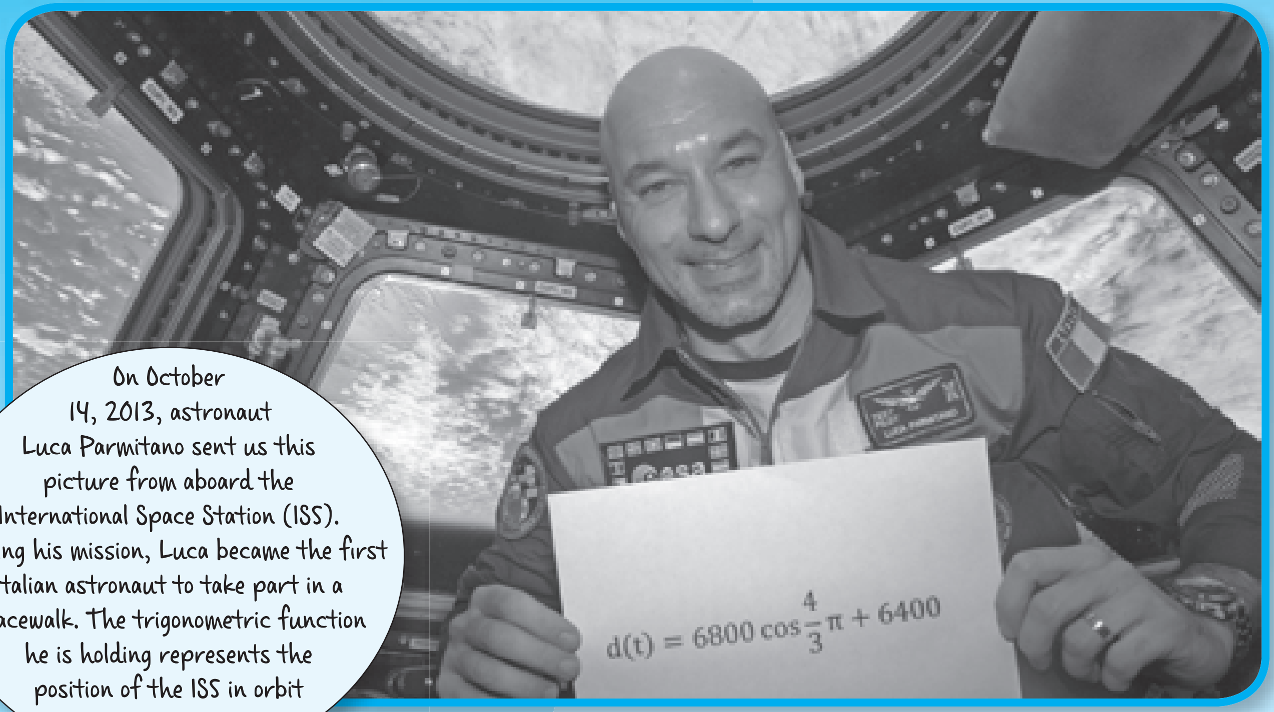
Trigonometric Equations

16



On October 14, 2013, astronaut Luca Parmitano sent us this picture from aboard the International Space Station (ISS). During his mission, Luca became the first Italian astronaut to take part in a spacewalk. The trigonometric function he is holding represents the position of the ISS in orbit over time.

16.1 Chasing Theta
Solving Trigonometric Equations 1133

16.2 Rabbits and Seasonal Affective Disorder
Modeling with Periodic Functions 1149

16.3 Behind the Wheel
Modeling Motion with a Trigonometric Function 1159

16.4 Springs Eternal
The Damping Function 1167

Chapter 16 Overview

In this chapter, students are introduced to solving trigonometric equations. They use their knowledge of the unit circle, radian measures, and the graphical behaviors of trigonometric functions to solve sine, cosine, and tangent equations. Students then apply all that they have learned to model various situations with trigonometric functions, including circular motion. Finally, students explore the damping function and modeling with trigonometric transformations.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
16.1	Solving Trigonometric Equations	F.TF.1 F.TF.2 F.TF.8	2	This lesson begins by highlighting what students already know that will allow them to solve trigonometric equations, such as graphical behaviors, periodicity identities, and the unit circle. Students then use techniques they have used to solve other equations to solve trigonometric equations, including using inverse trigonometric functions on a calculator. Students end the lesson by demonstrating a Pythagorean identity, which they then use to solve trigonometric equations.	X	X	X		X
16.2	Modeling with Periodic Functions	F.TF.5	1	Students use their knowledge of trigonometric functions and solving trigonometric equations to model real-world situations involving periodic behavior.	X				X
16.3	Modeling Motion with a Trigonometric Function	F.TF.5	1	In this lesson, students model the motion of a wheel using a trigonometric function. Students deconstruct the problem by first modeling parts of the situation using the basic sine function. Then, students apply what they know about transformations to write a function that models the real-world scenario.	X				
16.4	The Damping Function	F.TF.5	1	In this lesson, students build a damping function by constructing two separate functions—a trigonometric function and an exponential function—and then multiplying them to model the trigonometric function with decreasing amplitude.	X		X		

Skills Practice Correlation for Chapter 16

Lesson		Problem Set	Objectives
16.1	Solving Trigonometric Equations		Vocabulary
		1 – 6	Use the graph of a trigonometric function and domain restrictions to solve an equation
		7 – 15	Solve a trigonometric function over a given domain
		16 – 21	Use a periodicity identity to list four solutions to a trigonometric equation
		22 – 27	Solve a trigonometric equation over the set of all real numbers
		28 – 33	Determine the period of a trigonometric equation and then solve the equation over all real numbers
		34 – 39	Solve a trigonometric equation of quadratic form over the set of all real numbers
		40 – 45	Use the Pythagorean identity to determine the exact value of a trigonometric function
16.2	Modeling with Periodic Functions	1 – 6	Sketch the graph of a population model
		7 – 12	Determine the amplitude, period, phase shift, and vertical shift of a function given its equation
		13 – 18	Identify the amplitude, period, phase shift, and vertical shift of a function in the context of a problem situation
		19 – 24	Write the equation of a trigonometric function given its graph
		25 – 30	Write a sinusoidal regression equation for a set of data
		31 – 36	Compare and contrast the characteristics of two sinusoidal models in terms of a problem situation
16.3	Modeling Motion with a Trigonometric Function	1 – 6	Determine the amplitude of a function that models that motion of a wheel
		7 – 12	Determine the period and *B*-value of a function that models that motion of a wheel
		13 – 18	Use the *B*-value to determine the phase shift and *C*-value of a function that models that motion of a wheel
		19 – 24	Use a sinusoidal model to solve a real-world problem situation
16.4	The Damping Function		Vocabulary
		1 – 6	Identify the equation for the midline of a graph, based on a contextual situation
		7 – 12	Determine the minimum, maximum, and amplitude of the function of a problem situation
		13 – 18	Determine the period and *B*-value of the function of a problem situation
		19 – 24	Write a damping function for a problem situation

16.1

Chasing Theta

Solving Trigonometric Equations

LEARNING GOALS

In this lesson, you will:

- Write and solve trigonometric equations.
- Use periodicity identities to identify multiple solutions to trigonometric equations.
- Solve trigonometric equations using inverse trigonometric functions.
- Solve second-degree trigonometric equations.
- Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$.
- Use the Pythagorean identity to determine other trigonometric values.

KEY TERMS

- trigonometric equation
- inverse sine ($\sin^{-1}$)
- inverse cosine ($\cos^{-1}$)
- inverse tangent ($\tan^{-1}$)
- Pythagorean identity

ESSENTIAL IDEAS

- A trigonometric equation is an equation in which the unknown is associated with a trigonometric function.
- When using periodicity identities to solve trigonometric equations, adding or subtracting integer multiples, *n*, of the period for each function generates solutions to trigonometric equations.
- The inverse of each of the trigonometric functions—inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) and inverse tangent ($\tan^{-1}$)—in conjunction with the graphing calculator can be used to determine solutions to equations.
- The Pythagorean identity is a trigonometric identity that expresses the Pythagorean Theorem in terms of trigonometric functions: $(\sin(\theta))^2 + (\cos(\theta))^2 = (1)^2$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-TF Trigonometric Functions

Extend the domain of trigonometric functions using the unit circle

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Prove and apply trigonometric identities

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Overview

Trigonometric equations are solved using a variety of strategies. Graphs and periodicity identities are used to determine multiple solutions. Worked examples are provided throughout the lesson. Students solve trigonometric equations involving transformations on the basic function, and inverse trigonometric functions are used in conjunction with a graphing calculator to solve equations. Trigonometric equations for all real numbers written in quadratic form are solved by factoring or using the Quadratic Formula. Students prove the Pythagorean Identity and use it to determine other trigonometric values.

Warm Up

Solve this quadratic equation at least three different ways: $2x^2 + 5x = 12$.

$2x^2 + 5x = 12$

$2x^2 + 5x - 12 = 0$

$(2x - 3)(x + 4) = 0$

$2x - 3 = 0$ or $x + 4 = 0$

$x = \frac{3}{2}$ or $x = -4$

$2x^2 + 5x = 12$

$2x^2 + 5x - 12 = 0$

$a = 2 \quad b = 5 \quad c = -12$

$$x = \frac{-5 \pm \sqrt{25 - (4)(2)(-12)}}{(2)(2)}$$

$$= \frac{-5 \pm \sqrt{121}}{4}$$

$$= \frac{-5 \pm 11}{4}$$

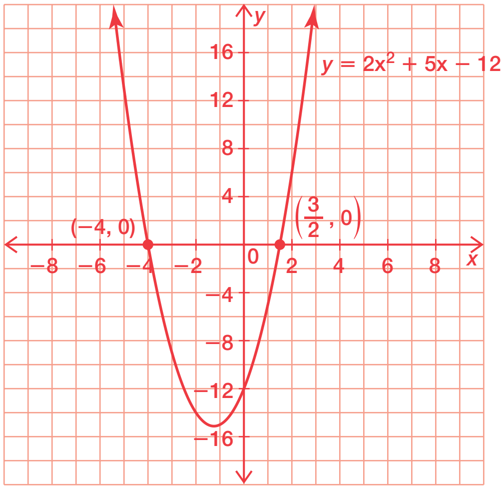
$x = \frac{3}{2}$ or $x = -4$

$2x^2 + 5x = 12$

$2x^2 + 5x - 12 = 0$

$2x^2 + 5x - 12 = y$

Graph the function and determine the zeros.



16.1

Chasing Theta

Solving Trigonometric Equations

LEARNING GOALS

In this lesson, you will:

- Write and solve trigonometric equations.
- Use periodicity identities to identify multiple solutions to trigonometric equations.
- Solve trigonometric equations using inverse trigonometric functions.
- Solve second-degree trigonometric equations.
- Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$.
- Use the Pythagorean identity to determine other trigonometric values.

KEY TERMS

- trigonometric equation
- inverse sine ($\sin^{-1}$)
- inverse cosine ($\cos^{-1}$)
- inverse tangent ($\tan^{-1}$)
- Pythagorean identity

A ground track, or ground trace, is a path that can be created by an orbiting satellite, like the International Space Station, mapped onto the surface of the Earth. Ground tracks are useful to engineers—and amateur astronomers—in tracking the locations of these satellites.

Because orbits are circular, ground tracks are usually in the shape of the graphs of periodic functions.

However, if a satellite has what is called a geostationary orbit, its ground track will be a single point, because it rotates at the same rate as the Earth. That means that it is above the same place on Earth at all times.

Problem 1

The term *trigonometric equation* is defined. Worked examples show how restrictions on the domain affect the solution(s), and a graph is used to determine the solution(s). Periodicity identities are used to determine multiple solutions.

Grouping

- Ask a student to read the information and definition. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- In which quadrants does $\sin(x) = \frac{1}{2}$?
- Which quadrant is described by $0 \le x \le \frac{\pi}{2}$?
- Which quadrant is described by $-\pi \le x \le 0$?
- Where did Caleb get $\frac{2\pi}{3}$?
- Why would Caleb subtract $\frac{\pi}{6}$ from $\frac{5\pi}{6}$?
- Does this strategy work when determining a solution in the second quadrant, given a solution in the first quadrant?
- Is the value of $\frac{5\pi}{6} + \frac{2\pi}{3}$ a solution?

PROBLEM 1 You Know A Lot!

A **trigonometric equation** is an equation in which the unknown is associated with a trigonometric function. The number of solutions of a trigonometric equation can vary depending on how the domain of the function is restricted.

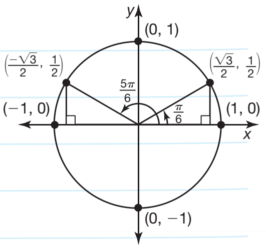
You can solve trigonometric equations using what you already know.

Consider the equation $\sin(x) = \frac{1}{2}$.

On the unit circle, you can see that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$. So, $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

When the domain is restricted to $0 \le x \le 2\pi$, these are the only 2 solutions to the equation.

When there are no domain restrictions, the equation has an infinite number of solutions.



