Trigonometric Functions

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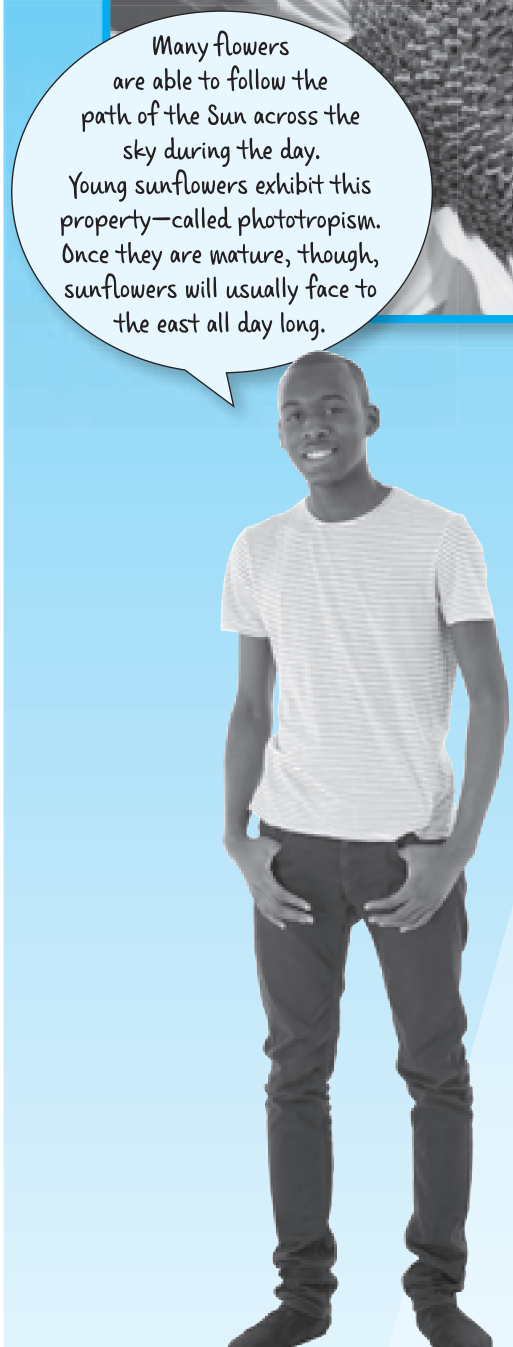

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Chapter 15 Overview

This chapter begins with a problem situation involving a Ferris wheel in which students explore how periodic functions are built. Lessons provide opportunities for students to analyze the graphs of periodic functions for characteristics such as the maximum, minimum, period, amplitude, and midline. Students will explore the unit circle to understand radian measure and convert between angle measures in degrees and radians. Using new understanding of the unit circle, radian measure, and periodic functions, students will investigate the sine and cosine functions as well as their characteristics and graphs.

In the later part of the chapter, students recall the transformational function form $g(x) = Af(B(x - C)) + D$ to graph and analyze transformations of the sine and cosine functions and build a graph of the tangent function using a context. Students will analyze the characteristics of the tangent graph, and apply their knowledge of transformations to sketch graphs of transformed tangent functions.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
15.1	Periodic Functions	F.IF.7e	1	This lesson begins by asking students to build and analyze two different periodic functions that model the height of a person on a Ferris wheel as a function of the number of revolutions of the wheel. Students then build and analyze a similar function using angle measures as inputs. Questions ask students to analyze the graph of each periodic function including the maximum, minimum, period, amplitude, and midline.	X				X
15.2	Radian Measure	F.TF.1 F.TF.2	1	This lesson provides opportunities for students to analyze the unit circle and develop the radian as a unit of angle measure that is equal in magnitude to the corresponding intercepted arc length on a unit circle. Questions ask students to convert between angle measures in degrees and angle measures in radians.	X		X		X

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
15.3	The Sine and Cosine Functions	F.TF.3(+) F.TF.5	2	This lesson provides opportunities for students to synthesize their knowledge of the unit circle, radian measure, and the behavior of the graphs of periodic functions to build an understanding of the sine and cosine functions and their specific characteristics. Students develop an understanding of sine and cosine as functions which take angle measure inputs and output real-number values that can be coordinates of points on a unit circle. Questions ask students to explore the periodicity identities for the sine and cosine functions.	X		X	X	
15.4	Transformations of Sine and Cosine Functions	F.TF.3(+) F.TF.5	2	The lesson provides opportunities for students to analyze transformations of sine and cosine functions individually. Questions ask students to modify the *A*-, *B*-, *C*-, and *D*-values of the transformational function form of sine and cosine functions. Questions then ask students to identify the characteristics of the sine and cosine function graphs that change as a result of each transformation. In the Talk the Talk, students summarize their knowledge of transformations.	X	X		X	
15.5	The Tangent Function	F.TF.3(+) F.TF.5	1	The lesson provides opportunities for students to build the graph of the tangent function, including the asymptotes, in context by modeling the change in the slope of a terminal ray as it traverses the unit circle in standard position. Questions ask students to analyze the characteristics of the tangent function and its graph and label its values for reference angles on the unit circle. Questions then ask students to apply their knowledge of transformations to sketch graphs of transformed tangent functions.	X	X	X		

Skills Practice Correlation for Chapter 15

Lesson		Problem Set	Objectives
15.1	Periodic Functions		Vocabulary
		1 – 6	Sketch graphs of periodic functions that represent given scenarios
		7 – 12	Determine whether graphs represent periodic functions
		13 – 20	Determine midlines and amplitudes of graphs of periodic functions
15.2	Radian Measure		Vocabulary
		1 – 10	Calculate radian measures of central angles given degree measures and radii measures
		11 – 20	Estimate degree measures of central angles given degree measures
		21 – 30	Convert radian measures to degree measures
		31 – 40	Convert degree measures to radian measures
15.3	The Sine and Cosine Functions		Vocabulary
		1 – 8	Use the unit circle to determine sine and cosine values of given radian measures
		9 – 18	Given radian angle measures, determine the coordinates of the point at which the terminal ray intersects the unit circle
		19 – 24	Evaluate the sine and cosine of the supplements of given radian measures
15.4	Transformations of Sine and Cosine Functions		Vocabulary
		1 – 6	Determine amplitudes of sine and cosine graphs
		7 – 12	Determine periods and frequencies of sine and cosine graphs
		13 – 20	Describe transformations of graphs given equations of basic sine and cosine functions and equations of transformed functions
		21 – 26	Use knowledge of transformations to sketch graphs of given equations of sine and cosine functions
15.5	The Tangent Function		Vocabulary
		1 – 8	Calculate tangent of angles given the cosine and sine of angles
		9 – 20	Evaluate tangent functions using sine and cosine functions
		21 – 26	Analyze graphs of transformed tangent functions and determine the equations of the transformed tangent functions
		27 – 32	Graph transformations of tangent functions

15.1

A Sense of Déjà Vu

Periodic Functions

LEARNING GOALS

In this lesson, you will:

- Model a situation with a periodic function.
- Analyze the period and amplitude of a periodic function.
- Determine the period, amplitude, and midline of a periodic function.

KEY TERMS

- periodic function
- period
- standard position
- initial ray
- terminal ray
- amplitude
- midline

ESSENTIAL IDEAS

- Periodic functions are used to model real-world situations.
- A periodic function is a function whose values repeat over regular intervals.
- The period of a periodic function is the length of the smallest interval over which the function repeats.
- An angle is in standard position when the vertex is at the origin and one ray of the angle is on the *x*-axis. The ray on the *x*-axis is the initial ray and the other ray is the terminal ray.
- The amplitude of a periodic function is one-half the absolute value of the difference between the maximum and minimum values of the function.
- The midline of a periodic function is a reference line whose equation is the average of the minimum and the maximum values of the function.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-IF Interpreting Functions

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

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Overview

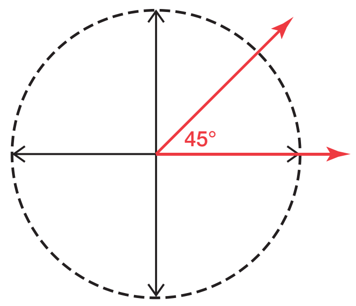
The term *periodic function* is defined. Periodic functions are used to model two situations related to a Ferris wheel: the height of a person on a Ferris wheel as a function of the number of revolutions of the wheel, and the height of a rider above ground with respect to the number of revolutions of an underground Ferris wheel. Students also use a protractor and graph to create a periodic function related to the height of a rider above ground on an underground Ferris wheel as a function of the measure of an angle in standard position and answer questions related to each problem situation.

Warm Up

Use a protractor and the axes to draw angles of the given measure in each circle.

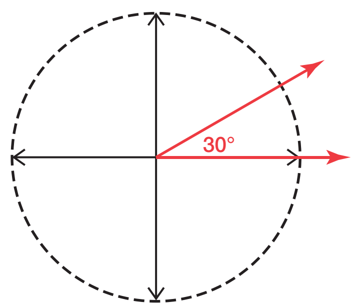
1. 45°

Drawings will vary.



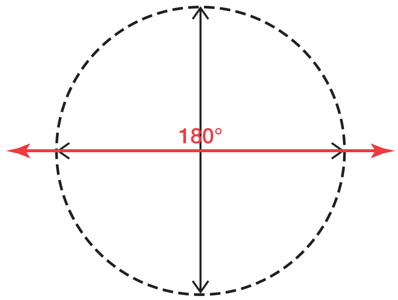
2. 30°

Drawings will vary.



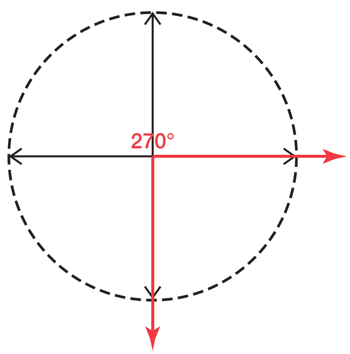
3. 180°

Drawings will vary.



4. 270°

Drawings will vary.



15.1

A Sense of Déjà Vu

Periodic Functions

LEARNING GOALS

In this lesson, you will:

- Model a situation with a periodic function.
- Analyze the period and amplitude of a periodic function.
- Determine the period, amplitude, and midline of a periodic function.

KEY TERMS

- periodic function
- period
- standard position
- initial ray
- terminal ray
- amplitude
- midline

The Ferris Wheel is named after George Washington Gale Ferris, Jr., a Pittsburgh bridge-builder. The first Ferris Wheel was designed and constructed by Ferris for the World's Columbian Exposition in Chicago in 1893. It stood 264 feet tall, contained 36 cars, took 20 minutes to complete 2 revolutions, and cost 50 cents per ride.

The record for the world's tallest Ferris Wheel has been broken many times since 1893. As of 2008, the Singapore Flyer, in Singapore holds the record. It stands a whopping 541 feet, over twice as tall as Ferris' original wheel.

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Problem 1

Students create a sketch to model the height of a rider above ground with respect to the number of revolutions of a Ferris wheel. They complete a table of values which relates the revolutions of the Ferris wheel to the height of a rider above the ground and describe the behavior of the graph. The terms periodic function and period are defined.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

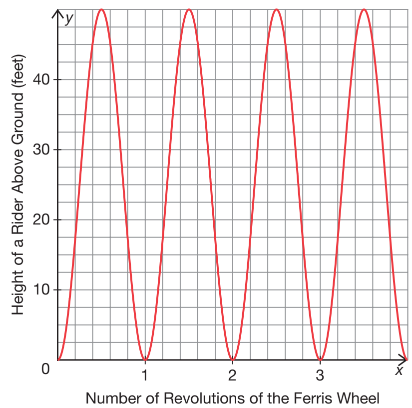
Guiding Questions for Share Phase, Question 1

- What points do you use to create the sketch modeling the height of a rider above ground?
- Which quadrants contain the graph of the situation?

PROBLEM 1 Who's Ready for a "Wheel" Good Time?

One of the most popular amusement park rides is the Ferris wheel. One Ferris wheel has a diameter of 50 feet. Riders board the cars at ground level and the wheel moves counterclockwise. Each ride consists of four revolutions and you can assume that the Ferris wheel rotates at a constant rate.

1. Create a sketch to model the height of a rider above ground with respect to the number of revolutions of the Ferris wheel. Include 4 revolutions.



Guiding Questions for Share Phase, Questions 2 through 4

- How do you determine the height of a rider when the Ferris wheel rotated $\frac{1}{8}$ of a revolution?
- How do you determine the height of a rider when the Ferris wheel rotated $\frac{1}{4}$ of a revolution?
- How do you determine the height of a rider when the Ferris wheel rotated $\frac{1}{2}$ of a revolution?
- Is the graph continuous?
- Does the graph contain a maximum?
- Does the graph contain a minimum?
- Where are the *x*-intercepts?
- Where is the *y*-intercept?
- When is the graph increasing, with respect to each revolution?
- When is the graph decreasing, with respect to each revolution?
- Does the graph maintain the same shape for each revolution?

Grouping

Ask a student to read the information and definitions and complete Question 5 as a class.

2. Compete the table to represent the height of a rider above ground as a function of the number of revolutions of the Ferris wheel.

Revolutions of the Ferris Wheel	Height of a Rider Above Ground (feet)
0	0
$\frac{1}{8}$	12.5
$\frac{1}{4}$	25
$\frac{3}{8}$	37.5
$\frac{1}{2}$	50
$\frac{5}{8}$	37.5
$\frac{3}{4}$	25
$\frac{7}{8}$	12.5
1	0

3. Describe the characteristics of your graph.

Answers will vary.

- The graph is continuous.
- The graph reaches a maximum at 50 feet and a minimum at 0 feet.
- The graph has an *x*-intercept at every whole number revolution.
- The graph has a *y*-intercept at $y = 0$.
- The graph is increasing in the first half of each revolution and decreasing in the second half of each revolution.

4. What do you notice about the shape of the graph for each revolution?

For each revolution, the shape of the graph is identical.

To model the height of a rider above ground on the Ferris wheel, you used a *periodic function*. A **periodic function** is a function whose values repeat over regular intervals. The **period** of a periodic function is the length of the smallest interval over which the function repeats.

5. Describe the period of the function that models the height of a rider above ground on the Ferris wheel.

The period is one revolution because the function repeats after each revolution.

Guiding Questions for Discuss Phase, Question 5

- Why is the period of the function modeling the height of a rider on the Ferris wheel one revolution?
- Is the period 2 revolutions? Why not?

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Problem 2

A second Ferris wheel is situated above and below ground level. Students create a sketch to model the height of a rider above ground with respect to the number of revolutions of the underground Ferris wheel. They complete a table of values which relates the revolutions of the Ferris wheel to the height of a rider above the ground and describe the behavior of the graph.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

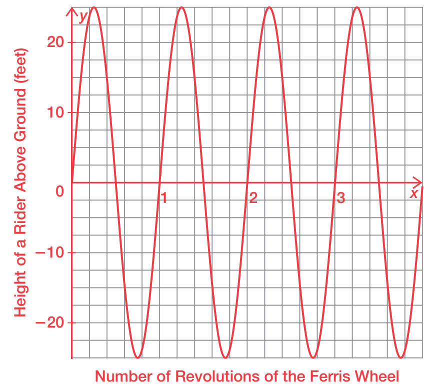
Guiding Questions for Share Phase, Question 1

- What points do you use to create the sketch modeling the height of a rider above ground?
- Which quadrants contain the graph of the situation?
- How is this graph different from the graph of the function in the previous problem?

PROBLEM 2 The Underground Ferris Wheel

At a different amusement park, a Ferris wheel was designed so that half of the wheel is actually below the ground. The diameter of this underground Ferris wheel is still 50 feet. The top of the ride reaches 25 feet above ground and the bottom of the ride reaches 25 feet below ground. Riders board the cars at ground level to the right and the Ferris wheel moves counterclockwise.

1. Create a sketch to model the height of a rider above ground with respect to the number of revolutions of the underground Ferris wheel. Include 4 revolutions.



Guiding Questions for Share Phase, Questions 2 through 4

- How do you determine the height of a rider when the Ferris wheel rotated $\frac{1}{8}$ of a revolution?
- How do you determine the height of a rider when the Ferris wheel rotated $\frac{1}{4}$ of a revolution?
- How do you determine the height of a rider when the Ferris wheel rotated $\frac{1}{2}$ of a revolution?
- Is the graph continuous?
- Does the graph contain a maximum?
- Does the graph contain a minimum?
- Where are the x-intercepts?
- Where is the y-intercept?
- When is the graph increasing, with respect to each revolution?
- When is the graph decreasing, with respect to each revolution?
- Does the graph maintain the same shape for each revolution?
- Why is the period of the function modeling the height of a rider on the Ferris wheel one revolution?

2. Complete the table to represent the height of a rider above ground as a function of the number of revolutions of the underground Ferris wheel.

Revolutions of the Ferris Wheel	Height of a Rider Above Ground (feet)
0	0
$\frac{1}{8}$	12.5
$\frac{1}{4}$	25
$\frac{3}{8}$	12.5
$\frac{1}{2}$	0
$\frac{5}{8}$	-12.5
$\frac{3}{4}$	-25
$\frac{7}{8}$	-12.5
1	0

3. Describe the characteristics of your graph.

Answers will vary.

- The graph is continuous.
- The graph reaches a maximum at 25 feet and a minimum at -25 feet.
- The graph has an x-intercept at every half revolution.
- The graph has a y-intercept at $y = 0$.
- The graph is increasing in the first quarter and the last quarter of each revolution and decreasing in the second and third quarter of each revolution.

4. Describe the period of the function that models the height of a rider above ground on the underground Ferris wheel.

The period is one revolution because the function repeats after each revolution.

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Problem 3

The terms related to an angle such as *standard position*, *initial ray*, and *terminal ray* are defined. Students use a protractor, a straightedge, and a graph to build a periodic function that models the height of a rider above the ground on the underground Ferris wheel as a function of the measure of an angle in standard position. Students answer questions related to the graph of the situation such as the location of minimum and maximum points, and symmetrical properties.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Which quadrants contain the graph of the situation?
- Which points on the graph have positive y-values?
- Which points on the graph have negative y-values?
- Why do all points on the graph have positive x-values?

PROBLEM 3 Periodic Functions

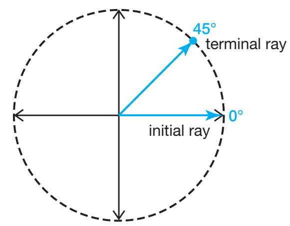
In the last two problems, you modeled the height of a rider above ground as a function of the number of revolutions of two different Ferris wheels. You can also model the height of a rider as a function using angle measures.

An angle is in **standard position** when the vertex is at the origin and one ray of the angle is on the x-axis. The ray on the x-axis is the **initial ray** and the other ray is the **terminal ray**.

1. Use a protractor, a straightedge, and the graph on the following page. Complete the steps shown to build a periodic function to model the height of a rider above ground on the underground Ferris wheel as a function of the measure of an angle in standard position. The position of the car is the intersection of the terminal ray and the circle.

Step 1: Analyze each axis label.

Step 2: Measure a 45° angle in *standard position*. Mark and label a point on the Ferris Wheel as shown.

