Exponential and Logarithmic Equations

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Chapter 13 Overview

In this chapter, students use their understanding of exponential and logarithmic functions to solve exponential and logarithmic equations. Students begin by building understanding and fluency with exponential and logarithmic expressions, including estimating the values of logarithms on a number line and then use this understanding to derive the properties of logarithms. Students explore alternative methods for solving logarithmic equations and solve exponential and logarithmic equations in context.

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
13.1	Exponential and Logarithmic Forms	F.BF.5(+)	2	Students convert between exponential and logarithmic equations, estimate the values of logarithmic expressions with different bases, and solve simple exponential and logarithmic equations. Students develop different strategies for solving logarithmic equations by identifying the different unknowns.	X	X	X	X	
13.2	Properties of Logarithms	F.BF.5(+)	1	In this lesson, students derive the Zero Property, the Product Rule, the Quotient Rule, and the Power Rule of Logarithms and identify the value of a logarithm with the same base and argument. Questions ask students to use the properties to rewrite logarithmic expressions.	X				
13.3	Solving Exponential Equations	F.BF.5(+) F.LE.4	1	Students derive the Change of Base Formula and use this formula as a strategy to solve exponential equations. Students also explore taking the logarithm of both sides of an equation to solve it. Questions ask students to analyze these solution strategies and to apply them to problems with and without context.			X		X

Lesson		CCSS	Pacing	Highlights	Models	Worked Examples	Peer Analysis	Talk the Talk	Technology
13.4	Solving Logarithmic Equations	F.BF.5(+) F.LE.4	2	In this lesson, students solve logarithmic equations for different unknowns—base, argument, or exponent using the strategies and properties they have learned in previous lessons. Questions ask students to solve these equations in context and to analyze the efficiency of solution strategies for different equations.	X		X	X	
13.5	Applications of Exponential and Logarithmic Equations	F.BF.5(+) F.LE.4	1	In this lesson, students use exponential and logarithmic models to solve problems involving logarithmic equations. Students explore problems with pH levels, population growth, and Newton's Law of Cooling.			X		

Skills Practice Correlation for Chapter 13

Lesson		Problem Set	Objectives
13.1	Exponential and Logarithmic Forms		Vocabulary
		1 – 8	Arrange terms to create true exponential and logarithmic equations
		9 – 16	Solve for unknowns in logarithmic equations
		17 – 24	Estimate logarithms to the tenths place
		25 – 30	Determine the appropriate unknown base in logarithmic equations
13.2	Properties of Logarithms	1 – 16	Use properties of logarithms to write logarithms in expanded form
		17 – 24	Write logarithmic expressions as single logarithms
		25 – 30	Write logarithmic expressions as algebraic expressions
13.3	Solving Exponential Equations		Vocabulary
		1 – 10	Use the Change of Base Formula to solve exponential equations
		11 – 20	Solve exponential equations using properties of logarithms
		21 – 26	Solve exponential equations and explain methods of solving
13.4	Solving Logarithmic Equations	1 – 8	Solve logarithmic equations and check answers
		9 – 16	Use properties of logarithms to solve equations and check answers
13.5	Applications of Exponential and Logarithmic Equations	1 – 6	Use logarithms to solve half-life problems
		7 – 12	Use exponential equations and formulas to solve problems in context
		13 – 18	Use the logarithmic formula for the magnitude of an earthquake to solve problems
		19 – 24	Use various logarithmic formulas to solve problems

13.1

All the Pieces of the Puzzle

Exponential and Logarithmic Forms

LEARNING GOALS

In this lesson, you will:

- Convert exponential equations into logarithmic equations.
- Convert logarithmic equations into exponential equations.
- Solve exponential and simple logarithmic equations.
- Estimate the values of logarithms on a number line.
- Evaluate logarithmic expressions.

KEY TERM

- logarithmic equation

ESSENTIAL IDEAS

- The value of a logarithmic expression is equal to the value of the exponent in the corresponding exponential expression.
- To estimate a logarithm that is not an integer, use a number line as a guide and identify the closest logarithms with the same base whose argument is less than and greater than the non-integer.
- For a fixed base greater than 1, as the value of the argument gets larger, the value of the logarithm gets larger as well.
- For a fixed argument, when the value of the base is greater than 1 and increasing, the value of the logarithm is decreasing.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-BF Building Functions

Build new functions from existing functions

5. (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Overview

Students convert between exponential and logarithmic forms of an equation, and then use this relationship to solve for an unknown base, exponent, or argument in a logarithmic equation. Students estimate logarithms that are irrational numbers by using a number line as a guide. An always, sometimes, never activity is used to summarize the lesson.

Warm Up

1. Convert the logarithmic equation to an exponential equation.

$\log_2(8) = x$

$2^x = 8$

2. Solve for the unknown in the exponential equation you wrote in Question 1.

$2^x = 8$

$x = 3$

3. Convert the logarithmic equation to an exponential equation.

$\log_2(16) = x$

$2^x = 16$

4. Solve for the unknown in the exponential equation you wrote in Question 3.

$2^x = 16$

$x = 4$

5. Write a logarithmic equation for a logarithm of base 2 that has a value which is between the values of the logarithms in Questions 2 and 4.

$\log_2(12) \approx 3.5$ or $\log_2(12) \approx 3.6$

6. Convert the logarithmic equation you wrote in Question 5 to an exponential equation.

$\log_2(12) \approx 3.5$

$2^{3.5} \approx 12$

or

$\log_2(12) \approx 3.6$

$2^{3.6} \approx 12$

13.1

All the Pieces of the Puzzle

Exponential and Logarithmic Forms

LEARNING GOALS

In this lesson, you will:

- Convert exponential equations into logarithmic equations.
- Convert logarithmic equations into exponential equations.
- Solve exponential and simple logarithmic equations.
- Estimate the values of logarithms on a number line.
- Evaluate logarithmic expressions.

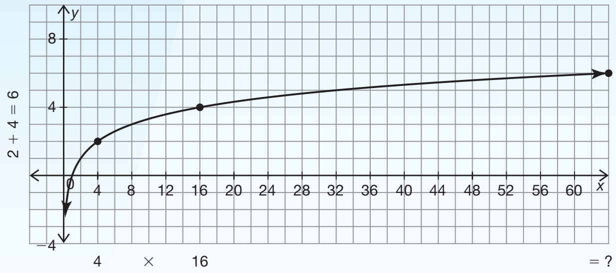
KEY TERM

- logarithmic equation

In the past, logarithms helped mathematicians and other scientists save a lot of time. Astronomers especially benefited from logarithms because of the enormous numbers—representing distances to other bodies in the solar system—they were required to calculate with.

The advantage to using logarithms was that they could essentially turn multiplication into addition and division into subtraction.

For example, what's 4×16? If you know (or can look up) some logarithms, you can turn this multiplication, $2^2 \cdot 2^4$, into addition: $2 + 4 = 6$.



The graph above shows the function $\log_2(x)$. What other multiplications can you turn into additions on this graph? What divisions can you turn into subtractions?

Problem 1

Students write expressions and equations in logarithmic form, given the corresponding expressions and equations in exponential form and vice versa.

Grouping

- Have students complete the table in Question 1 with a partner. Then have students share their responses as a class.
- Ask a student to read the information and definition of *logarithmic equation*. Discuss as a class.

Guiding Questions for Share Phase, Question 1

- Which number represents the base of the logarithm?
- Which number represents the argument?
- Which number represents the logarithm or exponent?
- Why is it helpful to consider the exponential relationship before converting it into logarithmic form?

PROBLEM 1 It's All Coming Together . . .

Recall that a logarithmic function is the inverse of an exponential function.

1. Write the equivalent form of the given exponential or logarithmic expression.

Exponential Form $y = b^x$	⇔	**Logarithmic Form** $x = \log_b (y)$
$12^2 = 144$	⇔	$\log_{12} (144) = 2$
$16^{\frac{1}{2}} = 4$	⇔	$\log_{16} (4) = \frac{1}{2}$
$10^5 = 100{,}000$	⇔	$\log 100{,}000 = 5$
$e^3 \approx 20.086$	⇔	$\text{In } 20.086 \approx 3$
$\left(\frac{2}{3}\right)^3 = \frac{8}{27}$	⇔	$\log_{\frac{2}{3}} \left(\frac{8}{27}\right) = 3$
$9^{\frac{3}{2}} = 27$	⇔	$\log_9 (27) = \frac{3}{2}$
$2^8 = x$	⇔	$\log_2 (x) = 8$
$6^x = 36$	⇔	$\log_6 (36) = x$
$n^5 = 243$	⇔	$\log_n (243) = 5$

When you evaluate a logarithmic expression (logarithm), you are determining the value of the exponent in the corresponding exponential expression.

$$\text{base}^{\text{exponent}} = \text{argument} \Leftrightarrow \log_{\text{base}} (\text{argument}) = \text{exponent}$$

The variables of the logarithmic equation have the same restrictions as the corresponding variables of the exponential equation. The base, b, must be greater than 0 but not equal to 1; the argument must be greater than 0; and the value of the exponent has no restrictions.

It is important to become familiar with how the base, argument, and exponent fit into a *logarithmic equation*. A **logarithmic equation** is an equation that contains a logarithm. To write a logarithmic equation, sometimes it is helpful to consider the exponential form first and then convert it to logarithmic form.

Grouping

- Have students complete all parts of Question 2 with a partner. Then have students share their responses as a class.
- As a class, discuss the worked example demonstrating how to solve for any unknown in a logarithmic equation.

Guiding Questions for Share Phase, Question 2

- Did you consider the exponential relationship before creating a logarithmic equation?
- How did you decide which number represented the exponent?
- How did you decide which number represented the base?
- How did you decide which number represented the argument?

Grouping

Have students complete all parts of Question 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3

- Is it easier to solve for the base, the exponent, or the argument? Why?
- What strategy was used to solve for the argument?
- What strategy was used to solve for the exponent?
- What strategy was used to solve for the base?
- If the exponents are equal, are the bases always equal?
- If the bases are equal, are the exponents always equal?

2. Arrange the given terms to create a true logarithmic equation.

a. 49, 2, 7

$7^2 = 49$

$\log_7 (49) = 2$

b. $-3, 6, \frac{1}{216}$

$6^{-3} = \frac{1}{216}$

$\log_6 \left(\frac{1}{216}\right) = -3$

c. 4, 4, 1

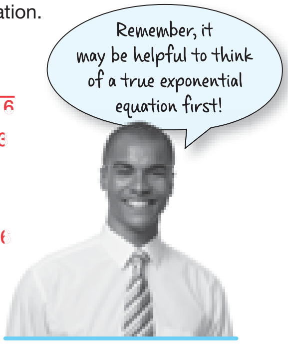
$4^1 = 4$

$\log_4 (4) = 1$

d. 256, 4, 4

$4^4 = 256$

$\log_4 (256) = 4$



Let's consider the relationship between the base, argument, and exponent. You can use that relationship to solve for any unknown in a logarithmic equation.

To solve for any unknown in a simple logarithmic equation, begin by converting it to an exponential equation.

Argument Is Unknown	Exponent Is Unknown	Base Is Unknown
$\log_4 (y) = 3$	$\log_4 (64) = x$	$\log_b (64) = 3$
$4^3 = y$	$4^x = 64$	$b^3 = 64$
$64 = y$	$4^x = 4^3$	$b^3 = 4^3$
	$x = 3$	$b = 4$

It is important to note that you can convert a logarithmic equation to an exponential equation regardless of which term is unknown.

3. Justify the last step of each case in the worked example.

a. If $4^3 = y$, why does $y = 64$?

In the equation $4^3 = y$, the variable is isolated so I can calculate the value of 4^3 to determine my answer.

b. If $4^x = 4^3$, why does $x = 3$?

In the equation $4^x = 4^3$, the bases are equal so I know that the exponents must also be equal.

c. If $b^3 = 4^3$, why does $b = 4$?

In the equation $b^3 = 4^3$, the exponents are equal so I know that the bases must also be equal.

Grouping

Have students complete all parts of Questions 4 and 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- What is unknown?
- How can you rewrite the equation in exponential form?

Guiding Questions for Share Phase, Question 5

- Did you first determine the value of $\log_5(625)$?
- What is the exponential form of $\log_5(625)$?
- What is the value of $\log_5(625)$?
- Is $\log_2(16)$ equivalent to $\log_5(625)$? How do you know?
- What is the value of $\log_2(16)$?
- Is $\log_3(81)$ equivalent to $\log_5(625)$? How do you know?
- What is the value of $\log_3(81)$?
- Is $\log_4(256)$ equivalent to $\log_5(625)$? How do you know?
- What is the value of $\log_4(256)$?
- Did you first determine the value of $\log_7\left(\frac{1}{7}\right)$?
- What is the exponential form of $\log_7\left(\frac{1}{7}\right)$?
- What is the value of $\log_7\left(\frac{1}{7}\right)$?
- Is $\log_2\left(\frac{1}{2}\right)$ equivalent to $\log_7\left(\frac{1}{7}\right)$? How do you know?
- What is the value of $\log_2\left(\frac{1}{2}\right)$?
- Is $\log_3\left(\frac{1}{3}\right)$ equivalent to $\log_7\left(\frac{1}{7}\right)$? How do you know?
- What is the value of $\log_3\left(\frac{1}{3}\right)$?
- Did you first determine the value of $\log_{64} 8$?
- What is the exponential form of $\log_{64}(8)$?
- What is the value of $\log_{64}(8)$?

4. Solve for the unknown in each logarithmic equation.

a. $\log_8(64) = n$

$8^n = 64$

$8^n = 8^2$

$n = 2$

b. $\log_n\left(\frac{1}{16}\right) = -2$

$n^{-2} = \frac{1}{16}$

$n^{-2} = \frac{1}{4^2}$

$n^{-2} = 4^{-2}$

$n = 4$

c. $\log_{\frac{1}{2}}(64) = n$

$\left(\frac{1}{2}\right)^n = 64$

$\left(\frac{1}{2}\right)^n = 2^6$

$\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{-6}$

$n = -6$

d. $\log n = -3$

$10^{-3} = n$

$\frac{1}{1000} = n$

$0.001 = n$

e. $\log_n(\sqrt[3]{49}) = \frac{2}{3}$

$n^{\frac{2}{3}} = \sqrt[3]{49}$

$n^{\frac{2}{3}} = 49^{\frac{1}{3}}$

$n^{\frac{2}{3}} = (7^2)^{\frac{1}{3}}$

$n^{\frac{2}{3}} = 7^{\frac{2}{3}}$

$n = 7$

f. $\log_9(27) = n$

$9^n = 27$

$3^{2n} = 3^3$

$n = \frac{3}{2}$

5. Write three logarithmic expressions that are equivalent to each given expression. Explain your strategy.

a. $\log_5(625)$

Answers will vary.

Sample responses include:

$\log_2(16)$, $\log_3(81)$, $\log_4(256)$, $\log_{\frac{1}{5}}\left(\frac{1}{625}\right)$, $\log 10{,}000$, $\ln(e^4)$

First, I evaluated the given logarithm.

$\log_5(625) = x$

$5^x = 625$

$x = 4$

Then I applied the exponent of 4 to three other bases, and wrote them in logarithmic form. (Example: $2^4 = 16$, so $\log_2(16)$ is equivalent to $\log_5(625)$.)

b. $\log_7\left(\frac{1}{7}\right)$

Answers will vary.

Sample responses include:

$\log_2\left(\frac{1}{2}\right)$, $\log_3\left(\frac{1}{3}\right)$, $\log_{\frac{1}{4}}(4)$, $\log 0.1$, $\ln\left(\frac{1}{e}\right)$

First, I evaluated the given logarithm.

$$\log_7\left(\frac{1}{7}\right) = x$$
$$7^x = \frac{1}{7}$$
$$x = -1$$

Then I applied the exponent of -1 to three other bases, and wrote them in logarithmic form. $\left(\text{Example: } 3^{-1} = \frac{1}{3}, \text{ so } \log_3\left(\frac{1}{3}\right) \text{ is equivalent to } \log_7\left(\frac{1}{7}\right).\right)$

c. $\log_{64}(8)$

Answers will vary.