1. Explain what error(s) Caleb made in his reasoning.

   Caleb subtracted: $\frac{5\pi}{6} - \frac{\pi}{6} = \frac{4\pi}{6}$, or $\frac{2\pi}{3}$.
   But this only works when determining a solution in the second quadrant, given that a solution in the first quadrant is known.
   The value $\frac{5\pi}{6} + \frac{2\pi}{3} = \frac{3\pi}{2}$ is not a solution.

Caleb

I can just keep adding or subtracting $\frac{2\pi}{3}$ to one solution to determine another solution to $\sin(x) = \frac{1}{2}$.

2. List the solution(s) of the trigonometric equation $\sin(x) = \frac{1}{2}$, given each of the domain restrictions.

   a. $0 \le x \le \frac{\pi}{2}$

   $x = \frac{\pi}{6}$

   b. $0 \le x \le 4\pi$

   $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$

   c. $-\pi \le x \le 0$

   No solution.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 3 through 5 with a partner. Then have students share their responses as a class.

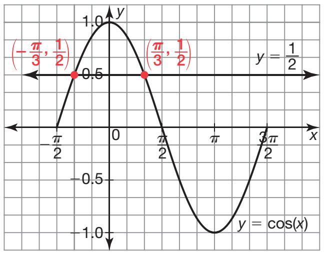
Guiding Questions for Share Phase, Questions 3 through 5

- How is the domain of the graphed function determined?
- Are the solutions to the equation $\cos(x) = \frac{1}{2}$ located at the intersections of the two graphs?
- Where on the graph does $\cos(x)$ have a value of 1?
- Where on the graph does $\cos(x)$ have a value of 0?
- Where on the graph does $\cos(x)$ have a value of -1?
- Where on the graph does $\cos(x)$ have a value of $-\frac{1}{2}$?
- Where on the graph does $\tan(x)$ have a value of 0?
- Where are the asymptotes on the graph of $y = \tan(x)$?

You can also use what you know about the graphs of trigonometric functions to solve trigonometric equations.

Let's consider the equation $\cos(x) = \frac{1}{2}$.

The equations $y = \cos(x)$ and $y = \frac{1}{2}$ are graphed on the coordinate plane.



3. Study the graph of $y = \cos(x)$.

a. Over what domain is the function graphed?

$-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

b. Write the solution(s) to the equation $\cos(x) = \frac{1}{2}$, given the domain restrictions. Then, plot and label the solution(s) on the coordinate plane.

$x = -\frac{\pi}{3}$ or $\frac{\pi}{3}$

4. Write the solution(s) to each equation, given the same domain restrictions in Question 3.

a. $\cos(x) = 1$

$x = 0$

b. $\cos(x) = 0$

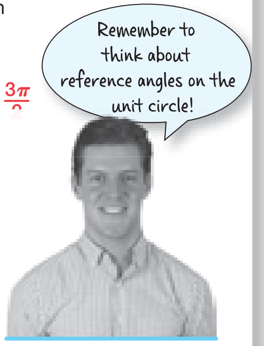
$x = -\frac{\pi}{2}, \frac{\pi}{2}$, or $\frac{3\pi}{2}$

c. $\cos(x) = -\frac{1}{2}$

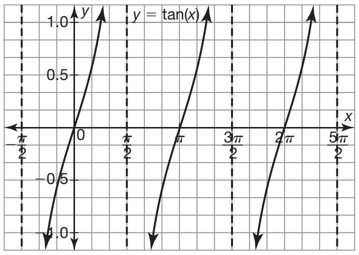
$x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

d. $\cos(x) = -1$

$x = \pi$



5. Use the graph of $y = \tan(x)$ over the domain $-\frac{\pi}{2} \le x \le \frac{5\pi}{2}$ to solve each equation.



a. $\tan(x) = 0$

$x = 0, \pi,$ or 2π

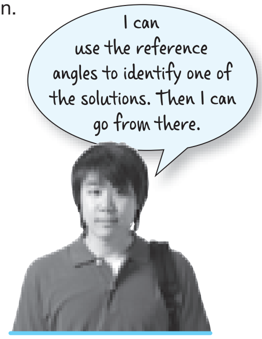
b. $\tan(x) =$ undefined

$x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2},$ or $\frac{5\pi}{2}$

You can use what you know about the periods of trigonometric functions to solve trigonometric equations. The periodicity identities you have learned are shown. Adding or subtracting integer multiples, n, of the period for each function generates solutions to trigonometric equations.

Periodicity Identities	
Sine	$\sin(x + 2\pi n) = \sin(x)$
Cosine	$\cos(x + 2\pi n) = \cos(x)$
Tangent	$\tan(x + \pi n) = \tan(x)$

6. Use a periodicity identity to list 4 solutions to each equation.



a. $\cos(x) = \frac{\sqrt{2}}{2}$

Answers will vary.

$x = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4} \ldots$

b. $\tan(x) = \sqrt{3}$

Answers will vary.

$x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3} \ldots$

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Question 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- Which periodic identity do you use to list the solutions in this situation?
- Which solution do you first determine?
- How is the first solution used to determine other solutions?
- How many solutions are possible?
- What is the period of this function?
- How can the sign of the value of the trigonometric function give you information about the quadrant in which the solution lies?
- How is the sign of the value of the trigonometric function related to the quadrant in which the solution lies?

Problem 2

A worked example solving a trigonometric equation involving transformations on the basic function is provided. Students list all of the solutions to the worked example over a restricted domain and discuss errors made in student work. Then they solve a trigonometric equation involving transformations on the basic function. Inverse trigonometric functions are introduced, and students use them in conjunction with a graphing calculator to solve equations. A change in the *B*-value of the transformed equation prompts students to take the change in period into account when determining solutions. Reference angles are used to determine the solutions for *x*. A worked example is provided and students determine solutions over different domains.

Grouping

- Ask a student to read the information. Discuss the worked example as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.
- Ask a student to read the information and complete Question 4 as a class.

PROBLEM 2 Same Ol', Same Ol'

When a trigonometric equation involves transformations on the basic function, solving the equation requires the same techniques you have used to solve other equations.

Solve $\sqrt{3}\tan(x) + 5 = 4$ over the domain $0 \le x \le \pi$.

$\sqrt{3}\tan(x) + 5 = 4$

$\sqrt{3}\tan(x) = -1$	Subtract 5 from both sides.
$\tan(x) = \frac{-1}{\sqrt{3}}$	Divide both sides by $\sqrt{3}$.
$= \frac{-\sqrt{3}}{3}$	Rewrite the radical expression.
$x = \frac{5\pi}{6}$	

1. List all of the solutions to the equation in the worked example over the domain of all real numbers. Show your work.

   $x = \frac{5\pi}{6} + \pi n$ for integer values of n

   I know that adding or subtracting the period of the function generates solutions to a trigonometric equation.

2. Explain why Amy is incorrect.

   The tangent of a value, *x*, is the ratio of the sine of *x* to the cosine of *x*, but that does not mean that sin(*x*) and cos(*x*) are equal to the specific numerator and denominator of the tangent ratio.

   If $\tan(x) = \frac{-\sqrt{3}}{3}$, then one value for *x* is $\frac{5\pi}{6}$.

   For the given angle, $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$,

   and $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

   So, $\frac{1}{2} \div -\frac{\sqrt{3}}{2} = \frac{-\sqrt{3}}{3}$, which is tan(*x*).

Amy

If $\tan(x) = \frac{-\sqrt{3}}{3}$, and $\tan(x) = \frac{\sin(x)}{\cos(x)}$, then I know that $\sin(x) = -\sqrt{3}$ and $\cos(x) = 3$.

Guiding Questions for Share Phase, Questions 1 through 3

- How many solutions are there for the worked example over the given domain?
- Is $-\frac{\pi}{6}$ a possible solution to the worked example? Why not?
- Is $-\frac{\pi}{6}$ in the domain $0 \le x \le 2\pi$?
- Is sin(x) equal to the value in the numerator of the tangent ratio? Why not?
- Is cos(x) equal to the value in the denominator of the tangent ratio? Why not?
- If $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$ and $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, does $\frac{1}{2} \div -\frac{\sqrt{3}}{2} = \frac{-\sqrt{3}}{3}$?
- Is $\frac{1}{2} \div -\frac{\sqrt{3}}{2} = \frac{-\sqrt{3}}{3}$ equal to tan(x)?
- What does sin(x) equal in this situation?
- What quadrant is associated with $\sin(x) = -\frac{\sqrt{3}}{2}$?

Guiding Questions for Discuss Phase, Question 4

- When using a graphing calculator to solve an equation, why must the mode be set to radians and not degrees?
- Are the values $\frac{\pi}{6}$ and 0.5235987756 . . . the same?
- Are there any domain restrictions in this situation?
- How many answers are possible?

3. Solve the equation $2\sin(x) + \sqrt{3} = 0$ over the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

$$2\sin(x) + \sqrt{3} = 0$$
$$2\sin(x) = -\sqrt{3}$$
$$\sin(x) = -\frac{\sqrt{3}}{2}$$
$$x = -\frac{\pi}{3}$$

You can use the inverse of each of the trigonometric functions to determine solutions to equations. The **inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$),** and **inverse tangent ($\tan^{-1}$)** functions are used to determine solutions to sine, cosine, and tangent equations, respectively.

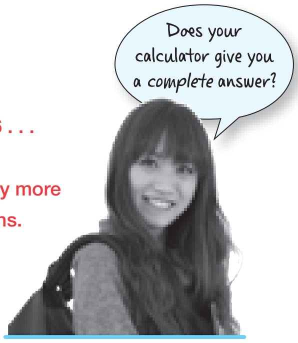
4. Use the steps shown to solve the equation $\sin(x) = \frac{1}{2}$ on a calculator.

Step 1: Press MODE and make sure it is set to RADIANS, not DEGREES.

Step 2: Press 2ND and SIN^{-1}. Type the value of sin(x) inside parentheses and press ENTER.

What value did the calculator return as the answer? Explain whether this value is or is not the correct answer and why.

The value returned by the calculator is 0.5235987756 . . .
This is correct, because it is the value of $\frac{\pi}{6}$.
However, the answer is not complete. There are many more solutions to the equation, given no domain restrictions.



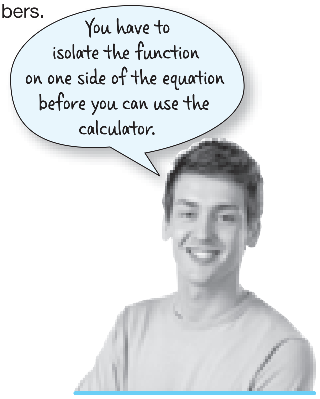
Grouping

Have students complete Question 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- What is the first step in the solution path?
- Is it necessary to rationalize the denominator?
- How is the inverse function helpful in solving this problem?
- Is the solution within the range of the basic function?
- How many solutions are possible?

5. Solve each equation over the domain of all real numbers.



a. $-5 + 2\sqrt{3}\cos(x) = -8$

$2\sqrt{3}\cos(x) = -3$

$\cos(x) = -\frac{3}{2\sqrt{3}}$

$\cos(x) = -\frac{\sqrt{3}}{2}$

$x = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$x = \frac{5\pi}{6} + 2\pi n$ or $x = \frac{7\pi}{6} + 2\pi n$

for integer values of n

b. $5\sin(x) + 9 = 3$

$5\sin(x) = -6$

$\sin(x) = -1.2$

There is no solution. The value -1.2 is not in the range of the sine function.

c. $6\tan(x) - 4 = -19$

$6\tan(x) = -15$

$\tan(x) = -2.5$

$x = \tan^{-1}(-2.5)$

$x \approx -1.1903 + \pi n$

for integer values of n

d. $5 - 8\cos(x) = 3$

$-8\cos(x) = -2$

$\cos(x) = 0.25$

$x = \cos^{-1}(0.25)$

$x \approx 1.3181 + 2\pi n$ or $x \approx 4.9651 + 2\pi n$

for integer values of n

Grouping

- Discuss the information and graph above Question 6 as a class.
- Have students complete Questions 6 and 7 with a partner. Then have students share their responses as a class.

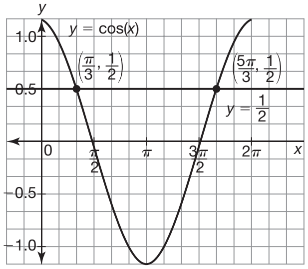
Guiding Questions for Share Phase, Questions 6 and 7

- If the *B*-value is 2, is the graph of the cosine function stretched or compressed by a factor of 2?
- What is the period of cos(2*x*)?
- How do you determine the period of cos(2*x*)?
- What is the frequency of cos(2*x*) over the domain $0 \le x \le 2\pi$?
- How does the frequency affect the number of solutions over the given domain?
- How is the reference angle helpful in determining the solution?
- What is the period of sin(4*x*)?
- How do you determine the period of sin(4*x*)?
- Where does sine have a value of −1?
- Does sine have a value of −1 at $\frac{3\pi}{2}$?

When the *B*-value is changed from a basic trigonometric function, you must take the change in period into account when determining solutions.

Let's consider the equation $\cos(x) = \frac{1}{2}$ over the domain $0 \le x \le 2\pi$.

The equations $y = \cos(x)$ and $y = \frac{1}{2}$ are graphed on the coordinate plane.