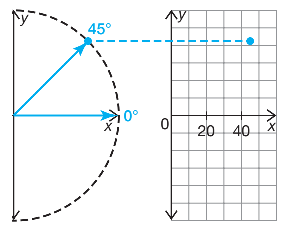


The measure of an angle in standard position is the amount of rotation from the initial ray to the terminal ray. When the rotation is counterclockwise, the angle measure is positive. When the rotation is clockwise, the angle measure is negative.

Step 3: Use a straightedge to line up the point on the Ferris wheel with the appropriate location on the coordinate plane. Plot the point.

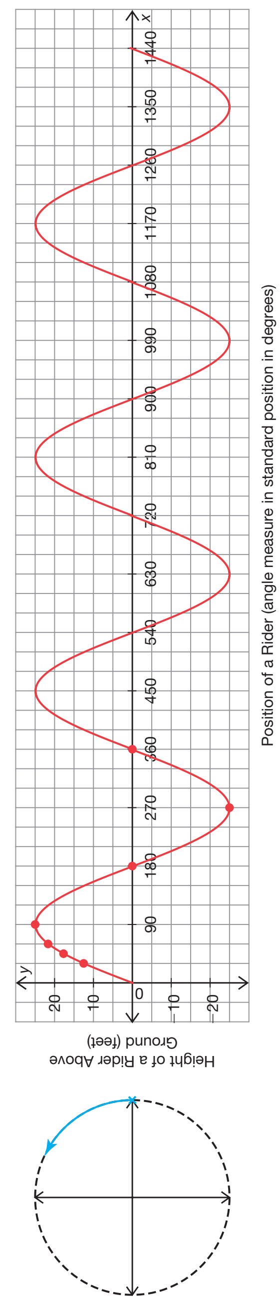
Step 4: Repeat Steps 2 and 3 for each angle measure:

0°, 30°, 60°, 90°, 180°, 270°, 360°



Step 5: Draw a smooth curve to connect the points of your graph.

Step 6: Continue the curve to represent angle measures greater than 360°.


Height of a Rider Above Ground (feet)
Position of a Rider (angle measure in standard position in degrees)
x
y
0
90
180
270
360
450
540
630
720
810
900
990
1080
1170
1260
1350
1440
20
10
−10
−20

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Guiding Questions for Share Phase, Questions 2 through 5

- Does the behavior of the graph repeat every 360°?
- Each seat travels a complete rotation every how many degrees?
- What is the maximum height a seat can be above the ground?
- What is the minimum height a seat can be below the ground?
- What is the greatest distance a rider can be from ground level?
- Is a rider at their highest point for an angle measure of 45°? 90°? 180°? 270°? 360°? Why or why not?
- How do you determine other angle measures at which the rider was at its highest point?
- Is a rider at its lowest point for an angle measure of 45°? 90°? 180°? 270°? 360°? Why or why not?
- How do you determine other angle measures at which the rider was at its lowest point?
- Which part of the graph is associated with the horizontal symmetry of the Ferris wheel?
- Which part of the graph is associated with the vertical symmetry of the Ferris wheel?
- Which part of the graph looks like a parabola?
- Does one-half of the graph appear to be a reflection? How so?

2. Determine the period of the function you graphed. What does this value represent in terms of this problem situation?

The graph repeats every 360°, so its period is 360°. Every 360°, the Ferris wheel has made another complete rotation starting from 0°.

3. Determine any maximum or minimum values of your graph. What does each value represent in terms of this problem situation?

The maximum value is 25, and the minimum value is −25. The absolute values of these quantities represent the greatest distance that a rider is from the ground.

4. At certain angle measures, a rider is at its highest or lowest point.

a. List 4 angle measures associated with a rider being at its highest point.

Answers will vary.

A rider is at its highest point for angle measures of 90°, 450°, 810°, 1170°, etc.

b. List 4 angle measures associated with a rider being at its lowest point.

Answers will vary.

A rider is at its lowest point for angle measures of 270°, 630°, 990°, 1350°, etc.

5. Describe the symmetries you see in the graph of the function. Explain how these are related to the symmetries associated with the Ferris wheel.

Answers will vary.

The graph of the function from 0° to 180° looks like a parabola that is symmetrical about the line $x = 90$. This symmetry comes from the horizontal symmetry of the Ferris wheel.

The graph of the function from 180° to 360° is a reflection of the graph from 0° to 180°. This symmetry comes from the vertical symmetry of the Ferris wheel.

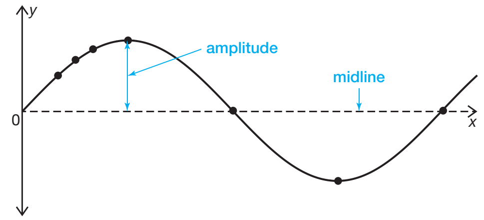
Grouping

- Ask a student to read the information and definitions. Discuss as a class.
- Have students complete Questions 6 and 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 6 and 7

- What are the maximum and minimum values of the situation described in Problem 1? Problem 2? Problem 3?
- What is one-half the absolute value of the difference between the maximum and minimum values of the situation described in Problem 1? Problem 2? Problem 3?
- Is the amplitude of the graph in Problem 1 equal to 0, 25, or 50? Why?
- Is the amplitude of the graph in Problem 2 equal to 0, 25, or 50? Why?
- Is the amplitude of the graph in Problem 3 equal to 0, 25, or 50? Why?
- How do you determine the midline for each situation?

The graphs of periodic functions have characteristics that are given special names, such as *amplitude* and *midline*.



The **amplitude** of a periodic function is one-half the absolute value of the difference between the maximum and minimum values of the function.

The **midline** of a periodic function is a reference line whose equation is the average of the minimum and maximum values of the function.

6. Determine the amplitude of each function you graphed in Problems 1 through 3. Show your work.

The function in Problem 1 has an amplitude of 25 because $\frac{1}{2}|50 - 0| = 25$.

The function in Problem 2 has an amplitude of 25 because $\frac{1}{2}|25 - (-25)| = 25$.

The function in Problem 2 has an amplitude of 25 because $\frac{1}{2}|25 - (-25)| = 25$.

7. Determine the midline of each function you graphed in Problems 1 through 3.

For the function in Problem 1, the equation of the midline is $y = 25$.

For the functions in Problem 2 and 3, the equation of the midline is $y = 0$.

Be prepared to share your methods and solutions.

15

Check for Students' Understanding

Consider an angle in standard position.

1. What degree measure is equivalent to each revolution?

   a. 1 revolution counterclockwise

   $360°$

   b. 2 revolutions counterclockwise

   $720°$

   c. $\frac{3}{4}$ of a revolution clockwise

   $-270°$

2. How many revolutions are equivalent to each degree measure? Be sure to indicate if each revolution is clockwise or counterclockwise.

   a. $45°$

   $\frac{1}{8}$ of a revolution counterclockwise

   b. $-60°$

   $\frac{1}{6}$ of a revolution clockwise

   c. $450°$

   $1\frac{1}{4}$ revolutions counterclockwise

15.2

Two Pi Radii

Radian Measure

LEARNING GOALS

In this lesson, you will:

- Determine the radian measure of angles.
- Convert between angle measures in degrees and angle measures in radians.
- Estimate the degree measure of central angle measures given in radians.
- Identify reference angles in radians.

KEY TERMS

- theta (θ)
- unit circle
- radians

ESSENTIAL IDEAS

- The measure of an arc of a circle is equal to the degree measure of the central angle that intercepts the arc.
- The length of an intercepted arc is given by: arc length $= 2\pi r \cdot \frac{\text{measure of central angle}}{360°}$
- A unit circle has a radius of 1 unit.
- A complete revolution of the terminal ray around the unit circle is equal to 360° or −360°.
- The arc length described by a complete revolution of the terminal ray around the unit circle is 2π units.
- There are 2π radians in 360° and π radians in 180°.
- The ratio of the intercepted arc length of a central angle to the radius is the measure of the central angle in radians.
- A radian is a unit that describes the measure of an angle in terms of the radius and arc length of a unit circle.
- The formula to convert radians to degrees is: $x \text{ radians} \cdot \frac{180°}{\pi \text{ radians}}$
- The formula to convert degrees to radians is: $x \text{ degrees} \cdot \frac{\pi \text{ radians}}{180°}$

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-TF Trigonometric Functions

Extend the domain of trigonometric functions using the unit circle

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

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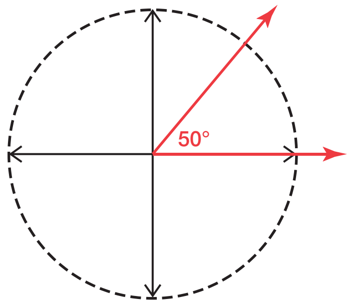
Overview

The terms *unit circle* and *radian measure* are introduced. Students develop the concept of radian measure and label the measures of several central angles of a unit circle in both degree measure and radian measure. Formulas are given and used to convert degree measure to radian measure and vice versa.

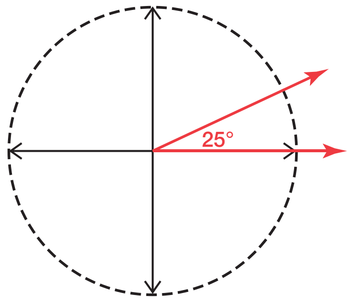
Warm Up

Use a protractor and the axes to draw an angle with the given measure in standard position.

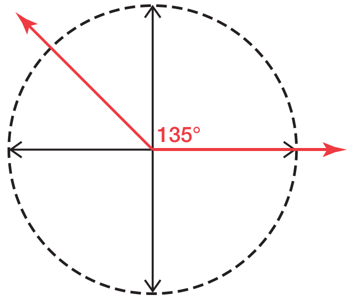
1. 50°



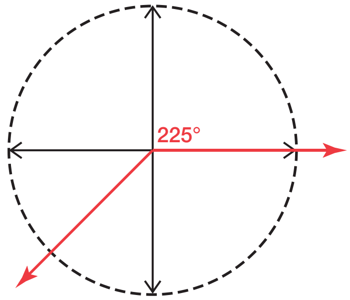
2. 25°



3. 135°



4. 225°



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15.2

Two Pi Radii

Radian Measure

LEARNING GOALS

In this lesson, you will:

- Determine the radian measure of angles.
- Convert between angle measures in degrees and angle measures in radians.
- Estimate the degree measure of central angle measures given in radians.
- Identify reference angles in radians.

KEY TERMS

- theta (θ)
- unit circle
- radians

There are many issues associated with using degrees as units for angle measures. For starters, angles measured in degrees can't tell you anything about the distance of a rotation. The speed of the orbiting International Space Station, for example, is understood in miles per hour, rather than degrees per hour.

Secondly, the number 360—representing the number of degrees in a circle—is an arbitrary number which has no logical relationship to circle measures.

Finally, computing with a logical angle measure, like the one you'll learn about in this lesson, is often more efficient—once you get the hang of it!

Can you think of another way to measure central angles?

Problem 1

Information for determining the measure of an arc of a circle and the length of its intercepted arc is reviewed. Students write expressions in terms of the radius to describe the arc length for a central angle measure. The term unit circle is introduced. Students use a protractor to determine the measure of central angles in the unit circle and their corresponding arc lengths. The term radian is defined and students label the measure of several central angles on a unit circle in degrees and radians.

Grouping

- Ask a student to read the information and complete Question 1 as a class.
- Have students complete all parts of Questions 2 and 3 with a partner. Then have students share their responses as a class.

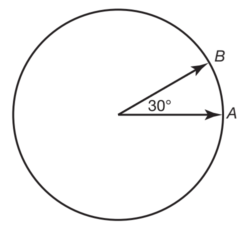
Guiding Questions for Discuss Phase, Question 1

- What fraction of the circumference of the circle is associated with $\frac{30^\circ}{360^\circ}$?
- What fraction of the circumference of the circle is associated with $\frac{45^\circ}{360^\circ}$?
- When is a circle considered a unit circle?
- How is a unit circle different than other circles?

PROBLEM 1 An Angle Measure by Another Name

Recall that the measure of an arc of a circle is equal to the degree measure of the central angle that intercepts the arc.

$$m\overset{\frown}{AB} = 30^\circ$$



The length of the intercepted arc is given by the expression:

$$\text{arc length} = 2\pi r \cdot \frac{\text{measure of central angle}}{360^\circ}$$

You can identify the central angle measures of a circle in standard position using the symbol **theta**, written as θ. For example, a central angle measure of 30° can be written as $\theta = 30^\circ$.

1. Given any circle with a radius of r units:

   a. Write an expression in terms of r to describe the arc length for a central angle measure of $\theta = 30^\circ$.

   $$\text{arc length} = 2\pi r \cdot \frac{30^\circ}{360^\circ} = \frac{2\pi r}{12} = \frac{\pi}{6}(r) \text{ units}$$

   b. Write an expression in terms of r to describe the arc length for a central angle measure of $\theta = 45^\circ$.

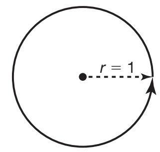
   $$\text{arc length} = 2\pi r \cdot \frac{45^\circ}{360^\circ} = \frac{2\pi r}{8} = \frac{\pi}{4}(r) \text{ units}$$

A powerful way to measure central angles of a circle is to identify arc lengths of the circle in terms of the radius of a *unit circle*. A **unit circle** has a radius of 1 unit.

Guiding Questions for Share Phase, Questions 2 and 3

- How many degrees are associated with a circle?
- What is the circumference formula?
- Why is the arc length of the circumference of the unit circle equal to 2π units?
- Why is a 180° angle in a unit circle associated with an arc length of π units?
- How does a 90° angle compare to a 180° angle?
- What is the arc length of a 180° angle?
- How does a 45° angle compare to a 90° angle?
- What is the arc length of a 90° angle?
- How does a 30° angle compare to a 90° angle?
- How does a 60° angle compare to a 30° angle?
- What is the arc length of a 30° angle?

2. Consider the unit circle shown.



a. Identify a central angle measure, θ, that represents a complete revolution of the terminal ray around the unit circle.

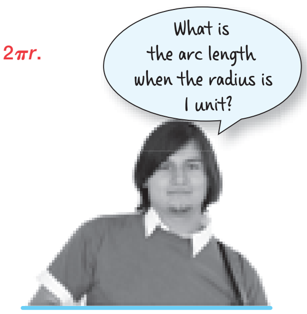
$\theta = 360°$ or $-360°$

A complete revolution measures 360° or −360°.

b. Identify a is the arc length of this central angle?

The arc length is the circumference of the circle, $2\pi r$.

Since the radius is 1, the arc length is 2π units.



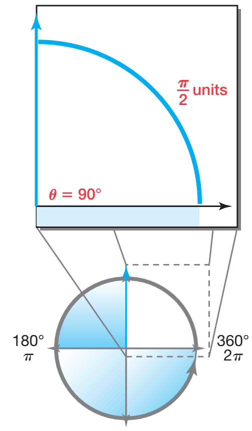
c. Identify a central angle measure, θ, and arc length that represent half of a revolution of the terminal ray around the unit circle.

$\theta = \frac{1}{2}(360°) = 180°$

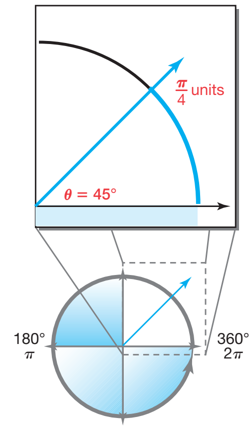
$\theta = \frac{1}{2}(-360°) = -180°$

The arc length of half of a revolution around the unit circle is $\frac{1}{2}(2\pi) = \pi$ units.

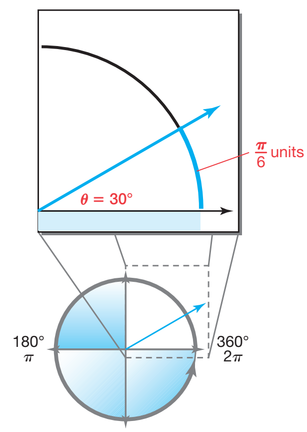
3. Use a protractor to determine each central angle measure, θ, in the unit circle. Then label the angle measures and their corresponding arc lengths in units. Explain how you determined your answers.



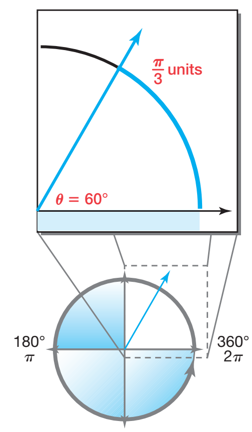
Because 90° is half of 180°, which has an arc length of π, the arc length of 90° is $\frac{\pi}{2}$ units.



Because 45° is half of 90°, its arc length is $\frac{\pi}{2} \times \frac{1}{2}$, or $\frac{\pi}{4}$ units.



Because 30° is $\frac{1}{3}$ of 90°, its arc length is $\frac{\pi}{2} \times \frac{1}{3}$, or $\frac{\pi}{6}$ units.



Because 60° is double 30°, its arc length is $\frac{\pi}{6} \times 2$, or $\frac{\pi}{3}$ units.

Grouping

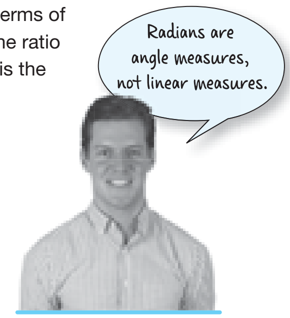
- Ask a student to read the information and definitions.
- Have students complete Questions 4 and 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- Did Josh use the arc length measure as the radian measure?
- If the circle has a radius of 2 units, what is the intercepted arc length for a central angle measure of 45°?
- Is the radian measure the ratio of the intercepted arc length to the radius?
- Why should Josh have divided the arc length measure by the radius of 2 to determine the radian measure of the angle?

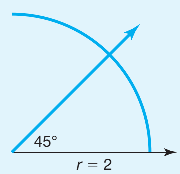
The unit that describes the measure of an angle theta, θ, in terms of the radius and arc length of a unit circle is called a *radian*. The ratio of the intercepted arc length of a central angle to the radius is the measure of the central angle in **radians**.

There are $\frac{2\pi r}{r}$, or 2π, radians in 360° and $\frac{\pi r}{r}$, or π, radians in 180°.



4. Sandy and Josh each determined the radian measure for a central angle measuring 45° in a circle with a radius of 2 units.

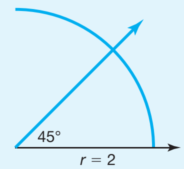
Sandy



$$\text{Arc length} = 2\pi(2) \cdot \frac{45°}{360°}$$
$$= 4\pi \cdot \frac{1}{8} = \frac{\pi}{2} \text{ units}$$

The radian measure of this angle is $\frac{\pi}{2} \div 2$, or $\frac{\pi}{4}$ rad.

Josh



$$\text{Arc length} = 2\pi(2) \cdot \frac{45°}{360°}$$
$$= 4\pi \cdot \frac{1}{8} = \frac{\pi}{2} \text{ units}$$

The radian measure of this angle is $\frac{\pi}{2}$ rad.

Explain why Josh's reasoning is incorrect.

Josh used the arc length measure as the radian measure. In a circle with a radius of 2 units, the intercepted arc length for a central angle measure of 45° is $\frac{\pi}{2}$ units. But radian measure is the ratio of the intercepted arc length to the radius, so Josh should have divided the arc length measure by the radius of 2 to determine the radian measure of the angle.