Sample responses include:

$\log_4(2)$, $\log_9(3)$, $\log_{16}(4)$, $\log_{25}(5)$, $\log_{\frac{1}{16}}\left(\frac{1}{4}\right)$, $\ln\left(e^{\frac{1}{2}}\right)$

First, I evaluated the given logarithm.

$$\log_{64}(8) = x$$
$$64^x = 8$$
$$x = \frac{1}{2}$$

Then I applied the exponent of $\frac{1}{2}$ to three other bases, and wrote them in logarithmic form. (Example: $(4)^{\frac{1}{2}} = 2$, so $\log_4(2)$ is equivalent to $\log_{64}(8)$.)

d. $\log_2(-2)$

This logarithm is not possible. The argument must be positive.

$2^x = -2$

There is no value of x that makes this equation true.

Problem 2

Students use a number line to estimate the value of logarithms to the tenths place when the argument is not a power of the base. A worked example is provided. Students generalize that $\log_b(1) = 0$ and $\log_b(b) = 1$. Next, given the argument and the exponent, students use a number line to estimate the base of logarithms.

Grouping

- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.
- Discuss the worked example as a class.

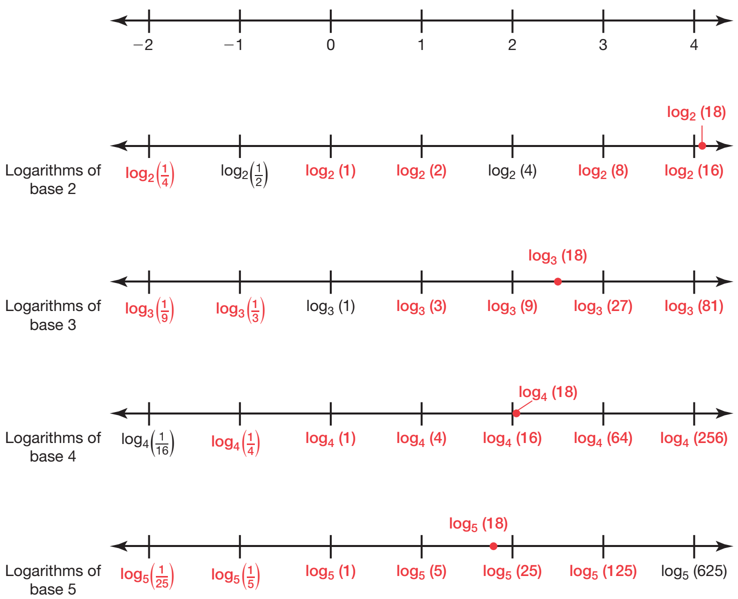
Guiding Questions for Share Phase, Questions 1 and 2

- How did you determine the labels on the number line for the logarithms of base 2?
- How did you determine the labels on the number line for the logarithms of base 3?
- How did you determine the labels on the number line for the logarithms of base 4?
- How did you determine the labels on the number line for the logarithms of base 5?
- What is true about the arguments of all logarithms equivalent to 0?
- Do the logarithms that are equivalent to 0 all have an argument of 1?
- What is true about the arguments of all logarithms equivalent to 1?
- Do the logarithms that are equivalent to 1 all have arguments that are equal to the bases?
- How is $b^0 = 1$ written as a logarithmic equation?
- How is $b^1 = b$ written as a logarithmic equation?

PROBLEM 2 Estimating Logarithms

In Problem 1, each logarithm was a rational number. However, a logarithm can be any real number, even an irrational number.

1. Label each number line using logarithmic expressions with the indicated base.



2. Compare the logarithms on the number lines.

a. Analyze all the logarithms that are equivalent to 0. Write a general statement using the base b to represent this relationship.

The logarithms that are equivalent to 0 all have an argument of 1, regardless of the base. Therefore, $\log_b(1) = 0$ for $b > 0$, $b \neq 1$.

b. Analyze all the logarithms that are equivalent to 1. Write a general statement using the base b to represent this relationship.

The logarithms that are equivalent to 1 all have an argument that is equal to the base. Therefore, $\log_b(b) = 1$ for $b > 0$, $b \neq 1$.

c. Rewrite the general statements from parts (a) and (b) in exponential form. Use exponent rules to verify that each statement is true.

The logarithmic equation $\log_b(1) = 0$ can be rewritten in exponential form as $b^0 = 1$. Any base raised to the 0 power is equal to 1, so the logarithmic statement is true.

The logarithmic equation $\log_b(b) = 1$ can be rewritten in exponential form as $b^1 = b$. Any base raised to the 1st power is equal to itself, so the logarithmic statement is true.

You can estimate the value of a logarithm that is not an integer by using a number line as a guide.

Estimate the value of $\log_3(33)$.

To estimate $\log_3(33)$ to the tenths place, identify the closest logarithm whose argument is less than 33 and the closest logarithm whose argument is greater than 33 on the number line that represents base 3.

The closest logarithm whose argument is less than 33:	The logarithm you are estimating:	The closest logarithm whose argument is greater than 33:
$\log_3(27)$	$\log_3(33)$	$\log_3(81)$

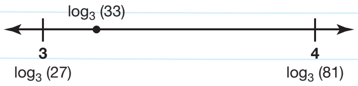
You know that $\log_3 27 = 3$ and $\log_3 81 = 4$. This means the estimate of $\log_3 33$ is between 3 and 4.

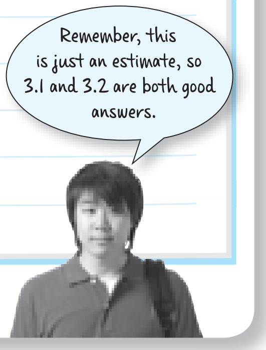
$$\log_3(27) < \log_3(33) < \log_3(81)$$

$$3 < x < 4$$

Next, estimate the decimal digit.

Because 33 is closer to 27 than to 81, the value of $\log_3 33$ is closer to 3 than to 4.





In this case, 3.2 is a good estimate for $\log_3(33)$.

Grouping

Have students complete Questions 3 and 4 with a partner. Then have students share their responses as a class.

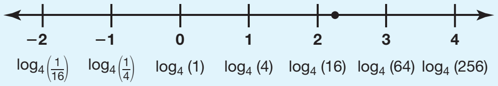
Guiding Questions for Share Phase, Question 3

- The logarithm $\log_4 (28)$ is located between which two logarithms on the number line?
- The value of the logarithm $\log_4 (28)$ is between which two logarithmic values on the number line?
- Did Sutton or Silas use the definition of *logarithm* when making their estimation?

3. Sutton and Silas were each asked to estimate $\log_4 (28)$.

Sutton

I estimated $\log_4 (28)$ by using the number line.



$$\log_4 (16) < \log_4 (28) < \log_4 (64)$$

$$2 < x < 3$$

$$\log_4 (28) \approx 2.3$$

Silas

I estimated $\log_4 (28)$ by converting the log into exponential form and estimating based on powers of 4.

$$\log_4 (28) = x$$

$$4^x = 28$$

I know that $4^2 = 16$ and $4^3 = 64$ so the estimate of $\log_4 (28)$ must be between 2 and 3.

$$\log_4 (28) \approx 2.4$$

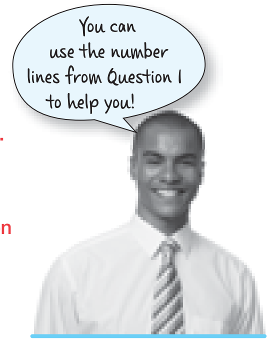
Silas did not use the number line, but his estimate was about the same as Sutton's. Will Silas's method always work?

The method Silas used will always work because he is using the definition of a logarithm when making his estimate.

Guiding Questions for Share Phase, Question 4

- The logarithm $\log_2 (10)$ is located between which two logarithms on the number line?
- The value of the logarithm $\log_2 (10)$ is between which two logarithmic values on the number line?
- The logarithm $\log_5 (4)$ is located between which two logarithms on the number line?
- The value of the logarithm $\log_5 (4)$ is between which two logarithmic values on the number line?
- The logarithm $\log_4 (300)$ is located between which two logarithms on the number line?
- The value of the logarithm $\log_4 (300)$ is between which two logarithmic values on the number line?
- The logarithm log 2500 is located between which two logarithms on the number line?
- The value of the logarithm log 2500 is between which two logarithmic values on the number line?

4. Estimate each logarithm to the tenths place and explain your reasoning.



a. $\log_2 (10)$

Answers will vary.

The logarithm $\log_2 10$ is approximately equal to 3.3.

$\log_2 (8) < \log_2 (10) < \log_2 (16)$

$3 < x < 4$

Since 10 is closer to 8 than to 16, the approximation should be closer to 3 than to 4.

b. $\log_5 (4)$

Answers will vary.

The logarithm $\log_5 4$ is approximately equal to 0.8.

$\log_5 (1) < \log_5 (4) < \log_5 (5)$

$0 < x < 1$

Since 4 is closer to 5 than to 1, the approximation should be closer to 1 than to 0.

c. $\log_4 (300)$

Answers will vary.

The logarithm $\log_4 300$ is approximately equal to 4.1.

$\log_4 (256) < \log_4 (300) < \log_4 (1024)$

$4 < x < 5$

Since 256 is closer to 300 than to 1024, the approximation should be closer to 4 than to 5.

d. log 2500

Answers will vary.

The logarithm log 2500 is approximately equal to 3.4.

$\log 1000 < \log 2500 < \log 10{,}000$

$3 < x < 4$

Since 2500 is closer to 1000 than to 10,000, the approximation should be closer to 3 than to 4.

Grouping

Have students complete Questions 5 through 11 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 5

- Can Mark and Scotty both be correct?
- Is there more than one correct answer in this situation?
- What is the value of the exponent in this situation?
- Because the value of the exponent is 2.9, does that mean the argument 58 should be closer to its upper limit?
- When the base is 4, is 58 close to the upper limit of 64?
- When the base is 5, is 58 close to the upper limit of 125?

5. Mark and Scotty were each asked to determine which base was used to estimate the value of $\log_b (58) = 2.9$.

Mark

The log of 58 falls between 2 and 3 when the base is 4.

$\log_4 (16) < \log_b (58) < \log_4 (64)$

$2 < 2.9 < 3$

So, $b \approx 4$.

Scotty

The log of 58 falls between 2 and 3 when the base is 5.

$\log_5 (25) < \log_b (58) < \log_5 (125)$

$2 < 2.9 < 3$

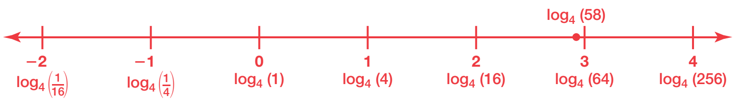
So, $b \approx 5$.

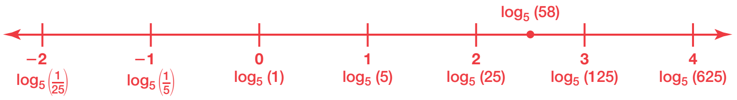
Who is correct? Explain your reasoning.

Mark is correct.

Because the value of the exponent is 2.9, that means that the argument should be close to its value when 3 is the exponent.

When the base is 4, 58 is very close to 4^3, or 64, whereas when the base is 5, 58 is not very close to 5^3, or 125.





The estimate of 2.9 makes more sense when the base is 4. Therefore, Mark is correct.

Guiding Questions for Share Phase, Question 6

- If the argument is 108, what two values does the log of 108 fall between on the number line?
- What base number is associated with the log of 108 if the exponent is 2.9?
- If the argument is 108, why won't the number line associated with the base of 4 be appropriate in this situation?
- If the argument is 0.4, what two values does the log of 0.4 fall between on the number line?
- What base number is associated with the log of 0.4 if the exponent is −1.3?
- If the argument is 0.4, why won't the number line associated with the base of 3 be appropriate in this situation?
- If the argument is 74, what two values does the log of 74 fall between on the number line?
- What base number is associated with the log of 74 if the exponent is 3.1?
- If the argument is 74, why won't the number line associated with the base of 5 be appropriate in this situation?

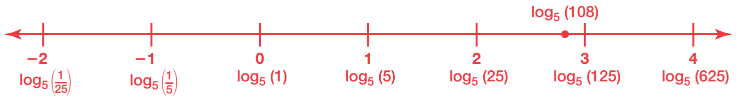
6. Use the number lines from Question 1 to determine the appropriate base of each logarithm.

a. $\log_b (108) = 2.9$

$b \approx 5$

$\log_5 (25) < \log_5 (108) < \log_5 (125)$

$2 < x < 3$

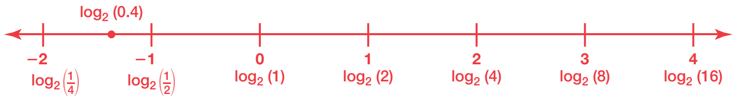


b. $\log_b (0.4) = -1.3$

$b \approx 2$

$\log_2 (0.25) < \log_2 (0.4) < \log_2 (0.5)$

$-2 < x < -1$

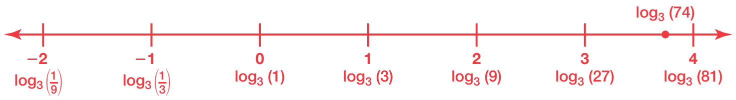


c. $\log_b (74) = 3.1$

$b \approx 4$

$\log_3 (27) < \log_3 (74) < \log_3 (81)$

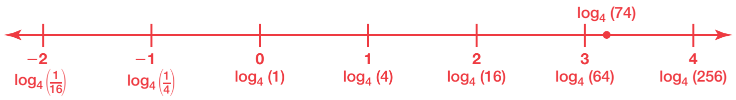
$3 < x < 4$



and

$\log_4 (64) < \log_4 (74) < \log_4 (256)$

$3 < x < 4$



Since the value is 3.1, the argument 74 should be close to the value when the exponent is 3. Therefore, $\log_3 (74)$ does not make sense. The base must be 4.

Guiding Questions for Share Phase, Questions 7 through 11

- For a fixed base, as the value of the argument gets larger, does the value of the logarithm get larger or smaller?
- Which two numeric values is $\log_2$ (18) between on the number line? How did you determine the estimate to the tenths place?
- Which two numeric values is $\log_3$ (18) between on the number line? How did you determine the estimate to the tenths place?
- Which two numeric values is $\log_4$ (18) between on the number line? How did you determine the estimate to the tenths place?
- Which two numeric values is $\log_5$ (18) between on the number line? How did you determine the estimate to the tenths place?
- For a fixed argument, as the value of the base gets larger, does the value of the logarithm get larger or smaller?
- What is the value of the base of In 18?
- What is In 18 written in exponential form?
- Which two log values is 2.718 between?
- What is the value of $\log_2$ (18)?
- What is the value of $\log_3$ (18)?
- Is the value of In 18 less than 4.2 but greater than 2.6?
- Is 2.718 closer to 2 or 3?

7. For a fixed base greater than 1, as the value of the argument gets larger, what happens to the value of the logarithm? Provide an example to illustrate your statement.

For a fixed base, as the value of the argument gets larger, the value of the logarithm gets larger as well.

For example, $\log_3 (3) = 1$, $\log_3 (9) = 2$, $\log_3 (27) = 3$. The base stayed fixed at 3, but as the argument increases, so does the value of the logarithm.

8. Plot $\log_2$ (18), $\log_3$ (18), $\log_4$ (18), and $\log_5$ (18) on the appropriate number lines in Question 1. Then use the number lines to estimate the numeric value of each logarithm to the tenths place. Verify your answers in exponential form.

See number lines.