The solutions for $\cos(x) = \frac{1}{2}$ over the domain $0 \le x \le 2\pi$ are $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

The solutions for $\cos(x) = \frac{1}{2}$ over the domain for all real numbers are $x = \frac{\pi}{3} + 2\pi n$ or $\frac{5\pi}{3} + 2\pi n$.

6. Now, let's consider the equation $\cos(2x) = \frac{1}{2}$ over the domain $0 \le x \le 2\pi$.

a. Determine the period of this function.

The period is $\frac{2\pi}{|B|} = \frac{2\pi}{2}$, or π.

b. The period of $y = \cos(2x)$ is different than $y = \cos(x)$. How does your answer to part (a) affect the number of possible solutions for $\cos(2x) = \frac{1}{2}$ over the domain $0 \le x \le 2\pi$?

Since the frequency of cos(2*x*) is 2 over the domain $0 \le x \le 2\pi$, the number of solutions will double. Therefore, there will be 4 possible solutions over the domain $0 \le x \le 2\pi$.

You can use what you know about reference angles to determine solutions for *x*.

To solve $\cos(2x) = \frac{1}{2}$, you know $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. So, to begin, let $\frac{\pi}{3} = 2x$ and solve for *x*.

$$2x = \frac{\pi}{3}$$

$$x = \frac{\pi}{6}$$

Because the period of cos(2*x*) is π, $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$ for $0 \le x \le 2\pi$.

c. Determine the remaining solutions for $\cos(2x) = \frac{1}{2}$ over the domain $0 \le x \le 2\pi$ given $\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$.

$$2x = \frac{5\pi}{3}$$
$$x = \frac{5\pi}{6}$$

Given the domain $0 \le x \le 2\pi$, $x = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$.

d. Write the solution for $\cos(2x) = \frac{1}{2}$ over the domain for all real numbers.

$x = \frac{\pi}{6} + \pi n$ or $x = \frac{5\pi}{6} + \pi n$ for integer values of n

7. Solve the equation $2\sin(4x) + 1 = -1$ over the set of real numbers.

The period of $\sin(4x)$ is $\frac{2\pi}{|B|} = \frac{2\pi}{4}$ or $\frac{\pi}{2}$.

$$2\sin(4x) + 1 = -1$$
$$2\sin(4x) = -2$$
$$\sin(4x) = -1$$

Because $\sin\left(\frac{3\pi}{2}\right) = -1$, let $4x = \frac{3\pi}{2}$.

$$x = \frac{3\pi}{8}$$

Therefore, $x = \frac{3\pi}{8} + \frac{\pi}{2}n$ for integer values of n.

16

Problem 3

Trigonometric equations are written in quadratic form where $ax^2 + bx + c = 0$ has x replaced with a trigonometric function. A worked example shows a trigonometric equation in quadratic form which is factorable, and the inverse sine function is used to solve the problem. Students solve trigonometric equations for all real numbers written in quadratic form by factoring or using the Quadratic Formula.

Grouping

- Ask a student to read the information and worked example. Complete Question 1 as a class.
- Have students complete Questions 2 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 2 through 5

- Is the quadratic equation factorable?
- What are the factors?
- What are the solutions?
- Is there an alternate solution path?
- Can the Quadratic Formula be used to solve this problem?
- Can the variable be isolated?
- Is the inverse function helpful in this situation?
- What is the range of the sine function?
- What is the range of the cosine function?
- If the value is out of range, is it still considered a solution?

PROBLEM 3 Power Trig

If an equation that can be written in the form $ax^2 + bx + c = 0$ has x replaced with a trigonometric function, the result is a trigonometric equation in quadratic form. These equations can be solved as you would solve other quadratic equations, by factoring or by using the Quadratic Formula.

Note that $(\sin(x))^2$ is usually written as $\sin^2 x$.

You can solve $2\sin^2(x) + 5\sin(x) = 3$ over the domain of all real numbers. Start with a substitution. This equation involves the sine function, so let $z = \sin(x)$.

$2z^2 + 5z = 3$

$2z^2 + 5z - 3 = 0$ Get equation in standard form.

$(2z - 1)(z + 3) = 0$ Factor the quadratic expression.

$2z - 1 = 0$ or $z + 3 = 0$ Set each factor equal to 0.

$2z = 1$

$z = \frac{1}{2}$ or $z = -3$ Solve each equation.

$\sin(x) = \frac{1}{2}$ or $\cancel{\sin(x) = -3}$ Replace z with the sine function.

$x = \frac{\pi}{6}, \frac{5\pi}{6} \ldots + 2\pi n$ Solve the equations using $\sin^{-1}$.

1. Explain why $\sin(x) = -3$ is crossed off in the worked example.

   The basic sine function has a minimum of -1. It is not possible for $\sin(x)$ to equal -3 without transformations on the function.

2. Solve $4\sin^2(x) - 1 = 0$ over the domain of all real numbers.

   Let $z = \sin(x)$.

   $4z^2 - 1 = 0$

   $(2z + 1)(2z - 1) = 0$

   $2z + 1 = 0$ or $2z - 1 = 0$

   $2z = -1$ $2z = 1$

   $z = -\frac{1}{2}$ $z = \frac{1}{2}$

   $\sin(x) = -\frac{1}{2}$ or $\sin(x) = \frac{1}{2}$

   $x = -\frac{\pi}{6} + 2\pi n$ or $x = -\frac{5\pi}{6} + 2\pi n$ or $x = \frac{\pi}{6} + 2\pi n$ or $x = \frac{5\pi}{6} + 2\pi n$

   Note: There is an alternate solution path. First, subtract one from both sides. Then divide both sides by four. Taking the square root of both sides, at this point, leads to two equations in the solution process.

3. Solve $2\cos^2(x) + \cos(x) = 1$ over the domain of all real numbers.

Let $z = \cos(x)$.

$2z^2 + z = 1$

$2z^2 + z - 1 = 0$

$(2z - 1)(z + 1) = 0$

$2z - 1 = 0$ or $z + 1 = 0$

$2z = 1$

$z = \frac{1}{2}$ $\qquad$ $z = -1$

$\cos(x) = \frac{1}{2}$ or $\cos(x) = -1$

$x = \frac{\pi}{3} + 2\pi n$ or $x = \frac{5\pi}{3} + 2\pi n$ or $x = \pi + 2\pi n$

4. Solve $2\tan^2(z) + 3\tan(z) - 1 = 0$ over the domain of all real numbers.

Let $x = \tan(z)$.

$2x^2 + 3x - 1 = 0$

$a = 2, \quad b = 3, \quad c = -1$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{17}}{4}$$

$x \approx 0.2808$ or $x \approx -1.7808$

$\tan(z) \approx 0.2808$ or $\tan(z) \approx -1.7808$

$z \approx 0.2737 + \pi n$ or $z \approx -1.0591 + \pi n$

5. Solve $6\sin^2(z) - 16\sin(z) - 33 = 0$ over the domain of all real numbers.

Let $x = \sin(z)$.

$6x^2 - 16x - 33 = 0$

$a = 6, \quad b = -16, \quad c = -33$

$$x = \frac{16 \pm \sqrt{(-16)^2 - 4(6)(-33)}}{2(6)}$$

$$x = \frac{16 \pm \sqrt{1048}}{12}$$

$x \approx 4.0311$ or $x \approx -1.3644$

$\sin(z) \approx 4.0311$ or $\sin(z) \approx -1.3644$

No solution.

Problem 4

Students demonstrate how the Pythagorean identity follows from the Pythagorean Theorem using a reference triangle in a unit circle. They use the Pythagorean identity to determine other trigonometric values.

Grouping

- Ask a student to read the information and definition. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

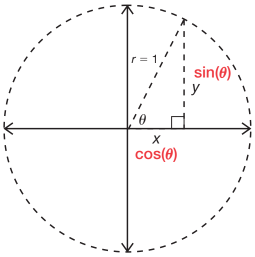
Guiding Questions for Share Phase, Questions 1 and 2

- Is the vertical distance from the x-axis describing the point where the hypotenuse intersects the unit circle associated with $\sin(\theta)$ or $\cos(\theta)$?
- Is the horizontal distance from the y-axis describing the point where the hypotenuse intersects the unit circle associated with $\sin(\theta)$ or $\cos(\theta)$?
- Why is $\sin^2(\theta) = 1 - \cos^2(\theta)$?
- Why is $\cos^2(\theta) = 1 - \sin^2(\theta)$?

PROBLEM 4 A Pythagorean Identity

You have learned about the periodicity identities, but there is another identity in trigonometry that can also help you solve trigonometric equations. A **Pythagorean identity** is a trigonometric identity that expresses the Pythagorean Theorem in terms of trigonometric functions. The Pythagorean identity states $(\sin(\theta))^2 + (\cos(\theta))^2 = (1)^2$.

You can prove this Pythagorean identity using your knowledge of the unit circle and the Pythagorean Theorem.



1. Demonstrate how the Pythagorean identity follows from the Pythagorean Theorem.

   a. Given the unit circle and the angle θ, label the side lengths of the right triangle in terms of $\sin(\theta)$ and $\cos(\theta)$.

   See the unit circle.

   b. State the Pythagorean Theorem.

   $a^2 + b^2 = c^2$

   c. Use substitution to demonstrate how the Pythagorean identity follows from the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$
$$(\sin(\theta))^2 + (\cos(\theta))^2 = (1)^2$$
$$\sin^2(\theta) + \cos^2(\theta) = 1$$

2. Write the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ in two other forms.

   a. Solve for $\sin^2(\theta)$.

   $\sin^2(\theta) + \cos^2(\theta) = 1$

   $\sin^2(\theta) + \cos^2(\theta) - \cos^2(\theta) = 1 - \cos^2(\theta)$

   $\sin^2(\theta) = 1 - \cos^2(\theta)$

   b. Solve for $\cos^2(\theta)$.

   $\sin^2(\theta) + \cos^2(\theta) = 1$

   $\sin^2(\theta) + \cos^2(\theta) - \sin^2(\theta) = 1 - \sin^2(\theta)$

   $\cos^2(\theta) = 1 - \sin^2(\theta)$

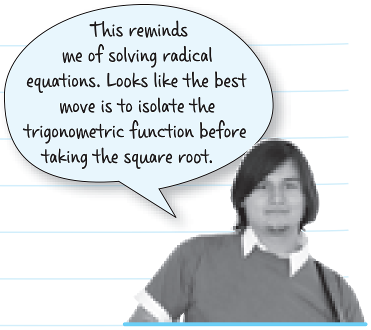
Grouping

- Ask a student to read the information and worked example. Discuss as a class.
- Have students complete Questions 3 through 6 with a partner. Then have students share their responses as a class.

You can use the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and what you know about solutions in different quadrants to determine values of trigonometric functions.

Determine the exact value of $\cos(\theta)$ in Quadrant II, given $\sin(\theta) = \frac{2}{3}$.

You can use a Pythagorean identity to determine $\cos(\theta)$.



$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\left(\frac{2}{3}\right)^2 + \cos^2(\theta) = 1$$

$$\frac{4}{9} + \cos^2(\theta) = 1$$

$$\cos^2(\theta) = \frac{5}{9}$$

$$\cos(\theta) = \pm\frac{\sqrt{5}}{3}$$

The solution is $\cos(\theta) = -\frac{\sqrt{5}}{3}$ because the solution is in Quadrant II $\left(\frac{\pi}{2} \le \theta \le \pi\right)$.

Guiding Questions for Share Phase, Questions 3 through 6

- What is the first step?
- What steps are necessary to isolate the trigonometric function?
- Are the numerator and denominator both perfect squares?
- Are both answers considered the solution?
- Is cos(θ) equal to a positive value or a negative value in Quadrant II?
- Is sin(θ) equal to a positive value or a negative value in Quadrant IV?
- Is cos(θ) equal to a positive value or a negative value in Quadrant IV?

3. Given sin(θ) $= \frac{3}{5}$ in Quadrant II, determine cos(θ).

$$\left(\frac{3}{5}\right)^2 + \cos^2(\theta) = 1$$

$$\frac{9}{25} + \cos^2(\theta) = 1$$

$$\cos^2(\theta) = \frac{16}{25}$$

$$\cos(\theta) = \pm\frac{4}{5}$$

$\cos(\theta) = -\frac{4}{5}$, because the solution is in Quadrant II.

4. Given cos(θ) $= -\frac{12}{13}$ in Quadrant III, determine sin(θ).

$$\left(-\frac{12}{13}\right)^2 + \sin^2(\theta) = 1$$

$$\frac{144}{169} + \sin^2(\theta) = 1$$

$$\sin^2(\theta) = \frac{25}{169}$$

$$\sin(\theta) = \pm\frac{5}{13}$$

$\sin(\theta) = -\frac{5}{13}$, because the solution is in Quadrant III.

5. Given cos(θ) $= \frac{1}{4}$ in Quadrant IV, determine sin(θ).

$$\left(\frac{1}{4}\right)^2 + \sin^2(\theta) = 1$$

$$\frac{1}{16} + \sin^2(\theta) = 1$$

$$\sin^2(\theta) = \frac{15}{16}$$

$$\sin^2(\theta) = \pm\frac{\sqrt{15}}{4}$$

$\sin(\theta) = -\frac{\sqrt{15}}{4}$, because the solution is in Quadrant IV.

