Modeling with Functions

14

You are looking at a picture of a cat looking at a picture of a cat looking at a picture of a cat . . . But you are also looking at a mathematical process of repetition and self-similarity called recursion.

Chapter 14 Overview

In this chapter, students explore various real-world and purely mathematical situations that are modeled with functions. Function composition is developed, and students apply function composition to solve contextual problems. Students also use functions to draw graphics, to model optimal solutions and self-similarity, and to study situations modeled by logistic growth, such as the spread of infectious diseases. Students end the chapter by choosing appropriate functions to model a variety of problem situations.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
14.1	Composition of Functions	F.IF.5 F.BF.1c F.BF.4b	2	Students are introduced to composition of functions in the context of combining coupons. Students then explore various characteristics of function composition, including using composition to verify that two functions are inverses. Students end the lesson by applying function composition to mathematical and real-world problems.	X		X		
14.2	Art and Transformations	F.IF.7.a F.IF.7.b F.IF.7.c F.IF.7.d F.IF.7.e	1	In this lesson, students use their knowledge of function families and transformation function form to create graphics and to identify functions used to create them. Students end the lesson by creating their own artwork on the coordinate plane using function transformation and domain restriction.	X				
14.3	Optimization	A.CED.3 A.REI.12 F.IF.1b F.IF.4	1	In this lesson, students determine optimal solutions in real-world situations by modeling systems of inequalities and quadratic functions. Questions ask students to graph and interpret functions modeling optimization situations.	X				X
14.4	Interpreting Graphs	F.IF.2 F.IF.4 F.IF.7d	1	In this lesson, students study logistic functions in the contexts of population growth and the spread of infectious disease. Students use technology to generate random data modeling the spread of an illness and the behavior of the function that models this situation.	X				X

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
14.5	Fractals	F.IF.3 F.BF.1a F.BF.2	2	Students explore fractals in this lesson, including the Sierpinski Triangle, the Menger Sponge, and the Koch Snowflake. Students explore the sequences that model the perimeter, area, or volume of these objects and derive an explicit formula for the perimeter of the Sierpinski Triangle using sigma notation. Questions ask students to build expressions and equations to model the characteristics of self-similar objects.	X				
14.6	Choosing Functions to Model Situations	A.CED.3 F.BF.1b F.BF.3 F.LE.2 F.LE.5	1	Students graph and analyze functions to model the speed of a rotor, the resistance of an electrical circuit, and the temperature of a liquid over time. Students also use technology to determine regression functions for data.			X		X

Skills Practice Correlation for Chapter 14

Lesson		Problem Set	Objectives
14.1	Composition of Functions		Vocabulary
		1 – 8	Determine the values of composition functions using graphs of $f(x)$ and $g(x)$
		9 – 14	Evaluate composition functions using the equations of $f(x)$ and $g(x)$
		15 – 20	Given a pair of functions, determine $f(g(x))$ and $g(f(x))$
		21 – 26	Use composition of functions to determine whether two functions are inverses of each other
		27 – 32	Determine the composition of two functions and state the domain
14.2	Art and Transformations	1 – 6	Determine the function family represented by a graph
		7 – 12	Identify the domain of a graphed function
		13 – 18	Given a domain, write the equation of a graphed function
		19 – 30	Graph a function over a given domain
14.3	Optimization	1 – 6	Write a system of inequalities to represent the constraints for given problem situations
		7 – 12	Graph a system of inequalities represented by given constraints
		13 – 18	Calculate the maximum and minimum values of a function given the intersection points of the constraints
		19 – 24	Maximize or minimize a function in quadratic form
14.4	Interpreting Graphs		Vocabulary
		1 – 6	Based on a graph, determine the initial growth stage, the exponential growth stage, the dampened growth stage, and the equilibrium stage, as well as the carrying capacity of a logistic function
		7 – 12	Use regression to determine the logistic equation for a set of data
		13 – 18	Use a logistic growth function to calculate the value of x or y
14.5	Fractals		Vocabulary
		1 – 6	Write an expression that represents the geometric sequence shown in a table
		7 – 12	Write a geometric sequence to represent the relationship between the stage of the fractal and the featured characteristic
		13 – 18	Calculate a specific component for each fractal, given its stage
14.6	Choosing Functions to Model Situations	1 – 6	Determine the indicated regression equation given a set of data
		7 – 12	Graph a set of data and then determine the appropriate regression equation
		13 – 22	Use a given function to calculate values in a real-world problem situation

It's Not New, It's Recycled

Composition of Functions

14.1

LEARNING GOALS

In this lesson, you will:

- Perform the composition of two functions graphically and algebraically.
- Use composition of functions to determine whether two functions are inverses of each other.
- Add, subtract, multiply, and divide with functions.
- Determine the restricted domain of a composite function.

KEY TERM

- identity function

ESSENTIAL IDEAS

- The process of evaluating one function inside of another function is called composition of functions.
- For two functions f and g, the composition of functions uses the output of $g(x)$ as the input of $f(x)$. It is notated as $(f \circ g)(x)$ or $(f(g(x)))$.
- If the functions $f(x)$ and $g(x)$ produce equivalent compositions $f(g(x))$ and $g(f(x))$ equal to x, then $f(x)$ and $g(x)$ are inverse functions.
- The identity function is the function that maps every real number x to the same real number x. The composition identity is defined as $f(x) = x$.
- The composition identity, x, results from the composition of two inverse functions.
- The domain of $(f \circ g)(x)$ is the set of all real numbers x in the domain of g such that $g(x)$ is in the domain of f.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F.IF Interpreting Functions

Interpret functions that arise in applications in terms of the context

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

F.BF Building Functions

Build a function that models a relationship between two quantities

1. Write a function that describes a relationship between two quantities.
   c. Compose functions.

Build new functions from existing functions

4. Find inverse functions.
   b. Verify by composition that one function is the inverse of another.

Overview

A flat rate coupon and a percentage off coupon are used in the context of a situation in which students write, graph, and compare functions. The composition of functions, operations associated with the composition of functions, and function notation are reviewed. Students use graphs to describe and evaluate several functions and the compositions of functions. Restrictions on the domain of composite functions are explored, and students write and analyze composite functions and their domains to model real-world problems.

Warm Up

Two common temperature scales are the Fahrenheit scale and the Celsius scale. Both of these scales are based on the freezing point of water, which is 32°F and 0°C. These formulas can be used to convert between the two scales:

$F = \frac{9}{5}C + 32$

$C = \frac{5}{9}(F - 32)$

1. Convert 50°C to degrees Fahrenheit.

$F = \frac{9}{5}(50) + 32$

$F = 90 + 32$

$F = 122$

2. Convert 131°F to degrees Celsius.

$C = \frac{5}{9}(F - 32)$

$C = \frac{5}{9}(99)$

$C = 55$

It’s Not New, It’s Recycled

Composition of Functions

14.1

LEARNING GOALS

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- Add, subtract, multiply, and divide with functions.
- Determine the restricted domain of a composite function.

KEY TERM

- identity function

A lot of people use coupons to save money on all kinds of items, especially groceries. And a few people really get into saving money with coupons. These strategic shoppers put a great deal of time and effort into spending as little money as possible.

How much can they save? Try $1100 in groceries for only $40.

But this is just a rare and extreme example. And it has been found that such “extreme couponing” isn’t worth the time required to make it extremely successful—especially when that time could be spent earning income at a job instead. Yet, the allure of saving money has led some to harmful habits and even criminal behavior!

Problem 1

Students write two different functions representing the discounted price of a flat screen television. Graphing the functions helps to determine the best method of discount with respect to the purchase price. Next, students write and compare functions that apply both discounts, 20% off the original price and a coupon for $100 off the original price, in different orders. The term *composition of functions* is reviewed and students rewrite the functions from the previous questions as a composition of functions. Real-life examples of function compositions are used and students complete a table by writing equivalent notations and verbal descriptions of operations associated with function composition. Graphs are used to evaluate and compare functions and the compositions of functions.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Does $p(x) = 0.20x$, or does $p(x) = 0.80x$? Why?
- Does $d(x) = x - 100$, or does $d(x) = 100 - x$? Why?
- How do the slopes of the two functions compare to each other?
- How do the y-intercepts of the two functions compare to each other?
- If the price of the television is greater than $500, which coupon should Betty use?
- If the price of the television is less than $500, which coupon should Betty use?
- If the price of the television is equal to $500, which coupon should Betty use?

PROBLEM 1 Extreme Couponing

1. Betty is in the market for a new flat screen TV. She has two coupons that she can use to purchase the television. One coupon is for 20% off the original price and the other coupon is for $100 off the original price.

   a. Write a function $p(x)$ to represent the discounted price of the television after the 20% off coupon is applied.

   $p(x) = 0.80x$

   b. Write a function $d(x)$ to represent the discounted price of the television after the $100 off coupon is applied.

   $d(x) = x - 100$

   c. Which coupon should Betty use? Justify your answer.

   Let x = the original price.

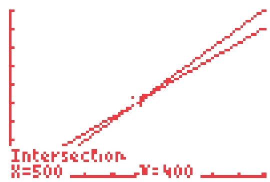
   Let y = the discounted price.

   Betty should use the 20% off coupon for a television that has an original price greater than $500 and the $100 off coupon for a television that has an original price less than $500.

   $0.80x = x - 100$

   $-0.2x = -100$

   $x = 500$



   When the original price is $500, both coupons provide an equal discounted price. As the original price gets larger than $500, the 20% off coupon will give Betty a lower discounted price. If the original price is less than $500, the $100 off coupon will give Betty a lower discounted price.

2. Suppose Betty is able to use both coupons. If the television that she wants costs $350, in which order should Betty apply the coupons to yield the lowest final price? Show your work.

   Betty should apply the 20% off coupon first and then the $100 off coupon.

   $p(350) = 0.80(350)$
   $= 280$
   $d(280) = 280 - 100$
   $= 180$

   $d(350) = 350 - 100$
   $= 250$
   $p(250) = 0.8(250)$
   $= 200$

14

Guiding Questions for Share Phase, Questions 3 and 4

- If Betty applies the 20% off coupon first, what will be the cost of the television?
- If Betty applies the $100 off coupon first, what will be the cost of the television?
- Does $t(x) = 0.8x - 100$, or does $t(x) = 0.8(x - 100)$? Why?
- Does $m(x) = 0.8x - 100$, or does $m(x) = 0.8(x - 100)$? Why?
- Will it always be better to apply a percentage off coupon before a flat rate coupon?

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 5 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 and 6

- Does $f(g(x))$ put the function $g(x)$ into the function $f(x)$ or put the function $f(x)$ into the function $g(x)$?
- Does $d(p(x))$ use the output of $p(x)$ as the input of $d(x)$ or the output of $d(x)$ as the input of $p(x)$?
- Is $t(x)$ associated with $d(p(x))$ or $p(d(x))$?

3. Suppose Betty uses the 20% off coupon first, followed by the $100 off coupon. Write the function $t(x)$ that represents the discounted price of a television that costs x dollars.

   $t(x) = 0.8x - 100$

4. Suppose Betty uses the $100 off coupon first, followed by the 20% off coupon. Write the function $m(x)$ that represents the discounted price of a television that costs x dollars.

   $m(x) = 0.8(x - 100)$
   $\quad = 0.8x - 80$

Recall that the process of evaluating one function inside of another function is called composition of functions. For two functions f and g, the composition of functions uses the output of $g(x)$ as the input of $f(x)$. It is notated as $(f \circ g)(x)$ or $f(g(x))$. The resulting function is called a composite function.

The two functions $t(x)$ and $m(x)$ that you wrote in Questions 3 and 4 can also be written as compositions of the original functions $p(x)$ and $d(x)$.

5. Rewrite $t(x)$ as $d(p(x))$ and $m(x)$ as $p(d(x))$.

   $m(x) = p(d(x)) = 0.80(x - 100)$
   The function $d(x) = x - 100$ is the input function into the equation $p(x) = 0.80x$.

   $t(x) = d(p(x)) = 0.80x - 100$
   The function $p(x) = 0.80x$ is the input function into the equation $d(x) = x - 100$.

6. Based on the functions $t(x)$ and $m(x)$, what can you conclude about the order to apply a percentage off coupon and a flat rate coupon? Explain your reasoning.

   It will always be better to apply a percentage off coupon before a flat rate coupon. The slope of the functions $t(x)$ and $m(x)$ are the same, but $t(x)$ has a lower y-intercept and will therefore always be lower.

- The function $d(x) = x - 100$ is the input function into what equation?
- The function $p(x) = 0.80x$ is the input function into what equation?

Guiding Questions for Share Phase, Question 7

- In what ways can depth affect volume?
- In what ways can a person's appetite affect cost?
- In what ways can time affect the area of a figure?

7. Determine the meaning of the composition of the two functions, $(f \circ g)(x)$. Give a possible real-life example for each.

   a. $f(x)$ = volume is a function of depth

   $g(x)$ = depth is a function of time

   Volume is a function of depth which depends on time.

   $f(g(x))$ = volume(depth(time))

   Answers will vary.

   As time increases, a hole is dug deeper, which increases its volume.

   b. $f(x)$ = cost is a function of consumption

   $g(x)$ = consumption is a function of appetite

   Cost is a function of consumption which depends on appetite.

   $f(g(x))$ = cost(consumption(appetite))

   Answers will vary.

   As a person's appetite increases, the amount of food they will consume also increases, which will affect the cost of their grocery bill.

   c. $f(x)$ = area is a function of radius

   $g(x)$ = radius is a function of time

   Area is a function of radius which depends on time.

   $f(g(x))$ = area(radius(time))

   Answers will vary.

   As time increases, the radius of a puddle gets larger, which also increases its area.

Grouping

- Have students complete Questions 8 through 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 8 through 10

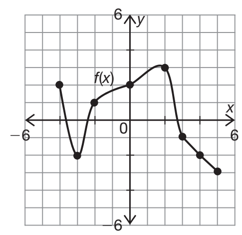
- Which notation describes the addition of like terms in the function?
- Which notation describes the difference of like terms in the functions?
- Which notation describes the product of the two functions?
- Which notation describes the quotient of the two functions?
- Which notation describes the substitution of one function into another?
- When the x-value of the point on the graph of $f(x)$ is equal to 3, what is the y-value?
- When the y-value of the point on the graph of $g(x)$ is equal to 4, what is the x-value?
- What did you evaluate first? Was the y-value or x-value given in this situation?
- What did you evaluate second? Was the y-value or x-value given in this situation?
- How did you know if the given value was an x-value or a y-value?
- Is the y-value always associated with the inverse function notation?
- Is the x-value always associated with the function notation?
- When evaluating a composition function, what is the first step that you perform?
- When the function is not a composition, what is the first step that you perform?

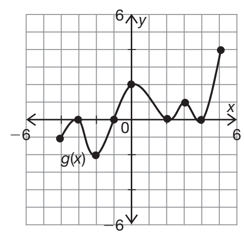
Just as the composition function $f(g(x))$ can also be written as $(f \circ g)(x)$, you can write all function operations with similar notation.

8. In each row of the table, write the equivalent notation and describe what the notation means.

Equivalent Function Notation		Description in Words
$f(x) + g(x)$	$(f + g)(x)$	Add the functions $f(x)$ and $g(x)$.
$f(x) - g(x)$	$(f - g)(x)$	Subtract the function $g(x)$ from the function $f(x)$.
$f(x) \cdot g(x)$	$(fg)(x)$	Multiply the functions $f(x)$ and $g(x)$.
$\frac{f(x)}{g(x)}$	$\left(\frac{f}{g}\right)(x)$	Divide the function $f(x)$ by the function $g(x)$.
$f(g(x))$	$(f \circ g)(x)$	Substitute the function $g(x)$ into the function $f(x)$.

9. Consider the graphs of the functions $f(x)$ and $g(x)$. Determine each value. Explain your reasoning.





a. $f(3)$

$f(3) = -1$

The y-value when $x = 3$ on the graph of $f(x)$ is -1.

b. $g^{-1}(4)$

$g^{-1}(4) = 5$

The x-value when $y = 4$ on the graph of $g(x)$ is 5.

c. $f(g(4))$

$f(g(4)) = 2$

First, I evaluated $g(4)$. The y-value when $x = 4$ on the graph of $g(x)$ is equal to 0. Then, I replaced $g(4)$ with 0 and evaluated $f(0)$. The y-value when $x = 0$ on the graph of $f(x)$ is 2.

d. $g(f(4))$

$g(f(4)) = -2$

First, I evaluated $f(4)$. The y-value when $x = 4$ on the graph of $f(x)$ is equal to -2. Then, I replaced $f(4)$ with -2 and evaluated $g(-2)$. The y-value when $x = -2$ on the graph of $g(x)$ is -2.

e. $g(f^{-1}(-3))$

$g(f^{-1}(-3)) = 4$

First, I evaluated $f^{-1}(-3)$. The x-value when $y = -3$ on the graph of $f(x)$ is 5. Then, I replaced $f^{-1}(-3)$ with 5 and evaluated $g(5)$. The y-value when $x = 5$ on the graph of $g(x)$ is 4.

f. $(fg)(3)$

$(fg)(3)$

$f(3) \cdot g(3) = -1$

First, I evaluated $f(3)$. The y-value when $x = 3$ on the graph of $f(x)$ is -1. Then, I evaluated $g(3)$. The y-value when $x = 3$ on the graph of $g(x)$ is 1. Last, I multiplied the results of $f(3)$ and $g(3)$ together.

g. $(f - g)(5)$

$(f - g)(5)$

$f(5) - g(5) = -7$

First, I evaluated $f(5)$. The y-value when $x = 5$ on the graph of $f(x)$ is -3. Then, I evaluated $g(5)$. The y-value when $x = 5$ on the graph of $g(x)$ is 4. Last, I subtracted the result of $g(5)$ from the result of $f(5)$.

h. $f(g^{-1}(0))$

$f(g^{-1}(0)) = \{-2, 1.7, 3\}$

First, I evaluated $g^{-1}(0)$. There are multiple x-values when $y = 0$ on the graph of $g(x)$. When $y = 0$, $x = \{-3, -1, 2, 4\}$. Then, I replaced $g^{-1}(0)$ with each of those x-values and evaluated them on the graph of $f(x)$.

$f(-3) = -2$ $\quad f(2) = 3$

$f(-1) \approx 1.7$ $\quad f(4) = -2$

i. $f(f^{-1}(3))$

$f(f^{-1}(3)) = 3$

First, I evaluated $f^{-1}(3)$. The x-value when $y = 3$ on the graph of $f(x)$ is 2. Then, I replaced $f^{-1}(3)$ with 2 and evaluated $f(2)$. The y-value when $x = 2$ on the graph of $f(x)$ is 3.

j. $(f + g)(0)$

$f(0) + g(0) = 4$

First, I evaluated $f(0)$. The y-value when $x = 0$ on the graph of $f(x)$ is 2. Then, I evaluated $g(0)$. The y-value when $x = 0$ on the graph of $g(x)$ is 2. Last, I added the result of $g(0)$ to the result of $f(0)$.

10. Consider the expressions in Question 9 that represent function composition.

a. Describe the similarities.

The expressions in parts (c), (d), (e), (h), and (i) represent function compositions.

In each of the composition functions, I evaluated one function first and then substituted the output value into another function.

b. What distinguished them from the functions that were not composition?

In each of the composition functions, the input value of one function was dependent on the output of another.

In each of the functions that were not composition, I evaluated the expressions separately and then performed the operation.

Problem 2

Students are given two functions, and they evaluate the composition of functions. Student work determining the composite function is used to identify errors in reasoning, and students determine the correct composite function. Given several pairs of functions, students determine composite functions. The identity function is defined and students use composition of functions to determine whether the given functions are inverses of each other. Composite functions are given and students determine two functions that, when composed, generate the composite function.

Grouping

Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- When evaluating a composition function what is the first step?
- Which equation is used first?
- What is the value of $g(-2)$?
- What value is substituted into $f(x)$?
- What is the value of $f(-2)$?
- What value is substituted into $g(x)$?
- What is the value of $g(0)$?
- What is the value of $g(2x)$?
- What is the value of $g(x)$?
- What is the value of $f(x)$?

PROBLEM 2 Calm, Cool, and Composed

1. Consider the functions $f(x) = 3x - 1$ and $g(x) = x^2 + 4$. Evaluate each composition of functions.

a. $f(g(-2))$

$$\begin{aligned} f(g(-2)) &= f((-2)^2 + 4) \\ &= f(4 + 4) \\ &= f(8) \\ &= 3(8) - 1 \\ &= 23 \end{aligned}$$

b. $g(f(-2))$

$$\begin{aligned} g(f(-2)) &= g(3(-2) - 1) \\ &= g(-6 - 1) \\ &= g(-7) \\ &= (-7)^2 + 4 \\ &= 53 \end{aligned}$$

c. $g(g(0))$

$$\begin{aligned} g(g(0)) &= g((0)^2 + 4) \\ &= g(0 + 4) \\ &= g(4) \\ &= (4)^2 + 4 \\ &= 20 \end{aligned}$$

d. $f(g(2x))$

$$\begin{aligned} f(g(2x)) &= f((2x)^2 + 4) \\ &= f(4x^2 + 4) \\ &= 3(4x^2 + 4) - 1 \\ &= 12x^2 + 12 - 1 \\ &= 12x^2 + 11 \end{aligned}$$

e. $f(g(x))$

$$\begin{aligned} f(g(x)) &= f(x^2 + 4) \\ &= 3(x^2 + 4) - 1 \\ &= 3x^2 + 12 - 1 \\ &= 3x^2 + 11 \end{aligned}$$

f. $g(f(x))$

$$\begin{aligned} g(f(x)) &= g(3x - 1) \\ &= (3x - 1)^2 + 4 \\ &= (3x - 1)(3x - 1) + 4 \\ &= 9x^2 - 6x + 1 + 4 \\ &= 9x^2 - 6x + 5 \end{aligned}$$

Guiding Questions for Share Phase, Question 2

- Did Jess evaluate the function using substitution?
- What operation did Jess use to evaluate the function?
- Is the composition of functions commutative?