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Guiding Questions for Share Phase, Question 5

- If $\theta = 90°$, which other central angle can easily be calculated?
- What is $\pi + \frac{\pi}{2}$ simplified?
- How are the angles located in the upper left quarter of the circle related to the angles in the upper right quarter?
- Are the angles located in the upper left quarter of the circle reflections of the angles in the upper right quarter?
- If $\theta = 30°$, which other central angle can easily be calculated?
- What is $\pi - \frac{\pi}{6}$ simplified?
- If $\theta = 45°$, which other central angle can easily be calculated?
- What is $\frac{\pi}{2} + \frac{\pi}{4}$ simplified?
- If $\theta = 60°$, which other central angle can easily be calculated?
- What is $\pi - \frac{\pi}{3}$ simplified?
- How can the angle measures in the bottom left quarter of the circle be determined using each of the theta values in the upper right quarter?
- If 180° is added to each of the theta values in the upper right quarter of the circle, will it determine the angle measures in the bottom left quarter?
- How can the angle measures in the bottom right quarter of the circle be determined using each of the theta values in the upper left quarter?

5. Use what you know about the symmetry of a circle to label each central angle measure in degrees and radians on the unit circle on the next page. Explain how you determined the measures and show your work.

See unit circle. Explanations will vary.

- $\theta = 90° \rightarrow 180° + 90°$, or $270°$

  $\pi + \frac{\pi}{2} = \frac{3\pi}{2}$ radians

- The angles in the upper left quarter of the circle are reflections of the angles in the upper right quarter.

  $\theta = 30° \rightarrow 180° - 30°$, or $150°$

  $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ radians

  $\theta = 45° \rightarrow 90° + 45°$, or $135°$

  $\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$ radians

  $\theta = 60° \rightarrow 180° - 60°$, or $120°$

  $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$ radians

- The angle measures in the bottom left quarter of the circle can be determined by adding 180° to each of the theta values in the upper right quarter.

  $\theta = 30° \rightarrow 180° + 30°$, or $210°$

  $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$ radians

  $\theta = 45° \rightarrow 180° + 45°$, or $225°$

  $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$ radians

  $\theta = 60° \rightarrow 180° + 60°$, or $240°$

  $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$ radians

- The angle measures in the bottom right quarter of the circle can be determined by adding 180° to each of the theta values in the upper left quarter.

  $\theta = 150° \rightarrow 180° + 150°$, or $330°$

  $\pi + \frac{5\pi}{6} = \frac{11\pi}{6}$ radians

  $\theta = 135° \rightarrow 180° + 135°$, or $315°$

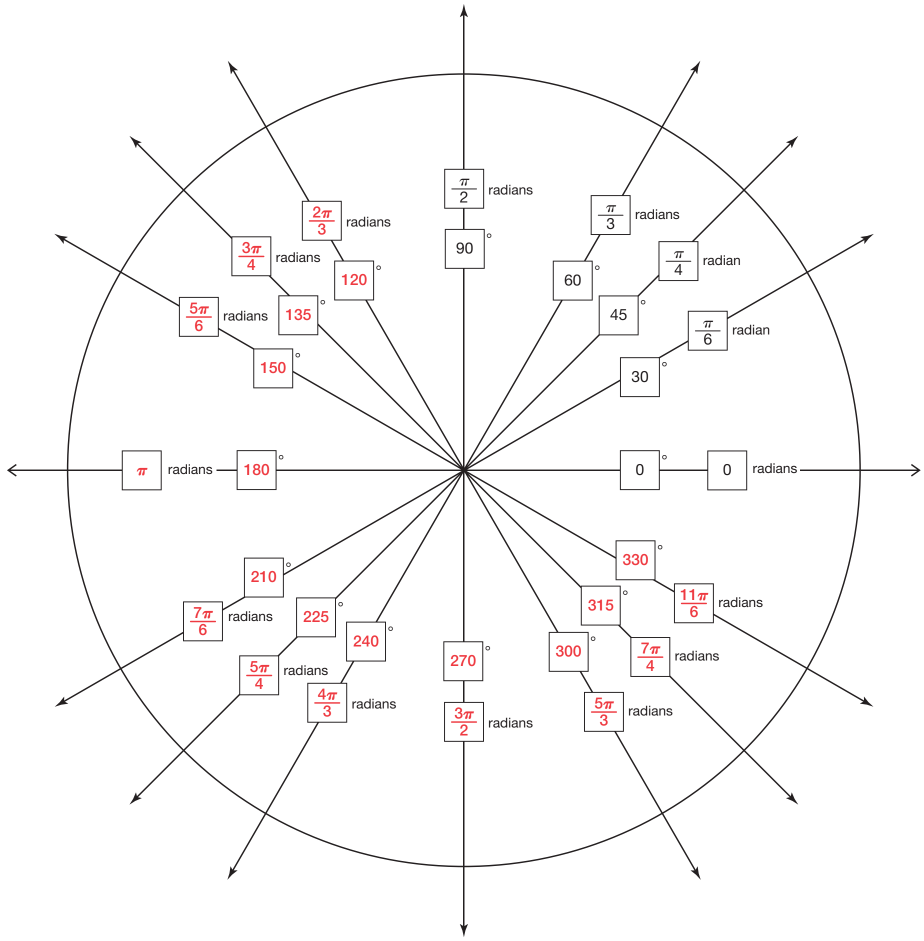
  $\pi + \frac{3\pi}{4} = \frac{7\pi}{4}$ radians

  $\theta = 120° \rightarrow 180° + 120°$, or $300°$

  $\pi + \frac{2\pi}{3} = \frac{5\pi}{3}$ radians

- If 180° is added to each of the theta values in the upper left quarter of the circle, will it determine the angle measures in the bottom right quarter?
- If $\theta = 150°$, which other central angle can easily be calculated?
- What is $\pi + \frac{5\pi}{6}$ simplified?
- If $\theta = 135°$, which other central angle can easily be calculated?
- If $\theta = 120°$, which other central angle can easily be calculated?

Degrees and Radians



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Problem 2

Students estimate the degree measure of several central angles given in radians. They are given the formulas needed to convert from radians to degrees and from degrees to radians. Students use the conversion formula for previously estimated angle measures and compare the results.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- How is the value $\frac{\pi}{4}$ considered a constant if π is irrational?
- How is the value $\frac{7\pi}{4}$ considered a constant if π is irrational?
- An angle measure of π radians is equal to how many degrees?
- An angle measure of π radians is approximately how many radians?
- How does 3 radians compare to 180°?
- An angle measure of 2π radians is equal to how many degrees?
- An angle measure of 2π radians is approximately how many radians?
- How does 6 radians compare to 360°?
- How does 2 radians compare to 120°?
- How does 4 radians compare to 240°?
- How does π radians compare to 60°?
- How does 1 radian compare to 60°?
- How does 5 radians compare to 300°?

PROBLEM 2 Pin the Tail on the Radian

It is important to keep in mind that values such as $\frac{\pi}{4}$ and $\frac{7\pi}{6}$ are constants. Each of these irrational numbers can be rewritten as non-terminating, non-repeating decimals.

$$\frac{\pi}{4} \approx \frac{3.14}{4} \approx 0.785 \quad \frac{7\pi}{6} \approx \frac{7(3.14)}{6} \approx 3.6633\ldots.$$

You can also write whole-number values for radians.

1. Estimate the degree measure of each central angle measure given in radians. Explain your reasoning.

   a. 3 radians

   Since π radians is equal to 180° and π radians is approximately 3.14 radians, 3 radians is a little less than 180°.

   b. 6 radians

   Since 2π radians is equal to 360° and 2π radians is approximately 6.28 radians, 6 radians is a little less than 360°.

   c. 2 radians

   Two radians is one third of 6 radians, which is a little less than 360°. So, 2 radians will be a little less than 120°.

   d. 4 radians

   Four radians is 2 times 2 radians, so 4 radians will be a little less than 240°.

   e. 1 radian

   Since π radians is equal to 180° and π radians is approximately 3.14 radians, 1 radian will be about one third of π radians, or a little less than 60°.

   f. 5 radians

   Since 1 radian is a little less than 60°, 5 radians will be close to but less than 300°.

Guiding Questions for Share Phase, Questions 2 through 4

- What is the radius of a unit circle?
- If the radius of a unit circle is 1, does the arc length also equal 1? Why?
- Does a central angle with a measure of 1 radian have an arc length equal to the radius of 1 unit? How do you know?
- How is the ratio $\frac{180^\circ}{\pi \text{ radians}}$ equal to 1?
- How is the ratio $\frac{\pi \text{ radians}}{180^\circ}$ equal to 1?
- Are the angle measures determined by the formulas accurate? Why or why not?
- An angle measure of $\frac{\pi}{2}$ radians is associated with what angle measure in degrees?
- If an angle measure is x degrees, what algebraic expression represents its complement?
- Is the complement of an angle measure of x degrees associated with the expression $90 - x$ or $x - 90$ degrees?

2. What is the arc length of a central angle that has a measure of 1 radian on the unit circle? Explain your reasoning.

A central angle with a measure of 1 radian has an arc length equal to the radius of 1 unit.

Because the measure of a central angle in radians is equal to $\frac{\text{arc length}}{\text{radius}}$,

arc length = (central angle measure in radians)(radius).

Since the radius of a unit circle is 1:

arc length = (1 radian)(1 unit) = 1

The formulas you can use to convert from radians to degrees and degrees to radians are shown.

Radians to Degrees: $x \text{ radians} \cdot \frac{180^\circ}{\pi \text{ radians}}$

Degrees to Radians: $x \text{ degrees} \cdot \frac{\pi \text{ radians}}{180^\circ}$

3. Use the formulas to convert each angle measure in Question 1 to degrees.
How close were your estimates?

3 radians ≈ 171.89°
6 radians ≈ 343.77°
2 radians ≈ 114.59°
4 radians ≈ 229.18°
5 radians ≈ 286.48°

4. Explain why Corinne is correct. Write a similar statement using degrees.

> **Corinne**
> *The complement of an angle measure θ in radians is $\left(\frac{\pi}{2} - \theta\right)$ radians.*

In radians, 90° is $\frac{\pi}{2}$ radians, so the complement of an unknown angle measure θ in radians would be $\left(\frac{\pi}{2} - \theta\right)$ radians.

The complement of an angle measure x in degrees is $90 - x$ degrees.

Be prepared to share your methods and solutions.

15

Check for Students' Understanding

1. Convert each angle in degree measure to radian measure.

a. 500°

$\frac{25\pi}{9}$

b. 390°

$\frac{13\pi}{6}$

c. 150°

$\frac{5\pi}{6}$

2. Convert each angle in radian measure to degree measure.

a. $\frac{\pi}{10}$

18°

b. $\frac{7\pi}{6}$

210°

c. $\frac{14\pi}{15}$

168°

Triangulation

The Sine and Cosine Functions

15.3

LEARNING GOALS

In this lesson, you will:

- Define the sine and cosine functions.
- Calculate values for the sine and cosine of reference angles.
- Define the sine and cosine of an angle as a coordinate of a point on the unit circle.
- Graph and compare the sine and cosine functions.

KEY TERMS

- sine function
- cosine function
- trigonometric function
- periodicity identity

ESSENTIAL IDEAS

- The sine ratio (sin) is the ratio of the length of the opposite side to the length of the hypotenuse in a right triangle:
  $\sin(\theta) = \frac{\text{opposite side}}{\text{hypotenuse}}$
- The cosine ratio (cos) is the ratio of the length of the adjacent side to the length of the hypotenuse in a right triangle:
  $\cos(\theta) = \frac{\text{adjacent side}}{\text{hypotenuse}}$
- The coordinates of the point where the terminal ray of a central angle θ intersects the unit circle can be written as $(\cos(\theta), \sin(\theta))$.
- The sine values are positive in the first and second quadrants and negative in the third and fourth quadrants. The cosine values are positive in the first and fourth quadrants and negative in the second and third quadrants.
- The period of the sine function is 2π and the period of the cosine function is 2π.
- The periodicity identity for the sine function is $\sin(x + 2\pi) = \sin(x)$, and the periodicity identity for the cosine function is $\cos(x + 2\pi) = \cos(x)$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-TF Trigonometric Functions

Extend the domain of trigonometric functions using the unit circle

3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\frac{\pi}{3}$, $\frac{\pi}{4}$ and $\frac{\pi}{6}$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

Model periodic phenomena with trigonometric functions

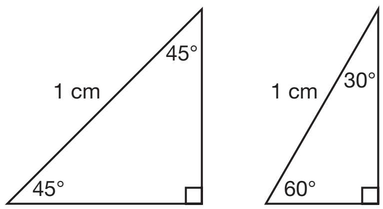
5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

15

Overview

The trigonometric functions sine and cosine and the periodicity identities are introduced in this lesson. Students explore the values of the sine and cosine functions using the unit circle and reference triangles centered at the origin in each of the four quadrants, such that $\theta = 30°, 45°, 60°$, etc. The sine and cosine of an angle are identified as the coordinates of any point on the unit circle and the coordinate values are used to graph the functions on a coordinate plane that is extended to 8π radians. Students calculate the values of the functions for different values of x and use the information to conclude that for all x, $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$. The measures of the angles in both radians and degrees and the value of each function related to each angle measure are summarized in a table. Students compare the functions and identify the key characteristics.

Warm Up



1. In the 45°-45°-90° triangle:

a. Describe the relationship between the length of the hypotenuse and the length of the side opposite the 45° angle.

The length of the hypotenuse is equal to the length of the side opposite the 45° angle times $\sqrt{2}$.

Hypotenuse $=$ leg $\cdot \sqrt{2}$

b. Determine the exact length of the side opposite the 45° angle.

$$\begin{aligned} \text{Hypotenuse} &= \text{leg} \cdot \sqrt{2} \\ 1 &= \text{leg} \cdot \sqrt{2} \\ \text{leg} &= \frac{1}{\sqrt{2}} \end{aligned}$$

2. In the 30°-60°-90° triangle:

a. Describe the relationship between the length of the hypotenuse and the length of the side opposite the 30° angle.

The length of the side opposite the 30° angle is equal to one-half the length of the hypotenuse.

Side opposite the 30° angle $= \frac{1}{2}h$

b. Determine the exact length of the side opposite the 30° angle.

$$\begin{aligned} \text{Side opposite the 30° angle} &= \frac{1}{2}h \\ &= \frac{1}{2}(1) \\ &= \frac{1}{2} \end{aligned}$$

c. Describe the relationship between the length of the hypotenuse and the length of the side opposite the 60° angle.

The length of the side opposite the 60° angle is equal to one-half the length of the hypotenuse times $\sqrt{3}$.

Side opposite the 60° angle $= \frac{1}{2}h\sqrt{3}$

d. Determine the exact length of the side opposite the 60° angle.

$$\begin{aligned}\text{Side opposite the 60° angle} &= \frac{1}{2}h\sqrt{3}\\ &= \frac{1}{2}(1)\sqrt{3}\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

15.3

Triangulation

The Sine and Cosine Functions

LEARNING GOALS

In this lesson, you will:

- Define the sine and cosine functions.
- Calculate values for the sine and cosine of reference angles.
- Define the sine and cosine of an angle as a coordinate of a point on the unit circle.
- Graph and compare the sine and cosine functions.

KEY TERMS

- sine function
- cosine function
- trigonometric function
- periodicity identity

Triangulation is a method that involves using triangles to identify the coordinates of an object or the distance of an object from a point. It is used in astrometry—a branch of astronomy involved in precisely measuring the locations and distances of astronomical bodies—navigation, and meteorology.

And of course, triangulation can be used to determine the coordinates of points on the unit circle!

Problem 1

Definitions of the sine and cosine ratio are given and side-length relationships for a 30°-60°-90° triangle and a 45°-45°-90° triangle are reviewed. Students label the side lengths of a 30°-60°-90° triangle and a 45°-45°-90° triangle centered at the origin and drawn in the first quadrant on a unit circle. Students use their answers to complete a table listing the sine and cosine values of the reference angle θ and the coordinates of the point where the terminal ray intersects the unit circle. They also write the coordinates of the intersection of the terminal ray and the unit circle at 0° and 90°. The values listed in the table are then used to label the coordinates of the points in the first quadrant on a unit circle.

Grouping

- Ask a student to read the information and definitions.
- Have students complete all parts of Questions 1 through 5 with a partner. Then have students share their responses as a class.

PROBLEM 1 Now You Will Determine the Right Triangle Connection

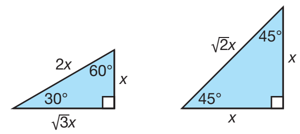
Recall that the sine ratio (sin) is the ratio of the length of the opposite side to the hypotenuse in a right triangle.

$$\sin(\theta) = \frac{\text{opposite side}}{\text{hypotenuse}}$$

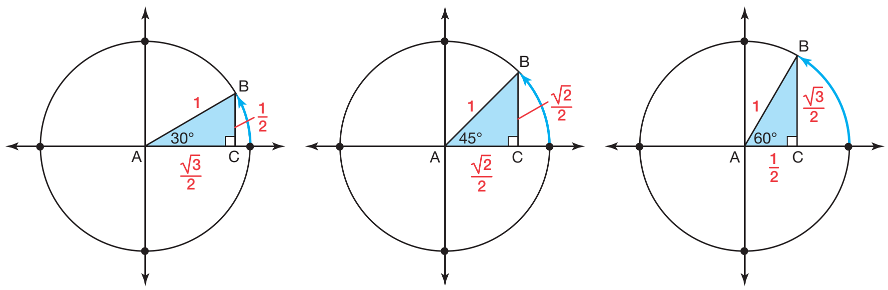
The cosine ratio (cos) is the ratio of the length of the adjacent side to the hypotenuse.

$$\cos(\theta) = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

The side-length relationships for a 30°–60°–90° triangle and a 45°–45°–90° triangle are shown.



The diagram shows a right triangle ABC placed on a unit circle centered at the origin. The central angle measures $\theta = 30°$, $\theta = 45°$, and $\theta = 60°$ are shown.



1. What is the length of the hypotenuse c in each circle? Label the measures on each triangle.
   Each circle is a unit circle, and the hypotenuse is a radius, so the length of the hypotenuse in each circle is 1 unit.

2. Label the side lengths of the triangles in each diagram in radical form.
   See diagram.

Guiding Questions for Share Phase, Questions 1 and 2

- What is the relationship between the hypotenuse of each reference triangle and the radius of the unit circle?
- Is the hypotenuse of the triangle and the radius of the unit circle the same line segment?
- Why is the length of the hypotenuse in each unit circle equal to 1?

Guiding Questions for Share Phase, Question 3

- Why is the sine of each angle measure equal to the length of the opposite side?
- What ratio represents the sine of each central angle measure?
- Why is the sine of each central angle measure equal to the length of the opposite side in each right triangle?
- Why is the cosine of each angle measure equal to the length of the adjacent side?
- What ratio represents the cosine of each central angle measure?
- Why is the cosine of each central angle measure equal to the length of the adjacent side in each right triangle?
- Is the cosine value at 0° equal to 0 or equal to 1? How do you know?
- Is the sine value at 0° equal to 0 or equal to 1? How do you know?
- Are the coordinates of the intersection of the terminal ray and the unit circle at 0° represented by the point (1, 0) or (0, 1)?
- Is the cosine value at 90° equal to 0 or equal to 1? How do you know?
- Is the sine value at 90° equal to 0 or equal to 1? How do you know?
- Are the coordinates of the intersection of the terminal ray and the unit circle at 90° represented by the point (1, 0) or (0, 1)?