6. Given $\sin(\theta) = -\frac{1}{10}$ in Quadrant IV, determine $\cos(\theta)$.

$$\left(-\frac{1}{10}\right)^2 + \cos^2(\theta) = 1$$

$$\frac{1}{100} + \cos^2(\theta) = 1$$

$$\cos^2(\theta) = \frac{99}{100}$$

$$\cos(\theta) = \pm\frac{3\sqrt{11}}{10}$$

$\cos(\theta) = \frac{3\sqrt{11}}{10}$, because the solution is in Quadrant IV.

Be prepared to share your methods and solutions.

Check for Students' Understanding

1. Solve $2\sin(x) + \sqrt{3} = 0$ over the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

$$2\sin(x) = -\sqrt{3}$$
$$\sin(x) = -\frac{\sqrt{3}}{2}$$
$$x = -\frac{\pi}{3}$$

2. Solve $\cos^2(x) + 2\cos(x) - 5 = -2$ over the domain of all real numbers.

$$\cos^2(x) + 2\cos(x) - 5 = -2$$

Let $z = \cos(x)$.

$$z^2 + 2z - 5 = -2$$
$$z^2 + 2z - 3 = 0$$
$$(z + 3)(z - 1) = 0$$
$$z = -3 \quad \text{or} \quad z = 1$$
$$\cancel{\cos(x) = -3} \quad \text{or} \quad \cos(x) = 1$$
$$x = 0 + 2\pi n$$

Rabbits and Seasonal Affective Disorder

16.2

Modeling with Periodic Functions

LEARNING GOALS

In this lesson, you will:

- Model real-world situations with periodic functions.
- Interpret key characteristics of periodic functions in terms of problem situations.

ESSENTIAL IDEAS

- Periodic functions are used to model real-world problems.
- The key characteristics of periodic functions, including period, amplitude, midline, and phase shift, are used to model components of real-world situations.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-TF Trigonometric Functions

Model periodic phenomena with trigonometric functions

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

Overview

A periodic function is used to model the seasonal fluctuation in the rabbit population. Students are given the model and equation and use them to construct a graph and answer questions related to the situation. In the second situation, students model the seasonal amount of daylight in various locations. They are given a table of data and use the values to graph the situation. The graph is then used to determine a model equation and that model is compared to a second model equation derived by the use of the graphing calculator. Students use the regression equation to answer related questions.

Warm Up

1. The first problem in this lesson involves a rabbit population. Why do you suppose a rabbit population can be modeled by a periodic function?

   Answers will vary.

   A rabbit population can be modeled by a periodic function because it grows and shrinks due to breeding and then dying off and predator behavior.

2. What time of the year do you suppose a rabbit population is at its maximum?

   Answers will vary.

   The rabbit population is at its maximum in the warmest months because there is more food available.

3. The second problem in this lesson involves seasonal affective disorder (SAD). SAD is a form of depression related to not having enough daylight for long periods of time. Why do you suppose SAD can be modeled by a periodic function?

   Answers will vary.

   SAD is related to the length of daylight hours, and the number of daylight hours is a periodic function.

4. What geographical location do you suppose can be associated with a high number of people with symptoms related to SAD?

   Answers will vary.

   SAD is more common in areas farther from the equator, such as Alaska.

16.2

Rabbits and Seasonal Affective Disorder

Modeling with Periodic Functions

LEARNING GOALS

In this lesson, you will:

- Model real-world situations with periodic functions.
- Interpret key characteristics of periodic functions in terms of problem situations.

Seasonal changes can affect a person's mood. You may have heard someone say that they have "the winter blues." Some people are so strongly affected by a lack of daylight that they experience a severe depression when the hours of daylight are shortest. This condition is called seasonal affective disorder (SAD).

A person with SAD experiences depression for several months when daylight hours are short. In general, the occurrence of SAD increases as the location gets farther from the equator. Symptoms of SAD include disruption of sleep patterns, feeling fatigued, overeating, craving carbohydrates, depression, and avoidance of social activity.

Problem

A trigonometric function modeling the rabbit population is given. Students complete a table of values using the model and sketch a graph of the function. They identify the translations, amplitude, period, and phase shift related to the function and determine when the population will reach 12,000 rabbits.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 7

- Is December considered Month 0 or Month 1? Why?
- If the value of x is 0 for December, will the result of the function equation with respect to the rabbit population be 6000?
- If January is Month 1, how was the rabbit population determined?
- If the value of x is 6 for June, will the result of the function equation with respect to the rabbit population be 16,000?
- If the value of x is 12 for December, will the result of the function equation with respect to the rabbit population be 6000?
- Has the basic sine function been translated up or down 11,000 units?
- How is the amplitude, period, and phase shift of the function determined?
- What is the average rabbit population over a year?
- Which key characteristic of the graph is associated with the average rabbit population over a year?

PROBLEM 1 Rabbits, Rabbits Everywhere!

The rabbit population in a national park rises and falls throughout the year. The population is at its approximate minimum of 6000 rabbits in December. As the weather gets warmer and food becomes more available, the population grows to its approximate maximum of 16,000 rabbits in June.

The function describing the rabbit population is

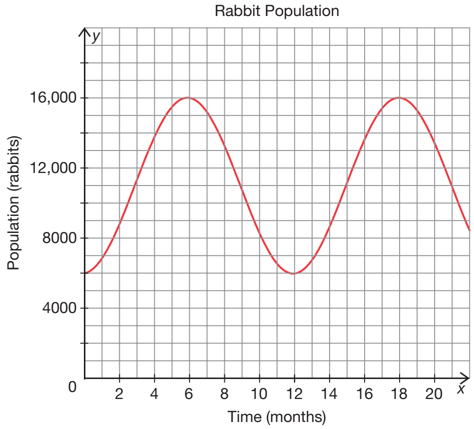
$f(x) = 5000 \sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) + 11{,}000$

where x is the time in months and $f(x)$ is the rabbit population.

1. Complete the table to show the rabbit population through one year.

Month	Time (months)	Rabbit Population (rabbits)
December	0	6000
January	1	6670
February	2	8500
March	3	11,000
April	4	13,500
May	5	15,330
June	6	16,000
July	7	15,330
August	8	13,500
September	9	11,000
October	10	8500
November	11	6670
December	12	6000

2. Sketch a graph of the function to show at least two periods of the function.



3. How has the function been translated vertically from the basic sine function?

The function has been translated up 11,000 units.

4. Determine the amplitude of the function.

The amplitude is 5000.

5. Determine the period of the function.

The period is 12 months.

6. Determine the phase shift of the function.

The phase shift is 3 months.

$5000 \sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) + 11{,}000 = 5000 \sin\frac{\pi}{6}(x - 3) + 11{,}000$

7. How is the vertical translation related to the algebraic function? What does it represent in terms of this problem situation?

The vertical translation is the number added to the function. It represents the average rabbit population over a year.

Grouping

Have students complete Questions 8 through 13 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 8 through 10

- Which key characteristic of the graph of the function is associated with the coefficient of the function?
- Which key characteristic of the graph of the function is associated with how far the rabbit population varies from the average over a year?
- Which key characteristic of the graph of the function is associated with the time it takes for one cycle of change in the rabbit population?
- What information in the function equation helps to identify the phase shift?
- Which key characteristic of the graph of the function is associated with the time it takes for the population to reach its average from the beginning of the year?

8. How is the amplitude related to the algebraic function? What does it represent in terms of this problem situation?

The amplitude is the coefficient on the function. It represents how far the population varies from the average over a year.

9. How is the period related to the algebraic function? What does it represent in terms of this problem situation?

The coefficient of the independent variable is 2π divided by the period. It represents the time for one cycle of change in the rabbit population.

10. How is the phase shift related to the algebraic function? What does it represent in terms of this problem situation?

The phase shift is the number subtracted from the independent variable. It represents the time it takes for the population to reach its average from the beginning of the year.

Guiding Questions for Share Phase, Questions 11 through 13

- If the rabbit population cycle occurred over six months, would the period of the function be halved?
- If the rabbit population cycle occurred over six months, would the graph show twice as many repetitions of the function over the same domain?
- If the rabbit population cycle occurred over six months, how many times would the graph repeat over the same domain?
- If the rabbit population cycle occurred over six months, how would if affect the coefficient of the independent variable?
- If the rabbit population cycle occurred over six months, would the coefficient of the independent variable double?
- If the minimum and maximum changed, would the amplitude and vertical translation change?
- What would the coefficient of the function or amplitude become?
- What would the value added to the function or vertical translation become?
- How much would the average increase with these new values?

11. If the rabbit population cycle occurred over six months instead of one year, how would the graph and equation change?

The period of the function would be halved. The graph would show twice as many repetitions of the function over the same domain. The coefficient of the independent variable would double.

12. If the rabbit population has a minimum of 4000 and a maximum of 20,000, how would the graph and equation change?

The amplitude and vertical translation would change. The amplitude (coefficient of the function) would become 8000, and the vertical translation (value added to the function) would become 12,000. The graph would have different maxima and minima, and the average would be increased by 1000 compared to the original rabbit population graph.

13. Describe the time(s) in months when the rabbit population is equal to 12,000. Show your work.

Each year, during the month of March and during the month of August, the rabbit population is equal to 12,000.

$$5000\sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) + 11{,}000 = 12{,}000$$

$$5000\sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) = 1000$$

$$\sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) = \frac{1}{5}$$

$$\sin^{-1}\left(\frac{1}{5}\right) \approx 0.20136 \quad \text{or} \quad \sin^{-1}\left(\frac{1}{5}\right) \approx \pi - 0.20136$$

$$\frac{\pi}{6}x - \frac{\pi}{2} \approx 0.20136 \quad \text{or} \quad \frac{\pi}{6}x - \frac{\pi}{2} \approx 2.94023$$

$$\frac{\pi}{6}x \approx 1.772156 \qquad \frac{\pi}{6}x \approx 4.51103$$

$$x \approx 3.38 + 12n \quad \text{or} \quad x \approx 8.62 + 12n$$

Problem 2

A table of values listing the number of approximate daylight hours on various days of the year in Chicago, Illinois, is given. Students use the data to sketch a graph of the function. They identify the minimum value, maximum value, amplitude, period, phase shift, and vertical shift related to the function described by the graph. The sine function is used to model this situation in the form of $f(x) = A \sin B(x - C) + D$. Students identify the A-, B-, C-, and D-values before writing the trigonometric function modeling the situation. Students enter the data into the calculator and perform a sinusoidal regression to determine a regression equation modeling the data. The regression equation is then compared to the first trigonometric equation written in the previous question and used to compute the times of the year when there are 12 hours of daylight.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

PROBLEM 2 Seasonal Affective Disorder

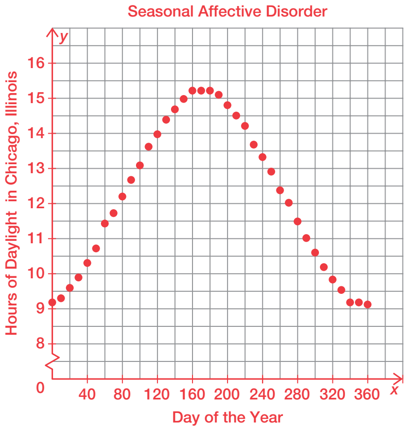
Patterns of daylight are related to seasonal affective disorder (SAD). The amount of daylight varies in a periodic manner and can be modeled by a sine function. The table shows the number of approximate daylight hours in Chicago, Illinois, which has latitude of 42° N.

Date	Day	Daylight Hours
Dec. 31	0	9.2
Jan. 10	10	9.3
Jan. 20	20	9.6
Jan. 30	30	9.9
Feb. 9	40	10.3
Feb. 19	50	10.7
Mar. 1	60	11.4
Mar. 11	70	11.7
Mar. 21	80	12.2
Mar. 31	90	12.7
Apr. 10	100	13.1
Apr. 20	110	13.6
Apr. 30	120	14.0
May 10	130	14.4
May 20	140	14.7
May 30	150	15.0
June 9	160	15.2
June 19	170	15.2
June 29	180	15.2
July 9	190	15.1
July 19	200	14.8
July 29	210	14.5
Aug. 8	220	14.2
Aug. 18	230	13.7
Aug. 28	240	13.3
Sept. 7	250	12.9
Sept. 17	260	12.4
Sept. 27	270	12.0
Oct. 7	280	11.5
Oct. 17	290	11.0
Oct. 27	300	10.6
Nov. 6	310	10.2
Nov. 16	320	9.8
Nov. 26	330	9.5
Dec. 6	340	9.2
Dec. 16	350	9.2
Dec. 26	360	9.1

Guiding Questions for Share Phase, Questions 1 through 6

- What does the minimum and maximum on the graph represent with respect to the problem situation?
- What does the amplitude represent with respect to the problem situation?
- How is the amplitude of the function determined?
- What does the period represent with respect to the problem situation?
- How is the period of the function determined?
- What does the phase shift represent with respect to the problem situation?
- How is the phase shift of the function determined?
- What does the vertical shift represent with respect to the problem situation?
- How is the vertical shift of the function determined?