2. Jess was asked to determine the composite function $(m \circ p)(x)$ given the functions $m(x) = 5 - x$ and $p(x) = 3x^2 + x$.

   a. Describe the error in Jess's method.

   Jess

   $$\begin{aligned}(m \circ p)(x) &= (5 - x)(3x^2 + x)\\ &= 15x^2 + 5x - 3x^3 - x^2\\ &= -3x^3 + 14x^2 + 5x\end{aligned}$$

   Jess did not substitute $p(x)$ for x in the function $m(x)$. She multiplied the functions.

   b. Determine the correct composite function $(m \circ p)(x)$.

   $$\begin{aligned}(m \circ p)(x) &= 5 - (3x^2 + x)\\ &= 5 - 3x^2 - x\\ &= -3x^2 - x + 5\end{aligned}$$

   c. Is $(p \circ m)(x)$ equivalent to $(m \circ p)(x)$? Explain your reasoning.

   No.

   $(m \circ p)(x) = -3x^2 - x + 5$

   $$\begin{aligned}(p \circ m)(x) &= 3(5 - x)^2 + (5 - x)\\ &= 3((5 - x)(5 - x)) + (5 - x)\\ &= 3(25 - 10x + x^2) + (5 - x)\\ &= 75 - 30x + 3x^2 + 5 - x\\ &= 3x^2 - 31x + 80\end{aligned}$$

Grouping

- Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.
- Ask a student to read the information about identities. Complete Question 5 as a class.

Guiding Questions for Share Phase, Questions 3 through 5

- Does $f(g(x))$ use $f(x)$ as the input function or $g(x)$ as the input function?
- Does $g(f(x))$ use $f(x)$ as the input function or $g(x)$ as the input function?
- Does $(f \circ g)(x)$ always equal $(g \circ f)(x)$?
- When are $(f \circ g)(x)$ and $(g \circ f)(x)$ equivalent?
- What always results from the composition of two inverse functions?

3. Determine $(f \circ g)(x)$ and $(g \circ f)(x)$ for each pair of functions $f(x)$ and $g(x)$.

a. $f(x) = 5^{x-1}$

$g(x) = x + 3$

$$(f \circ g)(x) = f(g(x)) = f(x+3) = 5^{x+3-1} = 5^{x+2}$$

$$(g \circ f)(x) = g(f(x)) = g(5^{x-1}) = 5^{x-1} + 2$$

b. $f(x) = 5^{x-1}$

$g(x) = \log_5 x + 1$

$$(f \circ g)(x) = f(g(x)) = f(\log_5 x + 1) = 5^{\log_5 x + 1 - 1} = 5^{\log_5 x} = x$$

$$(g \circ f)(x) = g(f(x)) = g(5^{x-1}) = \log_5(5^{x-1}) + 1 = x - 1 + 1 = x$$

c. $f(x) = 0.5x + 1.5$

$g(x) = 2x - 3$

$$(f \circ g)(x) = f(g(x)) = f(2x-3) = 0.5(2x-3) + 1.5 = x - 1.5 + 1.5 = x$$

$$(g \circ f)(x) = g(f(x)) = g(0.5x + 1.5) = 2(0.5x + 1.5) - 3 = x + 3 - 3 = x$$

d. $f(x) = 2x^2 - x$

$g(x) = \sqrt{x}$

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 2(\sqrt{x})^2 - \sqrt{x} = 2|x| - \sqrt{x}$$

$$(g \circ f)(x) = g(f(x)) = g(2x^2 - x) = \sqrt{2x^2 - x}$$

4. Compare the compositions $(f \circ g)(x)$ and $(g \circ f)(x)$ from Question 3.

   a. Identify the pairs of functions in which $f(g(x))$ and $g(f(x))$ are equivalent functions. Describe the similarities.

      In Question 3, $(f \circ g)(x) = (g \circ f)(x)$ for the functions $f(x) = 0.5x + 1.5$ and $g(x) = 2x - 3$. For this pair of functions, $f(g(x)) = x$ and $g(f(x)) = x$.

   b. Determine the relationship between the functions you identified in part (a).

      The functions $f(x)$ and $g(x)$ that have equivalent compositions $f(g(x))$ and $g(f(x))$ both equal to x are inverse functions.

Consider these two statements:

- The additive identity is 0. It results when two quantities that are additive inverses are added.
- The multiplicative identity is 1. It results when two quantities that are multiplicative inverses are multiplied.

5. How would you describe the composition identity? Provide an example to support your claim.

   The composition identity is x. It results from the composition of two inverse functions. For example, $f(x) = 3x + 1$ and $g(x) = \frac{x-1}{3}$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{x-1}{3}\right) \\ &= 3\left(\frac{x-1}{3}\right) + 1 \\ &= x - 1 + 1 \\ &= x\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x + 1) \\ &= \frac{3x + 1 - 1}{3} \\ &= \frac{3x}{3} \\ &= x\end{aligned}$$

Grouping

- Have students complete Questions 6 and 7 with a partner. Then have students share their responses as a class.
- Have students complete Question 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 6 and 7

- Does $(c \circ d)(x) = x$?
- Does $(d \circ c)(x) = x$?
- If $(c \circ d)(x) = x$ and $(d \circ c)(x) = x$, what does this imply about the functions $c(x)$ and $d(x)$?
- Does $(z \circ w)(x) = x$?
- Does $(w \circ z)(x) = x$?
- If $(z \circ w)(x) \neq x$ and $(w \circ z)(x) \neq x$, what does this imply about the functions $z(x)$ and $w(x)$?
- Does $(t \circ r)(x) = x$?
- Does $(r \circ t)(x) = x$?
- If $(t \circ r)(x) = x$ and $(r \circ t)(x) = x$, what does this imply about the functions $r(x)$ and $t(x)$?
- Did Maggie show $f(g(x)) = g(f(x))$ for all x-values?
- Does $f(g(1)) = g(f(1))$?
- Can two functions share a common (x, y) value and not have an inverse relationship?

The *identity function* is the function that maps every real number x to the same real number x. The composition identity, also known as the **identity function**, is defined as $f(x) = x$.

6. Use composition of functions to determine whether the given functions are inverses of each other.

a. $c(x) = \frac{1}{x} - 6$

$d(x) = \frac{1}{x + 6}$

Yes. The functions $c(x)$ and $d(x)$ are inverse functions because $(c \circ d)(x) = x$ and $(d \circ c)(x) = x$.

$$\begin{aligned}(c \circ d)(x) &= c(d(x)) \\ &= c\left(\frac{1}{x+6}\right) \\ &= \frac{1}{\frac{1}{x+6}} - 6 \\ &= x + 6 - 6 \\ &= x\end{aligned}$$

$$\begin{aligned}(d \circ c)(x) &= d(c(x)) \\ &= d\left(\frac{1}{x} - 6\right) \\ &= \frac{1}{\frac{1}{x} - 6 + 6} \\ &= \frac{1}{\frac{1}{x}} \\ &= x\end{aligned}$$

b. $z(x) = \frac{5}{4}x + 3$

$w(x) = \frac{4}{5}x - 3$

No. The functions $z(x)$ and $w(x)$ are not inverse functions because $(z \circ w)(x) \neq x$ and $(w \circ z)(x) \neq x$.

$$\begin{aligned}(z \circ w)(x) &= z(w(x)) \\ &= z\left(\frac{4}{5}x - 3\right) \\ &= \frac{5}{4}\left(\frac{4}{5}x - 3\right) + 3 \\ &= x - \frac{15}{4} + 3 \\ &= x - \frac{3}{4}\end{aligned}$$

$$\begin{aligned}(w \circ z)(x) &= w(z(x)) \\ &= w\left(\frac{5}{4}x + 3\right) \\ &= \frac{4}{5}\left(\frac{5}{4}x + 3\right) - 3 \\ &= x + \frac{12}{5} - 3 \\ &= x - \frac{3}{5}\end{aligned}$$

c. $r(x) = 2e^{x+4}$

$t(x) = \ln\left(\frac{x}{2}\right) - 4$

Yes. The functions $r(x)$ and $t(x)$ are inverse functions because $(r \circ t)(x) = x$ and $(t \circ r)(x) = x$.

$$\begin{aligned}(r \circ t)(x) &= r(t(x)) \\ &= r\left(\ln\left(\frac{x}{2}\right) - 4\right) \\ &= 2e^{\ln\left(\frac{x}{2}\right) - 4 + 4} \\ &= 2e^{\ln\left(\frac{x}{2}\right)} \\ &= 2\left(\frac{x}{2}\right) \\ &= x\end{aligned}$$

$$\begin{aligned}(t \circ r)(x) &= t(c(x)) \\ &= t(2e^{x+4}) \\ &= \ln\left(\frac{2e^{x+4}}{2}\right) - 4 \\ &= \ln(e^{x+4}) - 4 \\ &= x + 4 - 4 \\ &= x\end{aligned}$$

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7. Maggie is asked to determine whether $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x} + 1$ are inverse functions.

Maggie

The two functions are equivalent at the value $x = 0$, so they must be inverse functions.

$$\begin{aligned}(f \circ g)(0) &= f(g(0)) \\ &= f(\sqrt[3]{0} + 1) \\ &= f(1) \\ &= 1^3 + 1 \\ &= 2\end{aligned}$$

$$\begin{aligned}(g \circ f)(0) &= g(f(0)) \\ &= g(0^3 + 1) \\ &= g(1) \\ &= \sqrt[3]{1} + 1 \\ &= 2\end{aligned}$$

Explain why Maggie's method is insufficient, and provide a counterexample.

Maggie's method is insufficient because while $f(g(0)) = g(f(0))$ is true, $f(g(x)) \neq g(f(x))$ for all x values.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt[3]{x} + 1) \\ &= (\sqrt[3]{x} + 1)^3 + 1\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^3 + 1) \\ &= \sqrt[3]{x^3 + 1} + 1\end{aligned}$$

For example, $f(g(1)) \neq g(f(1))$.

$$\begin{aligned}(f \circ g)(1) &= f(g(1)) \\ &= f(\sqrt[3]{1} + 1) \\ &= f(2) \\ &= (2)^3 + 1 \\ &= 9\end{aligned}$$

$$\begin{aligned}(g \circ f)(1) &= g(f(1)) \\ &= g(1^3 + 1) \\ &= g(2) \\ &= \sqrt[3]{2} + 1 \\ &\approx 2.26\end{aligned}$$

Therefore even though they share a common (x, y) value, $f(x)$ and $g(x)$ are not inverse functions.

Guiding Questions for Share Phase, Question 8

- If $h(x) = \sqrt{x}$, what is $k(x)$?
- If $k(x) = 7x - 5$, what is $h(x)$?
- If $h(x) = \frac{11}{x} + 3$, what is $k(x)$?
- If $k(x) = x - 9$, what is $h(x)$?
- If $h(x) = 3^x + 5$, what is $k(x)$?
- If $k(x) = 2x$, what is $h(x)$?

8. For each composite function, determine two functions $h(x)$ and $k(x)$ that when composed will generate $(h \circ k)(x)$.

a. $(h \circ k)(x) = \sqrt{7x - 5}$

Answers will vary.

$h(x) = \sqrt{x}$

$k(x) = 7x - 5$

b. $(h \circ k)(x) = \frac{11}{x - 9} + 3$

Answers will vary.

$h(x) = \frac{11}{x} + 3$

$k(x) = x - 9$

c. $(h \circ k)(x) = 3^{2x} + 5$

Answers will vary.

$h(x) = 3^x + 5$

$k(x) = 2x$

Problem 3

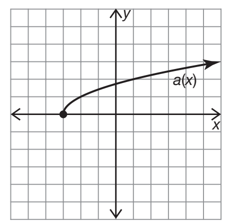
The graphs of two functions are given. Students determine the domain, sketch and write the composite function, and identify the domain of the composite function. They review student work looking for errors, sketch more graphs, write more composite functions, and then discuss domain restrictions that carry through to the composite function. In the last activity, several functions are given and students evaluate compositions, state the domain of the composite function, and explain their reasoning.

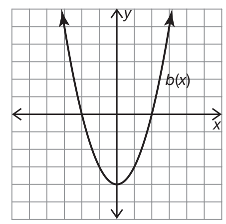
Grouping

Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

PROBLEM 3 Domain Restrictions

Consider the functions $a(x) = \sqrt{x + 3}$ and $b(x) = x^2 - 4$.





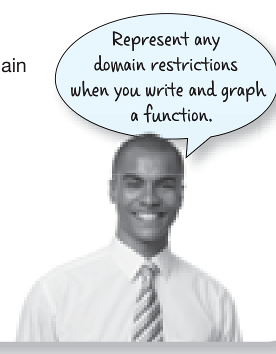
1. State the domain of $a(x)$ and of $b(x)$.

The domain of $a(x)$ is $[-3, \infty)$ and the domain of $b(x)$ is $(-\infty, \infty)$.

2. Write the composite function $(b \circ a)(x)$ and state the domain using interval notation.

$$\begin{aligned}(b \circ a)(x) &= b(a(x)) \\ &= b\left(\sqrt{x + 3}\right) \\ &= \left(\sqrt{x + 3}\right)^2 - 4 \\ &= x + 3 - 4 \\ &= x - 1, \text{ for } x \geq -3\end{aligned}$$

The domain is $[-3, \infty)$.



Guiding Questions for Share Phase, Questions 1 and 2

- Which function has a domain of $[-3, \infty)$?
- Which function has a domain of $(-\infty, \infty)$?
- How do you determine the domain of the function $(b \circ a)(x)$?
- Is the domain of $a(x)$ the same as the domain of $(b \circ a)(x)$? Why?
- Why is the domain of the composite function the same as the domain of the input function?

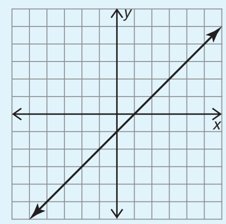
Guiding Questions for Share Phase, Questions 3 through 5

- Did Jacob restrict the domain?
- Are the values for x that are less than -3 defined in the input function?
- How did you determine the graph of the function $(a \circ b)(x)$?
- Is the domain of the function $(a \circ b)(x)$ all real numbers? Why or why not?
- What is the domain of the input function in this situation?
- If the domain of $(a \circ b)(x)$ is not the same as the domain of $(b \circ a)(x)$, what does this imply about the restrictions?
- Can the shape of the graph be determined by substituting the input function into the output function?
- Which function has a domain restriction that carried through to the composite function?
- If an element is in the domain of a composite function, must it first be defined in the domain of the input function?

3. Jacob simplified the composition function algebraically and then graphed the result. Describe the error Jacob made.

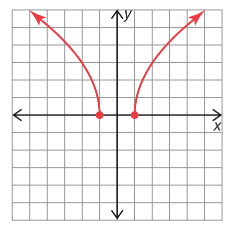
Jacob

$$\begin{aligned}(b \circ a)(x) &= b(a(x)) \\ &= b(\sqrt{x+3}) \\ &= (\sqrt{x+3})^2 - 4 \\ &= x + 3 - 4 \\ &= x - 1\end{aligned}$$



Jacob graphed the correct simplified function, but did not represent the domain restrictions. Values for x that are less than -3 are undefined because they are undefined in the input function.

4. Sketch a graph of the function $(a \circ b)(x)$. Show your work.



$$\begin{aligned}(a \circ b)(x) &= a(b(x)) \\ &= a(x^2 - 4) \\ &= \sqrt{(x^2 - 4 + 3)} \\ &= \sqrt{(x^2 - 1)}\end{aligned}$$

5. State the domain of $(a \circ b)(x)$. Is it the same as the domain of $(b \circ a)(x)$? Explain your reasoning.

The domain of $(a \circ b)(x)$ is $x \le -1$ or $x \ge 1$.

The domain of $(a \circ b)(x)$ is not the same as the domain of $(b \circ a)(x)$ because the input function of $(a \circ b)(x)$ does not have the same restrictions as the input function of $(b \circ a)(x)$.

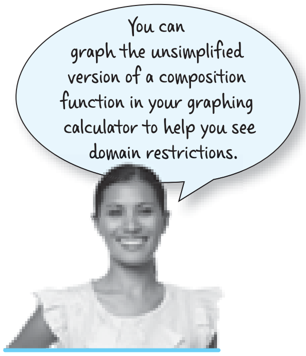
Grouping

- Discuss the information about domain restrictions of composite functions as a class.
- Have students complete Questions 6 and 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- Should Elise have considered the composite function when determining the domain?
- What is the denominator of the composite function?
- Is $x = 0$ defined for the composite function?
- Did Elise consider the domains of each function separately?

The domain of $(f \circ g)(x)$ is the set of all real numbers x in the domain of g such that $g(x)$ is in the domain of f. In order for an element to be in the domain of the function $(f \circ g)(x)$, it must first be defined in the domain of $g(x)$.



6. Elise and Camden were given the functions $m(x) = \frac{1}{x}$ and $n(x) = \sqrt{x + 5}$ and asked to determine the domain of $m(n(x))$. Determine whose method is correct and explain your reasoning.

> Elise
>
> I know that $m(n(x))$ means that $n(x) = \sqrt{x + 5}$ is the input function and it has a domain of $[-5, \infty)$.
>
> The output function is $m(x) = \frac{1}{x}$ and I know that the denominator can't be 0.
>
> Therefore, the domain of $m(n(x))$ is $x \geq -5,\ x \neq 0$.

> Camden
>
> I know that $m(n(x))$ means $m(\sqrt{x+5}) = \frac{1}{\sqrt{x+5}}$.
>
> The input function is $n(x) = \sqrt{x+5}$ and it has a domain of $[-5, \infty)$, but because it's in the denominator and the denominator can't be zero, -5 is not a part of the domain.
>
> Therefore, the domain of $m(n(x))$ is $(-5, \infty)$.

Camden is correct.

Elise is correct that for the function $m(x) = \frac{1}{x}$, 0 is undefined. However, she did not consider the composite function and that x is not the denominator of the composite function.

Guiding Questions for Share Phase, Question 7

- Is the domain of the composite function always the same as the domain of the input function? Why or why not?
- If the input function is linear, what is the domain?
- If the composite function is quadratic, what is the domain?
- Are there any restrictions to consider in this situation?
- If the input function is quadratic, is there always no restriction on the domain? Why or why not?
- If the composite function is radical, is the domain always restricted? Why or why not?
- What value creates an undefined function in this situation?
- Is the composite function restricted by the domain of the input function? How so?
- Does the restriction of the input function always carry through?

7. Use the functions $f(x) = 4x - 3$, $g(x) = x^2 - 5$, $h(x) = \sqrt{x + 1}$ and $j(x) = \frac{1}{x + 2}$ to evaluate each composition. Then state the domain of the composite function. Explain your reasoning.

a. $(g \circ f)(x)$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(4x - 3) \\ &= (4x - 3)^2 - 5 \\ &= 16x^2 - 24x + 9 - 5 \\ &= 16x^2 - 24x + 4\end{aligned}$$

The domain of $(g \circ f)(x)$ is $(-\infty, \infty)$.

The input function is linear and has a domain of $(-\infty, \infty)$. The composite function is quadratic. There are no restrictions to consider.

b. $(h \circ g)(x)$

$$\begin{aligned}(h \circ g)(x) &= h(g(x)) \\ &= h(x^2 - 5) \\ &= \sqrt{x^2 - 5 + 1} \\ &= \sqrt{x^2 - 4}\end{aligned}$$

The domain of $(h \circ g)(x)$ is $x < -2$ or $x > 2$.

The input function is quadratic and has a domain of $(-\infty, \infty)$ so there are no restrictions there. But the composite function is radical, so the radicand cannot be negative.

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c. $(j \circ f)(x)$

$$(j \circ f)(x) = j(f(x))$$
$$= j(4x - 3)$$
$$= \frac{1}{4x - 3 + 2}$$
$$= \frac{1}{4x - 1}$$

The domain of $(j \circ f)(x)$ is $x < \frac{1}{4}$ or $x > \frac{1}{4}$.

The input function is linear and has a domain of $(-\infty, \infty)$ so there are no restrictions there. But the composite function is rational, so the denominator cannot be 0.

d. $(g \circ h)(x)$

$$(g \circ h)(x) = g(h(x))$$
$$= g(\sqrt{x + 1})$$
$$= (\sqrt{x + 1})^2 - 5$$
$$= x + 1 - 5$$
$$= x - 4$$

The domain of $(g \circ h)(x)$ is $[-1, \infty)$.

The input function is radical and has a domain of $[-1, \infty)$. The composite function is linear, but the composite function is restricted by the domain of the input function $h(x)$.

e. $(j \circ j)(x)$

$$\begin{aligned}(j \circ j)(x) &= j(j(x)) \\ &= j\left(\frac{1}{x+2}\right) \\ &= \frac{1}{\frac{1}{x+2} + 2} \\ &= \frac{1}{\frac{1}{x+2} + \frac{2(x+2)}{x+2}} \\ &= \frac{1}{\frac{1+2x+4}{x+2}} \\ &= \frac{1}{\frac{2x+5}{x+2}} \\ &= \frac{x+2}{2x+5}\end{aligned}$$

The domain of $(j \circ j)(x)$ is all real numbers, $x \neq -\frac{5}{2}, -2$.