3. The hypotenuse of each right triangle represents the terminal ray of a central angle which intersects the unit circle at point *B*.

a. Complete the table to record the sine and cosine of each angle measure, θ, and the coordinates of the point where the terminal ray intersects the unit circle. Explain your reasoning.

θ	$\cos(\theta)$	$\sin(\theta)$	Coordinates of point *B*, Intersection of Terminal Ray and Unit Circle
30°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
60°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

- The sine of each angle measure is equal to the length of the opposite side.

  The sine of each central angle measure is equal to the ratio $\frac{\text{opposite side}}{\text{hypotenuse}}$.

  Since the hypotenuse of the unit circle is 1, the sine of each central angle measure is equal to the length of the opposite side in each right triangle.

- The cosine of each angle measure is equal to the length of the adjacent side.

  The cosine of each central angle measure is equal to the ratio $\frac{\text{adjacent side}}{\text{hypotenuse}}$.

  Since the hypotenuse of the unit circle is 1, the cosine of each central angle measure is equal to the length of the adjacent side in each right triangle.

b. Write the coordinates of the intersection of the terminal ray and the unit circle at 0°.

The coordinates of the intersection of the terminal ray and the unit circle at 0° are (1, 0).

$\cos(0°) = 1$, $\sin(0°) = 0$

c. Write the coordinates of the intersection of the terminal ray and the unit circle at 90°.

The coordinates of the intersection of the terminal ray and the unit circle at 90° are (0, 1).

$\cos(90°) = 0$, $\sin(90°) = 1$

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Guiding Questions for Share Phase, Questions 4 and 5

- Is the cosine of the central angle measure the same as the *x*-coordinate or *y*-coordinate of the point?
- Why is the cosine of the central angle measure the same as the *x*-coordinate of the point?
- Is the sine of the central angle measure the same as the *x*-coordinate or *y*-coordinate of the point?
- Why is the sine of the central angle measure the same as the *y*-coordinate of the point?
- When writing the coordinates of the points on the unit circle, why is it important to use radicals as opposed to their decimal equivalents?

4. Jorge conjectured that the coordinates of the point where the terminal ray of a central angle θ intersects the unit circle can always be written as $(\cos(\theta), \sin(\theta))$.

Do you think Jorge's conjecture is correct? Explain your reasoning.

Answers will vary.

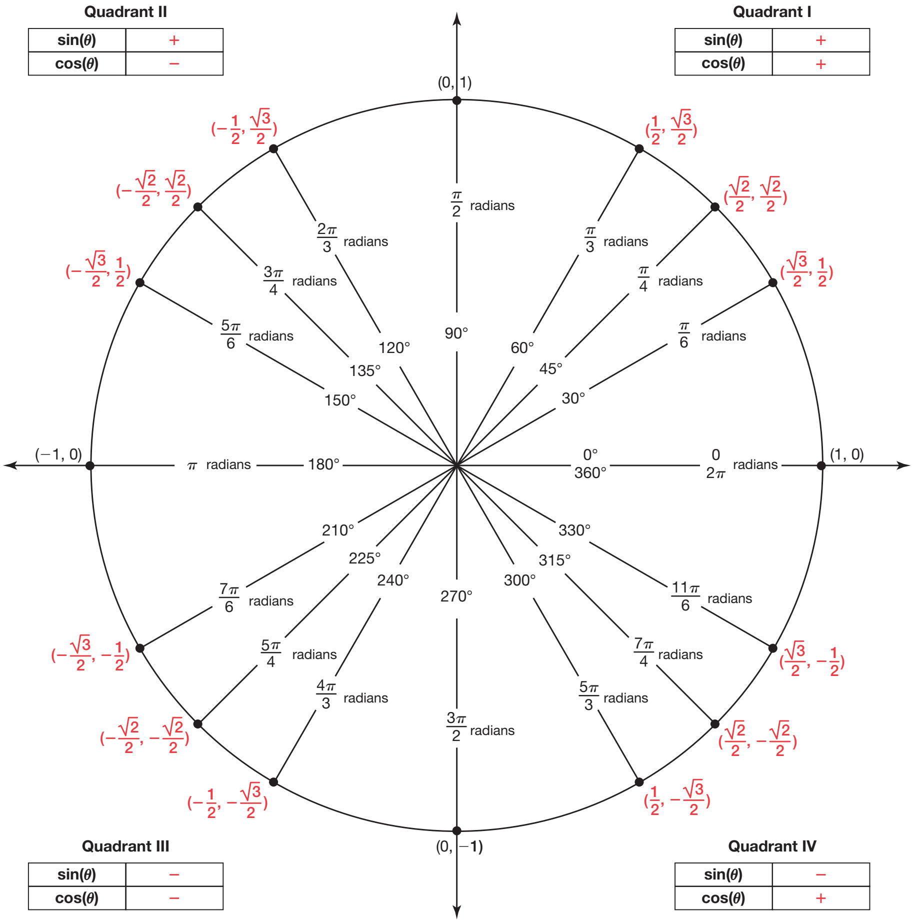
The ordered pair $(\cos(\theta), \sin(\theta))$ represents the location of any point on the unit circle, given an angle measure θ.

Because the hypotenuse is 1, the cosine of the central angle measure is the *x*-coordinate of the point and the sine is the *y*-coordinate of the point.

5. Use your answers in Questions 1 through 3 to determine the coordinates of the points in the first quadrant on the unit circle on the next page. Label the coordinates.

See unit circle.

Sine and Cosine on the Unit Circle



Grouping

Have students complete all parts of Questions 6 and 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 6 and 7

- Is sin(30°) and sin$\left(\frac{\pi}{6}\text{ radian}\right)$ the same value?
- Which coordinate of the point on the unit circle is used to determine sin$\left(\frac{\pi}{6}\text{ radian}\right)$?
- Is cos(30°) and cos$\left(\frac{\pi}{6}\text{ radian}\right)$ the same value?
- Which coordinate of the point on the unit circle is used to determine cos$\left(\frac{\pi}{6}\text{ radian}\right)$?
- Is sin(45°) and sin$\left(\frac{\pi}{4}\text{ radian}\right)$ the same value?
- How are the coordinates of the point associated with sin(30°) related to the coordinates of the point associated with cos(60°)?
- How are the coordinates of the point associated with sin(60°) related to the coordinates of the point associated with cos(30°)?
- How are the coordinates of the point associated with sin(45°) related to the coordinates of the point associated with cos(45°)?

6. Use the unit circle to evaluate each measure.

- $\sin\left(\frac{\pi}{6}\text{ radian}\right) = \frac{1}{2}$
- $\cos\left(\frac{\pi}{6}\text{ radian}\right) = \frac{\sqrt{3}}{2}$
- $\sin\left(\frac{\pi}{4}\text{ radian}\right) = \frac{\sqrt{2}}{2}$
- $\cos\left(\frac{\pi}{4}\text{ radian}\right) = \frac{\sqrt{2}}{2}$

7. For each angle measure in Question 6, evaluate the sine and cosine of the complement. Explain your reasoning.

The complement of $\frac{\pi}{6}$ radian, or 30°, is 90° − 30°, or 60°. This is equivalent to $\frac{\pi}{3}$ radians.

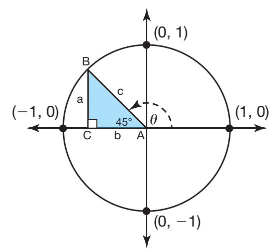
$\sin\left(\frac{\pi}{3}\text{ radians}\right) = \frac{\sqrt{3}}{2}$; $\cos\left(\frac{\pi}{3}\text{ radians}\right) = \frac{1}{2}$

The complement of $\frac{\pi}{4}$ radian, or 45°, is 90° − 45°, or 45°. This is equivalent to $\frac{\pi}{4}$ radian.

$\sin\left(\frac{\pi}{4}\text{ radian}\right) = \frac{\sqrt{2}}{2}$; $\cos\left(\frac{\pi}{4}\text{ radian}\right) = \frac{\sqrt{2}}{2}$

PROBLEM 2 To Infinity and Beyond the First Quadrant

1. The diagram shows a 45° central angle positioned in the second quadrant on the unit circle.



a. State the measure of θ in degrees and in radians. Explain how you determined your answer.

$\theta = \frac{3\pi}{4}$ radians, or 135°

The angle measure in degrees is 180° − 45°, or 135°.

The angle measure in radians is π radians $- \frac{\pi}{4}$ radian, or $\frac{3\pi}{4}$ radians.

b. Identify the coordinates of the point at which the terminal ray of the angle intercepts the circle. Explain how you determined your answer.

The coordinates of the point are $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Because the coordinates are $(\cos(\theta), \sin(\theta))$, I used the sine and cosine values for the 45°-45°-90° triangle to determine these coordinates. In this case, the cosine is negative because the point is located in the second quadrant.

Problem 2

A 45°-45°-90° triangle centered at the origin is drawn in the second quadrant on the unit circle. Students state the measure of θ in degrees and radians. They identify the coordinates of the point where the terminal ray intersects the unit circle. Students conclude the y-coordinates or sine values have remained unchanged when compared to the same angle in the first quadrant, but the x-coordinates or cosine values now have a negative value when compared to the x-values of the same angle in the first quadrant. Students use their prior knowledge to label the coordinates of the remaining points on the Sine and Cosine on the Unit Circle diagram provided.

Grouping

Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- What is the supplement of a 45° angle?
- What is π radians $-\frac{\pi}{4}$ radians?
- Is the cosine value positive or negative when the point on the circle is located in the second quadrant?
- Do the coordinates of this point and the coordinates of the symmetrical point in the first quadrant have the same y-coordinates?
- Are the sine values for 45° and 135° the same values?
- Do the coordinates of this point and the coordinates of the symmetrical point in the first quadrant have the same x-coordinates?
- Are the cosine values for 45° and 135° the same values?
- Are the coordinates of the point where the terminal ray of a central angle θ intersects the unit circle always written as $(\cos(\theta), \sin(\theta))$?
- Why is the value of sine positive in the first and second quadrants?
- Why is the value of sine negative in the third and fourth quadrants?

c. What do you notice about the coordinates of this point and the coordinates of the symmetrical point in the first quadrant?

The y-coordinates are the same, which means the sine values for 45° and 135°, or $\frac{\pi}{4}$ radians and $\frac{3\pi}{4}$ radians, are the same.

The x-coordinates, or cosine values, are the same except that in the second quadrant the cosine is a negative value because the x-coordinate is negative.

2. Use what you know about symmetry to label the coordinates of the remaining points on your Sine and Cosine on the Unit Circle diagram from Problem 1.

See unit circle.

3. Look back at Jorge's conjecture in Problem 1, Question 4. Is his conjecture correct? Explain your reasoning.

Yes. Jorge's conjecture is correct. The coordinates of the point where the terminal ray of a central angle θ intersects the unit circle can always be written as $(\cos \theta, \sin \theta)$.

4. Describe when the values of sine and cosine are positive and negative in the unit circle. Label this information on your Sine and Cosine on the Unit Circle diagram in Problem 1.

The sine values are positive in the first and second quadrants and negative in the third and fourth quadrants.

The cosine values are positive in the first and fourth quadrants and negative in the second and third quadrants.

5. Give examples to support Ray's conclusion.

Answers will vary.

The angles 0 and 2π radians are values of θ that have the same sine and cosine values.

Ray

Many different central angle measures have the same sine or cosine values.

- Why is the value of cosine positive in the first and fourth quadrants?
- Why is the value of cosine negative in the second and third quadrants?
- Are the angles 0 radians and 2π radians examples of values of θ that have the same sine values?
- Are the angles 0 radians and 2π radians examples of values of θ that have the same cosine values?

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Problem 3

Students use the y-values of the coordinates of the points where the terminal ray intersects the unit circle to create the graph of $y = \sin(x)$ and they use the x-values of the coordinates of the points where the terminal ray intersects the unit circle to create the graph of $y = \cos(x)$. Each function takes input values, θ, and outputs real number values which correspond to the coordinates of points on the unit circle. Students extend the graphs of the functions to $x = 8\pi$ and determine the values of each function at 0, 2π, 4π, 6π, and 8π. They conclude the period of each function is 2π radians and the two periodicity identities are given.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

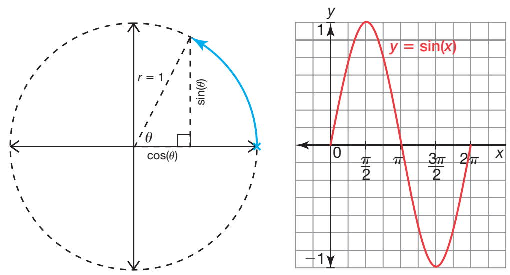
Guiding Questions for Share Phase, Questions 1 and 2

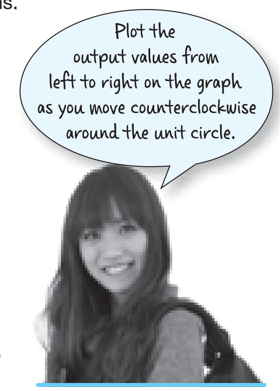
- What is the value of the function $y = \sin(x)$ when θ has the value of 0 radians? $\frac{\pi}{2}$ radians? π radians? $\frac{3\pi}{2}$ radians? 2π radians?
- Were the x-values or y-values of the coordinates of the points where the terminal ray intersects the unit circle used to create the graph of $y = \sin(x)$?

PROBLEM 3 Orbiting Back to the Functions

1. Use your completed unit circle from Problem 1 to graph the function $y = \sin(x)$.

   a. As the terminal ray traverses the unit circle counterclockwise in standard position, plot the output value, $\sin(\theta)$, that corresponds to the input value, θ, which is the radian measure of the central angle, from 0 to 2π radians.



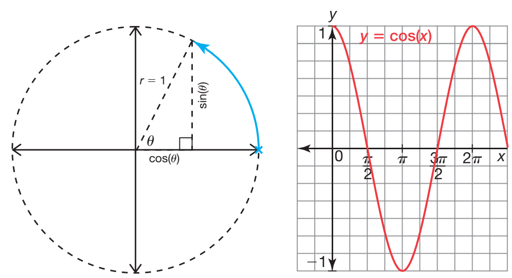


   b. What coordinate values on the unit circle did you use to create the graph of $y = \sin(x)$?

   I used the y-values of the coordinates of the points where the terminal ray intersects the unit circle to create the graph of $y = \sin(x)$.

2. Use your completed Sine and Cosine on the Unit Circle diagram from Problem 1 to graph the function $y = \cos(x)$.

   a. As the terminal ray traverses the unit circle counterclockwise in standard position, plot the output value, $\cos(\theta)$, that corresponds to the input value, θ, which is the radian measure of the central angle, from 0 to 2π radians.



   b. What coordinate values on the unit circle did you use to create the graph of $y = \cos(x)$?

   I used the x-values of the coordinates of the points where the terminal ray intersects the unit circle to create the graph of $y = \cos(x)$.

- What is the value of the function $y = \cos(x)$ when θ has the value of 0 radians? $\frac{\pi}{2}$ radians? π radians? $\frac{3\pi}{2}$ radians? 2π radians?
- Were the x-values or y-values of the coordinates of the points where the terminal ray intersects the unit circle used to create the graph of $y = \cos(x)$?

Grouping

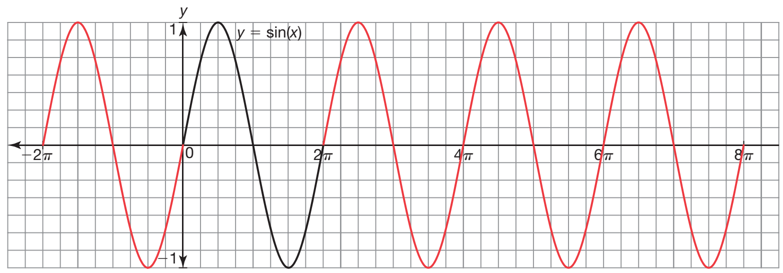
- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 3 through 6 with a partner. Then have students share their responses as a class.

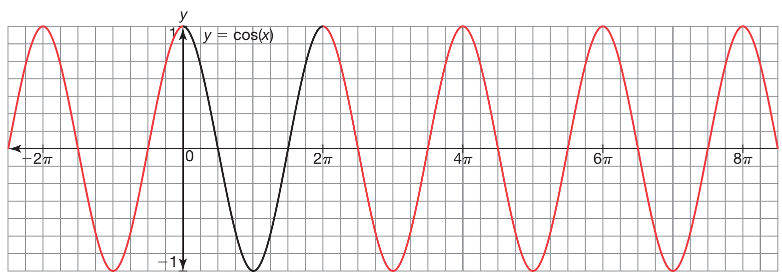
Guiding Questions for Share Phase, Questions 3 through 6

- If the graph of $y = \sin(x)$ is extended to $x = 8\pi$, how many instances does the value of the function equal 0? 1? -1? What are the values of θ when this occurs?
- If the graph of $y = \cos(x)$ is extended to $x = 8\pi$, how many instances does the value of the function equal 0? 1? -1? What are the values of θ when this occurs?
- Does each radian measure that is a multiple of 2π represent a complete revolution of the terminal ray of the central angle around the unit circle?
- Why is the value of each function at $x = 4\pi$, 6π, and 8π radians the same as the value of each function at 0 radians and 2π radians?
- Does $\sin(x + 2\pi) = \sin(x)$ for all values of x?
- Does $\cos(x + 2\pi) = \cos(x)$ for all values of x?
- What is the period of the sine function?
- What is the period of the cosine function?

You have graphed the *sine function* and *cosine function*. The **sine function** and **cosine function** are periodic **trigonometric functions**. Each of these trigonometric functions takes angle measures (θ values) as inputs and outputs real number values, which correspond to coordinates of points on the unit circle.

3. Extend the graphs of the functions $y = \sin(x)$ and $y = \cos(x)$ to $x = 8\pi$ radians.





a. Determine the values of $\sin(x)$ and $\cos(x)$ at 4π, 6π, and 8π radians.

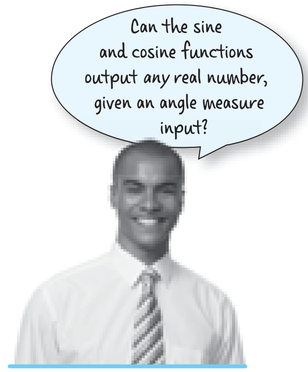
The value of the sine function at 4π, 6π, and 8π radians is 0.

The value of the cosine function at 4π, 6π, and 8π radians is 1.

b. Describe how you can determine each value from part (a) on the unit circle for each function.

Each radian measure that is a multiple of 2π represents a complete revolution of the terminal ray of the central angle around the unit circle, starting from $\theta = 0$ radians. So, the value of each function at $x = 4\pi$, 6π, and 8π radians is the same as the value of each function at 0 radians and 2π radians. For the sine function, this value is 0, and for the cosine function, this value is 1.

