two quantities of the same unit of measure.

Example

Television Land is a media service provider. They advertise that you can buy a monthly plan for as low as \$75 per month as long as you buy a DVR from them costing \$150. If you buy the monthly plan along with the DVR, how many months will it take for your average monthly cost of owning the DVR and the plan to be less than \$100?

In the 6th month the average monthly cost will be less than \$100. The average monthly cost of buying the plan along with the DVR for x months is $\frac{150 + 75x}{x}$.

$$100 = \frac{150 + 75x}{x}$$

$$100x = 150 + 75x$$

$$25x = 150$$

$$x = 6$$

9.5 Solving Rational Functions Graphically

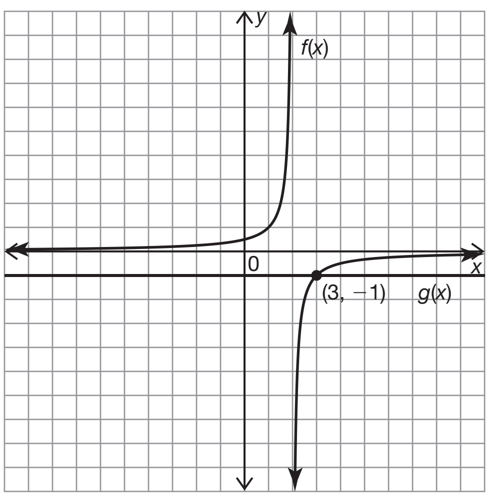
Equations of the form $f(x) = g(x)$, where $f(x)$ and $g(x)$ are rational functions, can be solved by graphing $y = f(x)$ and $y = g(x)$ on the same coordinate plane and identifying their intersection.

Example

$$\frac{-1}{x - 2} = -1$$

$$f(x) = \frac{-1}{x - 2}$$

$$g(x) = -1$$



$f(x) = g(x)$ at $x = 3$.

The solution is $x = 3$.

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