	In exponential form, this means that
$\log_2 (18) \approx 4.2$	$2^{4.2} \approx 18$
$\log_3 (18) \approx 2.6$	$3^{2.6} \approx 18$
$\log_4 (18) \approx 2.1$	$4^{2.1} \approx 18$
$\log_5 (18) \approx 1.8$	$5^{1.8} \approx 18$

9. For a fixed argument, when the value of the base is greater than 1 and increasing, what happens to the value of the logarithm?

For a fixed argument, as the value of the base gets larger, the value of the logarithm gets smaller.

10. How could you use the number lines from Question 1 to predict the value of In 18?

In $18 = x$

$e^x = 18$

$(2.718)^x \approx 18$

Since I know that $\log_2 (18) \approx 4.2$, and $\log_3 (18) \approx 2.6$, I also know that the value of the logarithm will have to be less than 4.2 but greater than 2.6.

11. Make a prediction for the value of In 18.

Answers will vary.

Since 2.718 is closer to 3 than to 2, the value of the logarithm must be closer to 2.6.

In $18 \approx 2.9$

Talk the Talk

Students choose from the words 'always, sometimes, or never' to make true statements.

Grouping

Have students complete Questions 1 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Talk the Talk

- What is the definition of *logarithm*?
- What is the domain of logarithms?
- What is the range of logarithms?
- What restrictions are associated with the base of a logarithm?
- Can the value of a logarithm be equal to any real number?
- Why can't the base of a logarithm equal 1?

Talk the Talk

Choose a word from the box that makes each statement true. Explain your reasoning.

always	sometimes	never

1. The value of a logarithm is __always__ equal to the exponent of the corresponding exponential equation.

This is the definition of a logarithm.

2. The argument of a logarithmic expression is __never__ a negative number.

A basic logarithmic function has a domain of $(0, \infty)$. The arguments therefore cannot be negative.

3. The value of a logarithm is __sometimes__ equal to a negative number.

The range of a logarithm is $(-\infty, \infty)$.

4. The base of a logarithm is __never__ a negative number.

By definition, the base of a logarithmic expression is greater than zero and not equal to 1.

5. A logarithm is __sometimes__ a value that is not a whole number.

Because the value of a logarithm is the exponent on the base, it can equal any real number from $(-\infty, \infty)$.

6. For a base greater than 1, if $b > c$ then the value of $\log_a b$ is __always__ greater than $\log_a c$.

When the bases are the same, as the argument gets larger, so does the value of the logarithm.

7. If $a > b$, then the value of $\log_a 1$ is __never__ less than $\log_b 1$.

Any logarithmic expression with an argument of 1, regardless of the base, will be equal to 0. Thus $\log_a 1$, will never be less than $\log_b 1$, because they are equal.

8. The base of a logarithm is __never__ equal to 1.

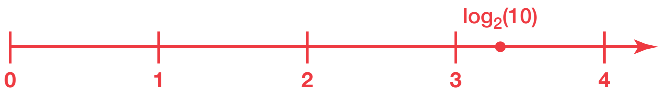
$\log_1 (y) = x \quad \rightarrow \quad 1^x = y$

This is basically a constant function because 1 to any power is equal to 1.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Use number lines to estimate the numeric value of $\log_2(10)$, $\log_3(10)$, $\log_4(10)$, and $\log_5(10)$ to the tenths place. Verify your answers in exponential form.



$\log_2(10) \approx 3.3$
$\log_3(10) \approx 2.1$
$\log_4(10) \approx 1.7$
$\log_5(10) \approx 1.4$

$2^{3.3} \approx 10$
$3^{2.1} \approx 10$
$4^{1.7} \approx 10$
$5^{1.4} \approx 10$

13.2

Mad Props

Properties of Logarithms

LEARNING GOALS

In this lesson, you will:

- Derive the properties of logarithms.
- Expand logarithmic expressions using the properties of logarithms.
- Rewrite multiple logarithmic expressions as a single logarithmic expression.

KEY TERMS

- Zero Property of Logarithms
- Logarithm with Same Base and Argument
- Product Rule of Logarithms
- Quotient Rule of Logarithms
- Power Rule of Logarithms

ESSENTIAL IDEAS

- The Zero Property of Logarithms states: “The logarithm of 1, with any base, is always equal to zero, or $\log_b 1 = 0$.”
- The Product Rule of Logarithms states: “The logarithm of a product is equal to the sum of the logarithms of the factors, or $\log_b (xy) = \log_b (x) + \log_b (y)$.”
- The Quotient Rule of Logarithms states: “The logarithm of a quotient is equal to the difference of the logarithms of the dividend and the divisor, or $\log_b \left(\frac{x}{y}\right) = \log_b (x) - \log_b (y)$.”
- The Power Rule of Logarithms states: “The logarithm of a power is equal to the product of the exponent and the logarithm of the base of the power, or $\log_b (x^n) = n \log_b (x)$.”

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-BF Building Functions

Build new functions from existing functions

5. (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Overview

Students develop rules and properties of logarithms based on their prior knowledge of various exponent rules and properties. They summarize the different properties by completing a table which defines each exponential and logarithmic property verbally and symbolically, providing examples for each instance.

Warm Up

Identify the property of exponents associated with each example.

1. $\left(\frac{2}{3}\right)^5 \cdot \left(\frac{2}{3}\right)^2 = \left(\frac{2}{3}\right)^7$

The Product Rule of Exponents

2. $\left(\frac{2}{3}\right)^1 = \left(\frac{2}{3}\right)$

Any number raised to the first power is equal to its base number.

3. $\left(\frac{2}{3}\right)^0 = 1$

Any number raised to the zero power is equal to one.

4. $\frac{2^{14}}{2^5} = 2^9$

The Quotient Rule of Exponents

13

13.2

Mad Props

Properties of Logarithms

LEARNING GOALS

In this lesson, you will:

- Derive the properties of logarithms.
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- Rewrite multiple logarithmic expressions as a single logarithmic expression.

KEY TERMS

- Zero Property of Logarithms
- Logarithm with Same Base and Argument
- Product Rule of Logarithms
- Quotient Rule of Logarithms
- Power Rule of Logarithms

Dimitri Mendeleev is best known for his work on the periodic table—arranging the 63 known elements into a periodic table based on atomic mass, which he published in *Principles of Chemistry* in 1869. His first periodic table was compiled on the basis of arranging the elements in ascending order of atomic weight and grouping them by similarity of properties. He predicted the existence and properties of new elements and pointed out accepted atomic weights that were in error. His table did not include any of the noble gases, however, which had not yet been discovered. Dmitri Mendeleev revolutionized our understanding of the properties of atoms and created a table that probably embellishes every chemistry classroom in the world.

Problem 1

Students develop the Zero Property of Logarithms using the Zero Property of Powers. Students also derive the logarithmic property $\log_b(b) = 1$ using the corresponding property $b^1 = b$ for exponents. The Product Rule of Powers is used to develop the Product Rule of Logarithms, and the Quotient Rule of Powers is used to develop the Quotient Rule of Logarithms.

Grouping

Have students complete all parts of Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Is any base raised to the zero power equal to 1?
- Is the logarithm of 1, with any base, always equal to 0?
- Is any number raised to the first power equal to the base?
- Is the logarithm of a number when the base and argument are equal, always equal to 1?

PROBLEM 1 Setting Ground Rules

Logarithms by definition are exponents, so they have properties that are similar to those of exponents and powers. In this lesson, you will develop logarithmic rules and properties that correspond to various exponential rules and properties you already know.

1. Let's consider the Zero Property of Powers to develop a corresponding logarithmic property.

   a. Write a sentence to summarize the Zero Property of Powers, $b^0 = 1$.

   Any base raised to the zero power is 1.

   b. Write the Zero Property of Powers in logarithmic form. This is a corresponding logarithmic property called the *Zero Property of Logarithms*.

   $\log_b(1) = 0$

   c. State the Zero Property of Logarithms in words.

   The logarithm of 1, with any base, is always equal to 0.

2. Let's consider the exponent rule that says that any number raised to the first power is equal to the base.

   a. Write an exponential equation to represent this rule. Use b as the base.

   $b^1 = b$

   b. Write your exponential equation from part (a) in logarithmic form.

   $\log_b(b) = 1$

   c. State this logarithmic relationship in words.

   When the base and argument are equal, the logarithm is always equal to 1.

13

Grouping

Have students complete all parts of Question 3 with a partner. Then have students share their responses as a class.

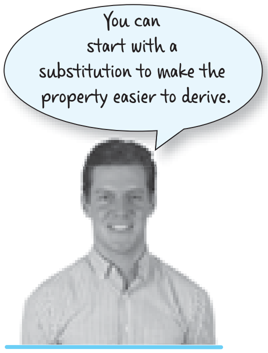
Guiding Questions for Share Phase, Question 3

- When multiplying powers with the same base, do you add, subtract, multiply, or divide the exponents?
- Is the logarithm of a product equal to the sum or difference of the logarithms of the factors?

3. Let's consider the Product Rule of Powers to derive a corresponding logarithmic property.

a. Write a sentence to summarize the Product Rule of Powers, $b^m \cdot b^n = b^{m+n}$.

When multiplying powers with the same base, you add the exponents.


You can start with a substitution to make the property easier to derive.

b. Analyze the steps that begin with the Product Rule of Powers to derive a corresponding logarithmic property called the *Product Rule of Logarithms*. Complete the last line of the diagram.

$$\begin{tikzcd}
 & \text{Given: } b^m \cdot b^n = b^{m+n} \arrow[dl] \arrow[d] & & \\
\text{Let } x = b^m. \arrow[d, leftrightarrow] & \text{Let } y = b^n. \arrow[r] \arrow[d, leftrightarrow] & \text{Use substitution to rewrite the given:} & xy = b^{m+n} \arrow[d, leftrightarrow] \\
\log_b(x) = m \arrow[rrr, bend left] & \log_b(y) = n \arrow[rr, bend right] & & \log_b(xy) = m + n
\end{tikzcd}$$

Rewrite each exponential equation as a logarithmic equation.

Finally, use substitution to write the property in terms of x and y: $\log_b(xy) = \underline{\log_b(x) + \log_b(y)}$

c. State the Product Rule of Logarithms in words.

The logarithm of a product is equal to the sum of the logarithms of the factors.

Grouping

Have students complete Questions 4 and 5 with a partner. Then have students share their responses as a class.

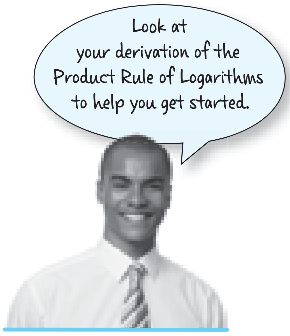
Guiding Questions for Share Phase, Questions 4 and 5

- When dividing powers with the same base, do you add, subtract, multiply, or divide the exponents?
- Is the logarithm of a quotient equal to the sum of the factors logarithms of the factors or the difference of the logarithms of the dividend and the divisor?
- When simplifying a power to a power, does the base change or remain the same?
- When simplifying a power to a power, are the exponents added together or multiplied?
- Is the logarithm of a power equal to the product of the exponent and the logarithm of the base of the power?

4. Let's consider the Quotient Rule of Powers to derive a corresponding logarithmic property.

a. Write a sentence to summarize the Quotient Rule of Powers, $\frac{b^m}{b^n} = b^{m-n}$, if $b^n \neq 0$.

When dividing powers with the same base, you subtract the exponents.



b. Complete the steps that begin with the Quotient Rule of Powers to derive a corresponding logarithmic property called the *Quotient Rule of Logarithms.*

Given: $\frac{b^m}{b^n} = b^{m-n}$, if $n \neq 0$

Let $x = b^m$. Let $y = b^n$. → Use substitution to rewrite the given: $\frac{x}{y} = b^{m-n}$

Rewrite each exponential equation as a logarithmic equation.

$\Updownarrow$ $\Updownarrow$ $\Updownarrow$

$\log_b x = m$ $\log_b y = n$ $\log_b \left(\frac{x}{y}\right) = m - n$

Finally, use substitution to write the property in terms of x and y: $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

c. Summarize the Quotient Rule of Logarithms in words.

The logarithm of a quotient is equal to the difference of the logarithms of the dividend and the divisor.

5. Let's consider the Power to a Power Rule to derive a corresponding logarithmic property.

a. Write a sentence to summarize the Power to a Power Rule, $(b^m)^n = b^{mn}$.

To simplify a power to a power, keep the base and multiply the exponents.

b. Complete the steps that begin with the Power to a Power Rule to derive a corresponding logarithmic property called the *Power Rule of Logarithms*.

Given: $(b^m)^n = b^{mn}$

Let $x = b^m$. ⟶ Use substitution to rewrite the given: $x^n = b^{mn}$

Rewrite each exponential equation as a logarithmic equation.

$\Updownarrow$ $\Updownarrow$

$\log_b x = m$ $\log_b x^n = mn$

Finally, use substitution to write the property in terms of x: $\log_b x^n = n \cdot \log_b x$

c. State the Power Rule of Logarithms in words.

The logarithm of a power is equal to the product of the exponent and the logarithm of the base of the power.

Grouping

Have students complete Question 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 6

- $15^0 = 1$ is an example of which property?
- $\log_6 (1) = 0$ is an example of which property?
- $7^1 = 7$ is an example of which property?
- $\log_6 (6) = 1$ is an example of which property?
- $x^2 \cdot x^6 \cdot x^5 = x^{13}$ is an example of which property?
- $\log_6 (5) + \log_6 (4) = \log_6 (20)$ is an example of which property?
- $\frac{x^9}{x^3} = x^6$ is an example of which property?
- $\log_6 (54) - \log_6 (6) = \log_6 (9)$ is an example of which property?
- $(x^3)^6 = x^{18}$ is an example of which property?
- $\log_6 (125) = 3 \log_6 (5)$ is an example of which property?

6. In this problem, you derived different properties of logarithms. Complete the tables to define each exponential and logarithmic property verbally and symbolically. Provide examples for each property.

Answers will vary.

Exponential Property	Logarithmic Property
Zero Property of Powers	**Zero Property of Logarithms**
Verbal: Any base raised to the zero power is 1.	**Verbal:** The logarithm of 1, with any base, is always equal to 0.
Symbolic: $b^0 = 1$	**Symbolic:** $\log_b (1) = 0$
Examples: $m^0 = 1$ $10^0 = 1$	**Examples:** $\log_a (1) = 0$ $\log_7 (1) = 0$
Base Raised to First Power	**Logarithm with Same Base and Argument**
Verbal: Any base raised to the first power is the base itself.	**Verbal:** The logarithm of a number, with the base equal to the same number, is always equal to 1.
Symbolic: $b^1 = b$	**Symbolic:** $\log_b (b) = 1$
Examples: $p^1 = p$ $6^1 = 6$	**Examples:** $\log_d (d) = 1$ $\log_2 (2) = 1$
Product Rule of Powers	**Product Rule of Logarithms**
Verbal: To multiply powers with the same base, you add the exponents.	**Verbal:** The logarithm of a product is equal to the sum of the logarithms of the factors.
Symbolic: $b^m \cdot b^n = b^{m+n}$	**Symbolic:** $\log_b (xy) = \log_b (x) + \log_b (y)$
Examples: $z^4 \cdot z^7 = z^{11}$ $3^m \cdot 3^n \cdot 3^p = 3^{m+n+p}$	**Examples:** $\log_a (mnp) = \log_a (m) + \log_a (n) + \log_a (p)$ $\log_2 (50) = \log_2 (5) + \log_2 (10)$

Exponential Property	Logarithmic Property
Quotient Rule of Powers	**Quotient Rule of Logarithms**
Verbal: To divide powers with the same base, you subtract the exponents.	**Verbal:** The logarithm of a quotient is equal to the difference of the logarithms of the dividend and the divisor.
Symbolic: $\frac{b^m}{b^n} = b^{m-n}$, if $n \neq 0$	**Symbolic:** $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
Examples: $\frac{r^8}{r^5} = r^3$ $\frac{5^d}{5^f} = 5^{d-f}$	**Examples:** $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$ $\log_2(5) = \log_2(45) - \log_2(9)$
Power to a Power Rule	**Power Rule of Logarithms**
Verbal: To simplify a power to a power, keep the base and multiply the exponents.	**Verbal:** The logarithm of a power is equal to the product of the exponent and the logarithm of the base of the power.
Symbolic: $(b^m)^n = b^{mn}$	**Symbolic:** $\log_b(x^n) = n \cdot \log_b(x)$
Examples: $(a^3)^4 = b^{12}$ $(7^w)^v = 7^{wv}$	**Examples:** $\log_a(r^s) = s \log_a(r)$ $\log_2(27) = 3 \log_2(3)$

PROBLEM 2 Don't Break the Rules!