1. Plot the points from the table using the day of the year for your independent variable and hours of daylight for your dependent variable.



2. Describe any minimum and maximum values on your graph.

The minimum value is 9.1 hours.

The maximum value is 15.2 hours.

3. Determine the amplitude of the function.

The amplitude of the function is 3.05 hours.

4. Determine the period of the function.

The period of the function is 360 days (as shown on the graph).

Note: Students may answer 365 days (one year).

5. Determine the phase shift of the graph.

The phase shift is $-\frac{\pi}{2}$ units.

6. Determine the vertical shift of the graph.

The vertical shift is 12.15 units.

Grouping

Have students complete Questions 7 through 12 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 7 through 10

- Which key characteristic of the graph of the function is associated with the number of hours the function varies from its average value?
- Which constant in the transformational form of the function is associated with the amplitude?
- Which key characteristic of the graph of the function is associated with the total number of days?
- How do you determine the value of *B*?
- How do you determine the value of *C*?
- Which constant in the transformational form of the function is associated with the average number of hours of daylight over a year?
- How do you determine the value of *D*?
- How does the amplitude in the regression equation compare to the amplitude in the algebraic function you used to model the situation?
- How does the vertical shift in the regression equation compare to the amplitude in the algebraic function you used to model the situation?

7. To model this situation with a sine function of the form $f(x) = A\sin B(x - C) + D$, you need to calculate the values of *A*, *B*, *C*, and *D*.

a. Determine the value of *A*. What does it represent in terms of this situation?

The value of *A* is 3.05. It represents the amplitude of the function or the number of hours the function varies (at most) from its average value.

b. Determine the value of *B*. Explain your reasoning.

$B = \frac{2\pi}{360} \approx 0.017$

There are 360 days represented on the graph (and in the table), so I used that value for the period. The value of *B* is 2π (the period of the basic function) divided by 360 (the period of the new function).

c. Determine the value of *C*.

The value of *C* is $-\frac{\pi}{2}$.

d. Determine the value of *D*. What does it represent in terms of this situation?

The value of *D* is 12.15. It represents the average number of hours of daylight over a year.

e. Write an algebraic function to model the data for the amount of daylight hours in Chicago, Illinois.

$f(x) = 3.05\sin\left(0.017x - \frac{\pi}{2}\right) + 12.15$

- How does the phase shift in the regression equation compare to the amplitude in the algebraic function you use to model the situation?
- What equation is used to determine the times of the year when there was exactly 12 hours of daylight?
- Which inverse function is used to solve this function equation?
- How is the distance of the location from the equator related to the amount of daylight over the winter?

8. Enter the data from the table into a calculator. Use the calculator to perform a sinusoidal regression for this data. Write the regression equation from the calculator. How does it compare to your equation?

$f(x) = 3.019 \sin(0.0167x - 1.3100) + 12.1237$

The values are very close to the values I used.

9. Use your function to describe times of the year when there are exactly 12 hours of daylight. Show your work.

Each year, during the 96th day of the year and during the 275th day of the year, there are exactly 12 hours of daylight.

$$3.05 \sin\left(0.017x - \frac{\pi}{2}\right) + 12.15 = 12$$

$$3.05 \sin\left(0.017x - \frac{\pi}{2}\right) = 0.15$$

$$\sin\left(0.017x - \frac{\pi}{2}\right) \approx 0.04918$$

$$\sin^{-1}(0.04918) \approx 0.0491998 \quad \text{or} \quad \sin^{-1}(0.04918) \approx \pi - 0.0491998$$

$$0.017x - \frac{\pi}{2} \approx 0.0491998 \quad \text{or} \quad 0.017x - \frac{\pi}{2} \approx 3.0923928$$

$$0.017x \approx 1.619996 \qquad 0.017x \approx 4.663189$$

$$x \approx 95.29 + 360n \quad \text{or} \quad x \approx 274.31 + 360n$$

10. Seasonal affective disorder appears to vary according to latitude. The farther a location is from the equator, the more prevalent cases of SAD become. Why might this happen?

The farther a location is from the equator, the less daylight in the location over the winter. The extremes in daylight are greater.

Guiding Questions for Share Phase, Questions 11 and 12

- Would the graph modeling the daylight hours in Anchorage be sinusoidal? Why?
- Would the graph modeling the daylight hours in Anchorage have the same period? Why?
- Would the graph modeling the daylight hours in Anchorage have the same phase shift? Why?
- Would the graph modeling the daylight hours in Anchorage have the same vertical shift? Why?
- Would the graph modeling the daylight hours in Anchorage have the same amplitude? Why not?
- Are the seasons reversed in the southern hemisphere?
- Does winter occur in the northern hemisphere when summer occurs in the southern hemisphere?

11. Anchorage, Alaska, is located at a latitude of 61° N. This is considerably farther north than Chicago. If we created a graph to model the daylight hours in Anchorage, how do you think it would compare to the graph for daylight hours in Chicago? In what ways would it be the same? In what ways would it be different?

The graph would still be sinusoidal and have the same period, phase shift, and vertical shift. The amplitude would be greater.

12. In locations like Chicago and Anchorage, SAD is most likely to occur around the month of January. In locations in the southern hemisphere, like Santiago, Chile (latitude 33.5° S), SAD occurs around the month of July. Why does this happen?

In the southern hemisphere, the seasons are reversed. For instance, winter occurs in the northern hemisphere when summer occurs in the southern hemisphere.

Be prepared to share your methods and solutions.

Check for Students' Understanding

The data in the table describes a function that is in transformational form: $f(x) = A \sin(Bx + C) + D$.

Write a transformed trigonometric equation that is described by these data.

x	y
0	200
1	170
2	110
3	80
4	110
5	170
6	200

The transformed trigonometric equation is $f(x) = 60 \sin(60x + 90) + 140$.

Students may or may not enter the data into the graphing calculator and perform a regression to determine the transformational form of the equation.

$f(x) = A \sin(Bx + C) + D$

The A- and D-value affect the output value after the sine of the function is evaluated.

$A: \frac{200 - 80}{2} = 60$

$D: 80 \text{ (min)} + 60 = 140 \text{ or } 200 \text{ (max)} - 60 = 140$

The B- and C-value affect the input value.

$B: \frac{2\pi}{6}$ **(number of increments, 0–5, until the pattern repeats)** $= \frac{\pi}{3}$**, or 60°**

C: The max of the sine function is at 90°. The max of the transformed function is at 0, that is, 90° before the initial max. The argument for the input should include +90°. You could also use the intercepts or min to identify the phase shift.

$f(x) = A \sin(Bx + C) + D$

$f(x) = 60 \sin(60x + 90) + 140$

16.3

Behind the Wheel

Modeling Motion with a Trigonometric Function

LEARNING GOALS

In this lesson, you will:

- Interpret characteristics of a graph of a trigonometric function in terms of a problem situation.
- Construct a trigonometric function to model a problem situation.

ESSENTIAL IDEAS

- Periodic functions are used to model real-world problems.
- Transformations of periodic functions can be used to map function behavior to the behavior of periodic phenomena.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-TF Trigonometric Functions

Model periodic phenomena with trigonometric functions

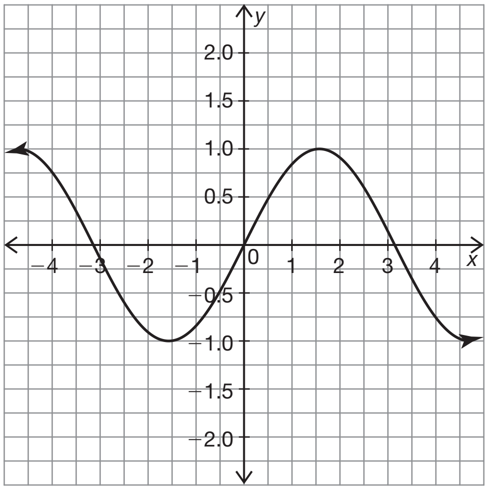
5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

Overview

A periodic function is used to model the height from the street of a point on the circumference of a wheel as a function of time. Students build a trigonometric function by first using the $y = \sin(x)$ function to represent a situation, then students use the $y = \cos(x)$ function to model the height from the street of a point on the circumference of a wheel as a function of time.

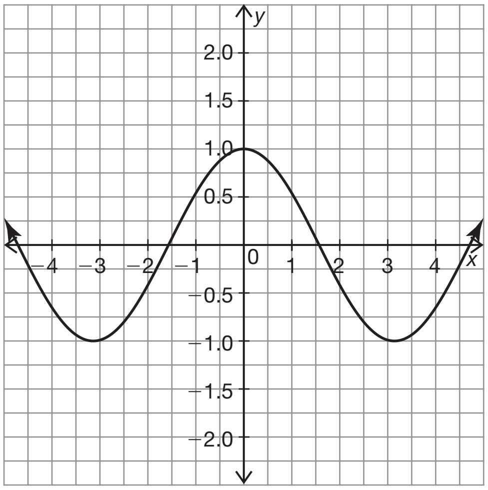
Warm Up

1. Identify the function graphed.



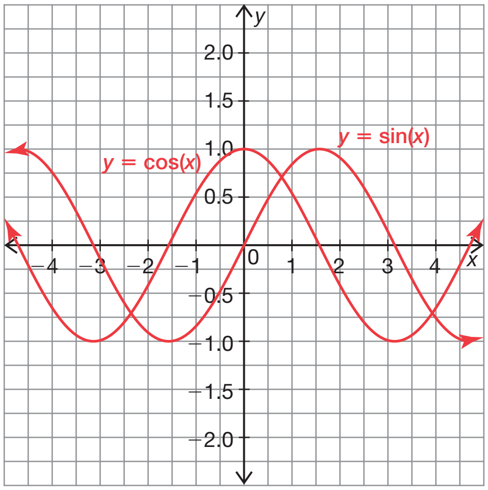
$y = \sin(x)$

2. Identify the function graphed.

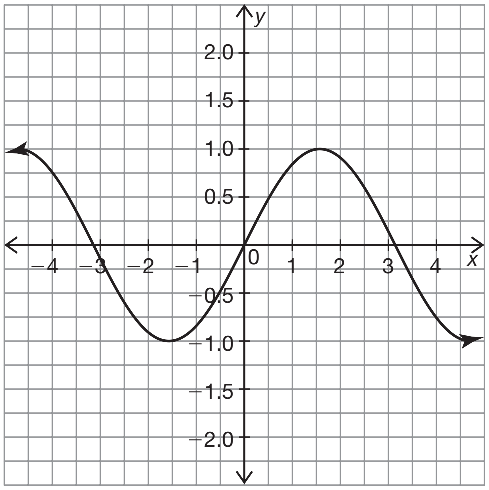


$y = \cos(x)$

3. Graph both functions in Questions 1 and 2 on the same coordinate plane.



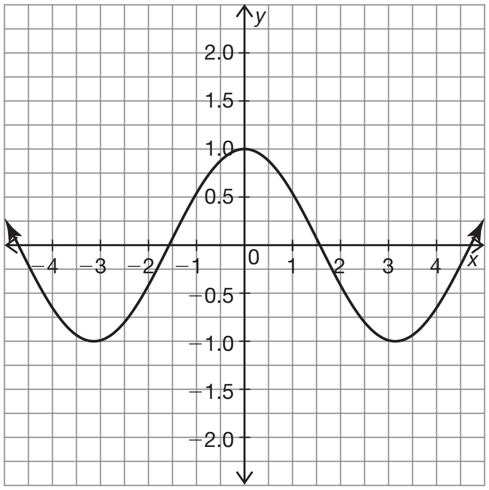
4. What transformed cosine function can be used to describe this graph? (Rewrite the sine function as a cosine function.)



Answers will vary.

$y = -\cos\left(x + \frac{\pi}{2}\right)$

5. What transformed sine function can be used to describe this graph? (Rewrite the cosine function as a sine function.)



Answers will vary.

$y = -\sin\left(x - \frac{\pi}{2}\right)$

16.3

Behind the Wheel

Modeling Motion with a Trigonometric Function

LEARNING GOALS

In this lesson, you will:

- Interpret characteristics of a graph of a trigonometric function in terms of a problem situation.
- Construct a trigonometric function to model a problem situation.

The invention of the wheel is something that can sometimes come up in everyday conversation. “Reinventing the wheel,” for example, is a phrase used to refer to coming up with a solution that already exists.

The phrase is likely used because the wheel is one of humankind’s oldest “inventions.” Evidence for vehicles that used wheels dates back to 3000 BCE.

In fact, the oldest wooden wheel, found in Slovenia, was determined to be over 5000 years old.

Problem 1

To model the height of a point on the circumference of a wheel moving clockwise, students first construct a model in which point *P* is located where a terminal ray in standard position intersects the circle at 0 radians. The point is moving counterclockwise and the wheel (circle) rotates in place. To model the height of the point from the street level, students use $y = \sin(x)$ to determine the *A*-, *B*-, *C*-, and *D*-values using 4 additional models of the wheel, and write the function in terms of radians and meters. Then the function is rewritten in terms of time in seconds per meter rather than radians per meter.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete all parts of Question 1 with a partner. Then have students share their responses as a class.