The input function is $j(x) = \frac{1}{x+2}$ and has a domain of all real numbers, $x \neq -2$.

The composite function is also linear and has a domain of all real numbers, $x \neq -\frac{5}{2}$, but the restriction of the input function carries through so the domain is all real numbers, $x \neq -\frac{5}{2}, -2$.

Problem 4

Formulas to calculate the number of bacteria in refrigerated food (quadratic) and the temperature of food (Celsius) when it is removed from refrigeration (linear) are given. Students use the formulas to calculate the number of bacteria in food and the number of hours it takes for food to become unsafe to eat. They determine the domain relevant to the context and write a composite function. Students also determine the domain and range of the composite function with respect to the problem situation. They write a similar function using the temperature in degrees Fahrenheit and identify the domain and range with respect to the problem situation.

Grouping

Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- How can 30 minutes be represented as a decimal?
- What is the value of $c(0.5)$?
- What is the value of $b(4)$?
- What does 500 mean with respect to the problem situation?
- What is the value of $c(2)$?
- What is the value of $b(10)$?

PROBLEM 4 Can I Still Eat This?

The number of bacteria B in a refrigerated food is given by $B(c) = 20c^2 - 80c + 500$, where c is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by $c(t) = 4t + 2$, where t is time in hours.

1. Calculate the number of bacteria that are in the food at each time interval.

   a. 30 minutes after removing the food from refrigeration

   There are 500 bacteria in the food 30 minutes after removing it from refrigeration.

   $$\begin{aligned}(B \circ c)(0.5) &= B(c(0.5))\\ &= B(4(0.5) + 2)\\ &= B(4)\\ &= 20(4)^2 - 80(4) + 500\\ &= 20(16) - 320 + 500\\ &= 500\end{aligned}$$

   b. 2 hours after removing the food from refrigeration

   There are 1700 bacteria in the food 2 hours after removing it from refrigeration.

   $$\begin{aligned}(B \circ c)(2) &= B(c(2))\\ &= B(4(2) + 2)\\ &= B(10)\\ &= 20(10)^2 - 80(10) + 500\\ &= 20(100) - 800 + 500\\ &= 1700\end{aligned}$$

- What does 1700 mean with respect to the problem situation?

Guiding Questions for Share Phase, Questions 2 through 7

- How is the value 14 degrees Celsius used in the input function?
- If the value of t is 0, what does that mean with respect to the problem situation?
- Can the value of t be equal to 3? How do you know?
- Why can't the value of t be greater than 3?
- Is $B(c)$ or $c(t)$ the input function in this situation?
- Is the composite function $(B \circ c)(t)$ or $(c \circ B)(t)$?
- Is the input function linear or quadratic?
- Is the composite function linear or quadratic?
- Does the input function have a restricted domain?
- Does the composition function have a restricted domain?
- How do you determine the range of the composite function?
- What equation is used to determine when the number of bacteria will reach 2000?
- What information is used to determine the function $F(t)$?
- To determine $F(t)$, do you use $F(c(t))$ or $c(F(t))$?
- To determine $F(t)$, which function is the input function?
- How does the domain of this function, $F(t)$, compare to the domain of the composite function $(B \circ c)(t)$?

2. Refrigerated food that has been removed long enough to reach a temperature of 14 degrees Celsius is considered unsafe to eat.

   a. Determine the number of hours it takes for a food to become unsafe to eat.

   The food is considered unsafe to eat once it is removed from refrigeration for 3 hours.

   $14 = 4t + 2$

   $12 = 4t$

   $3 = t$

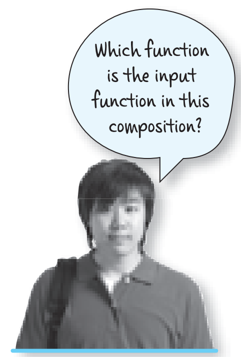
   b. Determine the domain of the function $c(t) = 4t + 2$ in the context of this situation.

   The domain of the function $c(t) = 4t + 2$ is $0 \le t < 3$.

   The function is defined from $t = 0$, when the food is removed from refrigeration, up to $t = 3$, when the food is considered unsafe to eat.

3. Write a function to determine the number of bacteria that are in a food that has been removed from refrigeration for t hours and is safe to eat.

$$
\begin{aligned}
(B \circ c)(t) &= B(c(t)) \\
&= B(4t + 2) \\
&= 20(4t + 2)^2 - 80(4t + 2) + 500 \\
&= 20((4t + 2)(4t + 2)) - 80(4t + 2) + 500 \\
&= 20(16t^2 + 16t + 4) - 320t - 160 + 500 \\
&= 320t^2 + 320t + 80 - 320t - 160 + 500 \\
&= 320t^2 + 420
\end{aligned}
$$



4. Determine the domain of the composite function.

   The domain of the function $(B \circ c)(t)$ is $0 \le t < 3$.

   The input function $c(t)$ has a domain of $0 \le t < 3$. The composite function is quadratic but the restricted domain of the input function carries through.

- How does the range of this function, $F(t)$, compare to the domain of the composite function $(B \circ c)(t)$?
- If the temperature of the food is 58°F, is it safe to eat?

5. Determine the range of the composite function. What does this mean in terms of this problem situation?

The range of the function is $420 \leq (B \circ c)(t) < 3300$.

This means that the range of the number of bacteria in food that has been removed from refrigeration from 0 to 3 hours is greater than or equal to 420 bacteria and less than 3300 bacteria.

$$\begin{aligned}(B \circ c)(t) &= 320t^2 + 420\\(B \circ c)(0) &= 320(0)^2 + 420\\&= 420\end{aligned}$$

$$\begin{aligned}(B \circ c)(t) &= 320t^2 + 420\\(B \circ c)(3) &= 320(3)^2 + 420\\&= 2880 + 420\\&= 3300\end{aligned}$$

6. When will the number of bacteria reach 2000?

The number of bacteria will reach 2000 when $t \approx 2.222$, or after approximately 2 hours and 13 minutes.

$$\begin{aligned}2000 &= 320t^2 + 420\\1580 &= 320t^2\\4.9375 &= t^2\\2.222 &\approx t\end{aligned}$$

7. Write a function $F(t)$ to determine the temperature in degrees Fahrenheit of a food that is safe to eat and has been left out of refrigeration for t hours. State the domain and range and explain what they mean in the context of this problem situation.

$$\begin{aligned}F(c(t)) &= \frac{9}{5}(4t + 2) + 32\\&= \frac{36}{5}t + \frac{18}{5} + 32\\&= 7.2t + 35.6\end{aligned}$$

The domain of the function is $0 \leq t < 3$ and the range of the function is $35.6 \leq F(c(t)) < 57.2$.

This means that the food is safe to eat up to three hours after it is removed from refrigeration. The temperature will be greater than 35.6° and less than 57.2° Fahrenheit.

$$\begin{aligned}F(c(t)) &= 7.2t + 35.6\\F(c(0)) &= 7.2(0) + 35.6\\&= 35.6\end{aligned}$$

$$\begin{aligned}F(c(t)) &= 7.2t + 35.6\\F(c(3)) &= 7.2(3) + 35.6\\&= 57.2\end{aligned}$$

Be prepared to share your solutions and methods.

Check for Students' Understanding

Define each function as the composition of two functions.

1. $f(x) = x^2 + 6x + 9$

   $f(x) = x^2 + 6x + 9$

   $g(x) = x^2$

   $h(x) = x + 3$

   $f(x) = (g \circ h)(x) = (x + 3)^2 = x^2 + 6x + 9$

2. $f(x) = x^2 - 10x + 25$

   $f(x) = x^2 - 10x + 25$

   $g(x) = x^2$

   $h(x) = x - 5$

   $f(x) = (g \circ h)(x) = (x - 5)^2 = x^2 - 10x + 25$

3. $f(x) = x^4 - 16x^2$

   $f(x) = x^4 - 16x^2$

   $g(x) = x^2$

   $h(x) = x^2 - 16x$

   $f(x) = (h \circ g)(x) = (x^2)^2 - 16(x^2) = x^4 - 16x^2$

14.2

Paint by Numbers

Art and Transformations

LEARNING GOALS

In this lesson, you will:

- Use transformations of functions and other relations to create artwork.
- Write equations for transformed functions and other relations given an image.

ESSENTIAL IDEAS

- Basic functions such as linear, quadratic, absolute value, rational, exponential, polynomial, and radical, along with the equation for a circle, are used to create graphics on the coordinate plane.
- Transformations of functions with restricted domains are associated with graphics created on the coordinate plane.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F.IF Interpreting Functions

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

d. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

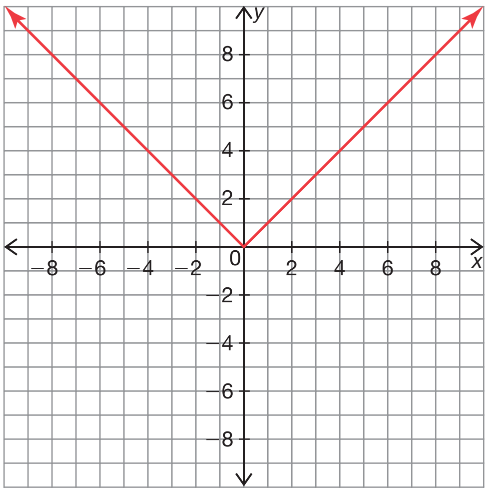
Overview

Students use their prior knowledge of transformational form function to create graphics on the coordinate plane. Ten different equations/relations are graphed, such as linear, quadratic, absolute value, rational, exponential, polynomial, radical, and circular. A chart reviews the descriptions of graphic transformations related to their equations written in transformational function form. Students use their prior knowledge of equations, relations, and transformational function form to first create artwork using given equations/relations and then to write the equations/relations associated with created artwork. Students also create their own artwork and list all associated equations/relations. Restricted domains are listed in each activity when appropriate.

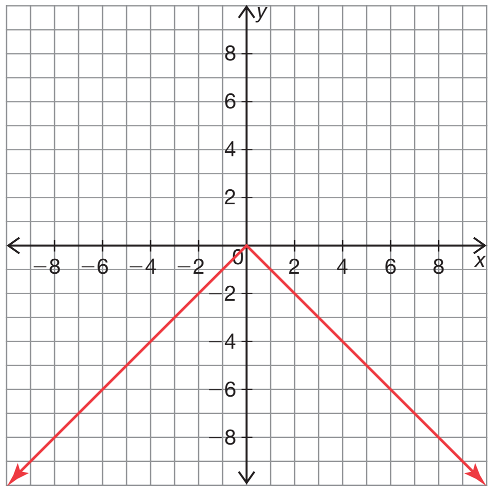
Warm Up

Sketch the graph of each function.

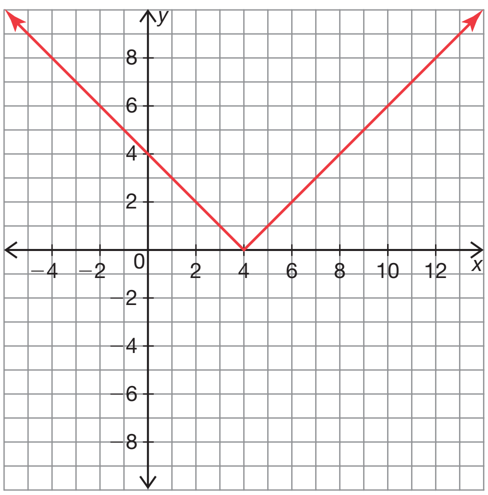
1. $f(x) = |x|$



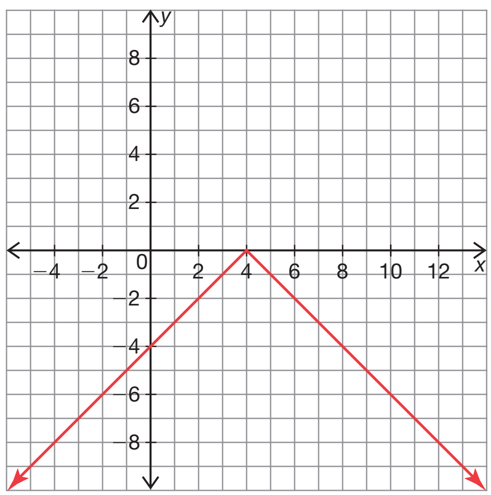
2. $f(x) = -|x|$



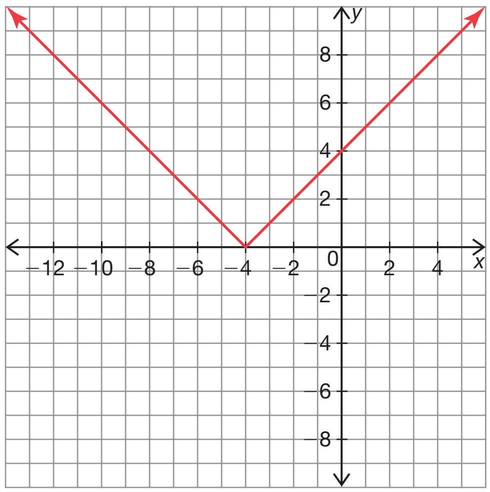
3. $f(x) = |x - 4|$



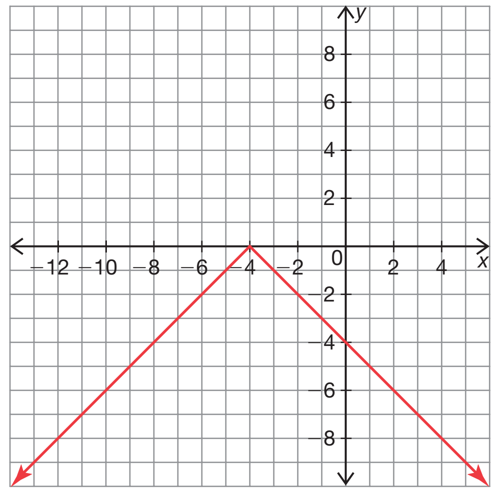
4. $f(x) = -|x - 4|$



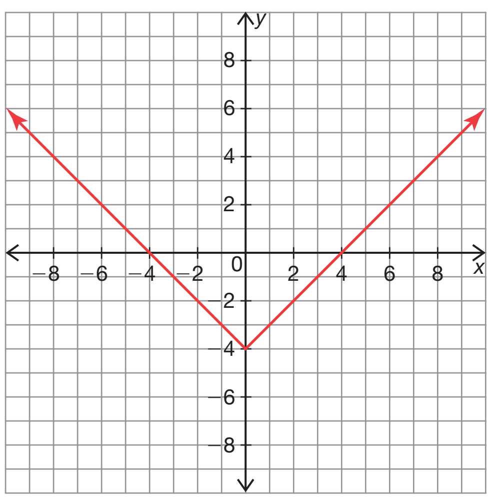
5. $f(x) = |x + 4|$



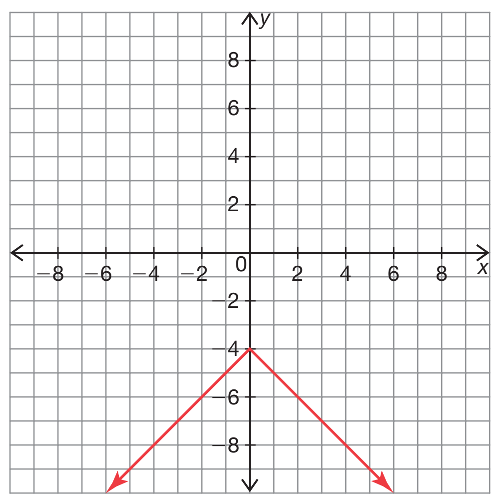
6. $f(x) = -|x + 4|$



7. $f(x) = |x| - 4$



8. $f(x) = -|x| - 4$



14.2

Paint by Numbers

Art and Transformations

LEARNING GOALS

In this lesson, you will:

- Use transformations of functions and other relations to create artwork.
- Write equations for transformed functions and other relations given an image.

Here's a picture that should make you smile:

$$y = -\sqrt{-x^2 + \frac{1}{2}}$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{32}$$

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{32}$$

Adorable, right?

Many contemporary artists like Helaman Ferguson and George Hart not only use mathematics to create their art but also use mathematics as a subject of their art.

Want to try your hand at creating some mathematical art?

Problem 1

Graphical representations of eight types of relations are shown: linear, quadratic, absolute value, polynomial, exponential, radical, and a circle. A chart reviews the symbolic and verbal descriptions of functions having undergone specified transformations. Students graph given equations using their prior knowledge of transformation function form and the graphic result is the image of the face of a cat. Next, the artwork on a graph is given and students list possible equations that created the art and include domain restrictions. In the last activity, students create their own artwork using a minimum of five different transformed functions or relations. Each equation is identified with a restricted domain when appropriate. Their work must include a linear equation, absolute value equation, quadratic equation, exponential equation, and radical equation.

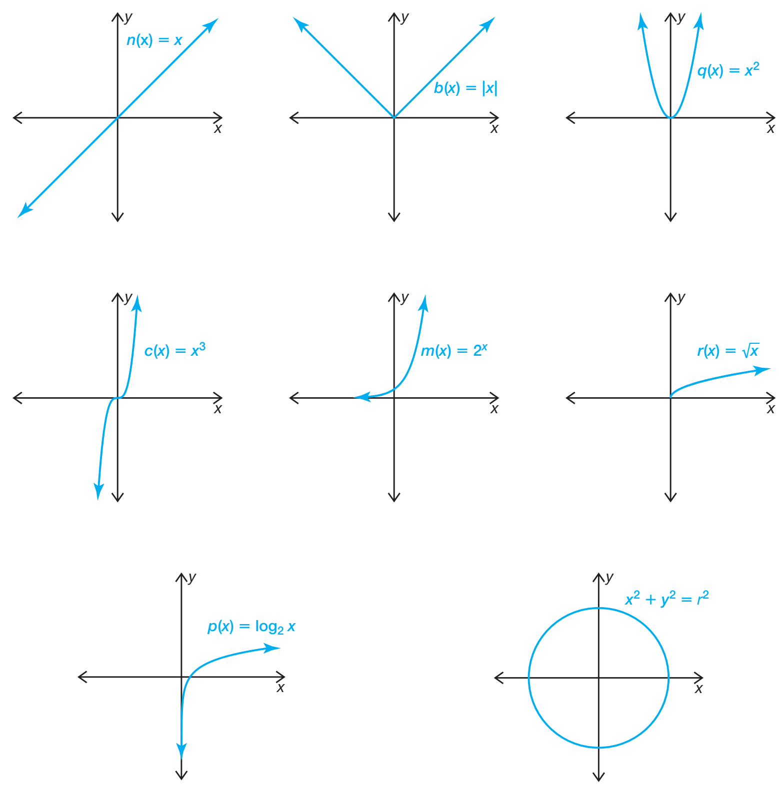
Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class after each question.

PROBLEM 1 It's Van Gogh Time!

You have studied many types of relations throughout your high school mathematics career. Now it's time to use all of them to create some art.

Recall the seven basic functions and the basic equation for a circle shown.



Recall the effect of each transformation on the graph of $y = f(x)$.

Function Form	Equation Information	Description of Transformation of Graph
$y = f(x) + D$	$D > 0$	Vertical shift up D units
	$D < 0$	Vertical shift down D units
$y = f(x - C)$	$C > 0$	Horizontal shift right C units
	$C < 0$	Horizontal shift left C units
$y = Af(x)$	$\lvert A \rvert > 1$	Vertical stretch by a factor of A units
	$0 < \lvert A \rvert < 1$	Vertical compression by a factor of A units
	$A < 0$	Reflection across the x-axis
$y = f(Bx)$	$\lvert B \rvert > 1$	Horizontal compression by a factor of $\frac{1}{\lvert B \rvert}$
	$0 < \lvert B \rvert < 1$	Horizontal stretch by a factor of $\frac{1}{\lvert B \rvert}$
	$B < 0$	Reflection across the y-axis

Guiding Questions for Share Phase, Question 1

- Which functions are linear?
- Which functions are quadratic?
- Which functions are exponential?
- Which functions are radical equations?
- Which equations are the equations of a circle?

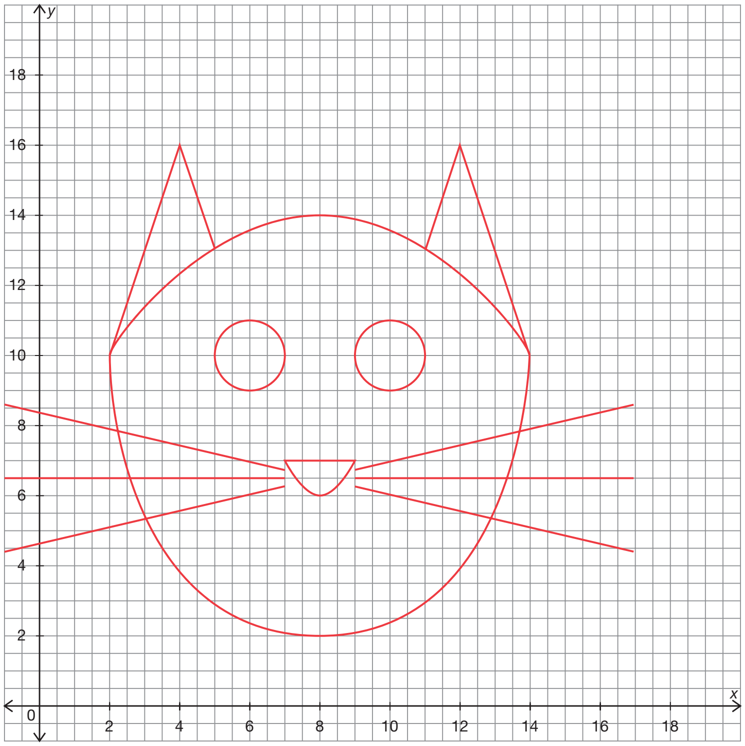
1. Use the seven basic functions, the basic equation for a circle, and your knowledge of the transformational function form to graph the given equations. What image do you see?