1. Use the properties of logarithms to rewrite each logarithmic expression in expanded form.

   a. $\log_4(6x^5)$

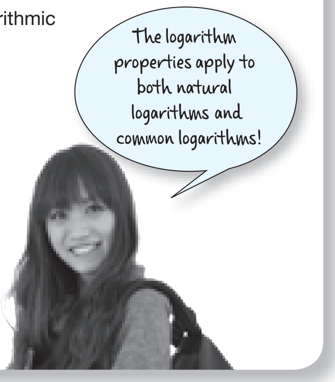
   $\log_4(6x^5) = \log_4(6) + 5\log_4(x)$

   b. $\log_7\left(\frac{3y^4}{x^3}\right)$

   $\log_7\left(\frac{3y^4}{x^3}\right) = \log_7(3) + 4\log_7(y) - 3\log_7(x)$

   c. $\ln(3xy^3)$

   $\ln(3xy^3) = \ln(3) + \ln(x) + 3\ln(y)$



Problem 2

Students use properties of logarithms to write logarithmic expressions in expanded form and to write logarithmic expressions as a single logarithm. Students write algebraic expressions for various logarithmic expressions.

Grouping

Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 3

- Does the expanded form of $\log_4(6x^5)$ involve the sum or difference of $\log_4(6)$ and $5\log_6(x)$?
- Does the expanded form of $\log_7\left(\frac{3y^4}{x^3}\right)$ involve the sum or difference of $\log_7(3)$, $4\log_7(y)$, and $3\log_7(x)$?
- Does the expanded form of $\ln(3xy^3)$ involve the sum or difference of $\ln 3$, $\ln x$, and $3\ln y$?
- Which rule or property was used to write the logarithmic expression as a single logarithm?

Guiding Questions for Share Phase, Questions 2 and 3

- Is $\log_a(50)$ equal to $(2p - r)$ or $(2p + r)$?
- Is $\log_a(0.3)$ equal to $(q + r - p)$ or $(q - r + p)$ or $(q - r - p)$?
- Is $\log_a\left(\frac{1}{27}\right)$ equal to $3q$ or $-3q$?

2. Use the properties of logarithms to rewrite each logarithmic expression as a single logarithm.

a. $\log_2(10) + 3\log_2(x)$

$\log_2(10) + 3\log_2(x) = \log_2(10x^3)$

b. $4\log(12) - 4\log(2)$

$4\log(12) - 4\log(2) = \log\left(\frac{12^4}{2^4}\right) = \log(1296) \approx 3.1126$

c. $3(\ln 3 - \ln x) + (\ln x - \ln 9)$

$3(\ln 3 - \ln x) + (\ln x - \ln 9) = \ln\left(\frac{3}{x}\right)^3\left(\frac{x}{9}\right) = \ln\left(\frac{3}{x^2}\right)$

3. Suppose $\log_a(5) = p$, $\log_a(3) = q$, and $\log_a(2) = r$. Write an algebraic expression for each logarithmic expression.

a. $\log_a(50)$

$$\begin{aligned}\log_a(50) &= \log_a(5^2 \cdot 2)\\ &= 2\log_a(5) + \log_a(2)\\ &= 2p + r\end{aligned}$$

b. $\log_a(0.3)$

$$\begin{aligned}\log_a(0.3) &= \log_a\left(\frac{3}{10}\right)\\ &= \log_a(3) - (\log_a(2) + \log_a(5))\\ &= q - r - p\end{aligned}$$

c. $\log_a\left(\frac{1}{27}\right)$

$$\begin{aligned}\log_a\left(\frac{1}{27}\right) &= \log_a\left(\frac{1}{3^3}\right)\\ &= \log_a(1) - 3\log_a(3)\\ &= -3q\end{aligned}$$

Be prepared to share your solutions and methods.

Check for Students' Understanding

Identify the property of logarithms associated with each example.

1. $\left(\frac{2}{3}\right)^5 \cdot \left(\frac{2}{3}\right)^2 = \left(\frac{2}{3}\right)^7$

   $\log_a(xy) = \log_a(x) + \log_a(y)$

2. $\left(\frac{2}{3}\right)^1 = \left(\frac{2}{3}\right)$

   $\log_a(a) = 1$

3. $\left(\frac{2}{3}\right)^0 = 1$

   $\log_a(1) = 0$

4. $\frac{2^{14}}{2^5} = 2^9$

   $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

13.3

What's Your Strategy?

Solving Exponential Equations

LEARNING GOALS

In this lesson, you will:

- Solve exponential equations using the Change of Base Formula.
- Solve exponential equations by taking the log of both sides.
- Analyze different solution strategies to solve exponential equations.

ESSENTIAL IDEAS

- The Change of Base Formula allows you to calculate an exact value for a logarithm by rewriting it in terms of a different base.
- Most calculators can only evaluate common logarithms and natural logarithms, so the Change of Base Formula may be necessary to evaluate logarithms when technology is involved.
- The Change of Base Formula states: $\log_b(c) = \frac{\log_a(c)}{\log_b(b)}$, where $a, b, c > 0$ and $a, b \neq 1$.
- If the bases are equal and the logarithms are equal, then the arguments must also be equal: If $\log_b(a) = \log_b(c)$, then $a = c$.
- If the logarithms are equal and the arguments are equal, then the bases must also be equal: If $a = c$, then $\log_b(a) = \log_b(c)$.

KEY TERM

- Change of Base Formula

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-BF Building Functions

Build new functions from existing functions

5. (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

F-LE Linear, Quadratic, and Exponential Models

Construct and compare linear, quadratic, and exponential models and solve problems

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Overview

Students use the context of subscribers to an online game to solve exponential equations by using the Change of Base Formula. Then students explore solving an exponential equation by taking the log of both sides of the equation. Students analyze alternative solution methods and common misconceptions throughout.

Warm Up

Match each logarithmic equation with its corresponding exponential expression.

Logarithmic Equation		Exponential Expression
1. $\log_2(1024) = 10$	(d)	**a.** 10^{10}
2. $\log_2(4) = 2$	(e)	**b.** 10^x
3. $\log_2(2) = x$	(h)	**c.** e^{10}
4. $\log 100 = 2$	(g)	**d.** 2^{10}
5. $\ln x = 10$	(c)	**e.** 2^2
6. $\log 10 = x$	(b)	**f.** 10^e
7. $\log x = 10$	(a)	**g.** 10^2
8. $\log x = e$	(f)	**h.** 2^x

13.3

What's Your Strategy?

Solving Exponential Equations

LEARNING GOALS

In this lesson, you will:

- Solve exponential equations using the Change of Base Formula.
- Solve exponential equations by taking the log of both sides.
- Analyze different solution strategies to solve exponential equations.

KEY TERM

- Change of Base Formula

It may seem like everyone you know is online. But of course not everyone in the world is online—yet. Here's a glimpse of how fast internet use was growing among the Earth's 7 billion people in June of 2012:

Region	Population (2012)	Internet Users (2000)	Internet Users (2012)
Africa	1,073,380,925	4,514,400	167,335,676
Asia	3,922,066,987	114,304,000	1,076,681,059
Europe	820,918,446	105,096,093	518,512,109
Middle East	223,608,203	3,284,800	90,000,455
North America	348,280,154	108,096,800	273,785,413
Latin America	593,668,638	18,068,919	254,915,745
Oceania/Australia	35,903,569	7,620,480	24,287,919

World Internet Usage and Population Statistics, June 30, 2012

How long do you think it will take before everyone on Earth is using the internet?

Problem 1

Students write an equation that calculates the total number of subscribers to an online game after a specified number of days and use the equation to solve for unknowns. The Change of Base Formula is introduced and used to revisit and evaluate the logarithmic situation. The value of the unknown resulting from the Change of Base Formula is compared to an answer that was previously estimated. A graphing calculator is used along with the Change of Base Formula to evaluate logarithmic expressions with large numbers.

Grouping

Have students complete all parts of Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- Is the equation that models the situation $P(t) = 50 + 3^t$ or is it $P(t) = 50 \cdot 3^t$?
- What equation is used to determine how many days it will take to have 4050 subscribers?
- Can 81 be rewritten as a power of 3?
- What equation is used to determine how many days it will take to have 20,000 subscribers?
- Can 400 be rewritten as a power of 3?
- Which two integer powers of 3 does the value 400 fall between?
- Which answer in this problem is exact and which answer is an estimate?

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PROBLEM 1 Don't Burst My Bubble

The newest online game is Bubblez Burst, a highly addictive game that runs on social media. On the first day of its release, 50 people subscribe. The creators estimate that everyone who subscribes will then send 3 more people to subscribe.

1. Write a function using the creator's estimate to model the total number of subscribers P who will be playing Bubblez Burst after t number of days.

   $P(t) = 50 \cdot 3^t$

2. How many days will it take for Bubblez Burst to have 4050 subscribers?

   It will take 3 days for Bubblez Burst to have 4050 subscribers.

   $P(t) = 50 \cdot 3^t$

   $4050 = 50 \cdot 3^t$

   $81 = 3^t$

   $3^4 = 3^t$

   $4 = t$

3. How many days will it take for Bubblez Burst to reach 20,000 subscribers?

   It will take approximately 5.4 days for Bubblez Burst to reach 20,000 subscribers.

   $P(t) = 50 \cdot 3^t$

   $20{,}000 = 50 \cdot 3^t$

   $400 = 3^t$

   I know that $3^5 = 243$ and $3^6 = 729$, so $5 < t < 6$. I would estimate t to be about 5.4.

4. How did your methods in Question 2 and Question 3 differ?

   In Question 2, I was able to use common bases to solve for t because 81 is a power of 3. In Question 3, I had to use estimation because 300 is not a power of 3.

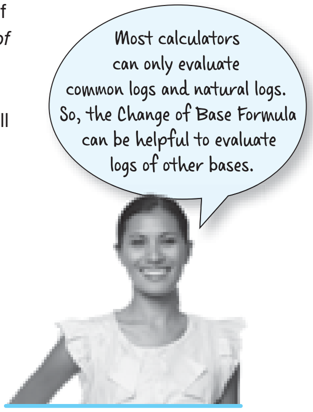
Grouping

- Ask a student to read the information and the Change of Base Formula.
- Have students complete Questions 5 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 7

- Which number is the base in the exponential equation $400 = 3^t$?
- Which number is the logarithm in the exponential equation $400 = 3^t$?
- Which number is the argument in the exponential equation $400 = 3^t$?
- Is a calculator needed to evaluate the logarithmic expression using the Change of Base Formula?
- How is a calculator used to evaluate the logarithmic expression?
- Which value is more accurate, the value resulting from the Change of Base Formula, or the value resulting from the estimation?
- Is $3^{5.4}$ or $3^{5.454}$ closer to 400?

So far, you have used estimation to determine the value of logarithms whose values were not integers. The *Change of Base Formula* allows you to calculate an exact value for a logarithm by rewriting it in terms of a different base. First, you will use the Change of Base Formula and then you will derive it.



The **Change of Base Formula** states:

$\log_b (c) = \frac{\log_a (c)}{\log_a (b)}$, where $a, b, c > 0$ and $a, b \neq 1$.

5. Rewrite the exponential equation you wrote in Question 3 as a logarithmic equation.

$400 = 3^t$

$\log_3 (400) = t$

6. Use the Change of Base Formula to evaluate the logarithmic expression using common logs. Round to the nearest thousandth.

$\log_3 (400) = x$

$\frac{\log 400}{\log 3} = x$

$5.454 \approx x$

7. Compare your estimate in Question 3 with the calculated value in Question 6 by substituting each value back into the original equation. What do you notice?

The calculated value of $\log_3 (400) \approx 5.454$ is more accurate.

The estimated value of $\log_3 (400)$ was 5.4, which yields 377.10 when checked in the equivalent exponential equation.

$3^{5.4} \overset{?}{=} 400$

$377.10 \approx 400$

The calculated value for $\log_3 (400)$ was approximately 5.454, which yields 400.15, or approximately 400, when checked in the equivalent exponential equation.

$3^{5.454} \overset{?}{=} 400$

$400.15 \approx 400$

Grouping

Have students complete Questions 8 through 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 8 through 10

- What equation is used to determine how many days it will take to have 1,000,000 subscribers?
- Can 20,000 be rewritten using a power of 3?
- How can the Change of Base Formula help to determine the number of days it will take to have 1,000,000 subscribers?
- Did Tammy perform the order of operations correctly?
- Did Tammy divide the arguments before taking the common log?
- Should Tammy have taken the common log of each argument first before dividing the arguments?
- The exponent 2.3 produces an argument between which two values?
- Does the argument of 600 fall between the same two values?
- Why should the logarithm be between the values 5 and 6?
- Is the argument 600 between the values 5 and 6? How do you know?
- What equation is used to determine how many days it will take to have 314,000,000 subscribers?

8. Use a calculator to determine how many days it will take for Bubblez Burst to reach one million subscribers.

It will take approximately 9 days to reach one million subscribers.

$$P = 50 \cdot 3^x$$
$$1{,}000{,}000 = 50 \cdot 3^x$$
$$20{,}000 = 3^x$$
$$\log_3 (20{,}000) = x$$
$$\frac{\log 20{,}000}{\log 3} = x$$
$$9.0145428 \approx x$$

9. Tammy was asked to approximate how many days it would take Bubblez Burst to reach 30,000 subscribers. Describe the calculation error Tammy made. Then, use your knowledge of estimation to explain why x could not equal 2.3.

Tammy

$$30{,}000 = 50 \cdot 3^x$$
$$600 = 3^x$$
$$\log_3 (600) = x$$
$$\frac{\log 600}{\log 3} = x$$
$$\log 200 = x$$
$$2.3 \approx x$$

Tammy's reasoning was incorrect because she performed the order of operations incorrectly. She divided the arguments before taking the common log, instead of taking the common log of each argument first.

Tammy's solution does not make sense. The exponent $x \approx 2.3$ would produce an argument between 9 and 27 because $3^2 = 9$ and $3^3 = 27$, but the argument of 600 does not fall between 9 and 27.

I know the logarithm should be between 5 and 6 because $3^5 = 243$ and $3^6 = 729$, and the argument 600 is between 243 and 729.

- Can 6,280,000 be rewritten using a power of 3?
- How can the Change of Base Formula help to determine the number of days it will take to have 314,000,000 subscribers?

10. In 2012, there were approximately 314 million people in the United States. In that year, how long would it take for everyone in the country to subscribe to Bubblez Burst?

It will take approximately 14.2 days for everyone in the United States to subscribe to Bubblez Burst.

$$\begin{aligned} P &= 50 \cdot 3^x \\ 314{,}000{,}000 &= 50 \cdot 3^x \\ 6{,}280{,}000 &= 3^x \\ \log_3 (6{,}280{,}000) &= x \\ \frac{\log 6{,}280{,}000}{\log 3} &= x \\ 14.24786588 &\approx x \end{aligned}$$

Problem 2

Students use two different strategies to solve logarithmic equations: equal bases and equal logarithms result in equal arguments, and equal logarithms and equal arguments result in equal bases. Students review and discuss student work that shows each step related to solving various exponential equations. In the last activity, students solve exponential equations and describe the reasoning behind the strategy they used in each.