PROBLEM 1 To Everything . . . Turn, Turn, Turn

Suppose a wheel with a radius of 0.2 meter rolls clockwise on a street at a rate of 2.4 m/s.

You can build a trigonometric function to model the height, *h*, from the street of a point, *P*, on the wheel as a function of time, *t*, in seconds. As the wheel rolls, the position of point *P* will move along the circle.

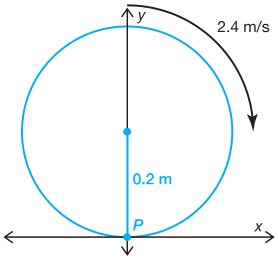


Figure 1

In order to build this trigonometric function, let's first think about point *P* from Figure 1 in standard position as point *P*′ in Figure 2 moving counterclockwise. In essence, think about the basic sine function because you are trying to model the vertical distance of point *P*′.

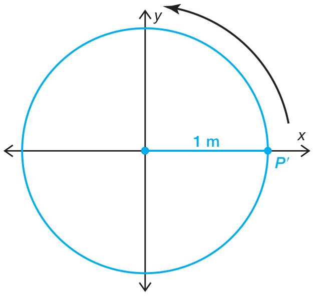


Figure 2

- Point *P*′ is located where a terminal ray in standard position intersects the circle at 0 radians.
- The point is moving counterclockwise instead of clockwise.
- The wheel is rotating in place.
- The *x*-axis represents the ground.
- You will use the sine function, $h(\theta) = \sin(\theta)$, to model the height of the point for each angle measure, θ, in radians.

Guiding Questions for Share Phase, Question 1

- How is the radius of the wheel related to the amplitude of the sine function?
- How is the amplitude determined?
- Does raising the wheel to rest on the ground affect the vertical shift of the function?
- If the location of the starting point is at 0 radians in standard position, is there a phase shift?
- How is the *B*-value determined?
- How do you represent a reflection across the *y*-axis?
- What identity can you use to get the negative sign outside of the function?

Let's consider each piece of information in the original problem situation and how you can use transformations to build an equation to model Figure 1.

1. Use the given information to sketch the next figure and write the corresponding equation. Describe the transformation.

a. Consider Figure 2 but the radius is 0.2 meter. Label point *P*′ on your graph.

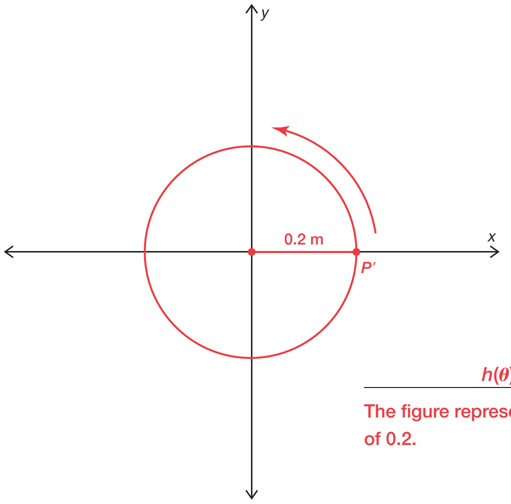


Figure 3

$h(\theta) = 0.2 \sin(\theta)$

The figure represents a dilation by a factor of 0.2.

b. Consider Figure 3 but the wheel rests on the ground. Label point *P*′ on your graph.

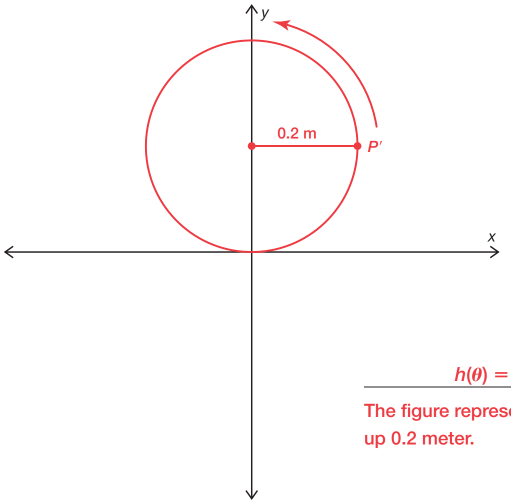
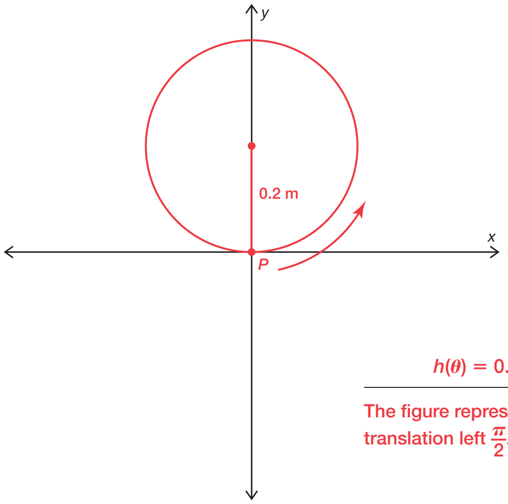


Figure 4

$h(\theta) = 0.2 \sin(\theta) + 0.2$

The figure represents a vertical translation up 0.2 meter.

c. Consider Figure 4 but translate point *P*′ to the original starting position, point *P*, in Figure 1. Label point *P* on your graph.

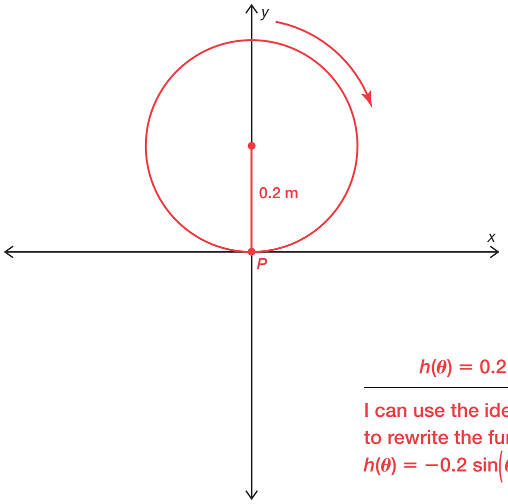


$h(\theta) = 0.2 \sin\left(\theta + \frac{\pi}{2}\right) + 0.2$

The figure represents a horizontal translation left $\frac{\pi}{2}$.

Figure 5

d. Consider Figure 5 but the wheel turns clockwise. Label point *P* on your graph.



$h(\theta) = 0.2 \sin\left(-\left(\theta + \frac{\pi}{2}\right)\right) + 0.2$

I can use the identity $\sin(-x) = -\sin(x)$ to rewrite the function as $h(\theta) = -0.2 \sin\left(\theta + \frac{\pi}{2}\right) + 0.2$.

The figure represents a reflection across the *y*-axis.

Figure 6

Grouping

Have students complete Question 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- For the Distance Formula $d = rt$, what values are known in this situation?
- How can you use the relationship between radian measure, circumference, and arc length to determine the distance?
- The ratio of the radian measure to 2π is equal to what ratio involving distance, d?
- How can you check to see if your equation is correct?

You have just written an equation that models the height of point *P* on the wheel with a radius of 0.2 meter in terms of θ.

Now let's consider the relationship between time and θ to write an equation for the height of point *P* on the wheel in terms of time.

2. Write an equation for the height of point *P* on the wheel in terms of time *t*.

a. Determine the relationship between time, *t*, and θ.

The relationship between time, *t*, and θ, is $12t = \theta$.

I know distance equals rate times time, so I can write distance in terms of time.

$d = rt$

$d = (2.4)(t)$

To write distance in terms of θ, I can set up a proportion.

The ratio of the distance to the arc length of the entire wheel (or circumference) is equal to the ratio of θ to the radian measure of the entire wheel (or 2π).

$$\frac{d}{2\pi r} = \frac{\theta}{2\pi}$$

$$\frac{d}{2\pi(0.2)} = \frac{\theta}{2\pi}$$

$$\frac{d}{0.4\pi} = \frac{\theta}{2\pi}$$

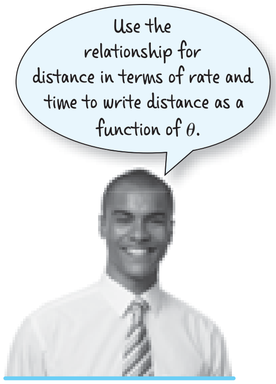
$$d = \frac{0.4\pi\theta}{2\pi}$$

$$d = 0.2\theta$$

Using substitution, I can write the relationship between time, *t*, and θ.

$(2.4)(t) = 0.2\theta$

$12t = \theta$



b. Write the final equation in terms of time *t*.

$h(t) = -0.2 \sin\left(12t + \frac{\pi}{2}\right) + 0.2$

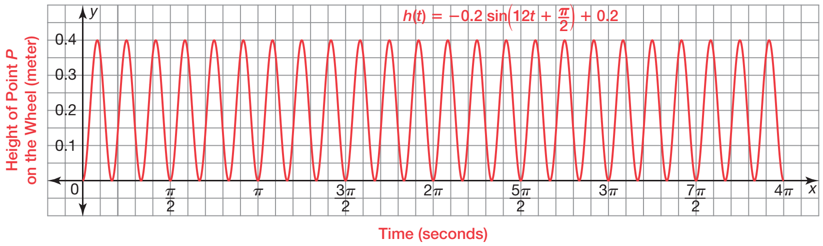
Grouping

Have students complete Questions 3 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 through 7

- Can you represent time values in terms of π on your graph?
- What does the y-axis of your graph measure?
- What equation is used to determine the height of the point at 1 second?
- How is the equation for a sine function related to the equation for a cosine function?
- Can you simplify the number of terms used by rewriting the sine function as a cosine function?
- What equation is used to determine when the point is at a height of 0.2 meter?
- Are there multiple solutions?

3. Sketch a graph of your function in Question 2. Label the axes.



4. Determine the height of the point at 1 second.

The height of the point at 1 second is approximately 0.031 meter.

$$h(t) = -0.2 \sin\left(12t + \frac{\pi}{2}\right) + 0.2$$

$$h(1) = -0.2 \sin\left(12 \cdot 1 + \frac{\pi}{2}\right) + 0.2$$

$$= -0.2 \sin\left(12 + \frac{\pi}{2}\right) + 0.2$$

$$\approx -0.2 \sin(13.570796) + 0.2$$

$$\approx -0.2(0.843854) + 0.2$$

$$h(1) \approx 0.0312$$

5. Rewrite your function as a cosine function. Explain your reasoning.

$h(t) = -0.2 \cos(12t) + 0.2$

I can use the identity $\sin(-x) = -\sin(x)$ to rewrite $h(t) = 0.2 \sin\left(-\left(12t + \frac{\pi}{2}\right)\right) + 0.2$ as $h(t) = -0.2 \sin\left(12t + \frac{\pi}{2}\right) + 0.2$.

Then, I can use the identity $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$ to rewrite the function as $h(t) = -0.2 \cos(12t) + 0.2$.

6. What are the advantages of rewriting your function as a cosine function?

Answers will vary.

By rewriting the equation as a cosine function, the equation is in simpler form because it has less transformations.

Also, the equation requires less computations when used to calculate a value.

7. At what time(s) is the height of the point at 0.2 meter?

At $t = -\frac{\pi}{24} + \frac{\pi n}{6}$ or $t = \frac{\pi}{24} + \frac{\pi n}{6}$ seconds, the point is at a height of 0.2 meter.

$$h(t) = -0.2\sin\left(12t + \frac{\pi}{2}\right) + 0.2$$
$$0.2 = -0.2\sin\left(12t + \frac{\pi}{2}\right) + 0.2$$
$$0 = -0.2\sin\left(12t + \frac{\pi}{2}\right)$$
$$0 = \sin\left(12t + \frac{\pi}{2}\right)$$
$$\sin(0) = 0 \quad \text{or} \quad \sin(\pi) = 0$$
$$12t + \frac{\pi}{2} = 0 \quad \text{or} \quad 12t + \frac{\pi}{2} = \pi$$
$$12t = -\frac{\pi}{2} \qquad 12t = \frac{\pi}{2}$$
$$t = -\frac{\pi}{24} \quad \text{or} \quad t = \frac{\pi}{24}$$

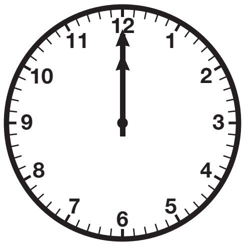
I know the period of the function, $\frac{2\pi}{|B|}$, is $\frac{2\pi}{|-12|}$ or $\frac{\pi}{6}$.

So, the point is at a height of 0.2 meter at $t = -\frac{\pi}{24} + \frac{\pi n}{6}$ or $t = \frac{\pi}{24} + \frac{\pi n}{6}$ seconds.

Be prepared to share your methods and solutions.

Check for Students' Understanding

Consider the second hand on the face of a clock. The length of the second hand and radius of the clock face are each 30 centimeters. Suppose the second hand begins its movement at exactly 12:00 midnight.



1. Complete the table describing the time in seconds and the distance in centimeters the tip of the second hand travels from where it begins. For each complete revolution, suppose that the distance resets to 0.