Equations	Restrictions
$y = -\frac{1}{10}q(x - 8) + 14$	$2 \le x \le 14$
$y = -3b(x - 4) + 16$	$2 \le x \le 5$
$y = -3b(x - 12) + 16$	$11 \le x \le 14$
$y = m(-x + 5) + 2$	$2 \le x \le 8$
$y = m(x - 11) + 2$	$8 \le x \le 14$
$(x - 6)^2 + (y - 10)^2 = 1$	
$(x - 10)^2 + (y - 10)^2 = 1$	
$y = q(x - 8) + 6$	$7 \le x \le 9$
$y = 7$	$7 \le x \le 9$
$y = -\frac{1}{4}n(x - 8) + 6.5$	$-1 \le x \le 7$
$y = -\frac{1}{4}n(x - 8) + 6.5$	$9 \le x \le 17$
$y = 6.5$	$-1 \le x \le 7$
$y = 6.5$	$9 \le x \le 17$
$y = \frac{1}{4}n(x - 8) + 6.5$	$-1 \le x \le 7$
$y = \frac{1}{4}n(x - 8) + 6.5$	$9 \le x \le 17$

See graph.

The image is the face of a cat.

14

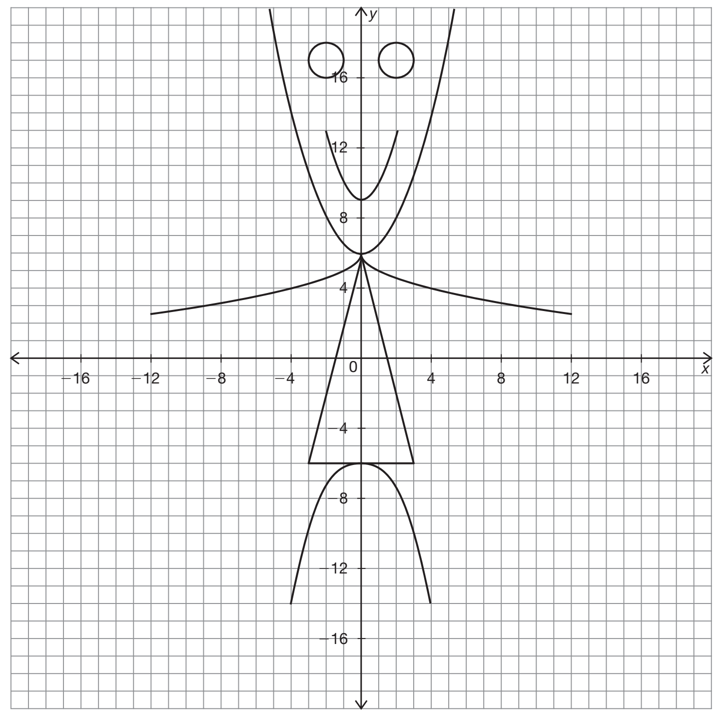


y
x
0
2
4
6
8
10
12
14
16
18

Guiding Questions for Share Phase, Question 2

- What are the coordinates of the center point of the circles in the artwork?
- How are the coordinates of the center point of a circle used to write the equation of the circle?
- What transformations are performed on the parabolas in the artwork?
- What transformation is performed on the absolute value function in the artwork?
- What part of the artwork is associated with the two exponential equations?
- What part of the artwork is associated with the two radical equations?
- What transformation is performed on the linear function? Where is it in the artwork?
- Do all of the functions in the artwork have restricted domains?
- Which functions do not have restricted domains?

2. Use the seven basic functions, the equation of a circle, and your knowledge of the transformational function form to determine the equation of each graph in the picture. Include corresponding domain restrictions where necessary. When possible, write each equation in terms of one of the basic functions or equations given at the beginning of this problem.



Equation(s) of a circle:

$(x - 2)^2 + (y - 17)^2 = 1$

$(x + 2)^2 + (y - 17)^2 = 1$

Exponential Equation(s):

$y = -m(-x - 1) - 5, -4 \le x \le -1$

$y = -m(x - 1) - 5, 1 \le x \le 4$

Quadratic Equation(s):

$y = \frac{1}{2}q(x) + 6$

$y = q(x) + 9, -2 \le x \le 2$

Absolute Value Equation(s):

$y = -4b(x) + 6, -3 \le x \le 3$

Radical Equation(s):

$y = -r(-x) + 6, -12 \le x \le 0$

$y = -r(x) + 6, 0 \le x \le 12$

Linear Equation(s):

$y = -6, -3 \le x \le 3$

Guiding Questions for Share Phase, Question 3

- When creating your own artwork, did you write the function before you created the artwork or did you create the artwork before writing the functions?
- Is your artwork symmetric to the x- or y-axis?

3. Create a picture or design using at least five transformed functions or equations. The transformations should include at least one stretch or compression, one translation, and one reflection. You must also include one of each of the following:

- linear equation
- absolute value equation
- quadratic equation
- exponential equation
- radical equation

When possible, write each equation in terms of one of the basic functions or equations given at the beginning of this problem, and then identify each type. Include corresponding domain restrictions where necessary.

Equation: **Type:**

See graph.

Answers will vary.

4. Ask a classmate to graph your equations. Does their graph match yours?

Be prepared to share your solutions and methods.

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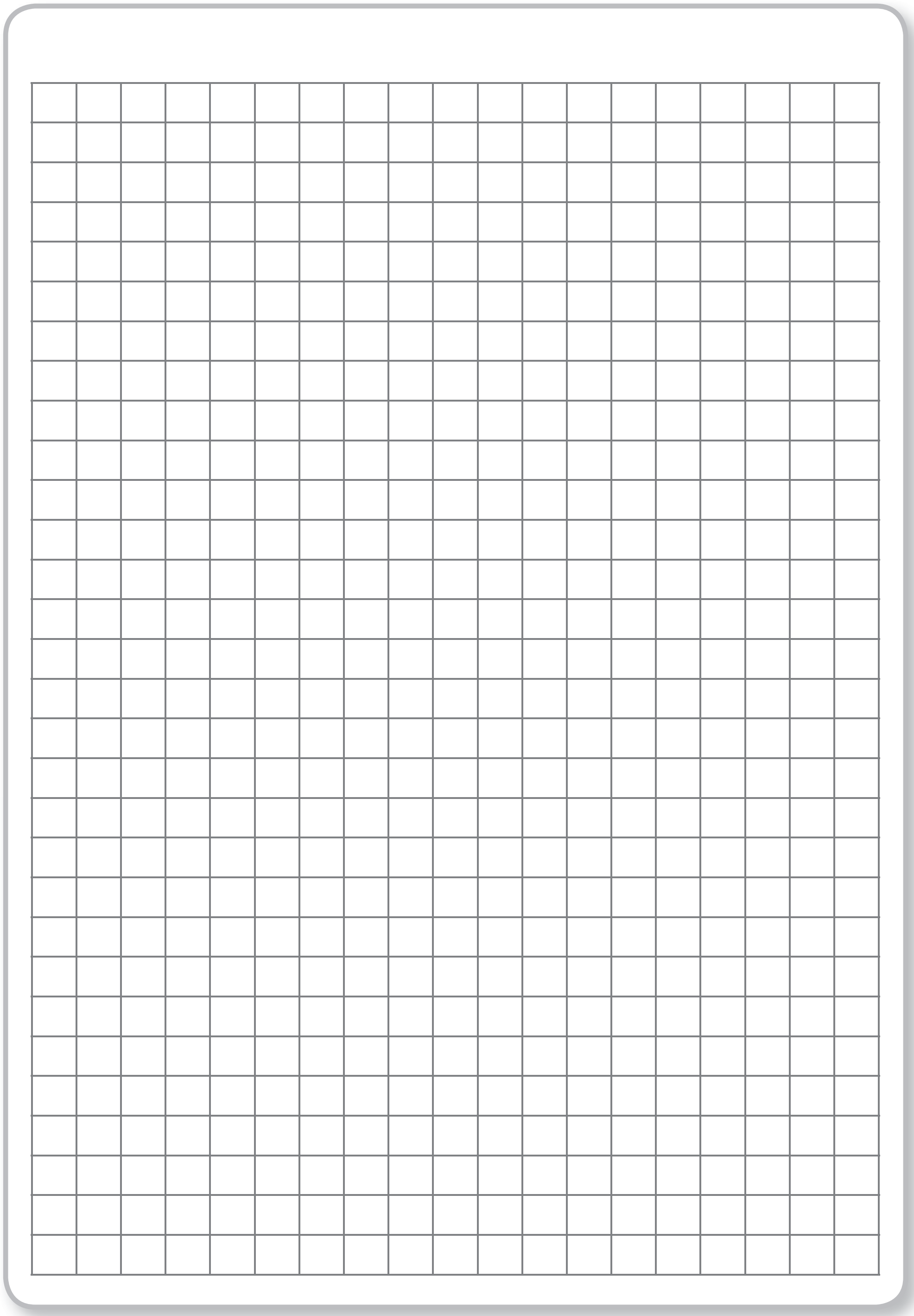
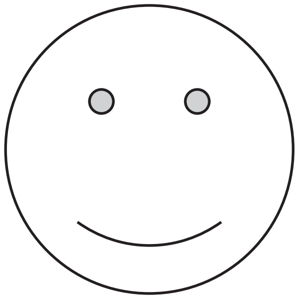


Check for Students' Understanding

Describe possible functions/relations you could use to graph the smiley face shown.



Answers will vary.

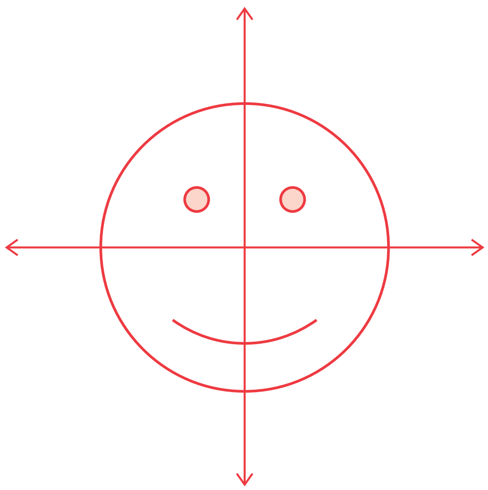
- Three equations for a circle

  One large circle with center at the origin

  Two small congruent circles (one circle reflected across the y-axis)

- One equation of a parabola with restricted domain

  with a negative y-intercept and symmetric to the y = axis



Make the Most of It

Optimization

14.3

LEARNING GOALS

In this lesson, you will:

- Determine constraints from a problem situation.
- Analyze a function to calculate maximum or minimum values.

ESSENTIAL IDEAS

- Systems of inequalities and quadratic functions can model real-world problems with optimal solutions.
- Constraints in a real-world situation are identified and used to compute optimal results.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A.CED Creating Equations

Create equations that describe numbers or relationships

3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*

A.REI Reasoning with Equations and Inequalities

Represent and solve equations and inequalities graphically

12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

F.IF Interpreting Functions

Understand the concept of a function and use function notation

1b. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

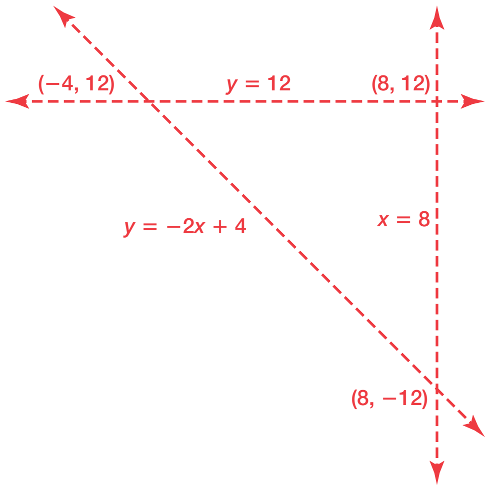
Overview

Students write and graph system of inequalities (linear programming) to describe a gym workout situation and shade the region associated with the solution. The points of intersection of the restraints on the coordinate plane are used to evaluate the equation written to test for optimization. In a second activity, students write a quadratic equation to model a ticket sale situation and use it to compute maximum revenue. Profit formulas are also used to compare different ticket prices.

Warm Up

Al is fencing part of his property. The boundaries for the fenced area are $y = 12$, $x = 8$, and $y = -2x + 4$.

- Graph the system of equations that models the area of the fenced land.
- Identify the vertices of the fenced area.



Vertices: (−4, 12), (8, −12), (8, 12)

14.3

Make the Most of It
Optimization

LEARNING GOALS

In this lesson, you will:

- Determine constraints from a problem situation.
- Analyze a function to calculate maximum or minimum values.

Optimization problems involve finding the best solution from a choice of solutions given the objective. One of the most famous optimization problems in mathematics is the Traveling Salesman Problem. This problem can be stated as follows:

> Given a number of cities connected by roads, describe the shortest route that can be taken by a traveling salesman in order to visit every city once and then return to his starting place.

Work on problems like these has potential applications in computer science and other fields. But a solution may forever remain elusive. In 1972, a computer scientist named Richard Karp showed that a solution to the Traveling Salesman problem might not even be possible!

Problem 1

Working out at a gym is the context and linear programming is used in this problem. Students write a system of inequalities to represent the constraints of a problem situation, graph the system on the coordinate plane, and shade the region that represents the solution set. Information about the calories burned related to two different physical activities is provided and students determine the number of hours needed to maximize a weekly workout by substituting the intersection points of the constraint into an equation they wrote using the calorie information. The equation is also used to determine whether a weight-loss goal is attainable.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- What are the variables in this situation?
- What do the variables represent in this situation?
- How many constraints are in this situation?
- How many of the constraints contain both variables?

PROBLEM 1 Getting in Shape

Brad is a distance runner who wants to design the most effective workout plan to prepare for his next race. To adequately prepare for the race, Brad must run between 3 and 6 hours per week, but he knows that he must also spend some time strength training to reduce the risk of injury. Brad wants to devote at least twice as much time to running as to strength training, but he can spend no more than 8 hours at the gym each week.

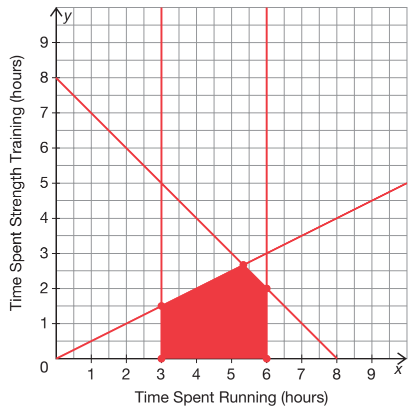
1. Write a system of inequalities to represent the constraints of this problem situation. Be sure to define your unknowns.

   Let x = the number of hours spent running

   Let y = the number of hours spent strength training

   $$\begin{cases} 0 \le y \le \frac{1}{2}x \\ x + y \le 8 \\ 3 \le x \le 6 \end{cases}$$

2. Graph the system of inequalities on the coordinate plane shown. Shade the region that represents the solution set.



- How many constraints are in terms of the hours spent running?
- How many constraints are in terms of the hours spent strength training?
- What shape is the region representing the solution set?
- How many vertices are associated with the region representing the solution set?

Guiding Questions for Share Phase, Questions 3 and 4

- What are the coordinates of the intersection points of the constraints?
- What equation is used to determine how Brad can maximize his workout?
- The coordinates of the intersection points of the constraints are used to evaluate what equation?
- How many calories did Brad burn running for 6 hours per week?
- How many calories did Brad burn strength training for 2 hours each week?

3. Running burns 600 calories per hour, while strength training burns 250 calories per hour. How many hours should Brad devote to each in order to maximize his weekly workout?

   Brad should devote 6 hours to running and 2 hours to strength training each week.

   The intersection points of the constraints are (3, 0), (6, 0), (3, 1.5), (5.33, 2.67), and (6, 2).

   $$600x + 250y = W$$
   $$600(3) + 250(0) = 1800$$
   $$600(6) + 250(0) = 3600$$
   $$600(3) + 250(1.5) = 2175$$
   $$600(5.33) + 250(2.67) = 3865.5$$
   $$600(6) + 250(2) = 4100$$

4. One of Brad's friends was interested in losing a little weight. He follows Brad's training program to try to lose one pound per week. In order to lose a pound a week, he needs to burn a total of 3500 calories each week. Will Brad's friend meet his goal of losing one pound per week? Explain your reasoning.

   Yes. By running for 6 hours per week and strength training for 2 hours each week, Brad is burning 4100 calories per week, which is more than 3500.

Problem 2

Attendance related to the cost of a prom ticket is the context in this problem. Students write a quadratic function that models the revenue generated from the sale of prom tickets for every x reduction in price. They calculate the x- and y-intercepts and explain their relevance to the problem situation. Next, students determine the ticket price that maximizes revenue and calculate the amount of revenue the new ticket price will generate. Overhead costs are provided and students use profit formulas to compare last year's ticket sale profit to the anticipated ticket sale profit resulting from the new ticket price.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- Is the function modeling the revenue generated from the sale of prom tickets for every x reduction in price a linear equation or a quadratic equation?
- What algebraic expression represents the price of prom tickets?

PROBLEM 2 It's Prom Time!

Last year, tickets to the prom cost \$50 per student and 120 students attended. A student survey found that for every \$5 reduction in price, 30 more students would attend.

1. Write a function to model the revenue generated from the sale of prom tickets for every x reduction in price.

   $R(x) = -150x^2 + 900x + 6000$

   The price of the prom tickets is equal to $(50 - 5x)$. The number of students that will attend the prom is equal to $(120 + 30x)$.

   Revenue = (price) (number of students)

   $$\begin{aligned} R(x) &= (50 - 5x)(120 + 30x) \\ &= 6000 + 1500x - 600x - 150x^2 \\ &= -150x^2 + 900x + 6000 \end{aligned}$$

2. Calculate the x- and y-intercept and explain what each means within the context of this problem situation.

   The y-intercept occurs where $x = 0$. Since x represents the number of \$5 reductions in price, when $x = 0$, it means that the ticket price remains unchanged.

   $$\begin{aligned} R(x) &= -150x^2 - 450x + 6000 \\ &= -150(0)^2 - 450(0) + 6000 \\ &= 6000 \end{aligned}$$

   If there were no reductions in price, the revenue would be the same as last year, which was \$50 each for 120 students, or \$6000.

   The x-intercept occurs where $y = 0$. Since y represents the amount of revenue generated, when $y = 0$, x represents the number of \$5 reductions in price that it would take for the prom to generate no money.

   $$\begin{aligned} R(x) &= -150x^2 + 900x + 6000 \\ 0 &= -150x^2 + 900x + 6000 \\ 0 &= x^2 - 6x - 40 \\ 0 &= (x - 10)(x + 4) \\ x &= 10 \qquad \cancel{x = -4} \end{aligned}$$

   If the price is reduced by \$5 ten times, that is equal to a total reduction of \$50, or the entire cost of last year's ticket. Therefore, the ticket price becomes \$0 and the prom would generate no revenue.

- What algebraic expression represents the number of students that will attend the prom?
- How is the revenue calculated in this situation?
- Is the price of the prom tickets times the number of students attending the prom equal to the revenue?

Guiding Questions for Share Phase, Question 2

- Graphically, where does the *y*-intercept occur?
- What does *x* represent?
- If the ticket price remains unchanged, what is the value of *x*?
- What was the revenue last year?
- Graphically, where does the *x*-intercept occur?
- What does *y* represent?
- If there is no revenue, what is the value of *y*?
- If the revenue is equal to 0, is the quadratic equation factorable?
- What are the solutions to the quadratic equation?
- Does the solution $x = -4$ make sense in this situation?
- If the price of the ticket is reduced by \$5 ten times, how much will the ticket cost?

Guiding Questions for Share Phase, Questions 3 and 4

- What is the formula for the vertex of a parabola?
- How is the vertex of the parabola helpful in this situation?
- If the value of *x* equals 3, what is the price of the ticket?
- What does the *x*-value of the maximum represent with respect to the problem situation?

3. Determine the ticket price that will maximize revenue.

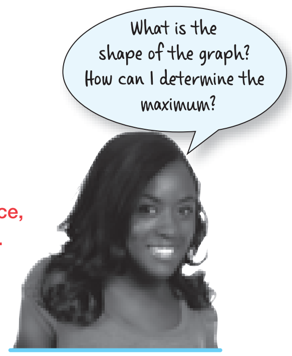
The prom ticket price should be \$35.

$x = \frac{-b}{2a}$

$x = \frac{-900}{2(-150)}$

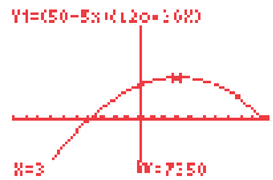
$x = 3$

Since *x* represents the number of \$5 reductions in price, when $x = 3$, the overall ticket price is reduced by \$15.



4. Calculate the amount of revenue that the new ticket price will generate.

The new ticket price will generate \$7350.