Grouping

Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- If the bases are equal, are the exponents equal?
- How can 64 be rewritten as a power of 2?
- If the exponents are equal, are the bases equal?
- How can 125 be rewritten as a power of 5?
- If the bases are equal and the logarithms are equal, are the arguments equal?
- If the logarithms are equal and the arguments are equal, are the bases equal?

PROBLEM 2 Log of Both Sides

Previously you solved exponential equations with common bases as well as exponential equations with common exponents.

1. Solve each exponential equation.

a. $2^x = 64$

$2^x = 2^6$

$x = 6$

b. $y^3 = 125$

$y^3 = 5^3$

$y = 5$

2. Explain the strategy you used to solve each exponential equation in Question 1.

For $2^x = 64$, I converted 64 to a power of base 2. Because the bases are equal, I know that the exponents must also be equal.

For $y^3 = 125$, I converted 125 to a base of power 3. Because the exponents are equal, I know that the bases must also be equal.

3. Solve each logarithmic equation.

a. $\log_3 (w) = \log_3 (20)$

$w = 20$

b. $\log_m (9) = \log_4 (9)$

$m = 4$

4. Explain the strategy you used to solve each logarithmic equation in Question 3.

For $\log_3 (w) = \log_3 (20)$, because the bases are equal and the logarithms are equal, I know that the arguments must also be equal.

For $\log_m (9) = \log_4 (9)$, because the logarithms are equal and the arguments are equal, I know that the bases must also be equal.

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Grouping

- Ask a student to read the information and formulas. Discuss as a class.
- Have students complete Questions 5 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 7

- Did Todd solve for x by taking the common log of the argument divided by the common log of the base?
- Did Danielle solve for x by taking the common log of the argument divided by the common log of the base?
- Does $x = \log_a (c)$?
- Does $x = \frac{\log c}{\log a}$?
- Does $\log_a (c) = \frac{\log c}{\log a}$?

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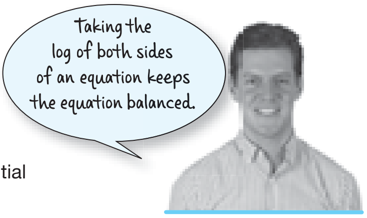
You just derived the relationship that if $\log_b (a) = \log_b (c)$, then $a = c$. The converse is also true. If $a = c$, then $\log_b (a) = \log_b (c)$. You can use this knowledge to now derive the Change of Base Formula.



5. Todd and Danielle each solved the exponential equation $4^{x-1} = 50$.

Todd

$$4^{x-1} = 50$$
$$x - 1 = \log_4 (50)$$
$$x - 1 = \frac{\log 50}{\log 4}$$
$$x - 1 \approx 2.822$$
$$x \approx 3.822$$

Danielle

$$4^{x-1} = 50$$
$$\log (4^{x-1}) = \log 50$$
$$(x - 1) \log 4 = \log 50$$
$$x - 1 = \frac{\log 50}{\log 4}$$
$$x = \frac{\log 50}{\log 4} + 1$$
$$x \approx 3.822$$

Describe how Todd's and Danielle's methods are different.

Todd solved for x by first rewriting the equation as a logarithmic equation. Then he used the Change of Base Formula.

Danielle solved for x by first taking the log of both sides. Then she used the properties of logarithms to solve.

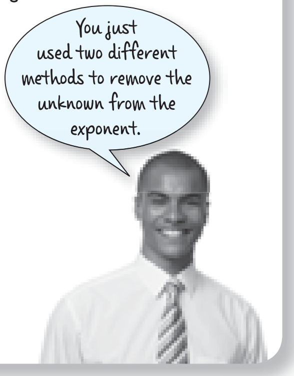
6. Consider the exponential equation $a^x = c$, where x is the unknown in the exponent and a and c are constants.

a. Solve the exponential equation for x by rewriting it in logarithmic form.

$x = \log_a (c)$

b. Solve the exponential equation for x by taking the log of both sides.

$$\log (a^x) = \log c$$
$$x \log a = \log c$$
$$x = \frac{\log c}{\log a}$$



c. How do the results from these two methods demonstrate the Change of Base Formula?

Both methods yield the same result.

I know that $x = \log_a (c)$ and $x = \frac{\log c}{\log a}$, therefore $\log_a (c) = \frac{\log c}{\log a}$.

7. Solve the exponential equation $8^x = 38.96$ using both Todd's and Danielle's methods. Round to the nearest thousandth and check your work.

$8^x = 38.96$

$\log (8^x) = \log 38.96$

$x \log 8 = \log 38.96$

$x = \frac{\log 38.96}{\log 8}$

$x \approx 1.761$

$8^x = 38.96$

$x = \log_8 (38.96)$

$x = \frac{\log 38.96}{\log 8}$

$x \approx 1.761$

Check: $8^{1.761} \overset{?}{=} 38.96$

$38.94 \approx 38.96$

Grouping

Have students complete Questions 8 and 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 8

- Which student solved the equation by converting it into a logarithmic equation first and then using the Change of Base Formula?
- Which student solved the equation by taking the log of both sides?
- Which student solved the equation by using the idea of equivalent bases and then set the exponents equal to each other?
- Whose method will only work if the equation is equal to a power of the base?

8. John, Bobbi, and Randy each solved the equation $9^{x+4} = 27$.

John

$$9^{x+4} = 27$$
$$x + 4 = \log_9 (27)$$
$$x + 4 = \frac{\log 27}{\log 9}$$
$$x + 4 = 1.5$$
$$x = -2.5$$

Bobbi

$$9^{x+4} = 27$$
$$\log (9^{x+4}) = \log 27$$
$$(x + 4)\log 9 = \log 27$$
$$x + 4 = \frac{\log 27}{\log 9}$$
$$x + 4 = 1.5$$
$$x = -2.5$$

Randy

$$9^{x+4} = 27$$
$$(3^2)^{x+4} = 3^3$$
$$3^{2(x+4)} = 3^3$$
$$2(x + 4) = 3$$
$$2x + 8 = 3$$
$$2x = -5$$
$$x = -2.5$$

a. Describe each method used.

John converted the exponential equation into a logarithmic equation and then used the Change of Base Formula.

Bobbi solved the equation by taking the log of both sides.

Randy solved the equation by using the idea of equivalent bases and setting the exponents equal to each other.

b. Will each method work for every logarithmic equation? Describe any limitations of each method.

John's method and Bobbi's method will each work for every exponential equation. Randy's method will work only if the bases are equal.

Guiding Questions for Share Phase, Question 9

- Did both Ameet and Neha use a method that changed the exponential equation to a logarithmic expression with a base which can be evaluated using a calculator?
- Which student wrote a logarithmic equation with base 10?
- Which student wrote a logarithmic equation with base *e*?
- Did both students use the Change of Base Formula to determine the approximate value of the unknown?

9. Ameet and Neha each took the logarithm of both sides to solve $24^x = 5$.

Ameet

$24^x = 5$

$\ln(24^x) = \ln 5$

$x \ln 24 = \ln 5$

$x = \frac{\ln 5}{\ln 24}$

$x \approx 0.506$

Check:

$24^x \overset{?}{=} 5$

$24^{0.506} \overset{?}{=} 5$

$4.99 \approx 5$

Neha

$24^x = 5$

$\log(24^x) = \log 5$

$x \log 24 = \log 5$

$x = \frac{\log 5}{\log 24}$

$x \approx 0.506$

Check:

$24^x \overset{?}{=} 5$

$24^{0.506} \overset{?}{=} 5$

$4.99 \approx 5$

Describe the similarities and differences in their methods. Explain why each student's method is correct.

Ameet and Neha's methods are similar in that they both rewrote $24^x = 5$ as a logarithmic equation with a base which can be evaluated using a calculator.

Their methods are different in that Ameet wrote an equivalent logarithmic expression with base *e*, whereas Neha wrote an equivalent logarithmic expression with base 10.

Each student's method is correct because they evaluated $\log_{24} 5$ using the Change of Base Formula and determined an approximate value of 0.506.

Grouping

Have students complete Questions 10 and 11 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 10 and 11

- Did Ashley perform the order of operations correctly?
- Did Ashley multiply 9 by $\frac{1}{3}$ before she worked with the exponent associated with $\frac{1}{3}$?
- Can 21 be rewritten as a power of 4?
- Can you use the Change of Base Formula in this situation?
- Can 36 be rewritten as a power of $\frac{3}{2}$?
- Can you take the log of both sides of the equation in this situation?
- Can 27 be rewritten as a power of 3?

10. Describe the error in Ashley's reasoning. Then solve the equation correctly.

Ashley

$9\left(\frac{1}{3}\right)^{2x} = 15$

$3^{2x} = 15$

$\log(3^{2x}) = \log 15$

$2x \log 3 = \log 15$

$2x = \frac{\log 15}{\log 3}$

$2x = 2.4649735 21\ldots$

$x \approx 1.232$

Ashley did not perform the order of operations correctly and multiplied 9 and $\frac{1}{3}$ together before working with the exponent on $\frac{1}{3}$.

$3^2\left(\frac{1}{3}\right)^{2x} = 15$

$3^2(3)^{-2x} = 15$

$3^{2-2x} = 15$

$2 - 2x = \log_3(15)$

$-2x = \log_3(15) - 2$

$x = \dfrac{\frac{\log 15}{\log 3} - 2}{-2}$

$x \approx -0.2325$

11. Solve each exponential equation. Explain why you chose the method that you used.

a. $4^{x-3} - 5 = 16$

$4^{x-3} = 21$

$x - 3 = \log_4(21)$

$x - 3 = \frac{\log 21}{\log 4}$

$x = \frac{\log 21}{\log 4} + 3$

$x \approx 5.20$

I used the Change of Base Formula because 21 cannot be written as a power of 4.

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b. $10 \cdot \left(\frac{3}{2}\right)^{2x} = 360$

$\left(\frac{3}{2}\right)^{2x} = 36$

$2x = \log_{\frac{3}{2}}(36)$

$2x = \frac{\log 36}{\log\left(\frac{3}{2}\right)}$

$x = \left(\frac{1}{2}\right)\frac{\log 36}{\log\left(\frac{3}{2}\right)}$

$x \approx 4.42$

I took the log of both sides because 36 cannot be written as a power of $\frac{3}{2}$.

c. $2 \cdot 3^{5x} + 1 = 55$

$2 \cdot 3^{5x} + 1 = 55$

$2 \cdot 3^{5x} = 54$

$3^{5x} = 27$

$3^{5x} = 3^3$

$5x = 3$

$x = \frac{3}{5}$

I used common bases because 27 can be rewritten as 3^3.

Be prepared to share your solutions and methods.

Check for Students' Understanding

Solve each exponential equation. Explain why you chose the method that you used.

1. $2^{x-5} + 6 = 30$

$$\begin{aligned} 2^{x-5} + 6 &= 30 \\ 2^{x-5} &= 24 \\ x - 5 &= \log_2 (24) \\ x - 5 &= \frac{\log 24}{\log 2} \\ x - 5 &\approx 4.58 \\ x &\approx 9.58 \end{aligned}$$

24 could not be rewritten as a power of 2, so I used the Change of Base Formula.

2. $7 \cdot (2)^{3x} = 840$

$$\begin{aligned} 7 \cdot (2)^{3x} &= 840 \\ (2)^{3x} &= 120 \\ 3x &= \log_2 (120) \\ 3x &= \frac{\log 120}{\log 2} \\ 3x &= 6.90689\ldots \\ x &\approx 2.30 \end{aligned}$$

120 could not be rewritten as a power of 2, so I used the Change of Base Formula.

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13.4

Logging On

Solving Logarithmic Equations

LEARNING GOALS

In this lesson, you will:

- Solve for the base, argument, and exponent of logarithmic equations.
- Solve logarithmic equations using logarithmic properties.
- Solve logarithmic equations arising from real-world situations.
- Complete a decision tree to determine efficient methods for solving exponential and logarithmic equations.

ESSENTIAL IDEAS

- It is more advantageous to solve a simple logarithmic equation by rewriting it in exponential form when the argument is unknown or the base is unknown.
- It is more advantageous to solve a simple logarithmic equation by applying the Change of Base Formula when the value of the logarithm is unknown.
- Strategies for solving logarithmic equations include: the use of properties of logarithms, setting bases or arguments equal, and converting to exponential form.
- Strategies for solving exponential equations include: taking the n^{th} root of both sides, using common bases or common exponents, taking the logarithm of both sides, and evaluating the power.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-BF Building Functions

Build new functions from existing functions

5. (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

F-LE Linear, Quadratic, and Exponential Models

Construct and compare linear, quadratic, and exponential models and solve problems

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Overview

Students solve logarithmic equations for the base, argument, or exponent by rewriting them as exponential equations or using the Change of Base Formula. Properties of logarithms are used to solve equations containing multiple logarithms, and students solve logarithmic equations in real-world contexts. A ‘decision tree’ is created describing the first step used to solve each type of exponential and logarithmic equation.

Warm Up

Use the properties of logarithms to solve each of the following.

1. Rewrite $\log_5 (7)$ using common logarithms and evaluate.

$$\log_5 (7) = \frac{\log 7}{\log 5} \approx 1.209$$

2. Rewrite $\log_3 (8x^4)$ using multiple logarithms.

$$\log_3 (8x^4) = \log_3 (8) + \log_3 (x)$$

3. Rewrite $\log_2 \left(\frac{5y^6}{x^4}\right)$ using multiple logarithms.

$$\log_2 \left(\frac{5y^6}{x^4}\right) = \log_2 (5) + 6 \log_2 (y) - 4 \log_2 (x)$$

4. Rewrite $2 \log 11 - 2 \log 5$ using a single logarithm and evaluate.

$$\begin{aligned} 2 \log 11 - 2 \log 5 &= \log \left(\frac{11^2}{5^2}\right) \\ &= \log 4.84 \\ &\approx 0.6848 \end{aligned}$$

13.4

Logging On

Solving Logarithmic Equations

LEARNING GOALS

In this lesson, you will:

- Solve for the base, argument, and exponent of logarithmic equations.
- Solve logarithmic equations using logarithmic properties.
- Solve logarithmic equations arising from real-world situations.
- Complete a decision tree to determine efficient methods for solving exponential and logarithmic equations.

Have you ever been asked to change your password?

Password strength is a measure of the effectiveness of a password in resisting guessing and brute-force attacks. In its usual form, it is an estimate of how many trials an attacker who does not have direct access to the password would need, on average, to guess it correctly. The strength of a password is a function of its length, complexity, and unpredictability.

Passwords are there to protect your private information, so make sure your password follows some common guidelines:

- A minimum password length of 12 to 14 characters, if permitted.
- Avoid passwords that use repetition, dictionary words, letter or number patterns, usernames, relative or pet names, or other biographical information.
- Include numbers, symbols, and capital and lowercase letters if allowed by the system.
- Avoid using the same password for multiple sites.

Can you recognize the difference between a strong and a weak password?

Problem

The formula for the decibel level, or loudness of sound, is modeled by a logarithmic equation. Students use the equation to solve for unknowns, such as the decibel level of a quiet library, the decibel level for city traffic, and the intensity of the sound of a baby crying. Students complete a chart by solving logarithmic equations two different ways when the argument is unknown, the exponent is unknown, and the base is unknown. Students conclude that it is more advantageous to solve a simple logarithmic equation by rewriting it in exponential form when the argument is unknown or the base is unknown. And when the value of the logarithm is unknown, it is more advantageous to apply the Change of Base Formula to solve the equation. Students choose the best method to solve several simple logarithmic equations.

Grouping

- Ask a student to read the information and formula. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- What equation is used to determine the decibel level of a quiet library?
- Is the base, argument, or exponent unknown in this situation?
- How is the number 1000 used in the equation?
- What happens to the quantities I_0 in the numerator and denominator of the log?
- What equation is used to determine the decibel level of city traffic?
- Is the base, argument, or exponent unknown in this situation?
- How is the number 500,000,000 used in the equation?
- What happens to the quantities I_0 in the numerator and denominator of the log?