4. Extend the graphs of the functions $y = \sin(x)$ and $y = \cos(x)$ in Question 3 through $x = -2\pi$.



a. Determine each sine value.

- $\sin\left(-\frac{\pi}{2}\right) =$ -1
- $\sin(-\pi) =$ 0
- $\sin\left(-\frac{3\pi}{2}\right) =$ 1
- $\sin(-2\pi) =$ 0

b. Determine each cosine value.

- $\cos\left(-\frac{\pi}{2}\right) =$ 0
- $\cos(-\pi) =$ -1
- $\cos\left(-\frac{3\pi}{2}\right) =$ 0
- $\cos(-2\pi) =$ 1

5. Consider the values of $\sin(x + 2\pi)$. How do these values compare to the values of $\sin(x)$?

For all x, $\sin(x + 2\pi)$ is equal to $\sin(x)$.

6. Consider the values of $\cos(x + 2\pi)$. How do these values compare to the values of $\cos(x)$?

For all x, $\cos(x + 2\pi)$ is equal to $\cos(x)$.

The period of the sine function is 2π radians, and the period of the cosine function is 2π radians. Thus, you can write two *periodicity identities*:

- $\sin(x + 2\pi) = \sin(x)$
- $\cos(x + 2\pi) = \cos(x)$

Each of these is called a **periodicity identity** because they are each based on the period of the function, 2π.

Talk the Talk

Students summarize the information in this lesson by using their labeled unit circle to complete a table which lists angle measures and their sine and cosine values. Students compare and contrast the two functions, and identify the key characteristics of the functions. Students discuss how the trigonometric functions are related to each other using the language of transformations.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Do the sine and cosine functions have the same maximum and minimum values?
- Do the sine and cosine functions both have the domain of all real numbers?
- Do the sine and cosine functions have the same range?
- Do the sine and cosine functions have the same y-intercepts?
- Do the sine and cosine functions have the same x-intercepts?
- Do the sine and cosine functions have maxima and minima at the same values for x?

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Talk the Talk

1. Use your completed Sine and Cosine on the Unit Circle diagram from Problem 1 to complete the table.

Angle Measure (θ)		$\cos(\theta)$	$\sin(\theta)$
radians	degrees		
0	0°	1	0
$\frac{\pi}{6}$	30°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	60°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	90°	0	1
$\frac{2\pi}{3}$	120°	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	135°	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	150°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
π	180°	-1	0

Angle Measure (θ)		$\cos(\theta)$	$\sin(\theta)$
radians	degrees		
$\frac{7\pi}{6}$	210°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	225°	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{4\pi}{3}$	240°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{3\pi}{2}$	270°	0	21
$\frac{5\pi}{3}$	300°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{7\pi}{4}$	315°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{11\pi}{6}$	330°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
2π	360°	1	0

2. Compare and contrast the functions $y = \sin(x)$ and $y = \cos(x)$. Describe the similarities and differences between the two functions.

Answers will vary.

Similarities:

- The period is 2π radians.
- The maximum and minimum values are 1 and -1, respectively.
- The domain is all real numbers.
- The range is real numbers between -1 and 1, inclusive.

Differences:

- They have different y-intercepts. The y-intercept for the sine function is (0, 0), and the y-intercept for the cosine function is (0, 1).
- They have different x-intercepts.
- The maximum and minimum values occur at different values for x.

15 Guiding Questions for Share Phase, Questions 3 through 6

- Is the amplitude of each function equal to 1 or 2?
- Is the midline of each function described by the equation $x = 0$ or $y = 0$?
- Which function increases between $-\pi$ and 0?
- Which function has x-intercepts at 0, π, 2π, 3π . . . radians?
- Is the cosine function the sine function translated to the left by $\frac{\pi}{2}$ radians or translated to the right by $\frac{\pi}{2}$ radians?
- Is $\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$ or is $\cos(x) = \sin\left(x - \frac{\pi}{2}\right)$?

3. Identify each of the characteristics for $y = \sin(x)$ and $y = \cos(x)$.

	$y = \sin(x)$	$y = \cos(x)$
y-intercept(s)	(0, 0)	(0, 1)
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$[-1, 1]$	$[-1, 1]$
Period	2π	2π
Minimum Output Value	-1	-1
Maximum Output Value	1	1
Amplitude	1	1
Midline	$y = 0$	$y = 0$

4. Describe the intervals of increase and decrease for both the sine and cosine functions. Explain your reasoning.

Answers will vary.

The sine function is increasing between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ radians, between $\frac{3\pi}{2}$ and $\frac{5\pi}{2}$ radians, and so on. It is decreasing between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ radians, between $\frac{5\pi}{2}$ and $\frac{7\pi}{2}$ radians, and so on.

The cosine function is increasing between $-\pi$ and 0 radians, between π and 2π radians, and so on. It is decreasing between 0 and π radians, between 2π and 3π radians, and so on.

5. Determine x-intercepts of the functions $y = \sin(x)$ and $y = \cos(x)$.

a. x-intercepts for $y = \sin(x)$:

Answers will vary.
0, π, 2π, 3π radians

b. x-intercepts for $y = \cos(x)$:

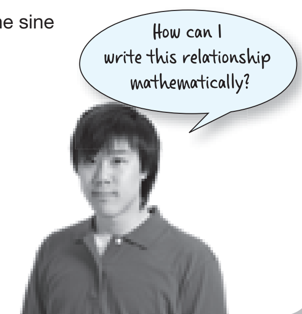
Answers will vary.
$\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ radians

6. Use the language of transformations to explain how the sine and cosine functions are related.

Answers will vary.

The cosine function is the sine function translated to the left by $\frac{\pi}{2}$ radians.

So, $\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$.



Be prepared to share your solutions and methods.

Check for Students' Understanding

Determine the measure of a positive angle and the measure of a negative angle that has the same sine and cosine value as the given angle measure.

1. $80°$

Answers will vary.

$440°, -280°$

2. $22°$

Answers will vary.

$382°, -338°$

3. $\frac{9\pi}{2}$

Answers will vary.

$\frac{\pi}{2}, -\frac{3\pi}{2}$

4. $\frac{28\pi}{2}$

Answers will vary.

$2\pi, -2\pi$

15

15.4

Pump Up the Amplitude

Transformations of Sine and Cosine Functions

LEARNING GOALS

In this lesson, you will:

- Transform the graphs of the sine and cosine functions.
- Determine the amplitude, frequency, and phase shift of transformed functions.
- Graph transformed sine and cosine functions using a description of the period, phase shift, and amplitude.

KEY TERMS

- frequency
- phase shift

ESSENTIAL IDEAS

- Multiplying a function $y = \sin(x)$ or $y = \cos(x)$ by a constant A such that $y = A\sin(x)$ or $y = A\cos(x)$ dilates the function vertically by a factor of $|A|$. If $|A| > 1$, the graph is stretched vertically by a factor of $|A|$, if $0 < |A| < 1$, the graph is compressed vertically by a factor of $|A|$, and if $A < 0$, the graph is reflected across the x-axis.
- The amplitude of a sine or cosine function is one-half the absolute value of the difference between the maximum and minimum values of the function.
- Multiplying the argument of a function $y = \sin(x)$ or $y = \cos(x)$ by a constant B such that $y = \sin(Bx)$ or $y = \cos(Bx)$ dilates the function horizontally by a factor of $|B|$. If $|B| > 1$, the graph is horizontally compressed a factor of $\frac{1}{|B|}$, if $0 < |B| < 1$, the graph is horizontally stretched by a factor of $\frac{1}{|B|}$, and if $B < 0$, the graph is reflected across the y-axis.
- The period of the functions $y = \sin(Bx)$ and $y = \cos(Bx)$ is $\frac{2\pi}{|B|}$.
- The frequency of a periodic function is the reciprocal of the period and specifies the number of repetitions of the graph of a periodic function per unit.
- Adding a constant C to the argument of a function $y = \sin(x)$ or $y = \cos(x)$ such that $y = \sin(x + C)$ or $y = \cos(x + C)$ translates the function horizontally. If $|C| > 0$, the graph is shifted to the left C units, and if $C < 0$, the graph is shifted to the right C units.
- Transforming a periodic function by subtracting a C-value from the argument of the function results in horizontal translations of the function called phase shifts.

15

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-TF Trigonometric Functions

Extend the domain of trigonometric functions using the unit circle

3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\frac{\pi}{3}$, $\frac{\pi}{4}$ and $\frac{\pi}{6}$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

Model periodic phenomena with trigonometric functions

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

Overview

The key characteristics of the sine and cosine functions as well as the general transformational function form of an equation are reviewed. Student explore the key characteristics and graphs of transformed functions such as $y = A\sin(x)$, $y = A\cos(x)$, $y = \sin(Bx)$, and $y = \cos(Bx)$, $y = \sin(x + C)$, $y = \cos(x + C)$, $y = \sin(x) + D$, and $y = \cos(x) + D$, where A, B, C and D are constants. They compare, contrast, and summarize the effects of each constant on both functions graphically and algebraically.

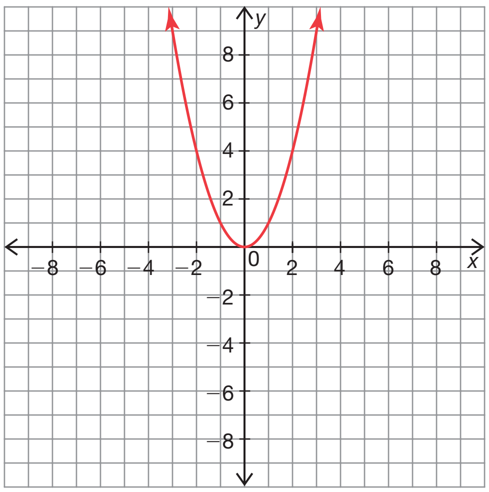
Warm Up

Consider the function $f(x) = 2(x - 3)^2 + 4$.

1. What is the basic function?

The basic function is $y = x^2$.

2. Graph the basic function.



3. Describe how you could transform the graph of the basic function in Question 2 to graph $f(x)$.

Answers will vary.

The vertex of $f(x)$ is (3, 4), and the vertex of the basic function is (0,0). The graph of $f(x)$ is a horizontal shift of the basic function 3 units to the right, and a vertical shift of the basic function 4 units up.

The function $f(x)$ is dilated by a factor of 2, which means all y-values of the basic function will be multiplied by 2. The graph of $f(x)$ will increase more quickly and look thinner than the graph of the basic function.

15.4

Pump Up the Amplitude

Transformations of Sine and Cosine Functions

LEARNING GOALS

In this lesson, you will:

- Transform the graphs of the sine and cosine functions.
- Determine the amplitude, frequency, and phase shift of transformed functions.
- Graph transformed sine and cosine functions using a description of the period, phase shift, and amplitude.

KEY TERMS

- frequency
- phase shift

The word *frequency* is used in many different ways—inside and outside of mathematics. When it comes to sound, a greater frequency means a higher pitch and a lesser frequency means a lower pitch. Higher pitch sound waves vibrate faster than lower pitch sound waves.

The frequencies of sound waves are measured in a unit called a Hertz (Hz). One Hertz is equal to 1 vibration per second. Humans can typically hear sounds with frequencies in the range of 20 Hz to 20,000 Hz.

Dogs, cats, bats, and dolphins can all hear sounds with very high frequencies—45,000 Hz, 85,000 Hz, 120,000 Hz, and 200,000 Hz, respectively. Elephants, on the other hand, can hear low-frequency sounds down to 5 Hz.

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Problem 1

The key characteristics of the sine and cosine functions are compared to the transformed function $y = A \sin(x)$ and $y = A \cos(x)$, where the A-value changes. Students determine how the A-value changes affect the minima, maxima, and amplitude. They sketch and compare graphs and key characteristics of the transformed functions noting similarities and differences.

Grouping

- Ask a student to read the information and definitions. Discuss as a class.
- Have students complete all parts of Questions 1 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Which constant is associated with the dilation of the basic function vertically?
- Under what circumstance does the constant A stretch the graph vertically by a factor of $|A|$?
- Under what circumstance does the constant A compress the graph vertically by a factor of $|A|$?
- Under what circumstance does the constant A reflect the graph across the x-axis?

PROBLEM 1 Amplitude

The table shows the characteristics of the graphs of the sine and cosine functions.

	$y = \sin(x)$	$y = \cos(x)$
y-intercept	(0, 0)	(0, 1)
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$[-1, 1]$	$[-1, 1]$
Period	2π	2π
Minimum Output Value	-1	-1
Maximum Output Value	1	1
Amplitude	1	1
Midline	$y = 0$	$y = 0$

Recall that transformations performed on any function $f(x)$ to form a new function $g(x)$ can be described by the transformational function form:

$$g(x) = Af(B(x - C)) + D$$

where A, B, C, and D represent constants.

1. In general, what effect does multiplying a function $y = f(x)$ by a constant, A, have on the graph of the function?

Multiplying the function by a constant dilates the basic function vertically by a factor of $|A|$.

- If $|A| > 1$, the graph is stretched vertically by a factor of $|A|$.
- If $0 < |A| < 1$, the graph is compressed vertically by a factor of $|A|$.
- If $A < 0$, the graph is reflected across the x-axis.

Guiding Questions for Share Phase, Questions 2 through 7

- How do you determine points on the graph of $g(x) = 2\sin(x)$?
- How do you determine points on the graph of $h(x) = \frac{1}{2}\sin(x)$?
- Do all of the graphs have the same period? What is the period?
- Do all of the graphs have the same x-intercepts? What are the x-intercepts?
- Do all of the graphs have the same y-intercepts? What are the y-intercepts?
- Why do all of the graphs have different maximum values?
- Why do all of the graphs have the different minimum values?
- How is the amplitude calculated for each function?
- Which part of the equation of the function is helpful when determining the maximum and minimum values?
- Which part of the equation of the function is helpful when determining the amplitude?

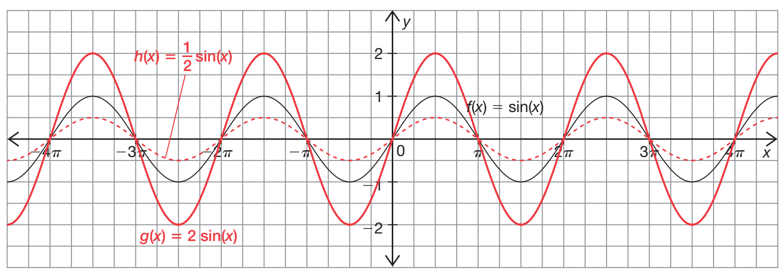
2. Which characteristics of the transformed function $y = A\sin(x)$ will differ from those of the basic function $y = \sin(x)$ if $|A| > 0$? Which characteristics will remain the same? Explain your predictions.

	$y = A\sin(x)$	
	Will Change	**Won't Change**
***y*-intercept**		×
Domain		×
Range	×	
Period		×
Minimum Output Value	×	
Maximum Output Value	×	
Amplitude	×	
Midline		×

Answers will vary.

I think the range and the amplitude will increase. The minimum will decrease, and the maximum will increase. The y-intercept, domain, period, and midline will not change.

3. A graph of the function $f(x) = \sin(x)$ is shown. Sketch the graphs of the functions $g(x) = 2\sin(x)$ and $h(x) = \frac{1}{2}\sin(x)$ on the same coordinate plane.



4. What similarities and differences do you notice about the three functions with respect to their periods, intercepts, and maximum and minimum values?

All of the graphs have the same period, *x*-intercepts, and *y*-intercepts.

The maximum and minimum values of each graph are different.

5. How do your graphs of the transformed functions compare with your predictions in Question 2?

Answers will vary.

Recall that the amplitude of a sine or cosine function is one-half the absolute value of the difference between the maximum and minimum values of the function.

6. Determine the maximum, minimum, and amplitude of each function you graphed.

a. $g(x) = 2\sin(x)$

maximum $= 2$

minimum $= -2$

$$\text{amplitude} = \frac{1}{2}|2 - (-2)| = \frac{1}{2}(4) = 2$$

b. $h(x) = \frac{1}{2}\sin(x)$

maximum $= \frac{1}{2}$

minimum $= -\frac{1}{2}$

$$\text{amplitude} = \frac{1}{2}\left|\frac{1}{2} - \left(-\frac{1}{2}\right)\right| = \frac{1}{2}(1) = \frac{1}{2}$$

7. Determine the maximum, minimum, and amplitude of each cosine function.

a. $f(x) = \cos(x)$

maximum $= 1$

minimum $= -1$

$$\text{amplitude} = \frac{1}{2}|1 - (-1)| = \frac{1}{2}(2) = 1$$

b. $g(x) = 3\cos(x)$

maximum $= 3$

minimum $= -3$

$$\text{amplitude} = \frac{1}{2}|3 - (-3)| = \frac{1}{2}(6) = 3$$

c. $h(x) = \frac{1}{4}\cos(x)$

maximum $= \frac{1}{4}$

minimum $= -\frac{1}{4}$

$$\text{amplitude} = \frac{1}{2}\left|\frac{1}{4} - \left(-\frac{1}{4}\right)\right| = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$$

Problem 2

The key characteristics of the sine and cosine functions are compared to the transformed function $y = \sin(Bx)$ and $y = \cos(Bx)$, where the B-value changes. Students determine how the B-value changes affect the minima, maxima, and amplitude. They sketch and compare graphs and key characteristics of the transformed functions noting similarities and differences.

Grouping

Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Which constant is associated with the horizontal stretch or compression of the basic function?
- Under what circumstance does the constant B stretch the graph horizontally by a factor of $\frac{1}{|B|}$?
- Under what circumstance does the constant B compress the graph horizontally by a factor of $\frac{1}{|B|}$?
- Under what circumstance does the constant B reflect the graph across the y-axis?

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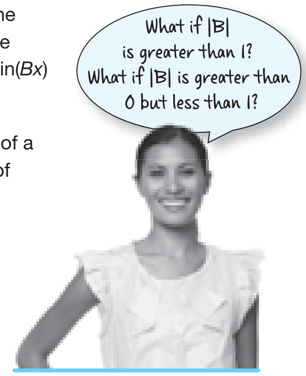
PROBLEM 2 Period and Frequency

Let's consider what effect multiplying the argument of a sine or cosine function by a constant, B, has on the graph of the function. The transformed function can be written as $y = \sin(Bx)$ or $y = \cos(Bx)$.



1. In general, what effect does multiplying the argument of a function $y = f(x)$ by a constant, B, have on the graph of the function?
   - If $|B| > 1$, then the graph is horizontally compressed by a factor of $\frac{1}{|B|}$.
   - If $0 < |B| < 1$, then the graph is horizontally stretched by a factor of $\frac{1}{|B|}$.
   - If $B < 0$, the graph is reflected across the y-axis.

2. Which characteristics of the transformed function $y = \cos(Bx)$ will differ from those of the basic function $y = \cos(x)$ if $|B| > 0$? Which characteristics will remain the same? Explain your predictions.

	$y = \cos(Bx)$	
	Will Change	**Won't Change**
y-intercept		×
Domain		×
Range		×
Period	×	
Minimum Output Value		×
Maximum Output Value		×
Amplitude		×
Midline		×

Answers will vary.