- What does the *y*-value of the maximum represent with respect to the problem situation?
- If the new ticket price is \$35, how many students will go to the prom?

Guiding Questions for Share Phase, Questions 5 and 6

- How many more students will attend the prom beyond what the flat rate covers?
- How much will the prom venue change the school? How was this determined?
- If the school is charged $2900 for the venue, how much profit will the school make?
- How many students attended the prom last year?
- How much was the revenue from the ticket price last year?
- Was any additional cost incurred last year due to the attendance? Why or why not?
- Will the increase in cost of the ticket price generate more or less revenue when compared to the previous year?
- How much more will be made if the cost of the ticket is reduced to $35?

5. The venue for the prom charges the school a flat rate of $2000 for up to 150 students, and then $15 for every additional student.

a. Calculate the profit that the prom made for the school last year.

Last year, 120 students attended the prom and the revenue from the ticket price was $6000. The prom venue charges $2000 for up to 150 students, so no additional cost was incurred.

Profit = Revenue − Cost

$P = 6000 - 2000$

$P = 4000$

b. Determine the profit that the school will make if they use the new ticket price.

The school's profit will be $4450.

According to the model, if the ticket price is $35, then 210 students will go to the prom. That is 60 more students than the flat rate covers, so the prom venue will charge the school $2000 + 60(15), or $2900.

Profit = Revenue − Cost

$P = 7350 - 2900$

$P = 4450$

6. Compare the profit that the prom made for the school last year with the profit that it will make this year if it uses the new ticket price. Is it worth it to change the ticket price? Explain your reasoning.

Yes. Last year the profit for the prom was $4000, and using the proposed ticket price for this year, the profit will be $4450. By reducing the ticket price, more students will attend and despite the increased cost, it will still generate a higher revenue.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Christy uses glass beads to make bracelets and necklaces. It takes her 30 minutes to make a bracelet and 45 minutes to make a necklace. She makes at least twice as many bracelets as necklaces. She works at most 40 hours a week. She wants to make at least 30 bracelets. The profit from a bracelet is \$10, and the profit from a necklace is \$18.

- Define the variables.
- Write the constraints as inequalities.
- Write the profit equation.
- Graph the region.
- Evaluate the profit equation at each vertex of the region to determine the number of bracelets and necklaces Christy should make to maximize her profit.

Let b represent the number of bracelets.

Let n represent the number of necklaces.

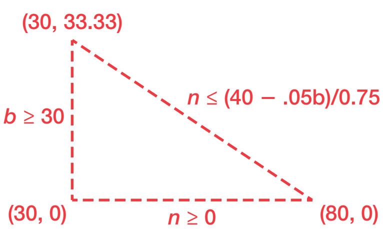
$0.5b + 0.75n \le 40$

$b \ge 30$

$n \ge 0$

$P = 10b + 18n$

Insert graph to include this region:



$P = 10(30) + 18(33) = 894$

$P = 10(80) + 18(0) = 800$

$P = 10(30) + 18(0) = 300$

14.4

A Graph Is Worth a Thousand Words

Interpreting Graphs

LEARNING GOALS

In this lesson, you will:

- Interpret the contextual meaning of a graph and analyze it in terms of a problem situation.
- Write a logistic growth function to model a data set.
- Use technology to generate random numbers in order to conduct an experiment modeling logistic growth.

KEY TERMS

- logistic functions
- carrying capacity

ESSENTIAL IDEAS

- A logistic function is a function which can be written in the form $y = \frac{C}{1 + Ae^{-Bx}}$, where the numerator represents the upper value of the function, or the carrying capacity.
- The four stages of a logistic function are the initial growth stage, the exponential growth stage, the dampened growth stage, and the equilibrium stage.
- The carrying capacity of a logistic function is the value the graph approaches in the equilibrium stage.
- Logistic functions are good models of biological population growth, the spread of diseases, the spread of information, and sales of new products over time.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F.IF Interpreting Functions

Understand the concept of a function and use function notation

2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

   d. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

Overview

Students analyze several graphs representing real world situations. They use verbal descriptions to identify the growth of a bacteria culture, and trends in weather with respect to temperature, solar radiation, and rainfall. In another activity, students match the appropriate graph with its verbal description. The terms logistic function and carrying capacity are defined. A table of values and a logistic function are given to model the population growth of a bug and students create a scatter plot which is used to answer questions related to the situation. In this lesson, the graphing calculator is used to; compare the behaviors of logistic functions having different carrying capacities, generate data for an experiment using the random number generator, and graph data from a scatter plot to perform a logistic regression to determine a logistic function.

Warm Up

1. What is the eventual consequence of an exponential growth situation?

   In an exponential growth situation, the output, or $f(x)$, increases indefinitely.

2. What is a real-world example of an exponential growth situation?

   The population growth of rabbits is an example of an exponential growth situation.

3. Regarding your real-world example of an exponential growth situation in Question 2, does this situation have a limit as to how large $f(x)$, or the output, can get? Explain your reasoning.

   Answers will vary.

   Yes, the growing rabbit population has a limit as to how large it can become. The rabbit population cannot become infinitely large for several reasons: the environment in which rabbits live is not infinitely large, rabbits have predators, rabbits die of disease or old age, food supply is limited, climate is too severe, natural disasters, not all rabbits are able to multiply.

4. Do all real-world examples of an exponential growth situation have a limit as to how large $f(x)$, or the output, can get? Explain your reasoning.

   Answers will vary.

A Graph Is Worth a Thousand Words

Interpreting Graphs

14.4

LEARNING GOALS

In this lesson, you will:

- Interpret the contextual meaning of a graph and analyze it in terms of a problem situation.
- Write a logistic growth function to model a data set.
- Use technology to generate random numbers in order to conduct an experiment modeling logistic growth.

KEY TERMS

- logistic functions
- carrying capacity

Suppose that you were given the choice between an instant gift of \$100 or flipping a coin for a chance to win \$200 or nothing. Which would you choose?

Now imagine another choice. Either you can lose \$100 instantly, or you can flip a coin for a chance of losing \$200 or nothing. Which would you choose?

Many studies have shown that people would choose to gamble in the second situation but not the first, even though the situations are mathematically equivalent. The results point to a general tendency known as loss aversion. Psychologists Daniel Kahnemann and Amos Tversky first demonstrated that this tendency can be modeled by a function similar to what is known as a logistic function.

Problem 1

Students are given a table of values that summarizes the population growth of a 'Hum Bug' in terms of days and number of bugs. They create a scatter plot for the data, sketch a curve through the points and use the graph to answer questions related to the situation. The terms *logistic function* and *carrying capacity* are defined, and a logistic function is given to model the population growth of the bug. Students use a graphing calculator to compare and discuss the changes in the graphical representation related to different values in the numerator of the logistic function.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Can the graph be described as S-shaped?
- Where is the graph close to zero?
- Where does the graph increase quickly?

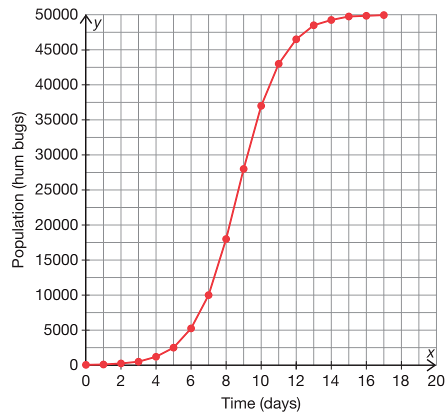
PROBLEM 1 The Hum Bug

Over the summer you noticed strange, noisy bugs in the field near your house. Each day, there seem to be more and more bugs in the field. You do a little research and find out that this insect is called a hum bug because of the humming sound it makes. You also find someone who has studied hum bug population growth in your area. The data they provide are summarized in the table shown.

Time (Days)	Population (Hum Bugs)
0	50
1	100
2	250
3	500
4	1200
5	2500
6	5250
7	10000
8	18000

Time (Days)	Population (Hum Bugs)
9	28000
10	37000
11	43000
12	46500
13	48500
14	49250
15	49750
16	49850
17	49950

1. Create a scatter plot for the data and sketch a curve through the points.



14

Guiding Questions for Share Phase, Questions 3 through 7

- As x gets large, what happens to the y-values?
- What is the behavior of the graph as x gets large?
- What on the graph indicates a growth in the hum bug population?
- How would you describe the population of the hum bug over the first few days?
- How would you describe the change in the population of the hum bug between Days 6 and 13?
- What characteristic on the graph indicates the hum bug population is slowing down?
- How would you describe the change in the population of the hum bug between Days 12 and 14?
- What characteristic on the graph indicates the hum bug population is leveling off?
- How would you describe the change in the population of the hum bug after Day 14?
- How could food sources affect the population of hum bugs?
- How could predators affect the population of hum bugs?
- How could introducing different bugs into the population affect the population of hum bugs?
- How can weather conditions adversely affect the population of hum bugs?

2. Describe the overall shape of your graph.

The graph is somewhat *s*-shaped. It is close to horizontal near zero and then increases quickly. As x gets larger, it flattens out again near 50000.

3. Where on the graph is the greatest increase in the hum bug population?

The greatest increase in the bug population is between Days 6 and 13.

4. Where on the graph does the hum bug population growth seem to slow down?

The population growth seems to slow down between Days 12 and 14.

5. Where on the graph does the hum bug population seem to level off?

The graph seems to level off after Day 14.

6. What factors in the environment might have caused the hum bug population to grow as it did?

The bug population growth may have been slow at first because the bugs were young or getting used to the environment. As the bugs mature and adjust to the environment , the population grows very quickly. As space and food become insufficient for the greater population, the growth may slow and the population may level off.

7. If insect populations can grow very quickly, why have they not just taken over the world?

The insect population is kept in check because other animals eat the insects. Also, weather conditions may kill off insects. If the insect population does grow too large, then they will run out of food or space for habitat. People control insect populations with bug sprays and insecticides on crops.

- What are some things people do to control bug populations near their homes?
- What are some businesses founded on the concept of bug control?

Grouping

- Read and discuss the information about logistic functions and carrying capacity as a class.
- Have students complete Questions 9 and 10 with a partner. Then have students share their responses as a class.

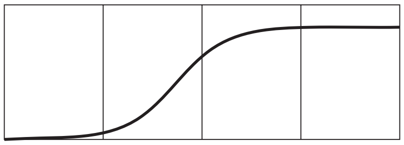
Guiding Questions for Share Phase, Questions 9 and 10

- What is the value of *A* in the logistic formula describing the hum bug population?
- What is the value of *B* in the logistic formula describing the hum bug population?
- What is the value of *C* in the logistic formula describing the hum bug population?
- What does $f(x)$ represent with respect to the problem situation?
- Can the graph of the logistic function be described as *S*-shaped?
- Does the graph of the given logistic function reach its upper limit at about the same time?
- What is the new upper limit or carrying capacity using this formula?
- What does the numerator of the logistic function represent?
- As the value of *x* increases, what value does the function approach?

The function graphed in this situation is closely modeled by the equation $f(x) = \frac{50{,}000}{1 + 1000e^{-0.8x}}$, which is part of the family of *logistic functions*.

Logistic functions are functions which can be written in the form $y = \frac{C}{1 + Ae^{-Bx}}$.

The S-shaped graph of a logistic function has four distinct intervals: the initial growth stage, the exponential growth stage, the dampened growth stage, and the equilibrium stage. The value the graph approaches in the equilibrium stage is called the **carrying capacity**.



Logistic functions are good models of biological population growth, the spread of diseases, the spread of information, and sales of new products over time.

9. Enter the function $f(x) = \frac{50{,}000}{1 + 1000e^{-0.8x}}$ into a graphing calculator and view the graph.

a. Change the numerator of the function so that it becomes $f(x) = \frac{40{,}000}{1 + 1000e^{-0.8x}}$. How did the graph change?

The graph has the same shape. It reaches its upper limit at about the same time, but the upper limit, or carrying capacity, is at 40,000 instead of 50,000.

b. Change the numerator again so the function becomes $f(x) = \frac{30{,}000}{1 + 1000e^{-0.8x}}$. How did the graph change?

The graph has the same shape. It reaches its upper limit at about the same time, but the upper limit or carrying capacity is at 30,000 instead of 50,000.

10. What does the value of *a* in the logistic function $y = \frac{a}{1 + be^{-cx}}$ represent on the graph?

The numerator of the logistic function represents the upper value of the logistic function or the carrying capacity. As the values of *x* increase, the function approaches the value of the numerator, which is the carrying capacity.

Problem 2

The spread of disease in a small school can be modeled by a logistic function. Students perform an experiment using a list of numbers and a random number generator function on a graphing calculator to generate data related to the spread of the disease. They enter the data in a table of values and use the data to create a scatter plot. The scatter plot is used to answer questions related to the stages of growth and the carrying capacity of the function. After entering the data in the graphing calculator, students perform a logistic regression and determine the approximate values of variables in a general logistic function. They also discuss how a change in the carrying capacity would affect the experiment and results.

Grouping

- Read and discuss the information about the scenario as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.
- Read the information about the experiment as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- How many students attend the school?

PROBLEM 2 Sick of School

Logistic functions can be used to model the spread of disease. The data you collect in this experiment will model the spread of a disease. After collecting the data, you will be able to analyze it by using a logistic function.

In a small school, there are 100 students. One day, Zelda Zero, a student, comes to the school infected with a virus that causes mild cold-like symptoms. On the first day that he has the virus, he has the potential to transmit the virus to one other person. On subsequent days, each infected person has the potential to transmit the virus to one other person. If the virus is transmitted to someone who has already had the virus, the person will not become ill again.

Before starting the experiment, take a few minutes to think about what will happen in the situation and answer each question.

1. How long will it take before most of the school is infected with the virus? Why?

   Answers will vary.

   I think it will take about 8 days for most of the school to be infected. If I double the number of infected people each day, on the eighth day all 100 people are infected.

2. Is there a limit to the number of people who can be infected? Explain your reasoning.

   Yes. There is a limit of 100 people who can be infected at the school.

To keep things simple for the experiment, you will represent the 100 students in the school with numbers from 0 to 99. Using the list of numbers on the next page, you can keep track of students infected with the virus by crossing off their number. You'll notice that 0 is already crossed off because Zelda Zero is already infected.

To simulate students getting infected with the virus, you can use the random number generator on a graphing calculator. Generate a random number between 0 and 99 by typing int(rand*100) and pressing ENTER. After getting the first random number, you can get more random numbers simply by pressing ENTER.

On the first day, you will potentially infect one more person by choosing one random number. On subsequent days, you will generate an amount of random numbers equal to the number of people already infected. If a random number that is generated is not already crossed off, cross it off on the list.

- How many students would be considered 'most of the school'?
- How do you determine the number of infected students on the second day? The third day? The fourth day?
- How many students would be infected the second day? The third day? The fourth day?
- How many students would be infected on the seventh day? The eighth day?
- Why can't more than 100 students be infected in the school?

Grouping

Have students complete Questions 3 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 3 through 6

- What number did you randomly generate the first day in the experiment?
- How many numbers did you randomly generate the second day in the experiment? What were the numbers?
- How many numbers did you randomly generate the third day in the experiment? What were the numbers?
- If you generate a random number that is already crossed off the list, why don't you need to generate an alternate number? What information in the setup of this experiment is associated with this rule?
- How many days did it take to cross off most of the numbers on your list? Is this what you expected? Why or why not?
- Is your scatter plot *S*-shaped?
- Which stage occurs from Day 0 to Day 2?
- Which stage occurs from Day 2 to Day 5?
- Which stage occurs from Day 5 to Day 8?
- Which stage occurs after Day 10?

Conduct the experiment following these directions.

- Use a graphing calculator to generate random numbers indicating who gets infected.
- Keep track of infected students by crossing out numbers as they are randomly selected. If a number is already crossed out, do not generate an alternate number.
- For each day, generate as many random numbers as there are currently infected students.
- At the end of each round of random numbers that represent one day, record the total number infected (crossed out numbers) in the table.
- Repeat the process. Work until most students are infected (you have most of your numbers crossed off).

3. Conduct the experiment.

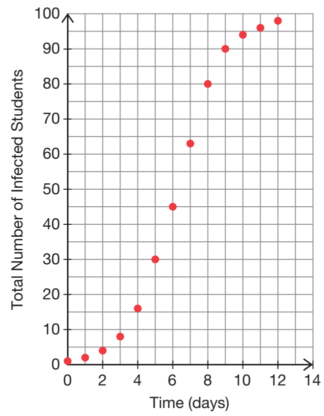
Sample results are shown.

~~0~~	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Time (Days)	Total Number of Infected Students (Students)
0	1
1	2
2	4
3	8
4	16
5	30
6	45
7	63
8	80
9	90
10	94
11	96
12	98

- What is the maximum number of students that can be infected?
- Is the carrying capacity in this situation the maximum number of students that can be infected?

4. Create a scatter plot of your data.



5. Using your graph and table, identify:

a. the initial growth stage.

Answers will vary.

The initial growth stage occurs from Day 0 to Day 2.

b. the exponential growth stage.

Answers will vary.

The exponential growth stage occurs from Day 2 to Day 5.

c. the dampened growth stage.

Answers will vary.

The dampened growth stage occurs from Day 5 to Day 8.

d. the equilibrium stage.

Answers will vary.

The equilibrium stages occurs after Day 10.

6. Identify the carrying capacity for this function. Explain your reasoning.

The carrying capacity is 100.

It makes sense that the carrying capacity is 100 because there are at most 100 students to infect.

Grouping

Have students complete Questions 7 through 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 7 through 9

- What is the value of *C* in your logistic function? How is this value determined?
- What is the value of *A* in your logistic function? How is this value determined?
- What is the value of *B* in your logistic function? How is this value determined?
- How is your logistic function different from your classmates' logistic functions?
- If there were 200 students in the school, would it take twice as long as it took to infect most students in a school with 100 students? Why or why not?
- How many extra days would it take to infect 200 students as opposed to 100 students?
- Would the shape of the graph for 200 students be *S*-shaped? How do you know?
- What would be the value of the carrying capacity in this situation?
- Are all linear functions associated with a constant difference?

7. Use a calculator to determine the equation of a logistic function that will model the data you collected for this problem. Determine the equation by performing a logistic regression.

Answers will vary.

$y = \frac{100}{(1 + 107e^{(0.754x)})}$

8. How do you think the experiment and results would be different if there were 200 people in the school?

Answers will vary. If there were 200 people in the school, it would take longer to infect almost everyone. It wouldn't take twice as long, but it would probably take a couple of extra days. The shape of the graph would stay the same, but the upper limit would go to 200 instead of 100. The equation would change, too. The numerator would be about 200.

9. Could you model the spread of the virus with another type of function (e.g. linear, quadratic, or exponential)? Why or why not?

Answers will vary. A linear function would not make sense because the scatter plot does not show a linear function. A quadratic or exponential might look right for part of the data while the virus is still spreading rapidly. However, a quadratic or exponential function would continue to increase rather than level off.

Be prepared to share your solutions and methods.

- Does the table of values or the scatter plot show a constant difference in this situation?
- Would a quadratic function eventually level off or increase indefinitely? How does that relate to this situation?
- Would an exponential function eventually level off or increase indefinitely? How does that relate to this situation?

Check for Students' Understanding

The logistic function $f(x) = \frac{C}{1 + Ae^{-Bx}}$ has three parameters, A, B, and C.

How do you think each of the positive parameters relate to each other and the value of the function?

Knowing the graph of e^{-x}, we know that e^{-Bx} approaches zero as x grows indefinitely. So $f(x)$ approaches C as x grows indefinitely.

The parameter C (carrying capacity) represents the limiting value of the output past which the output cannot grow.

If $x = 0$ in the formula for $f(x)$, then $(1 + A)\,f(0) = C$. So, A is the number of times that the initial output must grow to reach C.

The parameter B affects the steepness of the curve. As B increases, the curve approaches the horizontal line $y = C$ more rapidly. If $x = 0$, the y-intercept is $\left(0, \frac{C}{1 + A}\right)$.

This Is the Title of This Lesson

14.5

Fractals

LEARNING GOALS

In this lesson, you will:

- Build expressions and equations to model the characteristics of self-similar objects.
- Write sequences to model situations and use them to identify patterns.
- Analyze the counterintuitive aspects of fractals.

KEY TERMS

- fractal
- self-similar
- iterative process

ESSENTIAL IDEAS

- A self-similar object is exactly or approximately similar to a part of itself.
- A fractal is a complex geometric shape that is constructed by a mathematical pattern.
- The iterative process to create a Sierpinski Triangle is that for each stage an equilateral triangle is divided into smaller triangles by connecting the midpoints of the sides, and the center triangle is removed.
- The iterative process to create a Menger Sponge is that for each stage a cube is divided into 27 smaller cubes and the smaller cubes at the center of each face as well as the smaller cube at the center of the larger cube are removed.
- The iterative process to create a Koch Snowflake is that for each stage every side is an equilateral triangle and is trisected. The center one-third segment is removed and replaced with 2 segments of the same side, pointing outward.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F.IF Interpreting Functions

Understand the concept of a function and use function notation

3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ *for* $n \leq 1$.