PROBLEM 1 Keep It Down!

A decibel is a unit used to measure the loudness of sound. The formula for the loudness of a sound is given by

$$\text{dB} = 10 \log \left(\frac{I}{I_0}\right)$$

where dB is the decibel level. The quantity I_0 is the intensity of the threshold sound—a sound that can barely be perceived. The quantity I is the number of times more intense a sound is than the threshold sound.

1. The sound in a quiet library is about 1000 times as intense as the threshold sound: $I = 1000I_0$. Calculate the decibel level of a quiet library.

A quiet library has a decibel level of 30.

$$\begin{aligned} \text{dB} &= 10 \log \left(\frac{I}{I_0}\right) \\ &= 10 \log \left(\frac{1000I_0}{I_0}\right) \\ &= 10 \log 1000 \\ &= 30 \end{aligned}$$

2. The sound of traffic on a city street is calculated to be 500 million times as intense as the threshold sound. Calculate the decibel level for city traffic.

City traffic has a decibel level of approximately 87.

$$\begin{aligned} \text{dB} &= 10 \log \left(\frac{I}{I_0}\right) \\ &= 10 \log \left(\frac{500{,}000{,}000I_0}{I_0}\right) \\ &= 10 \log 500{,}000{,}000 \\ &\approx 86.9 \end{aligned}$$

Guiding Questions for Share Phase, Question 3

- What equation is used to determine the intensity of the sound of a crying baby?
- Is the base, argument, or exponent unknown in this situation?
- How is the value of 115 decibels used in the equation?

3. The sound of a crying baby registers at 115 decibels. How many times more intense is this sound than the threshold sound?

The sound of a crying baby is more than 300,000,000,000 times as intense as the threshold sound.

$$\text{dB} = 10 \log\left(\frac{I}{I_0}\right)$$

$$115 = 10 \log\left(\frac{I}{I_0}\right)$$

$$11.5 = \log\left(\frac{I}{I_0}\right)$$

$$10^{11.5} = \frac{I}{I_0}$$

$$I \approx 3.16 \times 10^{11}\ I_0$$

Each equation you wrote in Questions 1 through 3 is a logarithmic equation. To solve for an unknown in a simple logarithmic equation, you consider the relationship between the base, argument, and exponent.

Previously, you developed two strategies to solve for any unknown in a simple logarithmic equation:

- Rewrite the logarithmic equation as an exponential equation.
- Apply the Change of Base Formula.

When solving equations with a certain method, it is important to consider the number of steps involved. The more steps there are, the more chances you have to introduce an error in your calculations. Methods with fewer steps can be more accurate and efficient.

Grouping

Have students complete Questions 4 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 4

- Which number represents the base in the example $\log_5 (x) = 3.1$?
- Which number represents the log in the example $\log_5 (x) = 3.1$?
- Which number represents the argument in the example $\log_5 (x) = 3.1$?
- Is the base, log, or argument unknown in this situation?
- Is a calculator needed to evaluate the logarithmic expression after using the Change of Base Formula?
- Which value is more accurate, the value resulting from the Change of Base Formula, or the value resulting from the exponential equation?
- Is it easier to use the exponential equation or the Change of Base Formula in this situation?
- Which number represents the base in the example $\log_8 (145) = x$?
- Which number represents the log in the example $\log_8 (145) = x$?
- Which number represents the argument in the example $\log_8 (145) = x$?
- Is the base, log, or argument unknown in this situation?
- Is it easier to use the exponential equation or the Change of Base Formula in this situation?
- Which number represents the base in the example $\log_x (24) = 6.7$?
- Which number represents the log in the example $\log_x (24) = 6.7$?
- Which number represents the argument in the example $\log_x (24) = 6.7$?
- Is the base, log, or argument unknown in this situation?
- Is it easier to use the exponential equation or the Change of Base Formula in this situation?

4. Solve each logarithmic equation using two different methods.

	Example	First rewrite as an exponential equation. Then solve for *x*.	First apply the Change of Base Formula. Then solve for *x*.
Argument Is Unknown	$\log_5 (x) = 3.1$	$5^{3.1} = x$ $146.827 \approx x$	$\frac{\log x}{\log 5} = 3.1$ $\log x \approx 2.1668$ $10^{2.1668} \approx x$ $146.827 \approx x$
Exponent Is Unknown	$\log_8 (145) = x$	$8^x = 145$ $\log (8^x) = \log 145$ $x \log 8 = \log 145$ $x = \frac{\log 145}{\log 8}$ $x \approx 2.3933$	$\frac{\log 145}{\log 8} = x$ $2.3933 \approx x$
Base Is Unknown	$\log_x (24) = 6.7$	$x^{6.7} = 24$ $(x^{6.7})^{\frac{1}{6.7}} = (24)^{\frac{1}{6.7}}$ $x \approx 1.6069$	$\frac{\log 24}{\log x} = 6.7$ $\log 24 = 6.7 \log x$ $0.2060 \approx \log x$ $10^{0.2060} \approx x$ $1.6069 \approx x$

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Guiding Questions for Share Phase, Questions 5 through 7

- When the argument is unknown, why is rewriting the equation as an exponential equation a preferred method?
- When the exponent is unknown, why is applying the Change of Base Formula a preferred method?
- When the base is unknown, why is rewriting the equation as an exponential equation a preferred method?

5. Consider the position of the unknown for each logarithmic equation in Question 4. Circle your preferred method for solving. Explain your choice.

When the argument is unknown, rewriting as an exponential equation is my preferred method because the expression can be evaluated using a calculator.

When the exponent is unknown, applying the Change of Base Formula is my preferred method because the expression can be evaluated using a calculator.

When the exponent is unknown, rewriting as an exponential equation is my preferred method. You can take the n^{th} root of each side to solve for the unknown and evaluate using a calculator.

6. When might it be more efficient to solve a simple logarithmic equation by rewriting it in exponential form?

It is more advantageous to solve a simple logarithmic equation by rewriting it in exponential form when the argument is unknown or the base is unknown.

7. When might it be more efficient to solve a simple logarithmic equation by applying the Change of Base Formula?

It is more advantageous to solve a simple logarithmic equation by applying the Change of Base Formula when the value of the logarithm is unknown.

Grouping

Have students complete all parts of Question 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 8

- Is the base, log, or argument unknown in this situation?
- If the base or argument is unknown, which method is preferred?
- If the exponent is unknown, which method is preferred?

8. Circle the logarithmic equations that can be solved more efficiently when rewritten as exponential equations. Box the equations that can be solved more efficiently by applying the Change of Base Formula. Explain your choice.

a. $\log_4 (x + 3) = \frac{1}{2}$

Because the argument is unknown, rewriting as an exponential equation is the preferred method.

b. $\boxed{\log_{4.5} (9) = x - 1}$

Because the exponent is unknown, applying the Change of Base Formula is the preferred method.

c. $\log_{x+2} (7.1) = 3$

Because the base is unknown, rewriting as an exponential equation is the preferred method.

d. $\boxed{\log_3 (4.6) = 2 - x}$

Because the exponent is unknown, applying the Change of Base Formula is the preferred method.

e. $\ln (x + 4) = 3.8$

Because the argument is unknown, rewriting as an exponential equation is the preferred method.

f. $\boxed{\log_{11} (12) = x - 7}$

Because the exponent is unknown, applying the Change of Base Formula is the preferred method.

g. $\log_{1-x} (8) = 14.7$

Because the base is unknown, rewriting as an exponential equation is the preferred method.

h. $\log (4 - x) = 1.3$

Because the argument is unknown, rewriting as an exponential equation is the preferred method.

Grouping

Have students complete Question 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 9

- Do you move all of the terms to one side of the equation?
- Is the quadratic equation factorable?
- Which factors are associated with this quadratic equation?

9. Solve each logarithmic equation. Check your work.

a. $\log_2 (x^2 - 6x) = 4$

$2^4 = x^2 - 6x$

$16 = x^2 - 6x$

$0 = x^2 - 6x - 16$

$0 = (x - 8)(x + 2)$

$x = 8, -2$

Check: $\log_2 (8^2 - 6 \cdot 8) \overset{?}{=} 4$

$\log_2 (16) = 4$

$\log_2 ((-2)^2 - 6(-2)) \overset{?}{=} 4$

$\log_2 (16) = 4$

b. $\log_6 (x^2 + x) = 1$

$6^1 = x^2 + x$

$6 = x^2 + x$

$0 = x^2 + x - 6$

$0 = (x - 2)(x + 3)$

$x = 2, -3$

Check: $\log_6 (2^2 + 2) \overset{?}{=} 1$

$\log_6 (6) = 1$

$\log_6 ((-3)^2 + (-3)) \overset{?}{=} 1$

$\log_6 (6) = 1$

Problem 2

The formula for the amount of medication left in a patient's body is modeled by a logarithmic equation. Students use the equation to solve for unknowns such as the time it takes for a certain amount of medicine to remain in the body, the rate at which the medication leaves the body, and the amount of medicine that needs to be administered at the start of surgery. Properties of logarithms are used in logarithmic equations containing multiple logarithms to rewrite the equation to a familiar form. Students review student work to identify errors in solution paths and solve equations with multiple logarithms.

Grouping

- Ask a student to read the information and formula. Discuss as a class.
- Have students complete Questions 1 through 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- What equation is used to determine the amount of time it will take for 2 mg of the medicine to remain in the patient's body?
- How is the value 10 mg used in the equation?
- How is 20% used in the equation?
- What strategy was used to solve this problem?
- What equation is used to determine rate at which the medicine is leaving the patient's body?
- How is the value 6 hours used in the equation?
- How is 20 mg used in the equation?
- How is 5 mg used in the equation?
- The decimal 0.206 is equivalent to what percent?
- What strategy was used to solve this problem?

PROBLEM 2 Doctor, Doctor!

While most medications have guidelines for dosage amounts, doctors must also determine how long a medication will last when they write prescriptions. The amount of medicine left in a patient's body can be predicted by the formula

$$t = \frac{\log\left(\frac{C}{A}\right)}{\log(1 - r)}$$

where t is the time in hours since the medicine was administered, C is the current amount of medicine left in the patient's body in milligrams, A is the original dose of the medicine in milligrams, and r is the rate at which the medicine leaves the body.

1. A patient is given 10 milligrams of medicine which leaves the body at the rate of 20% per hour. How long will it take for 2 milligrams of the medicine to remain in the patient's body?

It will take about 7.2 hours for 2 milligrams of the medicine to remain in the patient's body.

$$t = \frac{\log\left(\frac{C}{A}\right)}{\log(1 - r)}$$

$$= \frac{\log\left(\frac{2}{10}\right)}{\log(1 - 0.20)}$$

$$t \approx 7.2$$

2. Six hours after administering a 20-milligram dose of medicine, 5 milligrams remain in a patient's body. At what rate is the medicine leaving the body?

The medicine is leaving the body at a rate of about 20.6% per hour.

$$t = \frac{\log\left(\frac{C}{A}\right)}{\log(1 - r)}$$

$$6 = \frac{\log\left(\frac{5}{20}\right)}{\log(1 - r)}$$

$$\log(1 - r) \approx -0.100$$

$$10^{-0.100} \approx 1 - r$$

$$r \approx 1 - 10^{-0.100}$$

$$r \approx 0.206$$

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Guiding Questions for Share Phase, Question 3

- What equation is used to determine the amount of medicine that must be administered at the start of surgery?
- How is the value 8 hours used in the equation?
- How is 4 mg used in the equation?
- How is 15% used in the equation?
- What strategy was used to solve this problem?

3. A patient is undergoing an 8-hour surgery. If 4 milligrams of medicine must be left in the patient's body at the end of surgery, and the medicine leaves the body at the rate of 15% per hour, how much medicine must be administered at the start of surgery?

The patient should receive about 14.7 milligrams of medicine before the 8-hour surgery.

$$t = \frac{\log\left(\frac{C}{A}\right)}{\log(1 - r)}$$

$$8 = \frac{\log\left(\frac{4}{A}\right)}{\log(1 - 0.15)}$$

$$\log\left(\frac{4}{A}\right) \approx -0.5646$$

$$10^{-0.5646} \approx \frac{4}{A}$$

$$A \approx \frac{4}{10^{-0.5646}}$$

$$A \approx 14.68$$

Grouping

Have students complete Questions 4 and 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 and 5

- Did Georgia correctly combine the logarithms on the left side of the equation?
- Did Santiago correctly combine the logarithms on the left side of the equation?
- Did Lorenzo correctly combine the logarithms on the left side of the equation?
- Which student used the Power Rule of Logarithms followed by converting the equation into exponential form to solve the equation?
- Which method was used to solve the equation with multiple logs?
- What rule or property can you use to combine the logarithms in this situation?
- What strategy can you use to solve this equation?

If a logarithmic equation involves multiple logarithms, you can use the properties of logarithms to rewrite the equation in a form you already know how to solve.

4. Analyze Georgia's, Santiago's, and Lorenzo's work.

Georgia

$\log 5 + \log x = 2$

$\log (5 + x) = 2$

$10^2 = 5 + x$

$95 = x$

Santiago

$\log 5 + \log x = 2$

$5x = 2$

$x = \frac{5}{2}$

Lorenzo

$\log 5 + \log x = 2$

$5 + x = 2$

$x = -3$

a. Explain the error in each student's reasoning.

Georgia, Santiago, and Lorenzo incorrectly combined the logarithms on the left side of the equation. Each student should have used the Power Rule of Logarithms to write log 5 + log x as log(5x) and then converted to exponential form to solve.

b. Solve log 5 + log x = 2. Check your work.

$\log 5 + \log x = 2$

$\log (5x) = 2$

$10^2 = 5x$

$100 = 5x$

$20 = x$

Check: $\log 5 + \log 20 \overset{?}{=} 2$

$\log 100 = 2$

5. Solve each logarithmic equation. Check your work.

a. $\log_5 (45x) - \log_5 (3) = 1$

$\log_5 (45x) - \log_5 (3) = 1$

$\log_5 \left(\frac{45x}{3}\right) = 1$

$5^1 = 15x$

$5 = 15x$

$\frac{1}{3} = x$

Check: $\log_5 \left(45 \cdot \frac{1}{3}\right) - \log_5 (3) \overset{?}{=} 1$

$\log_5 (5) = 1$

Don't forget the restrictions on the variables for the logarithmic equation, $y = \log_b x$. The variable y can be any real number, the base b must be greater than 0 and not equal to 1, and the argument x must be greater than zero.

b. $\log_2 (8) + 3 \log_2 (x) = 6$

$\log_2 (8) + \log_2 (x^3) = 6$

$\log_2 (8x^3) = 6$

$2^6 = 8x^3$

$64 = 8x^3$

$8 = x^3$

$x = 2$

Check: $\log_2 (8) + \log_2 (2^3) \overset{?}{=} 6$

$\log_2 (8 \cdot 2^3) \overset{?}{=} 6$

$\log_2 (64) \overset{?}{=} 6$

$2^6 \overset{?}{=} 64$

$64 = 64$

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Grouping

Have students complete Questions 6 through 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 6 and 7

- What is the difference between Pippa's solution and Kate's solution?
- Does the argument of the original logarithmic equation involve *x* or involve x^2? How does this affect possible values of *x*?
- Which method can you use to solve the equation with multiple logarithms?
- What rule or property can you use to combine the logs in this situation?
- What strategy can you use to solve this equation?
- Is the quadratic equation factorable?
- Is there an extraneous log in this situation? How do you know?