I think the period will either increase or decrease, depending on the value of $|B|$, because the graph of the transformed function will either be horizontally compressed or horizontally stretched. All of the other characteristics will remain the same.

If $|B| > 1$, the period will decrease, because the graph will be horizontally compressed, which means that the number of repetitions of the graph will increase across the same domain.

If $0 < |B| < 1$, the period will increase, because the graph will be horizontally stretched, which means that the number of repetitions of the graph will decrease across the same domain.

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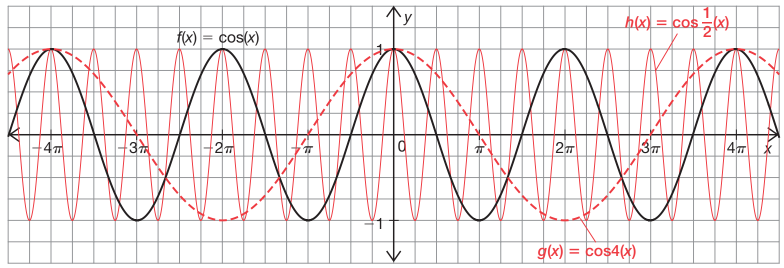
Guiding Questions for Share Phase, Questions 3 through 7

- How do you determine points on the graph of $g(x) = \cos(4x)$?
- How do you determine points on the graph of $h(x) = \cos\left(\frac{1}{2}x\right)$?
- Do all of the graphs have the same amplitude? What is the amplitude?
- Do all of the graphs have the same maximum values? What is the maximum value?
- Do all of the graphs have the same minimum values? What is the minimum value?
- Do all of the graphs have the same y-intercept? What is the y-intercept?
- Why do all of the graphs have different x-intercepts? What are the x-intercepts?
- Why do all of the graphs have different periods? What are the periods?
- How is the amplitude calculated for each function?
- Do both functions have the same coefficient? What is the coefficient in each function?
- What is the relationship between the coefficient of the function and the amplitude?
- Has the coefficient of each function's argument changed? Does this affect the period of the function?

3. A graph of the function $f(x) = \cos(x)$ is shown. Sketch the graphs of the functions $g(x) = \cos(4x)$ and $h(x) = \cos\left(\frac{1}{2}x\right)$ on the same coordinate plane.



4. What similarities and differences do you notice about the three functions with respect to their periods, intercepts, and maximum and minimum values?

All of the graphs have the same amplitude, and the maximum and minimum values are the same. All of the graphs have the same y-intercept.

The period of each graph is different, and the x-intercepts are different.

5. How do your graphs of the transformed functions compare with your predictions in Question 2?

Answers will vary.

6. How do the equations of the functions you graphed relate to the similarities and differences in the graphs?

The amplitudes are the same because each function has the same coefficient.

The period has changed because the coefficient of each function's argument has changed. The larger the absolute value of the coefficient of x, the shorter the period.

Grouping

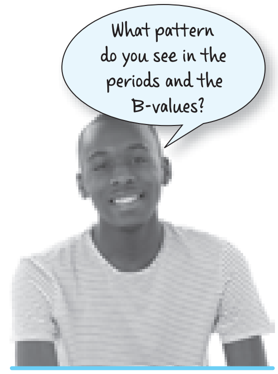
- Ask a student to read the worked example and discuss as a class.
- Complete Question 8 as a class.

Guiding Questions for Discuss Phase, Question 8

- What is the relationship between the coefficient of x and the length of the period of the function?
- How is the period of the function related to the frequency of the function?
- If the period of each function is $\frac{2\pi}{|B|}$, what is the reciprocal, or frequency of the function?

Recall that the period of a periodic function is the length of the smallest interval over which the function repeats.

7. Determine the period of each function from the graph.



a. $f(x) = \cos(x)$

The period of this function is 2π radians.

b. $g(x) = \cos(4x)$

The period of this function is $\frac{\pi}{2}$ radians.

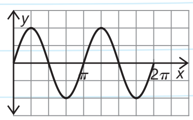
c. $h(x) = \cos\left(\frac{1}{2}x\right)$

The period of this function is 4π radians.

The B-value stretches or compresses a periodic function horizontally, so changes to the B-value have an effect on the period of the function.

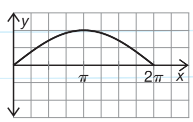
The parent function $y = \sin(x)$ has a period of 2π radians.

When the B-value is 2, there are 2 repetitions of the function in the original period, so the period is $\frac{1}{|B|} \cdot 2\pi$, or $\frac{1}{2} \cdot 2\pi = \pi$ radians.



$y = \sin(2x)$

When the B-value is $\frac{1}{2}$, there is $\frac{1}{2}$ of a repetition of the function in the original period, so the period is $\frac{1}{|B|} \cdot 2\pi$, or $2 \cdot 2\pi = 4\pi$ radians.



$y = \sin(\frac{1}{2}x)$

8. Write an expression to describe the period of the functions $y = \sin(Bx)$ and $y = \cos(Bx)$.

The period of each function is $\frac{2\pi}{|B|}$.

15

Grouping

- Ask a student to read the information and the definition. Discuss as a class.
- Have students complete Questions 9 and 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 9 and 10

- Does $\frac{|B|}{2\pi}$ describe the period or the frequency of the function?
- Does $\frac{2\pi}{|B|}$ describe the period or the frequency of the function?
- What is the reciprocal of $\frac{2\pi}{3}$?
- What is the reciprocal of 3π?
- What is the reciprocal of 8π?

Frequency is related to the period of the function. The **frequency** of a periodic function is the reciprocal of the period and specifies the number of repetitions of the graph of a periodic function per unit.

9. Write an expression to describe the frequency of the functions $y = \sin(Bx)$ and $y = \cos(Bx)$. Explain your reasoning.

Since the frequency is the reciprocal of the period, the frequency of a function $y = \sin(Bx)$ or $y = \cos(Bx)$ is $\frac{|B|}{2\pi}$.

10. Determine the period and frequency of each sine function.

a. $f(x) = \sin(3x)$

The period of this function is $\frac{2\pi}{3}$ radians. The frequency is $\frac{3}{2\pi}$ radians.

b. $g(x) = \sin\left(\frac{2}{3}x\right)$

The period of this function is $2\pi \div \frac{2}{3}$, or $2\pi \cdot \frac{3}{2} = 3\pi$ radians. The frequency is $\frac{1}{3\pi}$ radian.

c. $h(x) = \sin\left(\frac{1}{4}x\right)$

The period of this function is $2\pi \div \frac{1}{4}$, or $2\pi \cdot 4 = 8\pi$ radians. The frequency is $\frac{1}{8\pi}$ radian.

Problem 3

The key characteristics of the sine and cosine functions are compared to the transformed function $y = \sin(x - C)$ and $y = \cos(x - C)$, where the C-value changes. Students determine how the C-value changes affect the minima, maxima, and amplitude. They sketch and compare graphs and key characteristics of the transformed functions noting similarities and differences.

Grouping

- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.
- Ask a student to read the information and definition after Question 3. Discuss as a class.

Guiding Questions for Share Phase, Questions 1 through 3

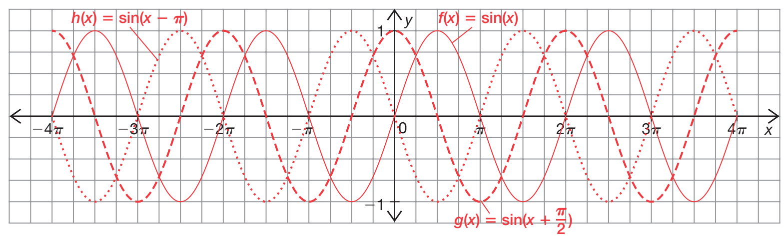
- Do all of the graphs have the same amplitude? What is the amplitude?
- Do all of the graphs have the same maximum values? What is the maximum value?
- Do all of the graphs have the same minimum values? What is the minimum value?
- Do all of the graphs have the same period? What is the period?
- Why do the graphs have different x-intercepts? What are the x-intercepts?
- Why do the graphs have different y-intercepts? What are the y-intercepts?
- Are the coefficients of the functions the same? What does this tell you about the function?
- Are the coefficients of the argument of the functions the same? What does this tell you about the function?
- What does adding a constant to the argument of the function do to the graph of the function?
- What does subtracting a constant from the argument of the function do to the graph of the function?

PROBLEM 3 Phase Shift

Now consider what effect subtracting a constant, C, to the argument of a sine or cosine function has on the graph of the function. The transformed function can be written as $y = \sin(x - C)$ or $y = \cos(x - C)$.

1. Sketch graphs of the functions shown over the domain $-4\pi \le x \le 4\pi$.

   a. $f(x) = \sin(x)$ b. $g(x) = \sin\left(x + \frac{\pi}{2}\right)$ c. $h(x) = \sin(x - \pi)$



2. What similarities and differences do you notice about the three functions in terms of their maximums, minimums, periods, and amplitudes?

   All of the graphs have the same amplitude, and the maximum and minimum y-values are the same. All of the graphs have the same period.

   The graphs have different x- and y-intercepts.

3. How do the equations of the functions you graphed relate to the similarities and differences in the graphs?

   The amplitudes and periods are the same because the coefficients on each function and on the argument of each function are the same.

   Adding a constant to the argument of the function (subtracting a negative C-value) shifts the graph to the left, while subtracting a constant (subtracting a positive C-value) shifts the graph to the right.

15

Guiding Questions for Discuss Phase

- What is the difference between a phase shift and a horizontal translation?
- What is the equation for a sine function with a phase shift of $\frac{\pi}{2}$?
- What is the equation for a cosine function with a phase shift of π?

Talk the Talk

Students predict the effect of adding a *D*-value to a sine and cosine function. Students also determine the effect of introducing a negative sign to the *A*- and *B*-values of a sine and cosine function using graphs. Students summarize the information in this lesson by completing a table listing the descriptions of various transformations on the sine and cosine functions and the effects each transformation has on the period, amplitude, midline, and phase shift.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- Which *D*-values cause the graph of the function to shift up *D* units?

Transforming a periodic function by subtracting a *C*-value from the argument of the function results in horizontal translations of the function. These transformations act just as they have on other functions you have studied. For periodic functions, horizontal translations are called **phase shifts**.

Talk the Talk

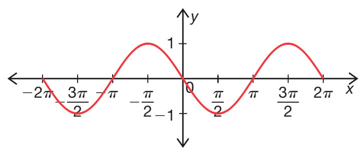
1. Predict the effect of adding a constant, *D*, to a sine or cosine function $y = f(x)$.

   If $D > 0$, the function will vertically shift up *D* units.

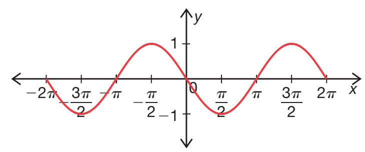
   If $D < 0$, the function will vertically shift down *D* units.

2. Use what you know about transformations to sketch the graph of each function.

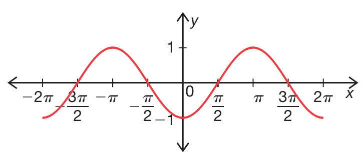
   a. $y = -\sin(x)$



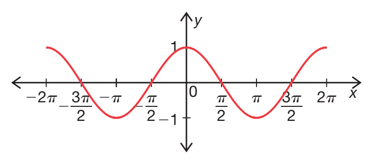
   b. $y = \sin(-x)$



   c. $y = -\cos(x)$



   d. $y = \cos(-x)$



3. Compare and contrast the graphs you sketched. What do you notice?

   Answers will vary.

   The graph of $y = \sin(-x)$ is the same as the graph of $y = -\sin(x)$.

   The graph of $y = \cos(-x)$ is the same as the graph of $y = \cos(x)$.

- Which *D*-values cause the graph of the function to shift down *D* units?
- What do you notice about the graph of $y = \sin(-x)$ and $y = -\sin(x)$?
- What do you notice about the graph of $y = \cos(-x)$ and $y = \cos(x)$?

Guiding Questions for Share Phase, Question 4

- Which constant is associated with the graph of the basic function shifting to the right or left?
- Which constant is associated with the graph of the basic function shifting up or down?
- Which constant is associated with the graph of the basic function stretching or compressing vertically?
- Which constant is associated with the graph of the basic function stretching or compressing horizontally?
- Which constant is associated with the period increasing?
- Which constant is associated with the period decreasing?
- Which constant is associated with a midline shift?
- Which constant is associated with a phase shift?
- Which constant is associated with the amplitude increasing?
- Which constant is associated with the amplitude decreasing?

4. Complete the table to describe the graph of each function as a transformation on $y = f(x)$.

Sine or Cosine Function	Equation Information	Description of Transformation of Sine or Cosine Graph	Effect on Period, Amplitude, Midline, Phase Shift
$y = f(x) + D$	$D > 0$	vertical shift up D units	midline shift up D units
	$D < 0$	vertical shift down D units	midline shift down D units
$y = f(x - C)$	$C > 0$	horizontal shift right C units	phase shift right C units
	$C < 0$	horizontal shift left C units	phase shift left C units
$y = Af(x)$	$\|A\| > 1$	vertical stretch by a factor of $\|A\|$ units	amplitude increase by a factor of $\|A\|$ units
	$0 < \|A\| < 1$	vertical compression by a factor of $\|A\|$ units	amplitude decrease by a factor of $\|A\|$ units
	$A < 0$	reflection across the x-axis	no change to period, amplitude, midline, or phase shift
$y = f(Bx)$	$\|B\| > 1$	horizontal compression by a factor of $\frac{1}{\|B\|}$	period decrease by a factor of $\frac{1}{\|B\|}$
	$0 < \|B\| < 1$	horizontal stretch by a factor of $\frac{1}{\|B\|}$	period increase by a factor of $\frac{1}{\|B\|}$
	$B < 0$	reflection across the y-axis	no change to period, amplitude, midline, or phase shift

Be prepared to share your methods and solutions.

15

Check for Students' Understanding

Consider the function $f(x) = -3\sin(2(x + \pi)) + 5$.

1. What is the basic function associated with this function?

The function $f(x) = \sin(x)$ is the basic function associated with $f(x) = -3\sin(2(x + \pi)) + 5$.

2. Identify the A-, B-, C-, and D-values and explain how they are related to the key characteristics of $f(x)$.

The A-value is -3 which is associated with the amplitude, which is the average of the maximum and minimum values of the function.

The B-value is 2 which is associated with the frequency, which is the reciprocal of the period.

The C-value is $-\pi$ which is associated with a phase shift, which is a horizontal shift.

The D-value is 5 which is associated with a vertical shift.

3. How are the x-values of the basic function affected by the transformations?

The function $f(x) = \sin(2x)$ compresses the function horizontally, dividing each x-value by a factor of 2.

The function $f(x) = \sin(2(x + \pi))$ horizontally shifts the function to the left, decreasing each x-value by π.

4. How are the y-values of the basic function affected by the transformations?

The function $f(x) = -\sin(2(x + \pi))$ reflects the output values across the y-axis.

The function $f(x) = -3\sin(2(x + \pi))$ vertically stretches the function, multiplying each y-value by a factor of 3.

The function $f(x) = -3\sin(2(x + \pi)) + 5$ vertically shifts the function up, increasing each y-value by 5.

15.5

Farmer's Tan

The Tangent Function

LEARNING GOALS

In this lesson, you will:

- Build the graph of the tangent function using the ratio $\frac{\sin(\theta)}{\cos(\theta)}$.
- Analyze characteristics of the tangent function, including period and asymptotes.
- Calculate values of the tangent function for common angles.
- Identify transformations of the tangent function.

KEY TERMS

- tangent function

ESSENTIAL IDEAS

- The tangent ratio is equal to the slope of the hypotenuse, which represents the terminal ray of the central angle on the unit circle.
- The tangent ratio (tan) is the ratio of the length of the opposite side to the length of the adjacent side in a right triangle: $\tan(\theta) = \frac{\text{opposite side}}{\text{adjacent side}}$
- On the unit circle the tangent ratio can be expressed as $\frac{\sin(\theta)}{\cos(\theta)}$.
- The tangent function is positive when $\sin(\theta)$ and $\cos(\theta)$ have the same sign, and the tangent function is negative when $\sin(\theta)$ and $\cos(\theta)$ have different signs.
- The period of the function $y = \tan(x)$ is π radians.
- The periodicity identity for the tangent function is written as $\tan(x + \pi) = \tan(x)$.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-TF Trigonometric Functions

Extend the domain of trigonometric functions using the unit circle

3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\frac{\pi}{3}$, $\frac{\pi}{4}$ and $\frac{\pi}{6}$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

Model periodic phenomena with trigonometric functions

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

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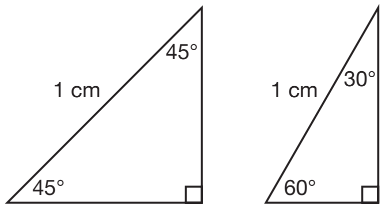
Overview

Students determine the slope for central angle θ formed as the terminal ray transverses in a unit circle. They use the information to graph the changing slope and conclude the graph represents a periodic function. The tangent ratio is reviewed and students explore the key characteristics of the tangent function and how it is defined in the unit circle in terms of the sine and cosine functions. Students will state the periodicity identity for the tangent function and label the tangent values on the unit circle. Students also graph transformed tangent functions and match functions with their appropriate graphs.

Warm Up



1. In the 45°-45°-90° triangle:

a. Describe the relationship between the line segment representing the hypotenuse and the ratio formed by the length of the side opposite the 45° angle and the length of the side adjacent to the 45° angle.

The ratio formed using length of the side opposite the 45° angle and the length of the side adjacent to the 45° angle is the slope of the line segment represented by the hypotenuse.

b. Determine the exact ratio formed by the length of the side opposite a 45° angle to the length of the side adjacent to the 45° angle.

$$\frac{\text{side opposite the 45° angle}}{\text{side adjacent to the 45° angle}} = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{\sqrt{2}} \div \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{1}$$

$$= 1$$

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2. In the 30°-60°-90° triangle:

   a. Describe the relationship between the line segment representing the hypotenuse and the ratio formed by the length of the side opposite the 30° angle and the length of the side adjacent to the 30° angle.

   The ratio formed using length of the side opposite the 30° angle and the length of the side adjacent to the 30° angle is the slope of the line segment represented by the hypotenuse.

   b. Determine the exact ratio formed by the length of the side opposite the 30° angle to the length of the side adjacent to the 30° angle.

$$\frac{\text{side opposite the } 30° \text{ angle}}{\text{side adjacent to the } 30° \text{ angle}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2} \div \frac{\sqrt{3}}{2}$$

$$= \frac{1}{2} \cdot \frac{2}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$

   c. Determine the exact ratio formed by the length of the side opposite the 60° angle to the length of the side adjacent to the 60° angle.

$$\frac{\text{side opposite the } 60° \text{ angle}}{\text{side adjacent to the } 60° \text{ angle}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$= \frac{\sqrt{3}}{2} \div \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{2}{1}$$

$$= \sqrt{3}$$

15.5

Farmer's Tan

The Tangent Function

LEARNING GOALS

In this lesson, you will:

- Build the graph of the tangent function using the ratio $\frac{\sin(\theta)}{\cos(\theta)}$.
- Analyze characteristics of the tangent function, including period and asymptotes.
- Calculate values of the tangent function for common angles.
- Identify transformations of the tangent function.