F.BF Building Functions

Build a function that models a relationship between two quantities

1. Write a function that describes a relationship between two quantities.
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

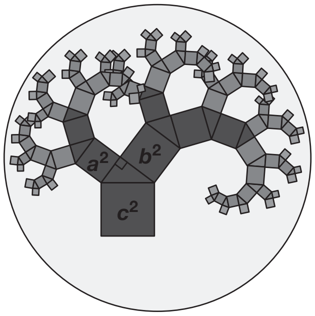
2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

Overview

The terms *self-similar, fractal,* and *iterative process* are defined. The Sierpinski Triangle, the Menger Sponge, and the Koch Snowflake are the self-similar objects in this lesson that are created by an iterative process. Students construct different stages of the models, use the images to complete tables of values, use the tables of values to identify infinite geometric sequences and patterns, describe end behaviors, write formulas, and make predictions.

Warm Up

The high school math club decided to use this image as a logo.



1. Describe how you might go about constructing this image.

   Answers will vary.

   The design of the image is repetitive. It begins with drawing a square of side length *c*, then using the upper side as the hypotenuse to draw a right triangle. Each side of the right triangle is then used to construct squares, and the uppermost side of each new square is used as the hypotenuses to construct two addition right triangles, and each side of the new right triangles are then used to construct squares, and the uppermost side of each new square is used as the hypotenuses to construct two additional right triangles, and so on.

2. Looking at this image, what geometric theorem comes to mind?

   This image is a visual representation of the Pythagorean Theorem.

This Is the Title of This Lesson

Fractals

14.5

LEARNING GOALS

In this lesson, you will:

- Build expressions and equations to model the characteristics of self-similar objects.
- Write sequences to model situations and use them to identify patterns.
- Analyze the counterintuitive aspects of fractals.

KEY TERMS

- fractal
- self-similar
- iterative process

If you like looking at pictures of cats, then the internet is for you. And if you like looking at pictures of cats looking at pictures of cats on the internet, then look into looking at the Infinite Cat Project.

As of late 2013, the project showcased about 1800 pictures of cats looking at pictures of cats looking at . . . you get the (very simple) idea.

Recursion like that featured on the Infinite Cat Project is an interesting mathematical topic, and a source of inspiration for self-referential lesson titles.

Problem 1

The terms *fractal, self-similar, iterative process,* and *Sierpinski Triangle* are introduced. The steps for constructing a Sierpinski Triangle are provided, and students describe the iterative process to create each stage, create the first three stages of the iteration, complete tables of values related to the unshaded triangles and the areas of the unshaded triangles, and identify the sequences formed in the tables as infinite geometric sequences. Students complete another table of values describing the stage, image, number of sides added, length of each added side, total length added, and total perimeter. Students conclude that as the number of stages approaches infinity, the area of the unshaded triangles in the Sierpinski Triangle approaches 0, which makes sense. They then write a formula in sigma notation for the perimeter of the Sierpinski Triangle at stage *n*.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 10 with a partner. Then have students share their responses as a class.

PROBLEM 1 The Sierpinski Triangle

A **fractal** is a complex geometric shape that is constructed by a mathematical pattern. Fractals are infinite and *self-similar* across different scales. A **self-similar** object is exactly or approximately similar to a part of itself. Many objects in the real world, such as coastlines, are self-similar, in that parts of them look roughly the same on any scale. Fractals often appear in nature. Examples of phenomena known to have fractal features include river networks, lightning bolts, ferns, snowflakes, and crystals.

The first fractal you will study is the Sierpinski Triangle, first described by Polish mathematician Wacław Sierpiński in 1915.

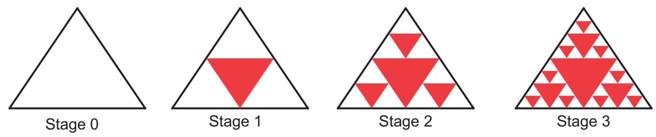
To construct the Sierpinski Triangle:

Stage 0: Begin with an equilateral triangle.

Stage 1: Connect the midpoints of the sides and remove the center triangle by shading it.

Stage 2: Repeat Stage 1 on the remaining triangles.

1. Complete each equilateral triangle to represent each stage of the Sierpinski Triangle.



Fractals are formed by an **iterative process**. The output from one iteration is used as the input for the next iteration. In many situations, this is also known as recursion.

2. Describe the iterative process to create the Sierpinski Triangle.

   The iterative process to create the Sierpinski Triangle is that for each stage an equilateral triangle is divided into smaller triangles by connecting the midpoints of the sides, and the center triangle is removed.

3. Determine the number of unshaded triangles at each stage and complete the table.

Stage (n)	0	1	2	3	4	5	n
Number of Unshaded Triangles	1	3	9	27	81	243	3^n

Guiding Questions for Share Phase, Questions 1 through 3

- What type of triangle is used when creating a Sierpinski Triangle?
- How is an equilateral triangle divided to create smaller triangles?
- Which triangle is removed in each stage of creating the Sierpinski Triangle?
- How do you determine the number of unshaded squares in each stage?

Guiding Questions for Share Phase, Questions 4 through 8

- What is the difference between a geometric sequence and an arithmetic sequence?
- Is the sequence associated with the output values in the table a geometric sequence or an arithmetic sequence? How do you know?
- What is the number of unshaded triangles a multiple of in each stage of the iteration?
- As the stage number approaches infinity, does the number of unshaded triangles also approach infinity?
- What fraction of the triangle was removed to form Stage 1?
- How many triangles were removed to form Stage 1? Out of how many triangles?
- Was one out of four triangles removed to form Stage 1?
- If one out of four triangles were removed to form Stage 1, what fraction of the area remains unshaded?
- How did you determine the total area of the unshaded triangles in Stage 2?
- How did you determine the total area of the unshaded triangles in Stage 3?
- How did you determine the total area of the unshaded triangles in Stage 4?

4. Identify the type of sequence represented by the number of unshaded triangles.
The number of unshaded triangles represents an infinite geometric sequence.

5. As the iterative process continues, what happens to the number of unshaded triangles in Sierpinski's Triangle?
For each iteration, the number of unshaded triangles is multiplied by 3. So, as n approaches infinity, the number of unshaded triangles in Sierpinski's Triangle approaches infinity.

6. Let the unshaded area at Stage 0 equal x square units. Determine the total area of unshaded triangles at each stage and complete the table.

Stage (n)	0	1	2	3	4	5	n
Area of Unshaded Triangles (square units)	x	$\left(\frac{3}{4}\right)^1 x$	$\left(\frac{3}{4}\right)^2 x$	$\left(\frac{3}{4}\right)^3 x$	$\left(\frac{3}{4}\right)^4 x$	$\left(\frac{3}{4}\right)^5 x$	$\left(\frac{3}{4}\right)^n x$

7. Identify the type of sequence represented by the area of unshaded triangles.
The area of unshaded triangles represents an infinite geometric sequence.

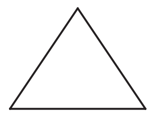
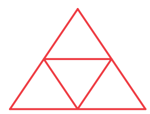
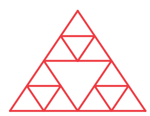
8. As the iterative process continues, what happens to the area of unshaded triangles in Sierpinski's Triangle?
For each iteration, the area of unshaded triangles is multiplied by $\frac{3}{4}$. So, as n approaches infinity, the area of unshaded triangles in Sierpinski's Triangle approaches zero.

- How did you determine the total area of the unshaded triangles in Stage 5?
- As the stage number approaches infinity, does the area of unshaded triangles also approach infinity? Why not?

Guiding Questions for Share Phase, Questions 9 and 10

- How did you determine the length of added sides for each stage?
- How did you determine the total length added for each stage?
- How did you determine the total perimeter for each stage?
- Why is the total perimeter for Stage 1 equal to $3s$?
- Why is the total perimeter for Stage 2 equal to $3s + \frac{3s}{2}$?
- Why is the total perimeter for Stage 3 equal to $3s + \frac{3s}{2} + \frac{9s}{4}$?

9. Complete the table to determine the perimeter of the Sierpinski Triangle through each iteration. Let s equal the side length of the equilateral triangle in Stage 0.

Stage	Image	Number of Sides Added	Length of Added Side	Total Length Added	Total Perimeter
0		0	N/A	N/A	$3s$
1		3	$\frac{s}{2}$	$\frac{3s}{2}$	$3s + \frac{3s}{2}$
2		9	$\frac{s}{4}$	$\frac{9s}{4}$	$3s + \frac{3s}{2} + \frac{9s}{4}$
n	N/A	3^n	$\frac{s}{2^n}$	$\frac{3^n s}{2^n} = \left(\frac{3}{2}\right)^n \cdot s$	$3s + \frac{3s}{2} + \frac{9s}{4} + \cdots + \left(\frac{3}{2}\right)^n \cdot s$

10. Write a formula using sigma notation for the perimeter of the Sierpinski Triangle at stage n.

$$3s + \sum_{i=1}^{n}\left(\frac{3}{2}\right)^i \cdot s$$

Problem 2

The term *Menger Sponge* is introduced. The steps for constructing a Menger Sponge are provided and students describe the iterative process to create each stage, create the first three stages of the iteration, complete tables of values related to the number of filled cubes and the area of unshaded squares, and identify the sequences formed in the tables as infinite geometric sequences. Students conclude that as the number of stages approaches infinity, the volume of filled cubes in the Menger Sponge approaches 0, which does not make sense.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- What type of solid figure is used when creating a Menger Sponge?
- How is a cube divided to create smaller cubes?
- Which cube is removed in each stage of creating the Menger Sponge?

PROBLEM 2 The Menger Sponge

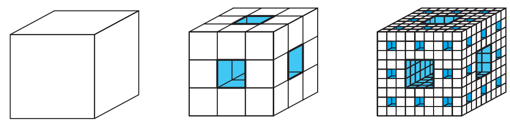
The next fractal you will study is the Menger Sponge, first described by Austrian mathematician Karl Menger in 1926.

To construct the Menger Sponge:

Stage 0: Begin with a cube.

Stage 1: Divide every face of the cube into 9 squares. This will subdivide the cube into 27 smaller cubes. Remove the smaller cube in the middle of each face, and remove the smaller cube in the very center of the larger cube, leaving 20 smaller cubes.

Stage 2: Repeat Stage 1 for the remaining squares on each face.



1. Describe the iterative process to create the Menger Sponge.

   The iterative process to create the Menger Sponge is that for each stage a cube is divided into 27 smaller cubes and the smaller cubes at the center of each face as well as the smaller cube at the center of the larger cube are removed.

2. Determine the number of filled cubes at each stage and complete the table.

Stage (n)	0	1	2	3	4	5	n
Number of Filled Cubes	1	20	400	8000	160,000	3,200,000	20^n

3. Identify the type of sequence represented by the number of filled cubes.

   The number of filled cubes represents an infinite geometric sequence.

- How did you determine the number of filled cubes in each stage?
- Is the sequence associated with the output values in the table a geometric sequence or an arithmetic sequence? How do you know?
- What is the number of filled cubes a multiple of in each stage of the iteration?

Guiding Questions for Share Phase, Questions 4 through 7

- As the stage number approaches infinity, does the number of filled cubes also approach infinity?
- What fraction of the cube was removed to form Stage 1?
- How many cubes were removed to form Stage 1? Out of how many cubes?
- Was one out of four triangles removed to form Stage 1?
- If 7 out of 27 were removed to form Stage 1, what fraction of the volume remains unshaded?
- How did you determine the total volume of filled cubes in Stage 2?
- How did you determine the total volume of filled cubes in Stage 3?
- How did you determine the total volume of filled cubes in Stage 4?
- How did you determine the total volume of filled cubes in Stage 5?
- As the stage number approaches infinity, does the volume of filled cubes also approach infinity? Why not?
- How can the number of filled cubes approach infinity as the volume of filled cubes approaches zero?

4. As the iterative process continues, what happens to the number of filled cubes in the Menger Sponge?

For each iteration, the number of filled cubes is multiplied by 20. So, as *n* approaches infinity, the number of filled cubes in the Menger Sponge approaches infinity.

5. Let the volume at Stage 0 equal 1 cubic unit. Determine the total volume of filled cubes at each stage and complete the table.

Stage (n)	0	1	2	3	4	5	n
Volume (cubic units)	1	$\left(\frac{20}{27}\right)^1$	$\left(\frac{20}{27}\right)^2$	$\left(\frac{20}{27}\right)^3$	$\left(\frac{20}{27}\right)^4$	$\left(\frac{20}{27}\right)^5$	$\left(\frac{20}{27}\right)^n$

6. Identify the type of sequence represented by the volume of filled cubes.

The volume of filled cubes represents an infinite geometric sequence.

7. As the iterative process continues, what happens to the volume of filled cubes in the Menger Sponge? Does this situation seem possible? Explain your reasoning.

For each iteration, the volume of filled cubes is multiplied by $\frac{20}{27}$. So, as *n* approaches infinity, the volume of filled cubes in the Menger Sponge approaches zero.

It does not seem possible that the number of filled cubes approaches infinity, but the volume approaches zero.

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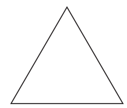
Problem 3

The term *Koch Snowflake* is introduced. The steps for constructing a Koch Snowflake are provided and the first two stages are drawn. Students describe the iterative process to create each stage and complete tables of values related to the stage, length of a side, number of sides, and total perimeter. They identify the sequences formed in the tables as infinite geometric sequences. Students conclude that as the number of stages approaches infinity, the length of a side in the Koch Snowflake approaches 0, the number of sides in the Koch Snowflake approaches infinity, and the total perimeter of the Koch Snowflake approaches infinity. As the length of a side approaches 0, the total perimeter approaches infinity which does not make sense. Next, students determine the areas of the equilateral triangle in Stage 0 and the triangle that was added to the Stage 0 figure to calculate the total area of the Stage 1 figure. This process is repeated to calculate the total area of the Stage 2 figure. The information is recorded in a table and students look for patterns which help make predictions about the number of new triangles in Stage 3, the area of one new triangle, and the total area of the figure in Stage 3. Students conclude that as the stage number increases, the number of new triangles increases, the area of one new triangle decreases, and the total area increases. All of these conclusions make sense.

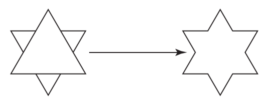
PROBLEM 3 The Koch Snowflake

The Koch Snowflake is a fractal that is created by using equilateral triangles. It is based on the Koch curve, first mentioned in a 1904 paper by the Swedish mathematician Helge von Koch.

In Stage 0, you begin with an equilateral triangle, such as the one shown. This is the first step in the creation of the Koch Snowflake.



In Stage 1, each side of the triangle is divided into thirds. Then, each middle segment becomes the base of a new equilateral triangle as shown. Finally, the middle segment is removed.



1. How does a side length of a new triangle compare to a side length of the Stage 0 triangle?

   The side length of a new triangle is one third of the side length of the Stage 0 triangle.

2. If a side length of the Stage 0 figure is 1 unit, what is a side length of the Stage 1 figure?

   A side length of the Stage 1 figure is $\frac{1}{3}$ unit.

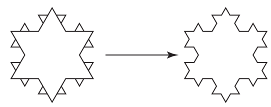
Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 11 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5

- What type of triangle is used when creating a Koch Snowflake?
- How is each side of the triangle divided to create a smaller triangle?
- Which line segment(s) are removed in each stage of creating the Koch Snowflake?
- How do you determine the length of side s in each stage?
- How do you determine the number of sides in each stage?
- How do you determine the total perimeter in each stage?
- Is the sequence associated with the output values in the table a geometric sequence or an arithmetic sequence? How do you know?
- What is the length of a side a multiple of in each stage of the iteration?

In Stage 2, each side of the figure from Stage 1 is divided into thirds, and the middle segments become the bases of new equilateral triangles. Then, the middle segments are removed.



This process is repeated on the remaining sides.

3. Describe the iterative process to create the Koch Snowflake.

The iterative process to create the Koch Snowflake is that for each stage every side of an equilateral triangle is divided into thirds and the center segment is removed and replaced with 2 segments of the same side, pointing outward.

4. Let the side length at Stage 0 equal s units. Complete the table.

Stage (n)	Length of a Side	Number of Sides	Total Perimeter
0	s	3	$3s$
1	$\left(\frac{1}{3}\right)^1 s$	12	$4s$
2	$\left(\frac{1}{3}\right)^2 s$	48	$\frac{48}{9}s$
3	$\left(\frac{1}{3}\right)^3 s$	192	$\frac{192}{27}s$
4	$\left(\frac{1}{3}\right)^4 s$	768	$\frac{768}{81}s$
5	$\left(\frac{1}{3}\right)^5 s$	3072	$\frac{3072}{243}s$
n	$\left(\frac{1}{3}\right)^n s$	$3 \cdot 4^n$	$3 \cdot \left(\frac{4}{3}\right)^n \cdot s$

5. Identify the type of sequence represented by:

a. the length of a side.

The length of a side represents an infinite geometric sequence.

b. the number of sides.

The number of sides represents an infinite geometric sequence.

c. the total perimeter.

The total perimeter represents an infinite geometric sequence.

Guiding Questions for Share Phase, Questions 6 and 7

- As the stage number approaches infinity, does the length of a side also approach infinity? Why not?
- What is the number of sides a multiple of in each stage of the iteration?
- As the stage number approaches infinity, does the number of sides also approach infinity? How do you know?
- What is the total perimeter a multiple of in each stage of the iteration?
- As the stage number approaches infinity, does the total perimeter also approach infinity? How do you know?
- How can the side lengths approach zero as the perimeter approaches infinity?

Guiding Questions for Share Phase, Question 8

- What is the length of each side of the equilateral triangle in Stage 0?
- Does the altitude in an equilateral triangle divide the triangle into two congruent triangles?
- Does the altitude in an equilateral triangle divide the triangle into two 30°-60°-90° triangles?
- Does the altitude in an equilateral triangle bisect the vertex angle?

6. As the iterative process continues, what happens to:

a. the length of a side.

For each iteration, the length of a side is multiplied by $\frac{1}{3}$. So, as n approaches infinity, the length of a side in the Koch Snowflake approaches 0.

b. the number of sides.

For each iteration, the number of sides is multiplied by 4. So, as n approaches infinity, the number of sides in the Koch Snowflake approaches infinity.

c. the total perimeter.

For each iteration, the total perimeter is multiplied by $\frac{4}{3}$. So, as n approaches infinity, the total perimeter in the Koch Snowflake approaches infinity.

7. Does this situation seem possible? Explain your reasoning.

It does not seem possible that the side lengths gets closer to zero, but the perimeter continues to grow.

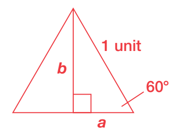
8. Consider the equilateral triangle in Stage 0.

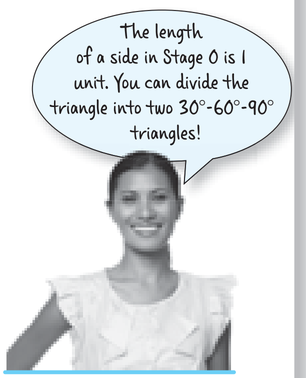
a. Calculate the altitude. (Hint: Use the fact that the altitude divides the triangle into two special right triangles.) Leave your answer in radical form.

$2a = 1$

$a = \frac{1}{2}$

$b = \frac{\sqrt{3}}{2}$ unit





b. Calculate the area of the equilateral triangle. Show all of your work and leave your answer in radical form.

$A = \frac{1}{2}(1)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{4}$ units²

c. What is the total area of the Stage 0 figure rounded to the nearest hundredth?

$A = \frac{\sqrt{3}}{4}$

≈ 0.43

- Does the altitude in an equilateral triangle bisect the base?
- What is the length of the base in this equilateral triangle?
- In a 30°-60°-90° triangle, what is the relationship between the length of the side opposite the 30° angle and the length of the hypotenuse?
- In a 30°-60°-90° triangle, what is the relationship between the length of the side opposite the 60° angle and the length of the hypotenuse?
- What is the area formula for any triangle?

Guiding Questions for Share Phase, Question 9

- What is the length of a side of one of the smaller triangles that was added to the Stage 0 figure?
- What is the length of the side opposite the 30° in this situation?
- What is the length of the side opposite the 60° in this situation?
- How does the area of this smaller triangle compare to the area of the triangle in the previous stage?
- How did you determine the total area of the Stage 1 figure?
- What is the length of a side of one of the smaller triangles that was added to the Stage 1 figure?
- What is the length of the side opposite the 30° in this situation?
- What is the length of the side opposite the 60° in this situation?
- How does the area of this smaller triangle compare to the area of the triangle in the previous stage?

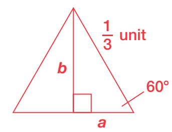
9. Consider one of the smaller triangles that was added to the Stage 0 figure.

a. Calculate the altitude. Leave your answer in radical form.

$2a = \frac{1}{3}$

$a = \frac{1}{6}$

$b = \frac{\sqrt{3}}{6}$ unit



b. Calculate the area of this triangle. Show all of your work and leave your answer in radical form.

$A = \frac{1}{2}\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{6}\right)$

$= \frac{\sqrt{3}}{36}$ units^2

c. Calculate the total area of the Stage 1 figure. Show all of your work and round your answer to the nearest hundredth.

$A = \frac{\sqrt{3}}{4} + 3\left(\frac{\sqrt{3}}{36}\right)$

$= \frac{9\sqrt{3}}{36} + \frac{3\sqrt{3}}{36}$

$= \frac{12\sqrt{3}}{36}$

$= \frac{\sqrt{3}}{3}$

≈ 0.58 units^2

Guiding Questions for Share Phase, Questions 10 and 11

- How did you determine the total area of the Stage 2 figure?
- What is the relationship between the number of new triangles in the Stage 3 figure and the number of new triangles in the Stage 2 figure?
- How did you determine the number of new triangles for Stage 3?
- What is the relationship between area of the new triangle in the Stage 3 figure and the area of the new triangle is the Stage 2 figure?
- How did you determine the area of one new triangle for Stage 3?
- How did you determine the total area for Stage 3?
- As the stage number approaches infinity, does the number of new triangles approach infinity? How do you know?
- As the stage number approaches infinity, does the area of one new triangle increase or decrease? How do you know?
- As the stage number approaches infinity, does the total area increase or decrease? How do you know?
- Does the total area increase as the area of each new triangle decreases?
- Does the total area increase as the number of new triangles increase?