6. Pippa and Kate disagree about the solution to the logarithmic equation $\log_5 x^2 - \log_5 4 = 2$.

Kate

$\log_5 (x^2) - \log_5 (4) = 2$

$\log_5 \left(\frac{x^2}{4}\right) = 2$

$5^2 = \frac{x^2}{4}$

$25 = \frac{x^2}{4}$

$100 = x^2$

$x = 10, -10$

Pippa

$\log_5 (x^2) - \log_5 (4) = 2$

$\log_5 \left(\frac{x^2}{4}\right) = 2$

$5^2 = \frac{x^2}{4}$

$25 = \frac{x^2}{4}$

$100 = x^2$

$x = 10, \cancel{-10}$

Reject −10 because the argument of a logarithm must be greater than zero.

Who is correct? Explain your reasoning.

Kate is correct. Both −10 and 10 are solutions of the original logarithmic equation because the argument is x^2, which is always positive.

Check: $\log_5 (x^2) - \log_5 (4) \stackrel{?}{=} 2$

$\log_5 (10^2) - \log_5 (4) \stackrel{?}{=} 2$

$\log_5 (100) - \log_5 (4) \stackrel{?}{=} 2$

$\log_5 (25) \stackrel{?}{=} 2$

$5^2 \stackrel{?}{=} 25$

$25 = 25$

$\log_5 ((-10)^2) - \log_5 (4) \stackrel{?}{=} 2$

$\log_5 (100) - \log_5 (4) \stackrel{?}{=} 2$

$\log_5 (25) \stackrel{?}{=} 2$

$5^2 \stackrel{?}{=} 25$

$25 = 25$

7. Solve $\log_3 (x - 4) + \log_3 (x + 2) = 3$. Check your work.

$\log_3 (x - 4) + \log_3 (x + 2) = 3$

$\log_3 (x^2 - 2x - 8) = 3$

$3^3 = x^2 - 2x - 8$

$27 = x^2 - 2x - 8$

$0 = x^2 - 2x - 35$

$0 = (x - 7)(x + 5)$

$x = 7, \cancel{-5}$

Check: $\log_3 (7 - 4) + \log_3 (7 + 2) \stackrel{?}{=} 3$

$\log_3 (3) + \log_3 (9) \stackrel{?}{=} 3$

$\log_3 (27) \stackrel{?}{=} 3$

$3^3 \stackrel{?}{=} 27$

$27 = 27$

$\log_3 (-5 + 3) + \log_3 (-5) = 3$

extraneous log undefined for −5

Guiding Questions for Share Phase, Questions 8 and 9

- Which student applied the properties of logarithms on the right side of the equation, then set the arguments of each equivalent logarithm equal to each other?
- Which student set one side of the equation equal to 0, applied the properties of logarithms on the left side of the equation, and then converted to exponential form?
- Which method can you use to solve the equation with multiple logarithms?
- What rule or property can you use to combine the logs in this situation?
- What strategy can you use to solve this equation?

8. Elijah and Zander each solved the logarithmic equation $2 \log 6 = \log x - \log 2$.

Elijah

$2 \log 6 = \log x - \log 2$

$\log (6^2) = \log \left(\frac{x}{2}\right)$

$36 = \frac{x}{2}$

$72 = x$

Check: $2 \log 6 \overset{?}{=} \log 72 - \log 2$

$\log (6^2) \overset{?}{=} \log \left(\frac{72}{2}\right)$

$\log 36 = \log 36$

Zander

$2 \log 6 = \log x - \log 2$

$\log (6^2) - \log x + \log 2 = 0$

$\log \left(\frac{36}{x}\right) + \log 2 = 0$

$\log \left(\frac{72}{x}\right) = 0$

$10^0 = \frac{72}{x}$

$1 = \frac{72}{x}$

$x = 72$

Check: $2 \log 6 \overset{?}{=} \log 72 - \log 2$

$\log (6^2) \overset{?}{=} \log \left(\frac{72}{2}\right)$

$\log 36 = \log 36$

Explain why each student's method is correct.

Elijah is correct because he applied the properties of logarithms on the right side of the equation and then set the arguments of each equivalent logarithm equal to each other to solve.

Zander is correct because he set one side of the equation equal to 0, applied the properties of logarithms on the left side of the equation, and then converted to exponential form to solve.

9. Solve each logarithmic equation. Check your work.

a. $2 \log (x + 1) = \log x + \log (x + 3)$

$2 \log (x + 1) = \log x + \log (x + 3)$

$\log ((x + 1)^2) = \log (x(x + 3))$

$(x + 1)^2 = x(x + 3)$

$x^2 + 2x + 1 = x^2 + 3x$

$1 = x$

Check: $2 \log (1 + 1) \overset{?}{=} \log 1 + \log (1 + 3)$

$\log ((2)^2) \overset{?}{=} \log (1(4))$

$\log 4 = \log 4$

b. $2 \log (x - 3) = \log 4 + \log \left(x - \frac{15}{4}\right)$

$2 \log (x - 3) = \log 4 + \log \left(x - \frac{15}{4}\right)$

$\log ((x - 3)^2) = \log (4x - 15)$

$(x - 3)^2 = 4x - 15$

$x^2 - 6x + 9 = 4x - 15$

$x^2 - 10x + 24 = 0$

$(x - 6)(x - 4) = 0$

$x = 4, 6$

Check: $2 \log (4 - 3) \overset{?}{=} \log 4 + \log \left(4 - \frac{15}{4}\right)$

$\log (1^2) \overset{?}{=} \log \left(4\left(\frac{1}{4}\right)\right)$

$\log 1 \overset{?}{=} \log 1$

$2 \log (6 - 3) \overset{?}{=} \log 4 + \log \left(6 - \frac{15}{4}\right)$

$\log (3^2) \overset{?}{=} \log \left(4\left(\frac{9}{4}\right)\right)$

$\log 9 = \log 9$

c. $\ln (x - 3) + \ln (x - 2) = \ln (2x + 24)$

$\ln (x - 3) + \ln (x - 2) = \ln (2x + 24)$

$\ln ((x - 3)(x - 2)) = \ln (2x + 24)$

$x^2 - 5x + 6 = 2x + 24$

$x^2 - 7x - 18 = 0$

$(x - 9)(x + 2) = 0$

$x = 9, \cancel{-2}$

Check: $\ln ((-2) - 3) + \ln ((-2) - 2) \overset{?}{=}$

$\ln (2(-2) + 24)$

extraneous ln undefined for -2

$\ln (9 - 3) + \ln (9 - 2) \overset{?}{=} \ln (2(9) + 24)$

$\ln 6 + \ln 7 \overset{?}{=} \ln 42$

$\ln 42 = \ln 42$

d. $2 \log_3 x = \log_3 4 + \log_3 16$

$2 \log_3 x = \log_3 4 + \log_3 16$

$\log_3 (x^2) = \log_3 (4(16))$

$x^2 = 64$

$x = 8$

Check: $2 \log_3 8 \overset{?}{=} \log_3 4 + \log_3 16$

$2 \log_3 8 \overset{?}{=} \log_3 64$

$\log_3 (8^2) \overset{?}{=} \log_3 64$

$\log_3 64 = \log_3 64$

Talk the Talk

A list containing an example of each type of exponential and logarithmic equation students have solved is provided. Students use the list to complete a 'decision tree' which demonstrates the most advantageous strategy for solving each type of equation and describes the first step students would use to solve each equation.

Grouping

Have students complete the decision tree with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Talk the Talk

- Which example has a first step in the solution associated with the use of the properties of logarithms?
- Which example has a first step in the solution associated with setting the bases equal or setting the arguments equal?
- Which example has a first step in the solution associated with converting to exponential form?
- Which example has a first step in the solution associated with taking the *n*th root of both sides?
- Which example has a first step in the solution associated with using common bases or common exponents?
- Which example has a first step in the solution associated with evaluating the power?

Talk the Talk

You have solved a variety of exponential and logarithmic equations. When doing so, you have had to consider the structure and characteristics of each equation.

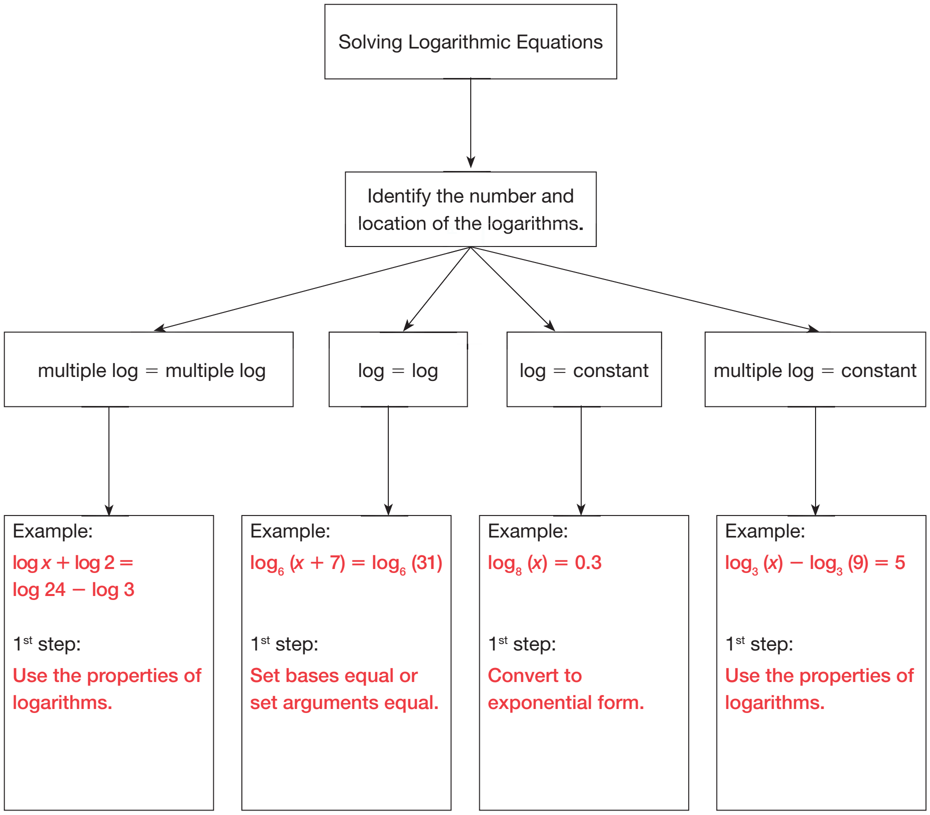
The list shown contains an example of each type of exponential and logarithmic equation you have solved.

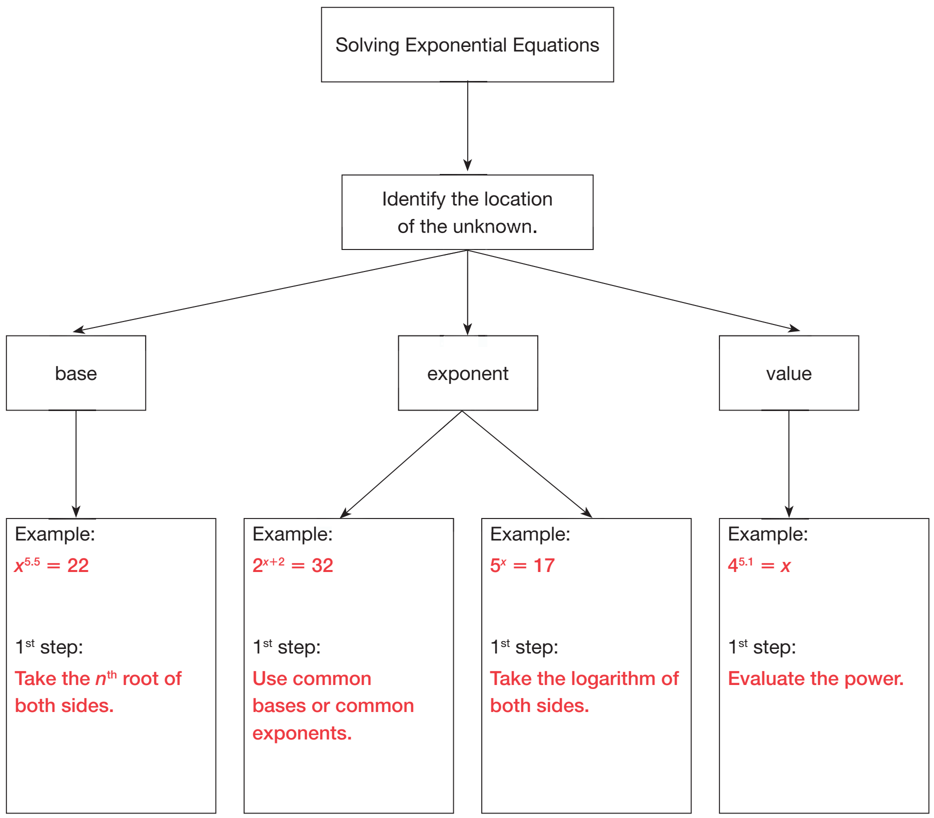
- $2^{x+2} = 32$
- $\log_6 (x + 7) = \log_6 31$
- $4^{5.1} = x$
- $\log_3 (x) - \log_3 (9) = 5$
- $5^x = 17$
- $\log x + \log 2 = \log 24 - \log 3$
- $x^{5.5} = 22$
- $\log_8 (x) = 0.3$

1. Complete the decision tree on the following pages to demonstrate the most advantageous strategy to solve each type of equation.
   a. For each branch of the decision tree, write the appropriate example equation from the list.
   b. Describe the first step you would use to solve each equation.

      See decision tree.

      Answers will vary.


Solving Logarithmic Equations
Identify the number and location of the logarithms.
multiple log = multiple log
log = log
log = constant
multiple log = constant
Example:
$\log x + \log 2 = \log 24 - \log 3$
1st step:
Use the properties of logarithms.
Example:
$\log_6 (x + 7) = \log_6 (31)$
1st step:
Set bases equal or set arguments equal.
Example:
$\log_8 (x) = 0.3$
1st step:
Convert to exponential form.
Example:
$\log_3 (x) - \log_3 (9) = 5$
1st step:
Use the properties of logarithms.



Be prepared to share your solutions and methods.

Check for Students' Understanding

Create an example problem to fit each first step.

The first step in the solution path is to:

1. Use common bases or common exponents.

   Answers will vary.

   $2^{x-3} = 128$

2. Convert to exponential form.

   Answers will vary.

   $\log_5(x) = 68$

3. Evaluate the power.

   Answers will vary.

   $2^{6.4} = x$

4. Take the logarithm of both sides.

   Answers will vary.

   $8^x = 510$

13.5

So When Will I Use This?

Applications of Exponential and Logarithmic Equations

LEARNING GOALS

In this lesson, you will:

- Use exponential models to analyze problem situations.
- Use logarithmic models to analyze problem situations.

ESSENTIAL IDEAS

- Exponential and logarithmic equations are used to model situations in the real world.

COMMON CORE STATE STANDARDS FOR MATHEMATICS

F-BF Building Functions

Build new functions from existing functions

5. (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

F-LE Linear, Quadratic, and Exponential Models

Construct and compare linear, quadratic, and exponential models and solve problems

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Overview

Exponential and logarithmic equations are used to model situations. Applications of exponential equations and logarithmic equations include: using the pH formula to analyze the type of crops that would grow best in a certain type of soil, using the continuous population growth formula to project the growth of online fan bases, and using a formula derived from Newton's Law to determine the elapsed time since a person died in hours.

Warm Up

The formula for the loudness of a sound is given by

$$dB = 10 \log \left(\frac{I}{I_0}\right)$$

where dB is the decibel level, I_0 is the intensity of the threshold sound—a sound that can barely be perceived, and I is the number of times more intense the sound is than the threshold sound.

It takes a sound of approximately 45 decibels to wake a sleeping person. This sound is how many times more intense than the threshold sound?

$$dB = 10 \log \left(\frac{I}{I_0}\right)$$

$$45 = 10 \log \left(\frac{I}{I_0}\right)$$

$$4.5 = \log \left(\frac{I}{I_0}\right)$$

$$10^{4.5} = \left(\frac{I}{I_0}\right)$$

$$I = 31{,}622.7766 I_0$$

The sound it takes to wake a sleeping person is about 31,623 times more intense than the threshold sound.