KEY TERM

- tangent function

Farming is all about cycles. Producing crops requires—among many other things—preparing the soil, seeding, watering, and harvesting. And this process is repeated over and over again.

Sometimes crops are rotated to help the soil. When the same crop occupies the same land for a long period of time, the crop often depletes the soil of a specific nutrient. When crops are rotated, different nutrients are taken from and returned to the soil, making the land usable for a longer period of time.

Agricultural cycles often depend on other cycles, such as cyclical patterns in the weather and the Sun.

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Problem 1

Students determine the slope within the context of a problem situation. Given the central angle measure θ in radians, they calculate the ratio for several central angles as the terminal ray traverses the unit circle, noting patterns when the value of the slope increases and decreases. Symmetry is used to graph slope values from $-\frac{\pi}{2}$ radians to 2π radians. A worked example shows the slope of the terminal ray in a reference triangle in Quadrant I is the same as the slope of the terminal ray in a reference triangle in Quadrant III because both rays are part of the same line. The graph and the worked example are used to determine the values of θ that have a slope equal to 1 and equal to -1, and why the relation is a periodic function.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete all parts of Question 1 with a partner. Then have students share their responses as a class.

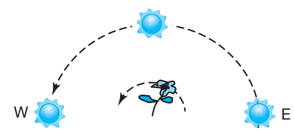
Guiding Questions for Share Phase, Question 1, part (a)

- For what value(s) of θ does the rise measure 0 units while the run measures 1 unit?
- If the rise measures 0 units and the run measures 1 unit, what is the value of the slope?
- For what value(s) of θ does the rise measure 1 unit while the run measures 1 unit?
- If the rise measures 1 unit and the run measures 1 unit, what is the value of the slope?
- For what value(s) of θ does the rise measure 1 unit while the run measures 0 units?
- If the rise measures 1 unit and the run measures 0 units, what is the value of the slope?

PROBLEM 1 Phototropism

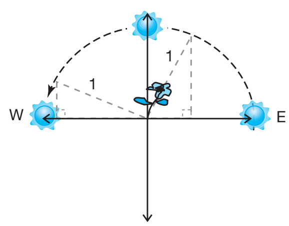
Many plants have evolved the ability to track sunlight as the Sun moves across the sky during the day. This movement is called phototropism.

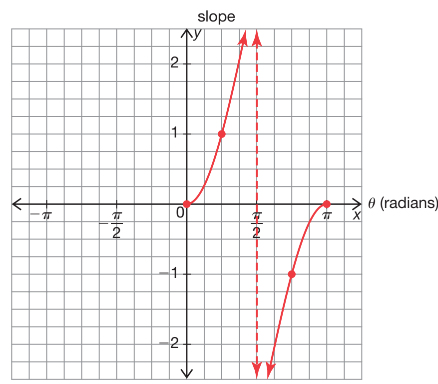
Imagine that flowers face due east in the morning where the Sun rises, and they track the sunlight throughout the day as the Sun moves directly overhead and then to the west.



1. Suppose you track the slope of the angle that a flower makes with the ground over the course of a day. Create a visual interpretation of the changing slope on the graph as you answer each question.

   a. What is the value of the slope at $\theta = 0$ radians, $\frac{\pi}{4}$ radian, and $\frac{\pi}{2}$ radians on the unit circle? Explain your reasoning.





At $\theta = 0$ radians, the rise measures 0 units and the run measures 1 unit. So, the slope at $\theta = 0$ radians is 0.

At $\theta = \frac{\pi}{4}$ radian, the rise measures $\frac{\sqrt{2}}{2}$ unit and the run measures $\frac{\sqrt{2}}{2}$ unit. So, the slope at $\theta = \frac{\pi}{4}$ radian is 1.

At $\theta = \frac{\pi}{2}$ radians, the rise measures 1 unit and the run measures 0 units. So, the slope at $\theta = \frac{\pi}{2}$ radians is undefined.

Guiding Questions for Share Phase, Question 1, parts (b) through (d)

- At which value(s) of θ does the slope increase from 0 to positive infinity?
- At which value(s) of θ does the slope decrease from 0 to negative infinity?
- At which value(s) of θ does the denominator of the slope equal 0?
- At which value(s) of θ does the numerator of the slope equal 0?

b. Describe the value of the slope as θ increases from 0 radians and approaches $\frac{\pi}{2}$ radians.

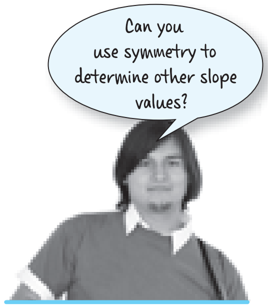
As θ increases from 0 radians and approaches $\frac{\pi}{2}$ radians, the value of the slope increases from 0 to positive infinity. At $\frac{\pi}{2}$ radians, the slope value is undefined.

c. What is the value of the slope at $\theta = \frac{3\pi}{4}$ radians and π radians? Explain how you determined each value.

At $\theta = \frac{3\pi}{4}$ radians, the slope is -1. At $\theta = \pi$ radians, the slope is 0.

The central angle that measures $\frac{3\pi}{4}$ radians on the unit circle is a reflection across the y-axis of the central angle that measures $\frac{\pi}{4}$ radian. So, its slope is -1.

At $\theta = \pi$ radians, the rise is 0, and the run is -1, so the slope is 0.



d. Describe the value of the slope as θ decreases from π radians and approaches $\frac{\pi}{2}$ radians.

As θ decreases from π radians and approaches $\frac{\pi}{2}$ radians, the value of the slope decreases from 0 to negative infinity. At $\frac{\pi}{2}$ radians, the slope value is undefined.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete all parts of Questions 2 through 5 with a partner. Then have students share their responses as a class.

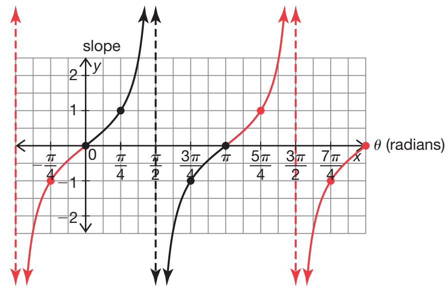
Guiding Questions for Share Phase, Questions 2 through 5

- Whenever the rise is 0 and the run is not 0, what is the value of the slope?
- When θ has a value of 0 radians, is the slope undefined or equal to 0?
- Whenever the rise is not 0 and the run is 0, what is the value of the slope?
- What transformations were used to complete the graph?
- If θ has a value of 45°, is the slope equal to 0, 1, or -1?
- If 180° is added to or subtracted from the angle measure of θ, does the slope remain the same? Why?
- If θ has a value of 135°, is the slope equal to 0, 1, or -1?
- Is there one and only one slope value for each value of θ in the unit circle?
- Does the graph of the relation pass the Vertical Line Test?
- Why is the period of the tangent function π radians?

At night, flowers do not continue to follow the Sun after it sets.

But suppose the flower represents the terminal ray of a central angle in standard position. Let's continue to model the change in the slope of the terminal ray as it traverses the unit circle.

2. Use your answers to Question 1 and what you know about symmetry to answer each question and complete the graph of the slope values from $-\frac{\pi}{2}$ radians to 2π radians.



a. For what value(s) of θ is the slope equal to 0?

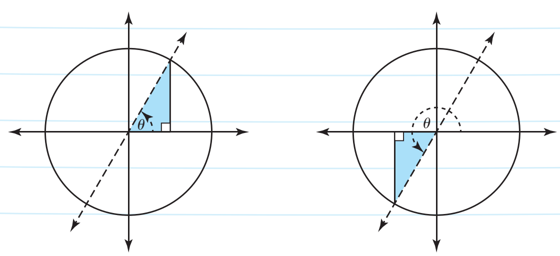
The slope is equal to 0 whenever the rise is 0 but the run is not 0.
So, the slope is 0 at $\theta = 0$ radians, π radians, 2π radians, etc.

b. For what value(s) of θ is the slope undefined?

The slope is undefined whenever the run is equal to 0.
So, the slope is undefined at $\theta = -\frac{\pi}{2}$ radians, $\frac{\pi}{2}$ radians, $\frac{3\pi}{2}$ radians, etc.

To help you think about slope values, you can remember that the terminal ray is a part of a line.

The triangles shown in the diagram are congruent. The hypotenuse of each triangle represents a terminal ray of a central angle with measure θ.



The slope of the terminal ray shown in Quadrant I is the same as the slope of the terminal ray shown in Quadrant III, because both rays are part of the same line. Both slopes are positive.

- Does the function repeat at 2π radians? Is this the smallest interval over which the function repeats?
- Are there asymptotes on the graph of the tangent function at all multiples of $\frac{\pi}{2}$ radians?

3. Use the worked example to help you answer each question and your completed graph in Question 2.

   a. For what value(s) of θ is the slope equal to 1?

   The slope is equal to 1 at $\theta = \frac{\pi}{4}$ radian (45°).

   Another slope of 1 can be calculated by adding or subtracting 180°, or π radians.

   So, the slope is 1 at $\theta = \frac{\pi}{4}$ radian, $\frac{5\pi}{4}$ radians, etc.

   b. For what value(s) of θ is the slope equal to -1?

   The slope is equal to -1 at $\theta = \frac{3\pi}{4}$ radians (135°), because this is a reflection across the y-axis of the angle measuring $\frac{\pi}{4}$ radian. Another slope of -1 can be calculated by adding or subtracting 180°, or π radians.

   So, the slope is -1 at $\theta = -\frac{\pi}{4}$ radian, $\frac{3\pi}{4}$ radians, $\frac{7\pi}{4}$ radians, etc.

4. Use your completed graph to answer each question.

   a. Explain why the relation you graphed is a function.

   For each θ value in radians, there is one and only one slope value. The graph of the relation also passes the Vertical Line Test.

   b. Is the function periodic? If so, determine the period of the function. If not, explain why not.

   Yes. The function is periodic. The period of the function is π radians.

5. James said that the period of the tangent function is 2π radians because the graph starting at 2π radians repeats the same values as it does starting at 0 radians. Juli says that the period of the tangent function is $\frac{\pi}{2}$ radians, because there is an asymptote at multiples of $\frac{\pi}{2}$ radians.

   Who is correct? Explain your reasoning.

   Neither James nor Juli is correct. The period of the tangent function is π radians.

   James is correct that the function repeats at 2π radians, but this is not the smallest interval over which the function repeats.

   Juli is not correct. There are not asymptotes at all multiples of $\frac{\pi}{2}$ radians. The asymptotes appear only at odd multiples of $\frac{\pi}{2}$ radians.

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Problem 2

The tangent ratio is reviewed and the tangent function is written in terms of the unit circle as $\frac{\sin(\theta)}{\cos(\theta)}$. Students determine the sign of the function in each quadrant, the equations of the vertical asymptotes that result from instances when the function is undefined, and the periodicity identity for the tangent function. Key characteristics of the tangent function are identified and tangent values are labeled on the unit circle.

Grouping

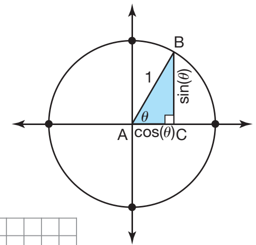
- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

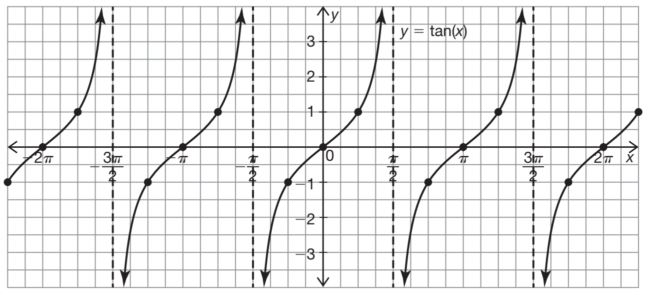
Guiding Questions for Share Phase, Questions 1 through 3

- What ratio represents $\sin(x)$?
- What ratio represents $\cos(x)$?
- What ratio represents $\frac{\sin(x)}{\cos(x)}$?
- If the tangent ratio is $\frac{\text{opposite side}}{\text{adjacent side}}$, how is it related to $\tan(x) = \frac{\sin(x)}{\cos(x)}$?
- If $\sin(\theta)$ and $\cos(\theta)$ are both associated with positive values, what is the sign of $\frac{\sin(\theta)}{\cos(\theta)}$? In which quadrant(s) does this occur?

PROBLEM 2 Tangent Waves of Grain

The function that you graphed in the previous problem is the **tangent function**. Recall that the tangent ratio (tan) is the ratio of the lengths of the opposite side and the adjacent side in a right triangle. The tangent ratio is equal to the slope of the hypotenuse, which represents the terminal ray of the central angle on the unit circle.





1. How can you write the tangent function in terms of sine and cosine, using the unit circle?

   The tangent ratio is the ratio $\frac{\text{opposite side}}{\text{adjacent side}}$, and on the unit circle this ratio is $\frac{\sin(\theta)}{\cos(\theta)}$. So, the tangent function can be written as $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

2. In which quadrants is the tangent function positive and negative? Explain your reasoning. Record this information on the Sine, Cosine, and Tangent on the Unit Circle diagram at the end of this Problem.

   The tangent function is positive when $\sin(\theta)$ and $\cos(\theta)$ have the same sign (Quadrants I and III), and the tangent function is negative when $\sin(\theta)$ and $\cos(\theta)$ have different signs (Quadrants II and IV).

3. Use what you know about rational functions to describe the discontinuities in the graph of the tangent function.

   Because $\tan(x) = \frac{\sin(x)}{\cos(x)}$, the graph of $\tan(x)$ will have vertical asymptotes whenever the denominator, $\cos(x)$, is equal to 0.

- If the $\sin(\theta)$ and the $\cos(\theta)$ are both associated with negative values, what is the sign of $\frac{\sin(\theta)}{\cos(\theta)}$? In which quadrant(s) does this occur?
- What will occur on the graph of $\tan(x) = \frac{\sin(x)}{\cos(x)}$ when $\cos(x) = 0$

Guiding Questions for Share Phase, Questions 4 through 7

- Why is the function tan $\left(\frac{n\pi}{2}\right)$ undefined for all odd integer values of n?
- Why is the periodicity identity for the tangent function written as $\tan(x + \pi) = \tan(x)$?
- Why doesn't the tangent function have any minimum or maximum output values?
- Why can't the amplitude of the tangent function be calculated?
- Is the domain of the tangent function all real numbers? Why not?
- How are the values of the tangent function in the first quadrant used to determine the values of the tangent function on the unit circle in the second quadrant? third quadrant? fourth quadrant?

4. What is the value of $\tan\left(\frac{n\pi}{2}\right)$ for any odd integer value of n?

For any odd integer n, the value of $\tan\left(\frac{n\pi}{2}\right)$ is undefined.

5. In previous lessons, you identified periodicity identities for both the sine function and cosine function. What is the periodicity identity for the tangent function? Explain your reasoning.

The period of the function $y = \tan(x)$ is π radians, so the periodicity identity for the tangent function is written as $\tan(x + \pi) = \tan(x)$.

6. The table shows some of the characteristics of the sine and cosine functions that you have identified. Complete the table for the tangent function.

	$y = \sin(x)$	$y = \cos(x)$	$y = \tan(x)$
y-intercept	(0, 0)	(0, 1)	(0, 0)
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$	all real numbers, except $\frac{n\pi}{2}$, where n is an odd integer
Range	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$
Period	2π	2π	π
Minimum Output Value	-1	-1	None
Maximum Output Value	1	1	None
Amplitude	1	1	None
Midline	$y = 0$	$y = 0$	$y = 0$

7. Complete the Sine, Cosine, and Tangent on the Unit Circle diagram by labeling the tangent values for each of the angle measures.

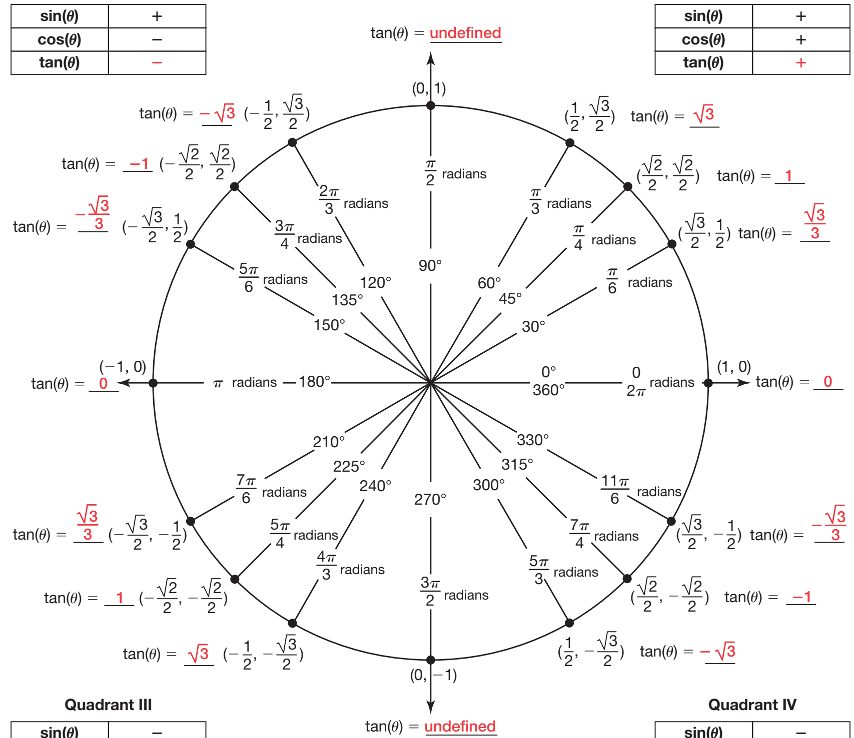
Sine, Cosine, and Tangent on the Unit Circle

Quadrant II

sin(θ)	+
cos(θ)	−
tan(θ)	−

Quadrant I

sin(θ)	+
cos(θ)	+
tan(θ)	+



Quadrant III

sin(θ)	−
cos(θ)	−
tan(θ)	+

Quadrant IV

sin(θ)	−
cos(θ)	+
tan(θ)	−

Problem 3

Students use equations to graph transformed tangent functions that have been reflected. They also match transformed tangent functions with their corresponding graphs.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

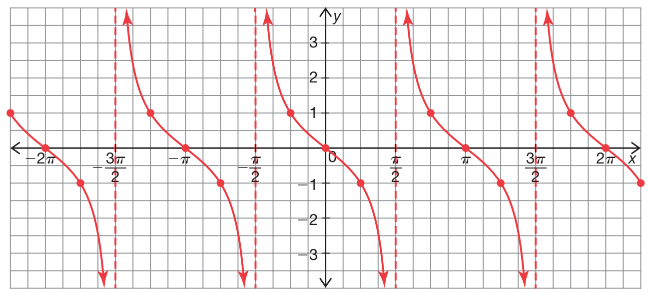
Guiding Questions for Share Phase, Question 1

- Which constant in the transformational function form is associated with $f(x) = -\tan(x)$? What is its effect on the basic function?
- Which constant in the transformational function form is associated with $g(x) = \tan(-x)$? What is its effect on the basic function?
- Which function, $f(x) = -\tan(x)$ or $g(x) = \tan(-x)$, is a reflection of the basic function across the x-axis?
- Which function, $f(x) = -\tan(x)$ or $g(x) = \tan(-x)$, is a reflection of the basic function across the y-axis?
- When a line is reflected across either the x- or y-axis, what is the relationship between the slope of the initial line and the slope of the reflected line?