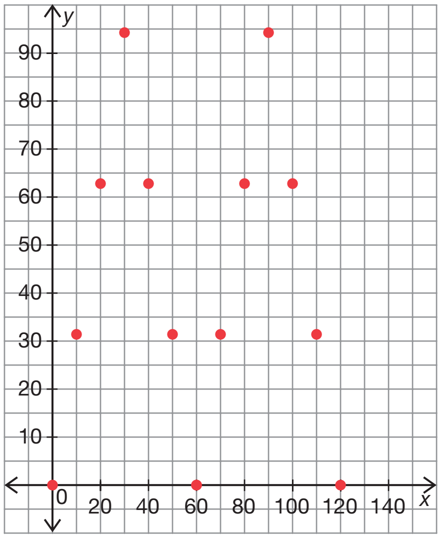
Time (seconds)	Distance (centimeters)
0	0
10	$10\pi \approx 31.4$
20	$20\pi \approx 62.8$
30	$30\pi \approx 94.25$
40	$20\pi \approx 62.8$
50	$10\pi \approx 31.4$
60	0
70	$10\pi \approx 31.4$
80	$20\pi \approx 62.8$
90	$30\pi \approx 94.25$
100	$20\pi \approx 62.8$
110	$10\pi \approx 31.4$
120	0

$C = 2\pi r$

$C = (2)(\pi)(30)$

$C = 60\pi \approx 188.5$

2. Use the table of values to create a graph.



3. Describe the domain and range of the graph.

The domain is all positive real numbers from 0 to 120.

The range is all positive real numbers from 0 to 30π.

4. Time stops for no one! Can this situation be described as periodic? Explain your reasoning.

The situation is periodic because as the number of seconds increases to infinity, the distance from the tip of the second hand to the initial location continues to increase and decrease, fluctuating between the values of 0 and 30π.

16.4

Springs Eternal

The Damping Function

LEARNING GOALS

In this lesson, you will:

- Choose a trigonometric function to model a periodic phenomenon.
- Determine the graphical attributes (amplitude, midline, frequency) of a periodic function from a description of a problem situation.
- Build a function that is a combination of a trigonometric function and an exponential function.

KEY TERM

- damping function

ESSENTIAL IDEAS

- Periodic functions are used to model real-world problems.
- A damping function is a function used to multiply a periodic function to decrease the amplitude over a time.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-TF Trigonometric Functions

Model periodic phenomena with trigonometric functions

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

Overview

A damping function is used to model the height of an object suspended on a spring bouncing up and down. Students build a trigonometric function by first using the $y = \cos(x)$ function to represent a situation where the object connected to the string bounces up and down the same amount forever. Then the situation is altered to be more realistic, and students develop the equation $y = -5(0.9)t \cdot \cos(2\pi t) + 16$ to model the height of the object on the spring over time.

Warm Up

A trigonometric function represents the changes in high and low tides at particular locations. The gravitational force of both the moon and sun affect the height of the tide. On a particular day at a specific location, high and low tides can be modeled by the function $H(t) = 5.3 \sin\left(\frac{\pi}{6}(t + 2)\right) + 4.2$, where $H(t)$ represents the height of the tide and t represents the number of hours since midnight.

1. What is the basic function?

   The basic function is $y = \sin(x)$.

2. Interpret the A-, B-, C-, and D-values in the function.

   $A = 5.3$, the amplitude

   $B = \frac{\pi}{6}$, the vertical stretch/compression factor, $\frac{2\pi}{\frac{\pi}{6}}$ is the period

   $C = 2$, the phase shift

   $D = 4.2$, the vertical shift

16

16.4

Springs Eternal

The Damping Function

16

LEARNING GOALS

In this lesson, you will:

- Choose a trigonometric function to model a periodic phenomenon.
- Determine the graphical attributes (amplitude, midline, frequency) of a periodic function from a description of a problem situation.
- Build a function that is a combination of a trigonometric function and an exponential function.

KEY TERM

- damping function

In this lesson, you will learn about a function that you probably use every day—to turn the volume of your music up or down. The volume dial alters the sound wave function by increasing or decreasing its amplitude.

Mathematical functions are often used to digitally modify sound waves, which can be encoded as periodic functions to be 'read' later by a sound-producing device. In fact, many digital sound effects—like the fade, which you will learn about in this lesson—are produced by mathematically adjusting sound wave functions.

Problem 1

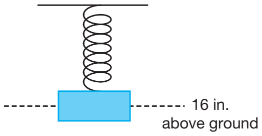
An object is suspended from a spring and is pulled below its resting position and is released. Information is given about the distances the object repeatedly bounces up and down. Students identify the independent and dependent quantities, the equation of the midline of the graph, the minimum and maximum values, and the amplitude of the periodic function modeling the situation. They sketch a graph of the function, write the values of the constants *A*, *B*, *C*, and *D*, interpret them with respect to the problem situation, determine the period of the function, and finally write an equation for the function. Students use the equation to answer questions related to the problem situation.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

PROBLEM 1 Bouncing Back

An object suspended from a spring is pulled 5 inches below its resting position and released, causing the object to bounce up and down once every second. At rest, the object's height above the ground is 16 inches.

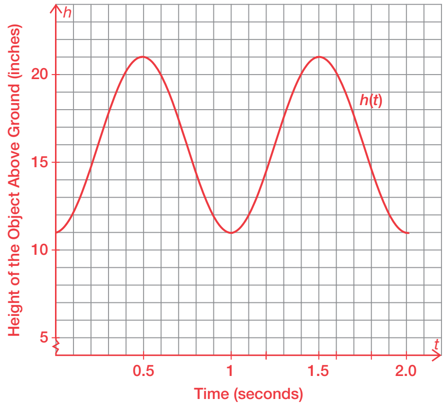


Suppose that the object bounces up 5 inches above its resting height and then back down to 5 inches below its resting height without stopping on every bounce. Let's build a periodic function to model the bouncing of the object on the spring over time.

1. Determine the independent and dependent quantities for this situation.

The independent quantity is time in seconds, and the dependent quantity is the height of the object above the ground in inches.

2. Sketch and label the graph of the function modeling the bouncing object over time *h*(*t*), given what you know about the height of the object. Represent at least two bounces of the object on the graph.



Guiding Questions for Share Phase, Questions 1 and 2

- Is the time in seconds the independent quantity or the dependent quantity?
- Which quantity is represented on the *x*-axis?
- Is the height of the object above the ground in inches the independent quantity or the dependent quantity?
- Which quantity is represented on the *y*-axis?
- What is the object's height above the ground when it is at rest?

Guiding Questions for Share Phase, Questions 3 through 5

- If the object bounces 5 inches up and 5 inches down from its at-rest position, what height is in the exact middle?
- Is the maximum considered 5 or 16 + 5? Why?
- Is the minimum considered −5 or 16 − 5? Why?
- Is the amplitude 5 or 10? Why?
- Where is the *y*-intercept on your graph?
- Where are the *x*-intercepts on your graph?
- Is the *y*-intercept also the minimum on the graph of the function? Why?
- Does the equation $y = -\cos(x)$ describe the graph?
- Is the amplitude of the periodic function 5 or −5? Why?
- Is the graph the graph of a cosine function?
- Is the graph of the cosine function reflected across the *x*-axis?
- Is the graph of the cosine function reflected across the *x*-axis and shifted horizontally?
- If there is no phase shift, what is the *C*-value?
- If the midline of the function is at $y = 16$, how many units is the graph vertically shifted?

3. Use your graph to determine each characteristic of the periodic function that will model this situation. Explain your reasoning.

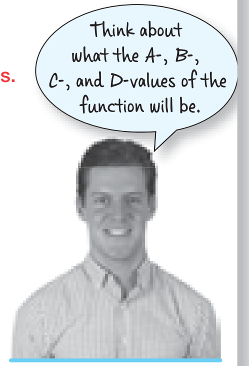
a. Determine the equation of the midline of the graph.

Since the object bounces 5 inches above and then 5 inches below 16 inches on each bounce, the midline is $y = 16$.

b. Determine the minimum, maximum, and amplitude of the function.

The minimum is −5 inches, and the maximum is +5 inches.

The amplitude is 5 inches: $\frac{1}{2}|5 - (-5)| = 5$.



4. Does your sketch model a sine curve or a cosine curve? Explain your reasoning.

My sketch models a cosine curve reflected across the midline of $y = 16$.

5. Write the values of *A*, *C*, and *D* for the function *h*(*t*). Explain how you determined each value.

- The value of *A* is −5.

  The amplitude of a periodic function is equal to $|A|$. The function *h*(*t*) has an amplitude of 5, and the function is reflected across the *x*-axis, so the *A*-value is −5.

- The value of *C* is 0.

  The *C*-value represents the phase shift of the graph of the function. The function *h*(*t*) is the graph of a cosine function reflected across the *x*-axis but not shifted horizontally.

- The value of *D* is 16.

  The *D*-value represents the vertical shift of the graph of the function. The basic cosine function has a midline at $y = 0$. The function *h*(*t*) has a midline at $y = 16$, so the function has been shifted up vertically 16 units.

Grouping

Have students complete Questions 6 through 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 6 through 10

- Is the period of the function 1 second? Why?
- What is the period of the cosine function?
- What equation is used to determine when the object on the spring is at its minimum height?
- What equation is used to determine when the object on the spring was at a height of 16 inches?
- Is there more than one solution?

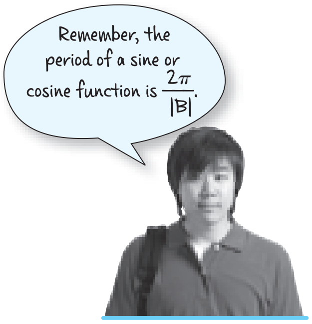
6. Determine the period of the function $h(t)$. Then write the B-value of the function. Show your work.

The period of the function $h(t)$ is 1 second.

Because $h(t)$ is a cosine function, the period can be written as $\frac{2\pi}{|B|}$.

$\frac{2\pi}{|B|} = 1$, so $|B| = 2\pi$.

So, the B-value is 2π.



7. Write the equation for the function $h(t)$ to represent the height of the object over time.

$h(t) = -5 \cos(2\pi t) + 16$

8. Explain why the sign of the B-value in this function can be either positive or negative.

The B-value, 2π, can be positive or negative because $\cos(x) = \cos(-x)$. On the unit circle, the cosine values of radian measures are symmetric across the y-axis, so reflecting the cosine function across the y-axis doesn't change the graph of the function.

9. Solve an equation to determine when the object on the spring is at its minimum height. Show your work.

The object is at its minimum height when $t = 0 + n$ seconds.

$$-5\cos(2\pi t) + 16 = 11$$
$$-5\cos(2\pi t) = -5$$
$$\cos(2\pi t) = \frac{-5}{-5}$$
$$\cos(x) = 1$$
$$x = 0, 2\pi \ldots$$
$$2\pi t = 2\pi$$
$$t = 1$$

The period is $\frac{2\pi}{2\pi}$, or 1, so $t = 0 + n$ seconds.

10. Solve an equation to determine when the object on the spring is at a height of 16 inches. Show your work.

The object is at a height of 16 inches when $t = 0.25 + n$ or $t = 0.75 + n$ seconds.

$$-5\cos(2\pi t) + 16 = 16$$
$$-5\cos(2\pi t) = 0$$
$$\cos(2\pi t) = 0$$
$$\cos(x) = 0$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2} \ldots$$
$$2\pi t = \frac{\pi}{2} \quad \text{or} \quad 2\pi t = \frac{3\pi}{2}$$
$$t = 0.25 \quad \text{or} \quad t = 0.75$$

The period is $\frac{2\pi}{2\pi}$, or 1, so $t = 0.25 + n$ seconds or $t = 0.75 + n$ seconds.

Problem 2

Using the situation from the previous problem, changes in details result in a more realistic model. When released, the object now bounces closer and closer to the midline until it comes to rest rather than continually bouncing up and down. Exponential equations are used to model the change in the height of the object above and below the midline. Students use the cosine function to write an equation describing the complete function, which represents the height of the object on the spring over time. The equation is used to answer questions related to the situation and the term *damping function* is defined.

Grouping

- Ask a student to read the information, and then complete Question 1 as a class.
- Have students complete Questions 2 through 4 with a partner. Then have students share their responses as a class.

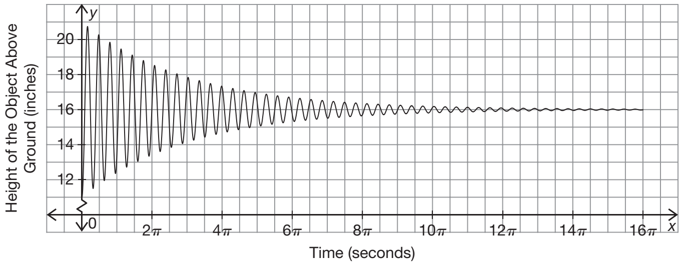
Guiding Questions for Discuss Phase, Question 1

- Does this function resemble the sine function or cosine function?
- Is the graph the result of a phase shift to a trigonometric function?
- Does the graphic behavior appear to be stretched or compressed?
- Would an exponential function describe this graphic behavior?
- Would an absolute value function help to describe this graphic behavior?
- How would you describe the graphic behavior of the amplitude of the function?