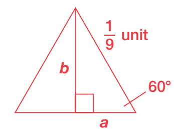
10. Consider one of the smaller triangles that was added to the Stage 1 figure.

a. Calculate the altitude. Leave your answer in radical form.

$2a = \frac{1}{9}$

$a = \frac{1}{18}$

$b = \frac{\sqrt{3}}{18}$ unit



b. Calculate the area of this triangle. Show all of your work and leave your answer in radical form.

$A = \frac{1}{2}\left(\frac{1}{9}\right)\left(\frac{\sqrt{3}}{18}\right)$

$= \frac{\sqrt{3}}{324}$ units²

c. Calculate the total area of the Stage 2 figure. Show all of your work and round your answer to the nearest hundredth.

$A = \frac{\sqrt{3}}{3} + 12\left(\frac{\sqrt{3}}{324}\right)$

$= \frac{108\sqrt{3}}{324} + \frac{12\sqrt{3}}{324}$

$= \frac{120\sqrt{3}}{324}$

$= \frac{10\sqrt{3}}{27}$

≈ 0.64 units²

11. Complete the table shown for Stage 0 through Stage 2.

Stage	Number of New Triangles	Area of One New Triangle	Total Area in Radical Form	Total Area (nearest hundredth)
0	1	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	0.43
1	3	$\frac{\sqrt{3}}{36}$	$\frac{\sqrt{3}}{3}$	0.58
2	12	$\frac{\sqrt{3}}{324}$	$\frac{10\sqrt{3}}{27}$	0.64
3	48	$\frac{\sqrt{3}}{2916}$	$\frac{94\sqrt{3}}{243}$	0.67

a. Predict the number of new triangles in the Stage 3 figure, the area of one new triangle, and the total area of the figure. Explain your reasoning. Record your results in the table.

The number of new triangles in the Stage 3 figure is four times the number of new triangles in the Stage 2 figure. So, there are 48 new triangles in the Stage 3 figure. The area of a new triangle is $\frac{1}{9}$ of the area of a new triangle in the Stage 2 figure. So, the area of a new triangle is $\frac{\sqrt{3}}{2916}$ square units. Add the area of each new triangle to the area of the Stage 2 figure to find the total area.

$$\begin{aligned} A &= \frac{10\sqrt{3}}{27} + 48\left(\frac{\sqrt{3}}{2916}\right) \\ &= \frac{1080\sqrt{3}}{2916} + \frac{48\sqrt{3}}{2916} \\ &= \frac{1128\sqrt{3}}{2916} \\ &= \frac{94\sqrt{3}}{243} \approx 0.67 \end{aligned}$$

b. What happens to the number of new triangles as the stage number increases?

The number of new triangles increases as the stage number increases.

c. What happens to the area of one new triangle as the stage number increases?

The area of one new triangle decreases as the stage number increases.

d. What happens to the total area as the stage number increases? Does this situation seem possible? Explain your reasoning.

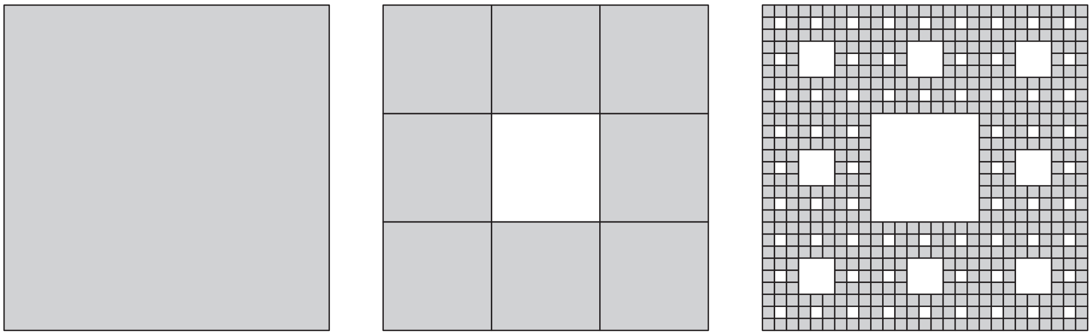
The total area increases as the stage number increases.

Yes. The total area grows slowly as the area of each new triangle gets smaller because the number of new triangles increases.

Be prepared to share your solutions and methods.

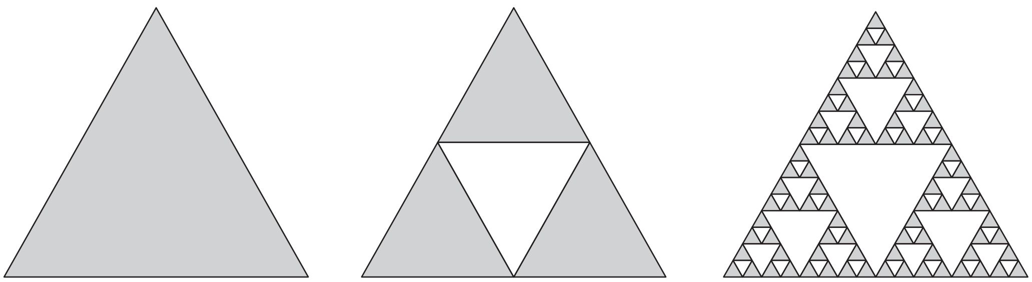
Check for Students' Understanding

The Sierpinski Carpet:



Stage 0 Stage 1 After 3 iterations

The Sierpinski Triangle:



Stage 0 Stage 1 After 4 iterations

- How is the Sierpinski Carpet different from the Sierpinski Triangle?
- Describe the iterative process used to create the Sierpinski Carpet.

In the Sierpinski Triangle, the central third of each triangle was removed. For the Sierpinski Carpet, the central square piece is removed. In each stage the squares are divided into thirds both horizontally and vertically. As with the Sierpinski Triangle, the area approaches zero as the total perimeter approaches infinity.

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14.6

Grab Bag

Choosing Functions to Model Situations

LEARNING GOALS

In this lesson, you will:

- Use technology to determine regression equations that model data.
- Choose functions to model problem situations.
- Graph and analyze function characteristics in terms of problem situations.

ESSENTIAL IDEAS

- Multiple representations such as tables, graphs, and equations are used to model real-world situations.
- Graphing calculators are used to perform an exponential regression.
- Radical functions, rational functions, and exponential functions, are used to model real-world situations.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

A.CED Creating Equations

Create equations that describe numbers or relationships

3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*

F.BF Building Functions

Build a function that models a relationship between two quantities

1. Write a function that describes a relationship between two quantities.

b. Combine standard function types using arithmetic operations.

Build new functions from existing functions

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

F.LE Linear, Quadratic, and Exponential Models

Construct and compare linear, quadratic, and exponential models and solve problems

2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

Interpret expressions for functions in terms of the situation they model

5. Interpret the parameters in a linear or exponential function in terms of a context.

Overview

Students work with tables, graphs, and inverse functions pertaining to 4 different situations. A table of values and scatter plot are given in the first situation and the students use a graphing calculator to determine an exponential regression equation needed to answer questions related to the situation. In other situations, they write functions and their inverse functions, graphing both. A radical equation and rational equation are used to model the situations.

Warm Up

The demand for the newest generation of reading tablets is modeled by the function

$price = 500 - 0.5e^{0.004x}$, where x is the demand and p is the price.

1. Determine the demand when the price is \$199.

The demand when the price is \$199 is 1600 reading tablets.

$$\begin{aligned} price &= 500 - 0.5e^{0.004x} \\ 199 &= 500 - 0.5e^{0.004x} \\ 0.5e^{0.004x} &= 301 \\ e^{0.004x} &= 602 \\ 0.004x &= \ln 602 \\ x &= \frac{\ln 602}{0.004} \\ x &\approx 1600 \end{aligned}$$

2. Determine the demand when the price is \$499.

The demand when the price is \$499 is 173 reading tablets.

$$\begin{aligned} price &= 500 - 0.5e^{0.004x} \\ 499 &= 500 - 0.5e^{0.004x} \\ 0.5e^{0.004x} &= 1 \\ e^{0.004x} &= 2 \\ 0.004x &= \ln 2 \\ x &= \frac{\ln 2}{0.004} \\ x &\approx 173 \end{aligned}$$

Grab Bag

14.6

Choosing Functions to Model Situations

LEARNING GOALS

In this lesson, you will:

- Use technology to determine regression equations that model data.
- Choose functions to model problem situations.
- Graph and analyze function characteristics in terms of problem situations.

At some amusement parks, rotor rides can spin you around fast enough to make you stick to the wall. The same force—centrifugal force—can also help to get your clothes dry.

In some clothes dryers, a combination of heat and spin "sucks" away the moisture from your clothes during the spin cycle. Because there are holes in the rotating tube inside the dryer, the water is able to exit the tube or be evaporated, leaving your clothes dry.

So, being in the rotor ride at the amusement park is like being stuck in a giant clothes dryer!

Problem 1

Students are given a table of values and scatter plot describing the consumer price index over a period of time. They determine a regression equation for the model and use the model to: create a graph, make future predictions, and calculate the consumer price index for a specified year. When using the model to solve for a consumer price index value listed in the table, students conclude the model should only be used to make predictions for other times.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- A dollar in 1960 was worth how many times what it was worth in 1982?
- What year was a dollar worth 0.58 what it was worth in 1982?
- How is 1960 represented on the x-axis?
- Will a linear regression equation fit the data on this scatter plot? Why or why not?

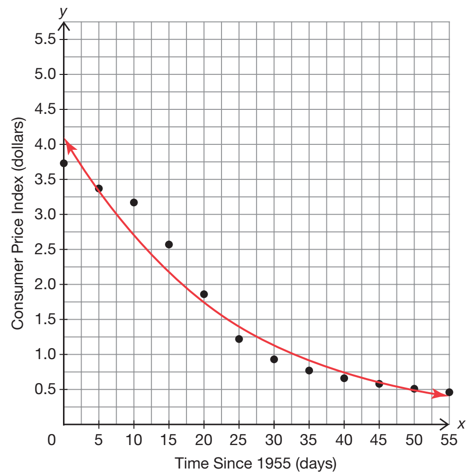
PROBLEM 1 Stretch That Dollar

The table shows the purchasing value of the dollar, or the consumer price index, for consumers in the United States from 1955 to 2010. The table uses the year 1982 as a base period, so the consumer price index written in dollars and cents in 1982 is 1.00.

In 1955, the consumer price index was 3.73. This means that a dollar in 1955 was worth 3.73 times what it was worth in 1982. Similarly, a dollar in 2010 was worth 0.46 times what it was worth in 1982.

Year	Consumer Price Index	Year	Consumer Price Index
1955	3.73	1985	0.93
1960	3.37	1990	0.77
1965	3.17	1995	0.66
1970	2.57	2000	0.58
1975	1.86	2005	0.51
1980	1.22	2010	0.46

The scatter plot shows the data in the table where x represents the number of years since 1955 and y represents the consumer price index.



- Does the consumer price index decrease at a constant rate? How do you know?
- How many data points on the scatter plot does your exponential regression equation actually pass through?
- Did your classmates arrive at the same exponential regression equation?
- Did you use a graph, table, or regression equation to predict the consumer price index in 2025?

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Guiding Questions for Share Phase, Questions 3 through 5

- What year is represented where $x = 0$ on the x-axis?
- What year is represented where $x = 5$ on the x-axis?
- What year is represented where $x = -5$ on the x-axis?
- How is five years before 1955 represented on the x-axis?
- Was the exponential regression equation used to calculate the consumer price index for 1950?
- Is the exponential regression equation an accurate model for calculating the years that are included in the table of values?
- Is the exponential regression equation an accurate model for calculating future predictions?

1. Determine the regression equation for the model that best represents the data. Explain how you determined your answer. Then, graph the model on the same coordinate plane as the scatter plot.

$y = 4.10(0.96)^x$ or $4.08392e^{-0.041x}$

Because the consumer price index does not decrease at a constant rate over time, I know that an exponential function is the best model for the data.

See graph.

2. Predict the consumer price index in 2025. Explain what your answer means in terms of this problem situation.

$y = 4.10(0.96)^{70}$

≈ 0.24

The consumer price index would be about 0.24 in 2025. This means that a dollar in 2025 is worth 0.24 times what it was worth in 1982.

3. Mr. Kratzer asks his students to calculate what model predicts the consumer price index was in 1950. Melina says that you must evaluate the function at $x = 5$ to determine the consumer price index in 1950. Dominique argues you must evaluate the function at $x = -5$ to determine the consumer price index in 1950. Who is correct? Explain your reasoning.

Dominique is correct, because 1950 is 5 years before 1955. Melina is incorrect, because evaluating the function at 5 would result in the consumer price index for 1960.

4. Calculate the consumer price index for 1950.

$4.10(0.96)^{-5} \approx 5.03$

I calculated the consumer price index for 1950 to be approximately 5.03.

5. The consumer price index in 1950 was actually 4.15. Compare this to the answer you calculated in Question 4. Explain why these answers differ.

Answers will vary.

However, students should recognize that the function is a model for the data from 1955 to 2010 and can only be used to make predictions for other times.

Problem 2

A radical equation describing the minimum speed required to keep a person safely on an amusement park ride is given. Students use the equation to answer questions related to the situation. They algebraically solve for the inverse of the function and graph both on a coordinate plane. Students discuss the initial function and the inverse function in terms of the parent function, domain, range, and end behavior. The circumference formula is used to determine the maximum number of people that can safely ride various designs. Students write a function related to the number of people the ride can hold in terms of its radius. Again, they determine the inverse function, use it to answer questions related to the situation, and graph both the initial function and the inverse function. Students discuss the initial function and the inverse function in terms of the parent function, domain, range, and end behavior.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 8 with a partner. Then have students share their responses as a class.

PROBLEM 2 The Rotor

The Rotor is a popular amusement park ride shaped like a cylindrical room. Riders stand against the circular wall of the room while the room spins. When The Rotor reaches the necessary speed, the floor drops out and centrifugal force leaves riders pinned up against the wall.

The minimum speed (measured in meters per second) required to keep a person pinned against the wall during the ride can be determined with the function $s(r) = 4.95\sqrt{r}$, where r is the radius of The Rotor measured in meters.

1. An amusement park designed a rotor ride with a radius of 2 meters. At what speed does it need to spin?

   $s(r) = 4.95\sqrt{2} \approx 7$

   The ride would need to spin at least 7 meters per second.

2. The same park decided to build a larger rotor ride with a radius of 4 meters. At what speed does it need to spin?

   $s(r) = 4.95\sqrt{4} = 9.9$

   The ride would need to spin at least 9.9 meters per second.

3. Designers at another park have a motor that could spin a rotor ride at 6 meters per second. How big can they make the ride?

   $6 = 4.95\sqrt{r}$

   $1.\overline{21} = \sqrt{r}$

   $1.47 \approx r$

   The designers could make a rotor ride with a radius of about 1.47 meters.

4. Algebraically determine the inverse of the function used to determine the speed of The Rotor.

   $s(r) = 4.95\sqrt{r}$

   $y = 4.95\sqrt{r}$

   $r = 4.95\sqrt{y}$

   $\frac{r}{4.95} = \sqrt{y}$

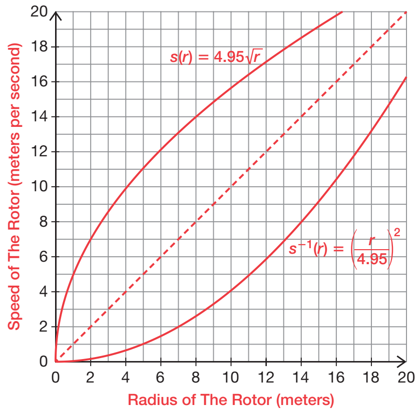
   $\left(\frac{r}{4.95}\right)^2 = y$

   $s^{-1}(r) = \left(\frac{r}{4.95}\right)^2 \qquad r \geq 0$

Guiding Questions for Share Phase, Questions 1 through 8

- What type of function is used to describe the radius of the Rotor and the speed at which it needs to spin?
- Given the radius of the Rotor, how do you determine the speed it needs to spin?
- If the radius of the Rotor doubles, does the speed it needs to spin double? Why not?
- What type of function is used to describe the inverse of the initial function?
- Is the graph of the function and the graph of inverse of the function symmetric to the line $y = x$?
- Is the parent function of the initial function a square root function?
- What transformation was performed on the square root function?
- Does the square root function have an x- or y-intercept? Where?
- Is the domain of the square root function restricted?
- Is the square root function increasing or decreasing?
- Does the square root function have any asymptotes?
- Is the parent function of the inverse function a quadratic function?
- What transformation was performed on the quadratic function?
- Does the quadratic function have an x- or y-intercept? Where?

5. Create a graph of the function and its inverse on coordinate plane shown.



6. Write a paragraph to describe the initial function, $s(r) = 4.95\sqrt{r}$.

Be sure to include information about the basic function, domain, range, and behavior of the function.

The function is a square root function that has been dilated by a factor of 4.95. There is an intercept at (0, 0). The domain is $r \geq 0$ and the range is $s(r) \geq 0$. The function increases over the entire domain. The end behavior is that as x goes to 0, y goes to 0, and as x goes to infinity, y goes to infinity. There are no asymptotes.

7. Write a paragraph to describe the inverse function.

Be sure to include information about the parent function, domain, range, and behavior of the function.

The inverse function is a quadratic function with a restricted domain. The function has been dilated by a factor of about 0.04. There is an intercept at (0, 0). The domain is $r \geq 0$ and the range is $s^{-1}(r) \geq 0$. The function increases over the entire domain. The end behavior is that as x goes to 0, y goes to 0, and as x goes to infinity, y goes to infinity. There are no asymptotes.

- Is the domain of the quadratic function restricted?
- Is the quadratic function increasing or decreasing?
- Does the quadratic function have any asymptotes?
- Why is the formula for the circumference of a circle important when calculating the maximum number of people who can safely ride the Rotor?
- How is the formula for the circumference used when calculating the maximum number of people who can safely ride the Rotor?
- What is the formula for the circumference?

Grouping

Have students complete Questions 9 through 14 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 9 through 14

- What type of function is used to describe the number of people a Rotor ride can hold based on its radius?
- Is the function used to describe the number of people a Rotor ride can hold based on its radius a linear function? How do you know?
- How is the formula for the circumference of a circle used to write the function describing the number of people a Rotor ride can hold based on its radius?
- What type of function is the inverse of the initial function?
- Is the inverse function of a linear function always another linear function?
- Is the number of riders associated with the range or domain of the initial function?
- Is the size associated with the range or domain of the initial function?
- Why is using the inverse of the initial function more appropriate in this situation?

8. The designers estimate that they must allow 60 cm (0.6 meter) of space along the wall for each person on the ride. For each of the designs from Questions 1 and 2, determine the maximum number of people who should be allowed on the ride at one time.

Circumference of first design $= 2\pi(2) \approx 12.566$ meters.

$12.566 \div 0.6 \approx 20.94$

About 20 people should be allowed on the first ride.

Circumference of second design $= 2\pi(4) \approx 25.133$ meters.

$25.133 \div 0.6 \approx 41.89$

About 41 people should be allowed on the second ride.

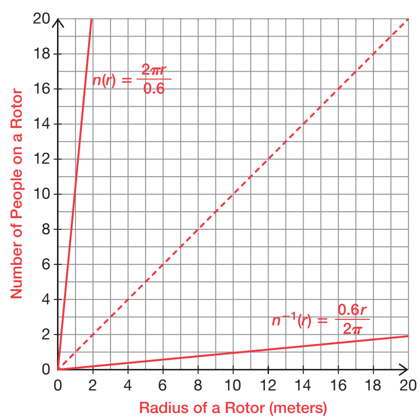
9. Write an algebraic function to determine the number of people a rotor ride can hold based on its radius.

$n(r) = \frac{2\pi r}{0.6}$

10. Determine the inverse of your function from Question 9 algebraically.

$$y = \frac{2\pi r}{0.6}$$
$$r = \frac{2\pi y}{0.6}$$
$$0.6r = 2\pi y$$
$$\frac{0.6r}{2\pi} = y$$
$$n^{-1}(r) = \frac{0.6r}{2\pi}$$

11. Graph your functions from Questions 9 and 10 on the coordinate plane shown.



- Is the domain and range of the initial function the same as the domain and range of the problem situation? Why not?
- Is the linear function increasing or decreasing?
- Is the domain and range of the inverse function the same as the domain and range of the problem situation? Why not?
- Is the inverse function increasing or decreasing?

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12. The ride designers at an amusement park want to create a rotor ride that can accommodate 25 riders at a time.

   a. Would you use the function given at the start of the problem or its inverse to determine the size of the ride? Why?

   Sample response: I would use the inverse. I am given the number of riders, which was the range of the original function. I need to find the size, which was the domain of the original function. The domain and range are switched.

   b. How big must they make the ride?

   $n^{-1}(25) = \frac{0.6(25)}{2\pi} \approx 2.387$

   The ride should be built with a radius approximately 2.387 meters long.