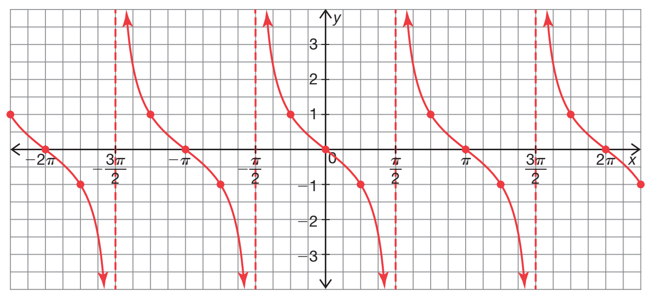
PROBLEM 3 Tan-sformations

1. Use what you know about transformations to sketch each graph.

   a. $f(x) = -\tan(x)$



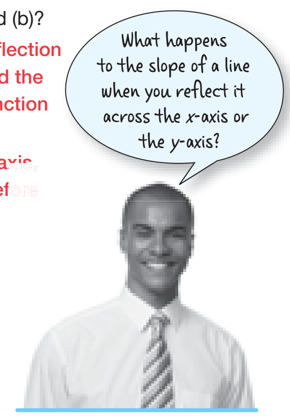
   b. $g(x) = \tan(-x)$



   c. What do you notice about the graphs in parts (a) and (b)?

   Both graphs are the same. The function $f(x)$ is a reflection of the basic tangent function across the x-axis, and the function $g(x)$ is a reflection of the basic tangent function across the y-axis.

   When a line is reflected across the x-axis or the y-axis, the slope of the line is the opposite of the slope before the reflection.



Guiding Questions for Share Phase, Question 2

- Which equation(s) describe a transformation associated with the *A*-value?
- Which equation(s) stretch or compress the basic function horizontally?
- Which equation(s) describe a transformation associated with the *B*-value?
- Which equation(s) stretch or compress the basic function vertically?
- Which equation(s) describe a transformation associated with the *C*-value?
- Which equation(s) shift the basic function horizontally?
- Which equation(s) describe a transformation associated with the *D*-value?
- Which equation(s) shift the basic function vertically?

2. Match each equation with its corresponding graph. Explain your reasoning.

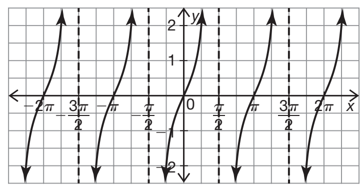
a. $y = \tan\left(\frac{1}{2}x\right)$

b. $y = \tan\left(x + \frac{\pi}{2}\right)$

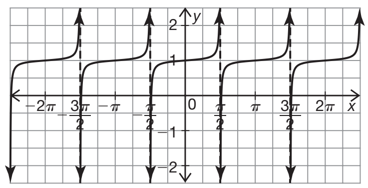
c. $y = \frac{1}{20}\tan(x) + 1$

d. $y = 2\tan(x)$

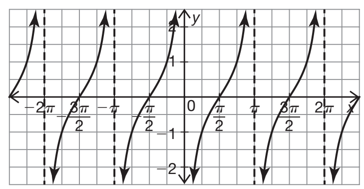
d



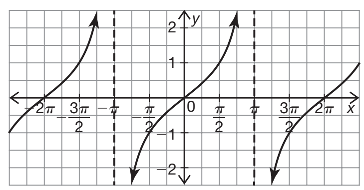
c



b



a



Explanations will vary.

Be prepared to share your methods and solutions.

Check for Students' Understanding

15

Consider the function $f(x) = -4\tan(6(x - \pi)) - 1$.

1. What is the basic function associated with this function?

The function $f(x) = \tan(x)$ is the basic function associated with $f(x) = -4\tan(6(x - \pi)) - 1$.

2. Identify the A-, B-, C-, and D-values and explain how they are related to the key characteristics of $f(x)$.

The A-value is -4, which is associated with the amplitude, which is the average of the maximum and minimum values of the function.

The B-value is 6, which is associated with the frequency, which is the reciprocal of the period.

The C-value is π, which is associated with a phase shift, which is a horizontal shift.

The D-value is -1, which is associated with a vertical shift.

3. How are the x-values of the basic function affected by the transformations?

The function $f(x) = \tan(6x)$ compresses the function horizontally, dividing each x-value by a factor of 6.

The function $f(x) = \tan(6(x - \pi))$ horizontally shifts the function to the right, increasing each x-value by π.

4. How are the y-values of the basic function affected by the transformations?

The function $f(x) = -\tan(6(x - \pi))$ reflects the output values across the y-axis.

The function $f(x) = -4\tan(6(x - \pi))$ vertically stretches the function, multiplying each y-value by a factor of 4.

The function $f(x) = -4\tan(6(x - \pi)) - 1$ vertically shifts the function down, decreasing each y-value by 1.

15

Chapter 15 Summary

KEY TERMS

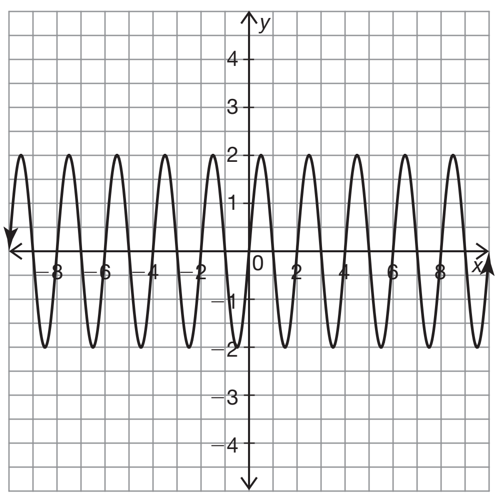
- periodic function (15.1)
- period (15.1)
- standard position (15.1)
- initial ray (15.1)
- terminal ray (15.1)
- amplitude (15.1)
- midline (15.1)
- theta (θ) (15.2)
- unit circle (15.2)
- radians (15.2)
- sine function (15.3)
- cosine function (15.3)
- trigonometric function (15.3)
- periodicity identity (15.3)
- frequency (15.4)
- phase shift (15.4)
- tangent function (15.5)

15.1 Identifying Periodic Functions and their Periods

A periodic function is a function whose values repeat over regular intervals. To identify a periodic function's period, determine the length of the smallest interval over which the function repeats.

Example

Determine whether the graph represents a periodic function over the interval shown. If so, identify the period.



The smallest interval over which this function repeats is 2. So, the period of the function is 2.

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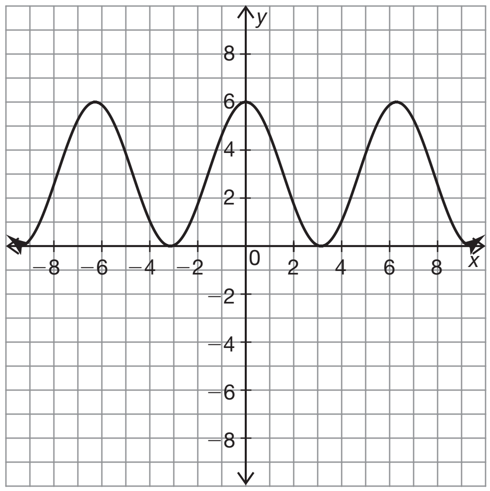
15.1 Determining the Amplitude and Midline of Periodic Functions

To determine the amplitude of a periodic function, calculate one-half of the absolute value of the difference between the maximum and minimum values of the function. The midline of a periodic function is a reference line whose equation is the average of the maximum and minimum values of a function.

Example

Determine the amplitude and midline of the given function.



The maximum value of the function is 6 and the minimum value is 0. So, the amplitude is $\frac{1}{2}|6 - 0|$ or 3. The midline is $y = \frac{6 + 0}{2} = 3$.

15.2 Converting Radian Measures to Degree Measures of Central Angles in a Unit Circle

To convert radian measures to degree measures of central angles in a unit circle, use the formula x radians $\cdot \frac{180°}{\pi \text{ radians}}$.

Example

Convert 3 radians to degrees.

$$\begin{aligned} x \text{ radians} \cdot \frac{180°}{\pi \text{ radians}} &= 3 \text{ radians} \cdot \frac{180°}{\pi \text{ radians}} \\ &= \frac{540°}{\pi} \\ &\approx 171.89° \end{aligned}$$

The degree measure of an angle with a radian measure of 3 radians is approximately 171.89°.

15.2 Converting Degree Measures to Radian Measures of Central Angles in a Unit Circle

To convert degree measures to radian measures of central angles in a unit circle, use the formula x degrees $\cdot \frac{\pi \text{ radians}}{180°}$.

Example

Convert $\theta = 225°$ to radians.

$$\begin{aligned} x \text{ degrees} \cdot \frac{\pi \text{ radians}}{180°} &= 225° \cdot \frac{\pi \text{ radians}}{180°} \\ &= \frac{225\pi}{180} \text{ radians} \\ &\approx \frac{5\pi}{4} \text{ radians} \end{aligned}$$

The radian measure of an angle with a degree measure of 225° is $\frac{5\pi}{4}$ radians.

15.3 Using the Unit Circle to Determine the Sine and Cosine of an Angle

In a unit circle, the ordered pair of the point where the terminal ray of an angle intersects the circle is the value of the cosine and sine of the angle.

Example

Use the unit circle to determine the sine and cosine of a central angle whose radian measure is $\frac{7\pi}{6}$.

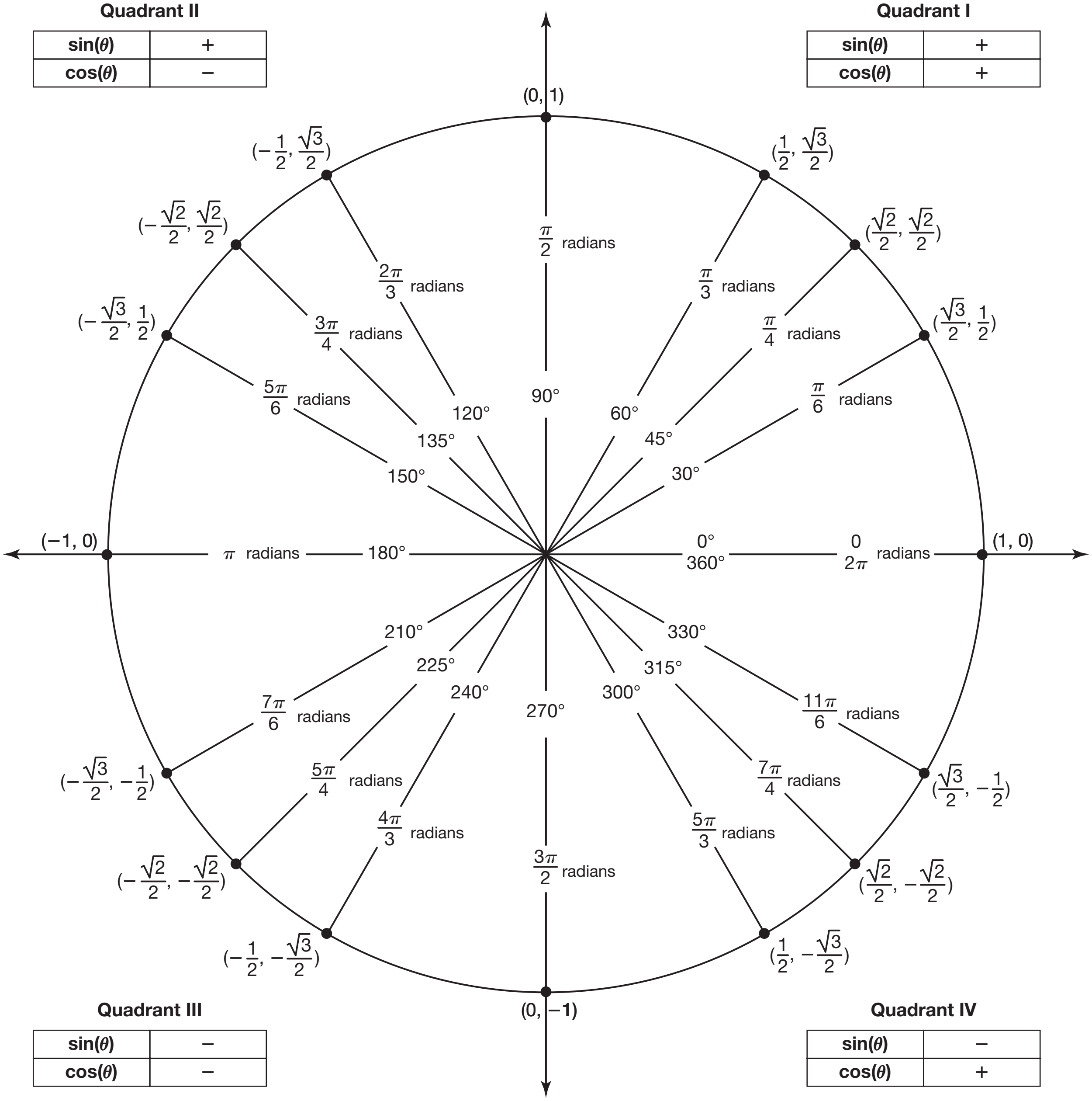
The ordered pair of the point where the terminal ray intersects the circle for a central angle with a measure of $\frac{7\pi}{6}$ radians is $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. Therefore, the sine of the angle is $-\frac{1}{2}$, and the cosine of the angle is $-\frac{\sqrt{3}}{2}$.

Quadrant II

sin(θ)	+
cos(θ)	–

Quadrant I

sin(θ)	+
cos(θ)	+



Quadrant III

sin(θ)	–
cos(θ)	–

Quadrant IV

sin(θ)	–
cos(θ)	+

15.4 Determining the Frequency of Sine and Cosine Functions

The frequency of a periodic function specifies the number of repetitions of the graph of a periodic function per unit. It is equal to the reciprocal of the period of the function or $\frac{|B|}{2\pi}$, where B is the coefficient of the argument.

Example

Determine the frequency of the function $y = 2\sin(4\pi) + 5$.

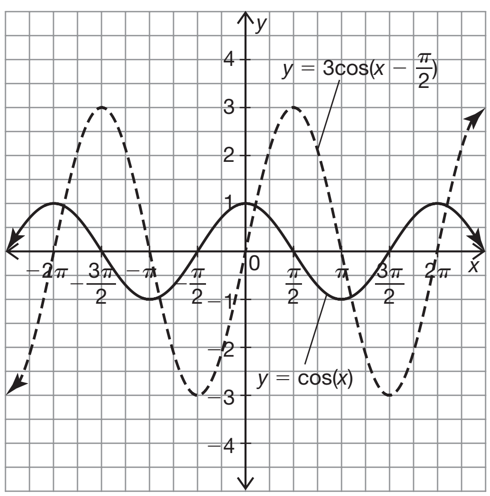
$$\text{frequency} = \frac{|B|}{2\pi} = \frac{|4|}{2\pi} = \frac{2}{\pi}$$

15.4 Graphing Transformations of Sine and Cosine Functions

To graph a transformation of a sine or cosine function, first graph the basic function. Then, use the equation of the transformed function to determine how to transform the basic function. For periodic functions of the form $g(x) = Af(B(x - C)) + D$, the A-value indicates the vertical stretch or compression, the B-value indicates the horizontal stretch or compression, the C-value indicates the phase shift, and the D-value indicates the vertical shift of the function.

Example

Describe how the basic function, $y = \cos(x)$, would be transformed to create the function $y = 3\cos\left(x - \frac{\pi}{2}\right)$. Then, graph the function.



The 3 indicates that the basic function, $y = \cos(x)$, must be vertically stretched by 3, and the $\frac{\pi}{2}$ indicates a phase shift to the right by $\frac{\pi}{2}$ units.

15.5 Evaluating the Tangent Function Using the Sine and Cosine Functions

The tangent function is the ratio of the sine to the cosine, or $\tan(x) = \frac{\sin(x)}{\cos(x)}$. To evaluate a tangent function for a given angle, use the unit circle to determine the sine and cosine of the angle. Then, calculate the ratio.

Example

Evaluate the tangent of $\frac{2\pi}{3}$ radians by using the relationship between the tangent function and the sine and cosine functions.

$$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$\begin{aligned}\tan\left(\frac{2\pi}{3}\right) &= \frac{\sin\left(\frac{2\pi}{3}\right)}{\cos\left(\frac{2\pi}{3}\right)}\\ &= \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\\ &= \frac{\sqrt{3}}{2}\cdot\left(-\frac{2}{1}\right)\\ &= -\sqrt{3}\end{aligned}$$

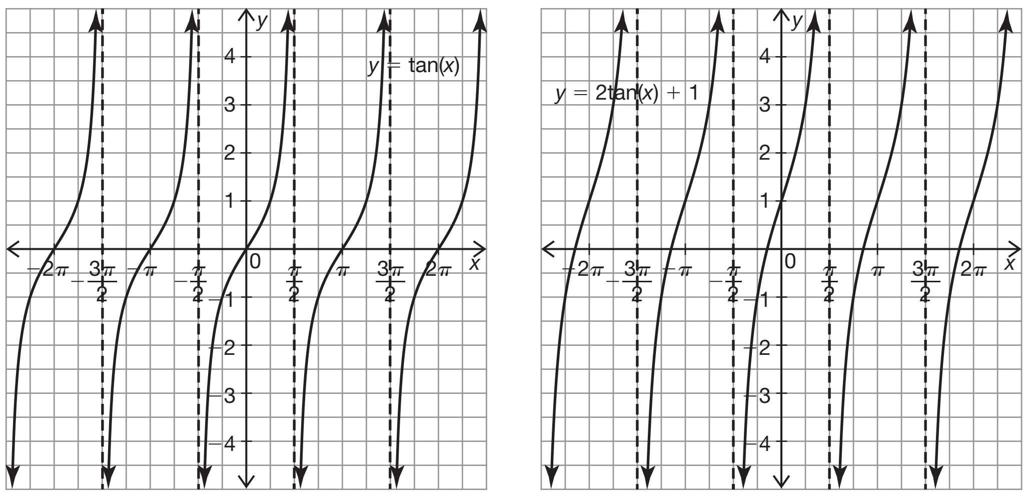
So, the tangent of $\frac{2\pi}{3}$ radians is $-\sqrt{3}$.

15.5 Graphing Transformations of the Tangent Function

To graph a transformation of a tangent function, first graph the basic function. Then, use the equation of the transformed function to determine how to transform the basic function. For periodic functions of the form $g(x) = Af(B(x - C)) + D$, the A-value indicates the vertical stretch or compression, the B-value indicates the horizontal stretch or compression, the C-value indicates the phase shift, and the D-value indicates the vertical shift of the function.

Example

Describe how the basic function, $y = \tan(x)$, would be transformed to create the function $y = 2\tan(x) + 1$. Then, graph the function.



The 2 indicates that the basic function, $y = \tan(x)$, must be vertically stretched by 2, and the 1 indicates a vertical shift up of 1 unit.