PROBLEM 2 And . . . Fade to Black

An object connected to a string and bouncing up and down the same amount forever is not realistic. Starting from when the object is released, the energy produced will eventually fade away. The object will bounce closer and closer to the midline until it once again comes to rest.

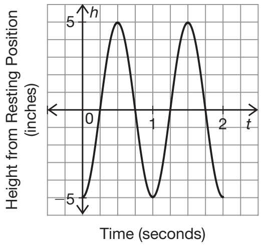
Let's consider the same situation from Problem 1. A more realistic model of the object's motion would look like this:



1. How do you think you can adjust the function $h(t)$ to create the shape of the graph shown? What is changing in each period of this function?

   Answers will vary.

A graph of the function $h(t) - 16$ is shown. This is the function relative to its resting position. Suppose that the distance the object bounces from its resting position decreases at a rate of 10% each second.



2. At $t = 0$, the object is at -5 inches from its resting position.

   a. Determine the object's new height at $t = 1$ second and $t = 2$ seconds.

   At $t = 1$ second, the object is at -4.5 inches.

   $-5 \cdot 0.9 = -4.5$

   At $t = 2$ seconds, the object is at -4.05 inches.

   $-5 \cdot (0.9)^2 = -4.05$

   b. Write an equation to describe the object's new height, n, over time, t. Explain your reasoning.

   I can write an exponential equation to show a decrease of 10% every second.

   At each second, the object bounces 90% of the distance of the previous bounce: $n(t) = -5 \cdot (0.9)^t$.

Guiding Questions for Share Phase, Questions 2 through 4

- What decimal represents a decrease of 10%?
- If the distance of the bounce decreases at a constant rate of 10% each second, how is the decrease calculated for the first second?
- If the distance of the bounce decreases at a constant rate of 10% each second, how is the decrease calculated for the second second?
- Is an exponential equation helpful in showing a decrease of 10% every second?
- Does the object bounce 90% of the distance of the previous bounce?
- At $t = \frac{1}{2}$, is the object above or below the midline? Should the height be described as positive or negative?
- How can your equation be adjusted to yield positive values when $t = \frac{1}{2}$ and $t = 1\frac{1}{2}$?
- What does the equation $h = 5(0.9)^t$ describe in terms of the height of the object?
- What does the equation $h = -5(0.9)^t$ describe in terms of the height of the object?
- Does the amplitude of the function constantly decrease over time?
- Does the amplitude of the function decrease exponentially over time?

c. Does your equation correctly describe the object's new height at $t = \frac{1}{2}$ second? at $t = 1\frac{1}{2}$ seconds? If not, what equation would be correct?

At $t = \frac{1}{2}$, my equation gives the height of the object as approximately -4.74 inches.

At $t = 1\frac{1}{2}$, my equation gives the height of the object as approximately -4.27 inches.

But this is incorrect. At $t = \frac{1}{2}$ and $t = 1\frac{1}{2}$, the object is above the midline.

The correct equation would be $b(t) = 5 \cdot (0.9)^t$. This would give the correct values of $b \approx +4.74$ in. and $b \approx +4.27$ in.

3. Explain why Kent is correct.

The equation $b(t) = 5(0.9)^t$ describes the change in the height of the object above the midline and the equation $b(t) = -5(0.9)^t$ describes the change in the height of the object below the midline.

In both cases, the amplitude of the function, $|A|$ is decreasing exponentially over time.

Kent

The equation $b(t) = |A| \cdot 0.9^t$ describes the change in the object's height over time, because $|A|$ represents the amplitude of the function.

4. Write the complete function that represents the height of the object on the spring over time.

$h(t) = -5(0.9)^t \cdot \cos(2\pi t) + 16$

Grouping

Have students complete Questions 5 and 6 with a partner. Then have students share their responses as a class.

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Guiding Questions for Share Phase, Questions 5 and 6

- What equation is used to determine the number of seconds it takes for the maximum height of the object on the spring to equal 18 inches?
- Why was the exponential function with a positive coefficient of 5 used in this situation?
- Why was a vertical shift of 16 used in this situation?
- Why wasn't the cosine part of the function used in this situation?

5. After how many seconds is the maximum height of the object on the spring equal to 18 inches? Explain how you determined your solution.

$5(0.9)^t + 16 = 18$

$5(0.9)^t = 2$

$0.9^t = 0.4$

$t = \log_{0.9}(0.4)$

$t \approx 8.69672$ seconds

I used the exponential function with the positive coefficient of 5 because this function contains all of the maximums of the cosine function. I also needed to include the vertical shift of 16. I did not need to use the cosine part of the function to solve this problem.

The function that you multiplied to the periodic function to decrease its amplitude over time is called a **damping function**. A damping function can be linear, quadratic, exponential, and on and on!

6. Write a function $g(t)$ to model the height of an object connected to a spring with decreased amplitude over time given the conditions:

- At rest, the object's height is 10 inches above the ground.
- The object bounces up and down once every 2 seconds.
- At $t = 0$, the object's height is 14 inches.
- The distance the object bounces from its resting position decreases at a rate of 15% each second.

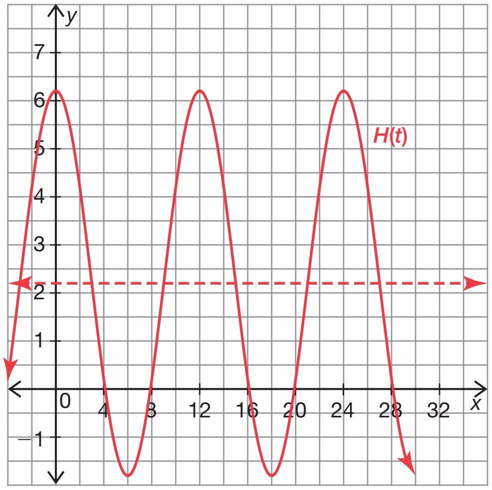
$g(t) = 4(0.85)^t \cdot \cos(\pi t) + 10$

Be prepared to share your methods and solutions.

Check for Students' Understanding

A trigonometric function represents the changes in high and low tides at particular locations. The gravitational force of both the moon and sun affect the height of the tide. High and low tides can be modeled by the function $H(t) = 4\sin\left(\frac{\pi}{6}(t+3)\right) + 2.2$, where $H(t)$ represents the height of the tide, and t represents the number of hours since midnight.

1. Graph the function.



2. What is the amplitude of this function, and what does it mean with respect to the problem situation?

   The amplitude is 4. It describes how far above and below the average amount the tide goes. In this situation the tide is 4 feet above/below the average amount.

3. What is the vertical shift of this function, and what does it mean with respect to the problem situation?

   The vertical shift is 2.2. It describes the average height for the tide.

4. What is the phase shift of this function, and what does it mean with respect to the problem situation?

   The horizontal shift is 3. It describes the times of each day the high tide and low tide occur.

5. What is the period of this function, and what does it mean with respect to the problem situation?

The period is $\frac{2\pi}{\frac{\pi}{6}} = 12$ hours. The tide goes through the entire cycle every 12 hours.

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6. What is the high tide at midnight?

The high tide at midnight is 6.2 feet.

Let $t = 0$

$$H(t) = 4\sin\left(\frac{\pi}{6}(t + 3)\right) + 2.2$$

$$= 4\sin\left(\frac{\pi}{6}(0 + 3)\right) + 2.2$$

$$= 4\sin\left(\frac{\pi}{6}(3)\right) + 2.2$$

$$= 4\sin\left(\frac{\pi}{2}\right) + 2.2$$

$$= 4 + 2.2$$

$$= 6.2$$

Chapter 16 Summary

KEY TERMS

- trigonometric equation (16.1)
- inverse sine ($\sin^{-1}$) (16.1)
- inverse cosine ($\cos^{-1}$) (16.1)
- inverse tangent ($\tan^{-1}$) (16.1)
- Pythagorean identity (16.1)
- damping function (16.4)

16.1 Solving Trigonometric Equations

Solutions to trigonometric equations can be determined using graphs, knowledge about periods and transformation, and inverses of the trigonometric functions. It is important to be aware of domain restrictions when solving trigonometric equations.

Example

Solve the equation over the domain of real numbers.

$$2\cos(x) + 5 = 6$$

$$2\cos(x) = 1$$

$$\cos(x) = \frac{1}{2}$$

$$x = \cos^{-1}\left(\frac{1}{2}\right)$$

$$x = \frac{\pi}{3} + 2\pi n \text{ or } \frac{5\pi}{3} + 2\pi n$$

Therefore, the values of x are $\frac{\pi}{3} + 2\pi n$ or $\frac{5\pi}{3} + 2\pi n$ for integer values of n.

16.1 Solving Trigonometric Equations in Quadratic Form

Trigonometric equations are in quadratic form when they are written in the form $ax^2 + bx + c = 0$, where x represents a trigonometric function. To solve trigonometric equations in quadratic form, use substitution and then follow the steps for solving quadratic equations.

Example

Solve the equation $4\sin^2(x) = 1$ over the domain of real numbers.

Let $a = \sin(x)$.

$$4a^2 = 1$$

$$4a^2 - 1 = 0$$

$$(2a - 1)(2a + 1) = 0$$

$$2a - 1 = 0 \quad \text{or} \quad 2a + 1 = 0$$

$$2a = 1 \quad \text{or} \quad 2a = -1$$

$$a = \frac{1}{2} \quad \text{or} \quad a = -\frac{1}{2}$$

$$\sin(x) = \frac{1}{2} \quad \text{or} \quad \sin(x) = -\frac{1}{2}$$

$$x = \sin^{-1}\left(\frac{1}{2}\right) \quad \text{or} \quad x = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6} \ldots + 2\pi n \quad \text{or} \quad x = \frac{7\pi}{6}, \frac{11\pi}{6} \ldots + 2\pi n$$

Therefore, the values of x are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} + 2\pi n$ for integer values of n.

16.1 Using the Pythagorean Identity to Determine Trigonometric Values

The Pythagorean identity states that $\sin^2(\theta) + \cos^2(\theta) = 1$. It can be used to determine the values of trigonometric functions.

Example

Given $\cos(\theta) = -\frac{3}{4}$ in Quadrant III, determine $\sin(\theta)$.

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\sin^2(\theta) + \left(-\frac{3}{4}\right)^2 = 1$$

$$\sin^2(\theta) + \frac{9}{16} = 1$$

$$\sin^2(\theta) = \frac{7}{16}$$

$$\sin(\theta) = \pm\frac{\sqrt{7}}{4}$$

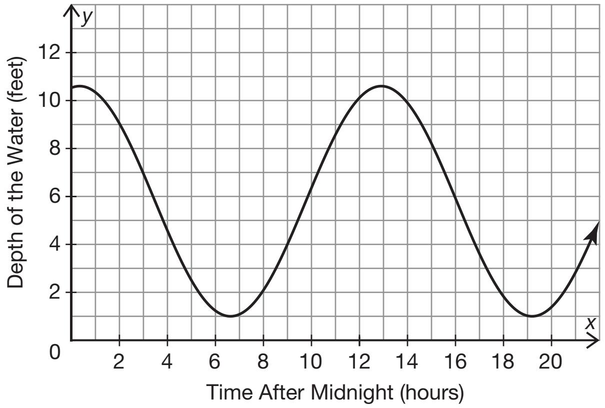
So, $\sin(\theta) = -\frac{\sqrt{7}}{4}$, because it is in Quadrant III.

16.2 Modeling with Periodic Functions

Periodic functions can be used to model real-world situations. Analyzing the functions in the context in which they are given can provide deeper understanding about the functions and the situation.

Example

As the tide changes over the course of the day, the depth of the water near the coast also changes. The change in the depth of the water in feet near the coast in Harbortown can be modeled by the function $d(t) = 4.8 \sin(0.5t + 1.4) + 5.8$, where t represents the number of hours after midnight. Graph the function. Then, identify and describe the amplitude, period, and vertical shift in terms of the situation.



The amplitude is approximately $\frac{|10.6 - 1.0|}{2}$ or 4.8. This represents the average change in depth of the water in feet over the course of the day. The period is $\frac{2\pi}{|0.5|}$ or 4π, which is approximately 12 hours.

This represents the time it takes for the depth to return to its starting point. The vertical shift is 5.8, and it represents the average depth of the water over the course of the day.

16.4 Understanding a Damping Trigonometric Function

A damping trigonometric function is a function in which the amplitude changes as the input changes. In a damping function, A is not a constant.

Example

A ball is dropped from a height of 12 feet and allowed to bounce until it comes to a rest.

The function that represents the height of the ball over time, in seconds, is given by the function $h(t) = 6\left(\frac{1}{2}\right)^t \cos(2\pi t) + 6$. Calculate the height of the ball at 3 seconds.

$$h(t) = 6\left(\frac{1}{2}\right)^t \cos(2\pi t) + 6$$

$$\begin{aligned} h(3) &= 6\left(\frac{1}{2}\right)^3 \cos(2\pi(3)) + 6 \\ &= 6\left(\frac{1}{8}\right) \cos(6\pi) + 6 \\ &= \frac{3}{4}(1) + 6 \\ &= 6.75 \end{aligned}$$

The height of the ball at 3 seconds is 6.75 feet.