13. Write a paragraph to describe your function from Question 9.

   Be sure to include information about the parent function, domain, range, and behavior of the function.

   The function is a linear function with a slope of about 1.047. It has an intercept at (0, 0). While the domain and range of the function is any real number, only non-negative numbers make sense for the problem situation. The function increases over the entire domain. Its end behavior is that as x goes to negative infinity, y goes to negative infinity. As x goes to infinity, y goes to infinity.

14. Write a paragraph to describe the inverse function from Question 10.

   Be sure to include information about the parent function, domain, range, and behavior of the function.

   The function is a linear function with a slope of about 0.955. It has an intercept at (0, 0). While the domain and range of the function is any real number, only non-negative numbers make sense for the problem situation. The function increases over the entire domain. Its end behavior is that as x goes to negative infinity, y goes to negative infinity. As x goes to infinity, y goes to infinity.

Problem 3

A rational equation describing the total resistance of an electrical circuit in terms of ohms is given. Students use the equation to answer questions related to the situation. They algebraically solve for the total resistance of an electrical circuit having two or three resistors connected in parallel and the resistance of a specified branch given the total resistance of the circuit.

Grouping

- Ask a student to read the information. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- What type of function describes the total resistance R_T of the circuit in ohms?
- Does $R_T = \frac{1}{5} + \frac{1}{8}$ or does $\frac{1}{R_T} = \frac{1}{5} + \frac{1}{8}$?
- What is the least common multiple in this situation?
- What operations were used to solve for R_T?
- Does $R_T = \frac{1}{4} + \frac{1}{6} + \frac{1}{10}$ or does $\frac{1}{R_T} = \frac{1}{4} + \frac{1}{6} + \frac{1}{10}$?
- What is the least common multiple in this situation?

PROBLEM 3 It's Electric!

In an electrical circuit, when resistors are connected in parallel, the total resistance R_T of the circuit in ohms is given by:

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$$

where n is the number of resistors and R_1 through R_n are the resistances of each resistor in ohms.

1. Determine the total resistance of an electrical circuit having two resistors connected in parallel if their resistances are 5 ohms and 8 ohms.

   The total resistance is approximately 3.1 ohms.

$$\frac{1}{R_T} = \frac{1}{5} + \frac{1}{8}$$
$$40R_T \cdot \left(\frac{1}{R_T}\right) = \left(\frac{1}{5} + \frac{1}{8}\right) \cdot 40R_T$$
$$40 = 8R_T + 5R_T$$
$$40 = 13R_T$$
$$\frac{40}{13} = R_T$$

2. Three resistors connected in parallel have resistances of 4 ohms, 6 ohms, and 10 ohms. What is the total resistance in this electric circuit?

   The total resistance is approximately 1.9 ohms.

$$\frac{1}{R_T} = \frac{1}{4} + \frac{1}{6} + \frac{1}{10}$$
$$240R_T \cdot \left(\frac{1}{R_T}\right) = \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{10}\right) \cdot 240R_T$$
$$240 = 60R_T + 40R_T + 24R_T$$
$$240 = 124R_T$$
$$\frac{240}{124} = R_T$$

- How is the problem different if the total resistance is given and the resistance of one branch is unknown?
- What steps are required to solve the problem in this situation?
- Does $\frac{1}{12} = \frac{1}{30} + \frac{1}{R_2}$ or does $12 = \frac{1}{30} + \frac{1}{R_2}$?
- Does $\frac{1}{10} = \frac{1}{20} + \frac{1}{30} + \frac{1}{R_3}$ or does $10 = \frac{1}{20} + \frac{1}{30} + \frac{1}{R_3}$?

3. The total resistance in a parallel wiring circuit is 12 ohms. If the resistance of one branch is 30 ohms, what is the resistance in the other branch?

The resistance of the other branch is 20 ohms.

$$\frac{1}{12} = \frac{1}{30} + \frac{1}{R_2}$$

$$60R_2 \cdot \left(\frac{1}{12}\right) = \left(\frac{1}{30} + \frac{1}{R_2}\right) \cdot 60R_2$$

$$5R_2 = 2R_2 + 60$$

$$3R_2 = 60$$

$$R_2 = 20$$

4. A three-resistor parallel wiring circuit has a total resistance of 10 ohms. If two of the branches of the circuit have resistances of 20 ohms and 30 ohms, what is the resistance in the third branch?

The resistance of the third branch is 10 ohms.

$$\frac{1}{10} = \frac{1}{20} + \frac{1}{30} + \frac{1}{R_3}$$

$$60R_3 \cdot \left(\frac{1}{10}\right) = \left(\frac{1}{20} + \frac{1}{30} + \frac{1}{R_3}\right) \cdot 60R_3$$

$$6R_3 = 3R_3 + 2R_3 + 60$$

$$6R_3 = 5R_3 + 60$$

$$R_3 = 60$$

Problem 4

Students determine the amount of time a bottle of soda can be left outside before it will freeze. They begin by listing everything that was given about the function needed to model the situation and then write an exponential function to model the temperature of a bottle of soda over time. The function is used to answer the related question.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

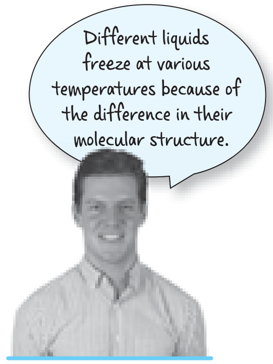
- What is the temperature of the initial bottle of soda, when $t = 0$?
- What does the x-value of each point on the function represent with respect to the problem situation?
- What does the y-value of each point on the function represent with respect to the problem situation?
- Is the point (0, 72) or the point (72, 0) on the function? What does it mean with respect to the problem situation?

PROBLEM 4 Cool It Down

Andre bought several 2-liter bottles of soda, but he did not have enough room in his refrigerator. He left the bottles outside to cool. Before he put each bottle outside, they were at room temperature, 72°F.

The outside temperature was 25°F. After 30 minutes, he brought one bottle of soda inside and measured the temperature. The temperature of the bottle of soda was 60°F.

A bottle of soda will freeze at 28°F. How long can Andre leave the bottles of soda outside before they will freeze?



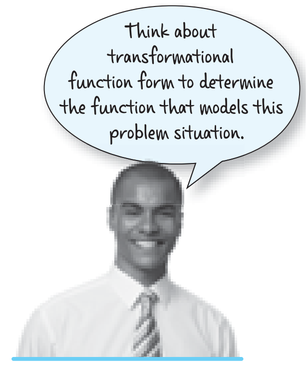
1. List everything you know about the function to model the temperature of a bottle of soda over time.
   - The initial bottle of soda at room temperature is 72°F. So, at $t = 0$, the soda is 72°F or there is a point on the function at (0, 72).
   - I also know that at $t = 30$, the soda is 60°F. So, there is a point on the function at (30, 60).
   - I also know that the temperature of the bottle of soda can never go below that of the outside temperature, which is 25°F. So, there is a horizontal asymptote at $y = 25$.
   - The bottle of soda cools continuously. I know there is a horizontal asymptote, therefore I can use the formula for exponential decay $N(t) = N_0e^{rt}$ to model this problem situation.

- Is the point (30, 60) or the point (60, 30) on the function? What does it mean with respect to the problem situation?
- Can the temperature of the bottle of soda ever go below the outside temperature or 25°F? Why not?
- Does the graph in this problem situation have a horizontal or vertical asymptote? How do you know?
- Where would the horizontal asymptote on the graph of the problem situation be? How do you know?

Guiding Questions for Share Phase, Questions 2 and 3

- Does the bottle of soda continuously cool?
- Can the formula for exponential decay be used to model this problem situation? Why or why not?
- What is the formula for exponential decay?
- What is the transformational function form of the formula for exponential decay?
- What is the *D*-value in this transformational function form?
- How is the *A*-value in this transformational function form determined?
- How can the point (0, 72) be used to solve for the *A*-value in this transformational function form?
- How is the *B*-value in this transformational function form determined?
- How can the point (30, 60) be used to solve for the *B*-value in this transformational function form?
- Why is $S(t) = 28$ used to determine how long it will take for a bottle of soda to freeze?
- Approximately how many hours and minutes is 280.8 minutes?

2. Write a function to model the temperature of a bottle of soda over time.



The function to model the temperature of a bottle of soda over time is $S(t) = 47e^{-0.0098t} + 25$.

I can use the transformational function form of the formula for exponential decay, or $S(t) = Ae^{Bt} + D$ to model this problem situation.

Because the graph has a horizontal asymptote at $y = 25$, the *D*-value is 25.

So, $S(t) = Ae^{Bt} + 25$.

To determine the *A*-value, I can substitute (0, 72) into the equation.

$72 = Ae^{B(0)} + 25$

$72 = A + 25$

$47 = A$

So, $S(t) = 47e^{Bt} + 25$.

To determine the *B*-value, I can substitute (30, 60) into the equation.

$$60 = 47e^{B(30)} + 25$$

$$35 = 47e^{30B}$$

$$\frac{35}{47} = e^{30B}$$

$$\ln\left(\frac{35}{47}\right) = 30B$$

$$\frac{\ln\left(\frac{35}{47}\right)}{30} = B$$

$$-0.0098 \approx B$$

So, $S(t) = 47e^{-0.0098t} + 25$.

14

3. How long can Andre leave the bottles of soda outside before they freeze?

It will take approximately 280.8 minutes, or 4 hours and 41 minutes for a bottle of soda to reach its freezing point.

To determine how long it will take the bottles of soda to freeze, I can solve $S(t) = 28$.

$$S(t) = 47e^{-0.0098t} + 25$$
$$28 = 47e^{-0.0098t} + 25$$
$$3 = 47e^{-0.0098t}$$
$$\frac{3}{47} = e^{-0.0098t}$$
$$\ln\left(\frac{3}{47}\right) = -0.0098t$$
$$\frac{\ln\left(\frac{3}{47}\right)}{-0.0098} = t$$
$$280.8 \approx t$$

Be prepared to share your solutions and methods.

14

Check for Students' Understanding

The number of people who have heard a rumor increases exponentially. Every person who hears a rumor repeats it to two people an hour. Ten people start the rumor, so the number of people who have heard the rumor is given by $N(t) = 10(3)^t$, where t is the time in hours.

1. How many people will have heard the rumor in 6 hours?

   In 6 hours, 7290 people will have heard the rumor.

   $$\begin{aligned} N(t) &= 10(3)^t \\ &= 10(3)^6 \\ &= 10(729) \\ &= 7290 \end{aligned}$$

2. How long will it take for 2000 people to have heard the rumor?

   It will take approximately 4.8 hours, or 4 hours and 48 minutes, for 2000 people to have heard the rumor.

   $$\begin{aligned} N(t) &= 10(3)^t \\ 2000 &= 10(3)^t \\ 200 &= (3)^t \\ \ln (3)^t &= \ln 200 \\ t \ln 3 &= \ln 200 \\ t &= \frac{\ln 200}{\ln 3} \\ t &\approx 4.8 \end{aligned}$$

Chapter 14 Summary

KEY TERMS

- identity function (14.1)
- logistic functions (14.4)
- carrying capacity (14.4)
- fractal (14.5)
- self-similar (14.5)
- iterative process (14.5)

14.1 Writing Composition Functions

In a composition of two functions f and g, written as $(f \circ g)(x)$, the output of $g(x)$ becomes the input of $f(x)$. That is, the function $g(x)$ is substituted for x into the function $f(x)$.

Example

Consider the functions $f(x) = 2x - 3$ and $g(x) = x^2 + 3x - 7$. Determine $(f \circ g)(x)$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(x^2 + 3x - 7)\\ &= 2(x^2 + 3x - 7) - 3\\ &= 2x^2 + 6x - 14 - 3\\ &= 2x^2 + 6x - 17\end{aligned}$$

14.1 Determining Inverse Functions

Two functions f and g are inverses of each other if the composite functions $(f \circ g)(x)$ and $(g \circ f)(x)$ both yield the identity function. That is, $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

Example

Use function composition to determine whether the functions $q(x) = \sqrt{x - 1}$ and $h(x) = x^2 + 1$, $x \geq 0$, are inverses of each other.

$$\begin{aligned}(q \circ h)(x) &= q(h(x))\\ &= q(x^2 + 1)\\ &= \sqrt{(x^2 + 1) - 1}\\ &= \sqrt{x^2}\\ &= x\end{aligned}$$

$$\begin{aligned}(h \circ q)(x) &= h(q(x))\\ &= h(\sqrt{x - 1})\\ &= \left(\sqrt{x - 1}\right)^2 + 1\\ &= (x - 1) + 1\\ &= x\end{aligned}$$

The functions $q(x) = \sqrt{x - 1}$ and $h(x) = x^2 + 1$, $x \geq 0$, are inverses of each other, because $(q \circ h)(x) = x$ and $(h \circ q)(x) = x$.

14.1 Determining the Domain of Composite Functions

The domain of $(f \circ g)(x)$ is the set of all real numbers x in the domain of g such that $g(x)$ is in the domain of f. In order for an element to be in the domain of the function $(f \circ g)(x)$, it must first be defined in the domain of $g(x)$.

Example

Let $r(x) = \frac{3}{6 - x}$ and $s(x) = \sqrt{x - 4}$. State the domain of $(r \circ s)(x)$. Explain how you determined your answer.

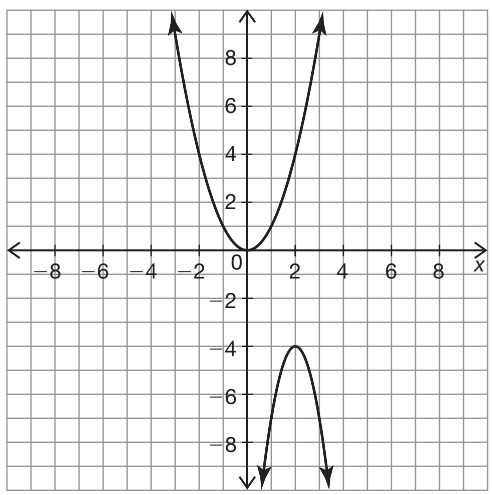
The composition of the functions $r(x)$ and $s(x)$ is $(r \circ s)(x) = \frac{3}{6 - \sqrt{x - 4}}$. The domain of the input is $[4, \infty)$, but because the denominator cannot be equal to zero, the domain of the composite function must be $[4, \infty)$, $x \neq 40$.

14.2 Reviewing Transformations of Functions

The function $y = f(x)$ is transformed when the function or its argument is multiplied, divided, increased, or decreased by a constant.

Example

Describe how the graph of the function $f(x) = x^2$ would be transformed to be the graph of the function $g(x) = -3(x - 2)^2 - 4$. Then, graph both functions.



The -3 indicates that the function would be stretched vertically by a factor of 3 and reflected across the x-axis. The 2 indicates that the function would be shifted to the right 2 units. The 4 indicates that the function would be shifted down 4 units.

14.3 Analyzing Functions to Maximize or Minimize a Value

Functions can be written and analyzed to determine the maximum or minimum in a real-world situation. To do this, write a function to model the situation and then determine the input value that will yield the maximum or minimum output value.

Example

The dimensions of a rectangular garden in feet are given by the expressions $27 - x$ and $x + 3$. Write an equation for the area of the garden. Then, determine the actual dimensions of the garden that would produce that greatest area. What is that area?

$$\begin{aligned} A &= bh \\ &= (27 - x)(x + 3) \\ &= 27x + 81 - x^2 - 3x \\ &= -x^2 + 24x + 81 \end{aligned}$$

So, the area can be represented by the function $A = -x^2 + 24x + 81$.

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{24}{2(-1)} \\ &= 12 \end{aligned}$$

The maximum area will occur when $x = 12$ feet. So, the dimensions that will produce the maximum area are $27 - 12$ or 15 feet and $12 + 3$ or 15 feet. A rectangular garden that is 15 feet by 15 feet will have an area of 225 square feet.

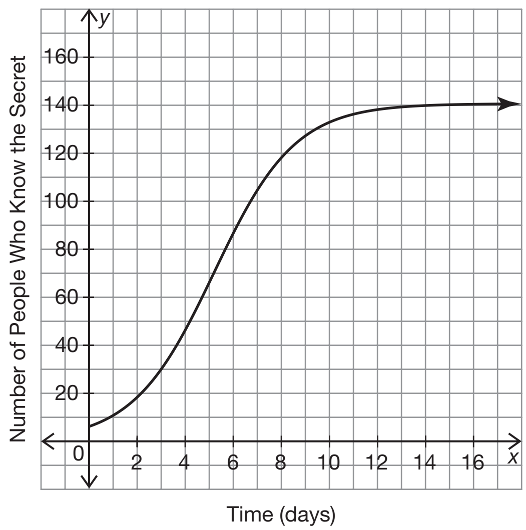
14.4 Modeling with Logistic Functions

Logistic functions are functions that can be written in the form $y = \frac{C}{1 + Ae^{-Bx}}$. They can be used to model biological population growth, the spread of diseases, the spread of information, and sales of new products over time. To model data with a logistic function, enter the data in a graphing calculator and then run a logistic regression on the data.

Example

On Monday, Alexa told several of her friends a secret. On Tuesday, each of her friends told a friend or two. The secret continued to spread over the following 10 days. The data representing the spread of the secret is shown in the table. Use a graphing calculator to run a logistic regression on the data. Write the function and identify the carrying capacity. Then, identify the initial growth stage, the exponential growth stage, the dampened growth stage, and the equilibrium stage.

Time (days)	Number of People who Know the Secret
0	4
1	8
2	16
3	35
4	49
5	64
6	85
7	103
8	119
9	130
10	134
11	135
12	137



The function that represents this relationship is $f(x) = \frac{140.6514}{(1 + 21.7768e^{-0.5923x})}$, and the carrying capacity is approximately 141. The initial growth stage occurs from Day 0 to Day 2. The exponential growth stage occurs from Day 2 to Day 5. The dampened growth stage occurs from Day 5 to Day 10, and the equilibrium stage occurs from Day 10 to Day 12.

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14.5

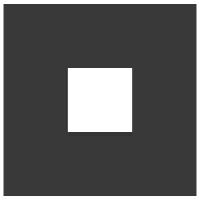
Understanding the Iterative Process Using Fractals

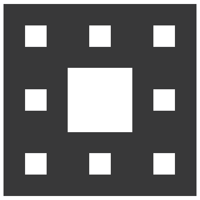
An iterative process is one in which the output of one iteration becomes the input of the next iteration. Relationships between the iterations and the area and perimeter of the fractal can often be represented as an infinite geometric sequence.

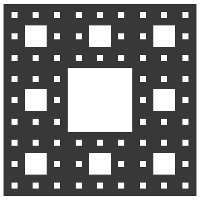
Example

The Sierpinski Carpet is formed in a manner that is similar to that of the Sierpinski Triangle. Begin with a square, and then divide it into 9 congruent squares. Remove the center square. Repeat this process. The initial stage and three additional stages of the Sierpinski Carpet are shown. Write a geometric sequence that represents the relationship between the stage and the number of shaded squares in the figure. Then, use the sequence to determine the number of shaded squares in the Stage 6 figure.


Stage 0


Stage 1


Stage 2


Stage 3

Stage (n)	0	1	2	3	n
Shaded squares	2^0 or 1	2^3 or 8	2^6 or 64	2^9 or 512	2^{3n}

The number of shaded squares in each figure represents an infinite geometric sequence. To determine the number of shaded squares in the Stage 6 figure, use the expression 2^{3n}.

$$2^{3n} = 2^{(3)(6)}$$

$$= 2^{18}$$

$$= 262{,}144$$

There are 262,144 shaded squares in the Stage 6 figure of the Sierpinski Carpet.

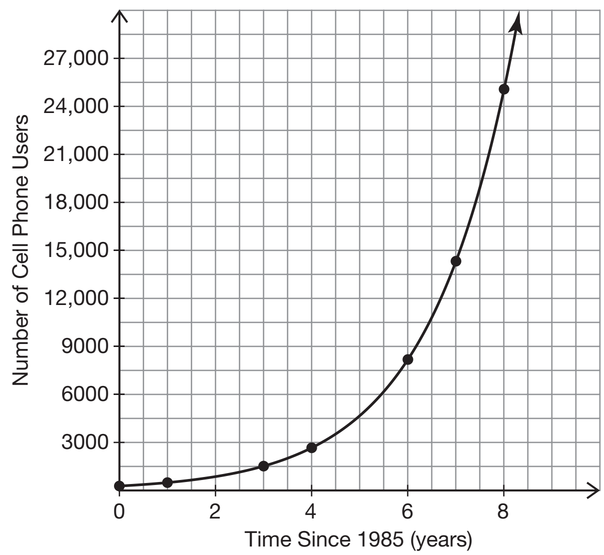
14.6 Choosing a Mathematical Model for a Real-World Situation

Real-life data can often be represented by a mathematical model. However, it can be difficult to determine which model works best. Plotting data on a scatter plot and then calculating and graphing regression equations on the same scatter plot can help determine which model is the best fit.

Example

The cell phone use of Americans has increased dramatically since 1985. The data table shows the number of cell phone subscribers in the small town of Springfield.

Time Since 1985 (years)	Number of Cell Phone Users
0	285
1	498
3	1527
4	2672
6	8186
7	14,325
8	25,069



The number of cell phone users increases with the increase of each year. Increasing functions can be modeled by a linear or exponential function. However, because the number of cell phone users does not increase at a constant rate, the data cannot be modeled by a linear function. The exponential regression equation for the data is $f(x) = 284.938e^{0.55964x}$. This function when graphed closely models the data.

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