13.5

So When Will I Use This?

Applications of Exponential and Logarithmic Equations

LEARNING GOALS

In this lesson, you will:

- Use exponential models to analyze problem situations.
- Use logarithmic models to analyze problem situations.

Crime investigators use logarithmic equations to estimate a body's time of death based on two temperature readings of the body. Specifically, investigators use what's known as Newton's Law of Cooling, which states that an object cools down at a rate that is proportional to the temperature difference between the object and the environment.

Coroners—government officials who are responsible for verifying deaths—often use a rule of thumb to estimate the time of death: subtract 2 degrees from normal body temperature for the first hour after death and then 1 degree for each hour after that.

Problem 1

The formula for calculating a pH level is modeled by a logarithmic equation. Students use the equation to solve for unknowns such as the pH level of orange juice, baking soda, and the concentration of hydrogen ions in vinegar and lime water. Optimal pH soil levels for a variety of fruits and vegetables are given and students use this information to list the crops a farmer should grow, determine the soil's optimal hydrogen ion range for growing onions, and decide if a farmer's choice of crops is feasible.

Grouping

- Ask a student to read the information and formula. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4

- What equation is used to determine the solution?
- Is the base, argument, or exponent unknown in this situation?

PROBLEM 1 Efficient pHarming

The pH scale is a scale for measuring the acidity or alkalinity of a substance, which is determined by the concentration of hydrogen ions. The formula for pH is

$$pH = -\log H^+$$

where H^+ is the concentration of hydrogen ions in moles per cubic liter. Solutions with a pH value less than 7 are acidic. Solutions with a pH value greater than 7 are alkaline. Solutions with a pH of 7 are neutral. For example, plain water has a pH of 7.

1. The H^+ concentration in orange juice is 0.000199 mole per cubic liter. Determine the pH level of orange juice, and then state whether it is acidic or alkaline.

$pH = -\log H^+$

$pH = -\log 0.000199$

$pH \approx 3.701$

The pH level of orange juice is about 3.7.

The pH level is less than 7, so the orange juice is acidic.

2. The concentration of hydrogen ions in baking soda is 5.012×10^{-9} mole per cubic liter. Determine the pH level of baking soda, and then state whether it is acidic or alkaline.

$pH = -\log H^+$

$pH = -\log (5.012 \times 10^{-9})$

$pH \approx 8.3$

The pH level of baking soda is about 8.3.

The pH level is greater than 7, so baking soda is alkaline.

3. Vinegar has a pH of 2.2. Determine the concentration of hydrogen ions in vinegar.

$pH = -\log H^+$

$2.2 = -\log H^+$

$10^{-2.2} = H^+$

$H^+ \approx 0.0063$

The concentration of hydrogen ions in vinegar is about 0.0063 mole per cubic liter.

4. Lime water has a pH of 12. Determine the concentration of hydrogen ions in lime water.

$$\text{pH} = -\log \text{H}^+$$
$$12 = -\log \text{H}^+$$
$$10^{-12} = \text{H}^+$$
$$\text{H}^+ = 0.000000000001$$

The concentration of hydrogen ions in lime water is about 0.000000000001 mole per cubic liter.

The pH level of soil is used to determine which plants will grow best in an area. Different types of plants and vegetables require varying degrees of soil acidity. Generally, the soil tends to be acidic in moist climates and alkaline in dry climates.

The chart shows the optimal pH soil levels for a variety of fruits and vegetables.

Asparagus 6.5 – 7.5	**Avocados** 6.0 – 7.0	**Beets** 5.6 – 6.6
Carrots 5.0 – 6.0	**Garlic** 5.0 – 6.0	**Lettuce** 6.5 – 7.0
Mushrooms 7.0 – 8.0	**Onions** 6.2 – 6.8	**Peanuts** 5.0 – 6.0
Peppers 6.0 – 8.0	**Potatoes** 5.8 – 6.5	**Raspberries** 6.0 – 6.5
Spinach 5.0 – 7.0	**Sweet Corn** 6.0 – 7.0	**Yams** 6.0 – 8.0

5. A farmer measures the hydrogen ion concentration of the soil to be 6×10^{-7} mole per cubic liter. List the crops that the farmer can grow in this type of soil.

$$\text{pH} = -\log (6 \times 10^{-7})$$
$$\text{pH} \approx 6.2$$

The farmer can grow avocados, beets, onions, peppers, potatoes, raspberries, spinach, sweet corn, and yams in this soil.

Grouping

- Ask a student to read the information about the pH level of soil. Discuss the chart as a class.
- Have students complete Questions 5 through 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 7

- What equation is used to determine the pH level of the soil?
- Is the base, argument, or exponent unknown in this situation?
- How was the pH level of the soil used to determine the list of crops?
- What equations are used to determine the soil's optimal hydrogen ion range for growing onions?
- Is the base, argument, or exponent unknown in this situation?
- What equation is used to determine which crops were feasible?
- Is the base, argument, or exponent unknown in this situation?
- Why won't beets grow in the farmer's garden?
- Can the farmer plant garlic?
- Can the farmer plant peanuts?

6. Determine the range of the soil's optimal hydrogen ion concentration for growing onions.

The optimal range of hydrogen ion concentration in the soil is 1.58×10^{-7} to 6.31×10^{-7} mole per cubic liter.

Onions grow best in a soil with pH levels between 6.2 and 6.8.

$$\begin{aligned} 6.2 &= -\log H^+ \\ -6.2 &= \log H^+ \\ 10^{-6.2} &= \log H^+ \\ 0.000000631 &\approx H^+ \end{aligned}$$

$$\begin{aligned} 6.8 &= -\log H^+ \\ -6.8 &= \log H^+ \\ 10^{-6.8} &= \log H^+ \\ 0.000000158 &\approx H^+ \end{aligned}$$

7. The farmer wants to plant spinach, carrots, and beets. She measures the hydrogen ion concentration of the soil to be 3.16×10^{-6} mole per cubic liter. Is her plan feasible? Explain your reasoning.

Spinach and carrots will grow in this soil, but beets will not. She can plant garlic or peanuts instead.

$\text{pH} = -\log(0.00000316)$

$\text{pH} \approx 5.5$

13

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Problem 2

The formula for calculating continuous population growth is modeled by an exponential equation. Students use the model to create additional exponential equations that calculate the number of online followers after a specified amount of time, how long it takes to get a specified number of followers, or the rate at which the number of followers grows. Fan bases include a boy band, a pro-football quarterback, a pro-football running back, and a talk show host.

Grouping

Have students complete Questions 1 through 6 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 and 2

- Which variable in the model for continuous population growth represents the initial number of followers?
- Which variable in the model for continuous population growth represents the rate of growth?
- Which variable in the model for continuous population growth represents the time in days?

PROBLEM 2 Follow Me!

Gina, a social media manager, uses the model for continuous population growth to project the monthly increase in the number of followers on her clients' social media sites.

1. Recall that the formula for continuous population growth is $N(t) = N_0e^{rt}$. State what each quantity represents in terms of this problem situation.
   - N_0 is the initial number of followers
   - r is the rate of growth
   - t is the time in months
   - $N(t)$ is the number of followers after t months.

2. Gina claims that when she started working with an up-and-coming boy band, they had 18,450 online followers and she was able to increase their followers by 26% per month.

   a. Use Gina's claim to write a function to represent the number of online followers that the boy band had after t months.

   $N(t) = 18{,}450e^{0.26t}$

   b. If Gina started working with the band on May 1st, how many online followers did they have by September 1st?

   The band would have had 52,199 online followers by September 1st.

   There are 4 months from May to September, so $t = 4$.

   $N(t) = 18{,}450e^{0.26t}$

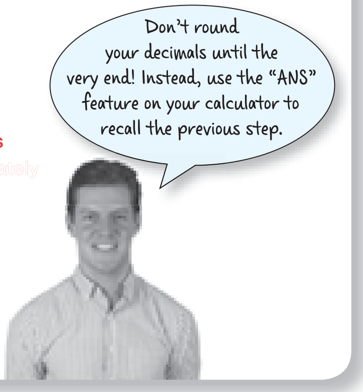
   $N(4) = 18{,}450e^{(0.26\cdot 4)}$

   ≈ 52199.054

   c. How long did it take for the band to surpass 100,000 online followers?

   The band surpassed 100,000 online followers after approximately 6.5 months, or approximately November 15th.

   $$100{,}000 = 18{,}450e^{0.26t}$$
   $$5.420054201 \approx e^{0.26t}$$
   $$\ln(5.420054201) \approx 0.26t$$
   $$1.690105816 \approx 0.26t$$
   $$6.500 \approx t$$



- Which variable in the model for continuous population growth represents the number of followers after t months?
- How is 18,450 used in Gina's equation?
- How is 26% used in Gina's equation?
- How many months are from May to September?
- Is the base, argument, or exponent unknown in this situation?
- How is 100,000 used in Gina's formula?

Guiding Questions for Share Phase, Question 3

- Is the base, argument, or exponent unknown in this situation?
- How is 105,326 used in Gina's equation?
- How is 4125 used in Gina's equation?
- Is the base, argument, or exponent unknown in this situation?
- How is the pro-football quarterback's equation different from the boy band's equation?
- How is the pro-football quarterback's equation similar to the boy band's equation?
- What is 105,326 doubled? How is it used in Gina's equation?

3. Gina's top client is a pro football quarterback. On her website, Gina claims that since she began managing his account three years ago, his online followers have grown to 105,326.

a. If he had 4125 online followers when he hired Gina, at what monthly rate did his number of online followers grow?

His number of online followers grew at approximately 9% per month.

$$\begin{aligned} N(t) &= N_0e^{rt} \\ 105{,}326 &= 4125e^{36r} \\ 25.53357576 &\approx e^{36r} \\ \text{In}(25.53357576) &\approx 36r \\ 3.239994283 &\approx 36r \\ 0.0899998412 &\approx r \end{aligned}$$

b. Write an exponential function to represent the number of online followers the quarterback will have in any given month, assuming the number of followers continues to grow at the same rate.

$N(t) = 105{,}326e^{0.09t}$

c. How long would it take the quarterback to double his current online following of 105,326?

It would take approximately 7.7 months, or 7 months and 21 days, to double his current online following.

$$\begin{aligned} N(t) &= 10{,}5326e^{0.09t} \\ 210{,}652 &= 105326e^{0.09t} \\ 2 &= e^{0.09t} \\ \text{In } 2 &= 0.09t \\ 0.6931471806 &\approx 0.09t \\ 7.702 &\approx t \end{aligned}$$

Guiding Questions for Share Phase, Questions 4 through 6

- How is the pro-football running back's equation different from the pro-football quarterback's equation?
- How is 62,100 used in the equation?
- How is 14% used in the equation?
- How is 15 months used in the equation?
- What equation is used to determine how many online followers the running back had when he hired Gina?
- Is the base, argument, or exponent unknown in this situation?
- What equation is used to determine whether the running back will have more online followers than the quarterback?
- What formula is used to determine the rate per month of increase online following for the talk show host?
- What rate did Josh use when checking Gina's work?
- Was the rate Josh used rounded off?
- If the rate was not rounded to the nearest hundredth, would it change the answer?
- What equation was used to predict the number of followers the talk show host would have in one year?
- Is the base, argument, or exponent unknown in this situation?

4. Gina also manages a running back on the same team who currently has 62,100 online followers. If he has seen a 14% increase in his followers, how many people followed him when he hired Gina 15 months ago?

The running back had 7605 followers when he hired Gina 15 months ago.

$62{,}100 = N_0 e^{0.14 \cdot 15}$

$\frac{62{,}100}{e^{0.14 \cdot 15}} = N_0$

$7604.544 \approx N_0$

5. Will the running back ever have the same number of online followers as the quarterback? If so, when? If not, explain your reasoning.

Yes. After approximately 10.6 months, or 10 months and 18 days, the running back will have the same number of followers as the quarterback.

$105{,}326e^{0.09t} = 62{,}100e^{0.14t}$

$(1.696070853)e^{0.09t} \approx e^{0.14t}$

$1.696070853 \approx e^{0.05t}$

$\ln(1.696070853) \approx 0.05t$

$t \approx 10.566$

6. Gina acquires a talk show host as a new client. The host currently has 5200 online followers.

a. Gina claims that she can triple the host's number of online followers in 6 months. Determine the rate per month of increased online following.

Gina is claiming approximately 18% per month growth in online followers.

$15{,}600 = 5200e^{6r}$

$3 = e^{6r}$

$\ln 3 = 6r$

$1.098612289 \approx 6r$

$0.1831020481 \approx r$

b. Josh decides to check Gina's work.

Josh

$N(t) = 5200e^{0.18t}$

$N(6) = 5200e^{0.18(6)}$

$N(6) \approx 15312.33367$

$15312 \neq 3(5200)$

Gina did not triple the host's followers.

What is the error in Josh's method? Check Gina's work correctly.

Josh used the rounded rate of 0.18 rather than a more precise decimal value.

$$N(t) = 5200e^{0.1831020481t}$$

$$N(6) = 5200e^{0.1831020481(6)}$$

$$\approx 15{,}600$$

c. Assuming that Gina's promised rate of growth is true, project how many followers the talk show host will have in a year.

The talk show host will have approximately 46,800 followers in a year.

$$N(t) = 5200e^{0.1831020481t}$$

$$N(12) = 5200e^{0.1831020481(12)}$$

$$\approx 46{,}800$$

13

Problem 3

The formula for calculating the elapsed time since a person has died is modeled by a logarithmic equation. Students use the model to calculate the time of death given the temperature of the body and the room temperature.

Grouping

- Ask a student to read the information and formula. Discuss as a class.
- Have students complete Questions 1 and 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 1

- What equation is used to estimate the time of death in this situation?
- How is the value of the numerator in the formula determined?
- What information is needed to determine the value of the numerator in the formula?
- How is the value of the denominator in the formula determined?
- What information is needed to determine the value of the denominator in the formula?
- Is the base, argument, or exponent unknown in this situation?

PROBLEM 3 Cracking the Case

A coroner uses a formula derived from Newton's Law of Cooling, a general cooling principle, to calculate the elapsed time since a person has died. The formula is

$$t = -10 \ln\left(\frac{T - R}{98.6 - R}\right)$$

where T is the body's measured temperature in °F, R is the constant room temperature in °F, and t is the elapsed time since death in hours.

1. At 8:30 AM a coroner was called to the home of a person who had died. The constant temperature of the room where the body was found is 70°F.

a. At 9:00 AM the body's measured temperature was 85.5°F. Use this body temperature to estimate the time of death.

Using the 9:00 AM body temperature, the estimated time of death is approximately 6 hours and 8 minutes earlier, or 2:52 AM.

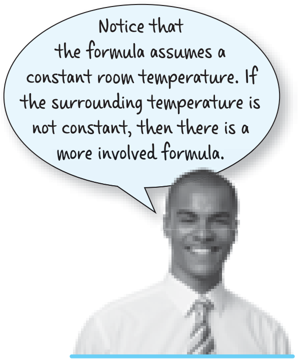
$t = -10 \ln\left(\frac{T - R}{98.6 - R}\right)$

$t = -10 \ln\left(\frac{85.5 - 70}{98.6 - 70}\right)$

$t \approx -10 \ln 0.542$

$t \approx 6.13$

$t \approx$ 6 hours and 8 minutes



A more accurate estimate of the time of death is found by taking two or more readings and averaging the calculated times of death.

b. At 9:30 AM the body's measured temperature was 82.9°F. Use this body temperature to estimate the time of death.

Using the 9:30 AM body temperature, the estimated time of death is approximately 7 hours and 58 minutes earlier, or 1:32 AM.

$t = -10 \ln\left(\frac{T - R}{98.6 - R}\right)$

$t = -10 \ln\left(\frac{82.9 - 70}{98.6 - 70}\right)$

$t \approx -10 \ln 0.451$

$t \approx 7.96$

$t \approx$ 7 hours and 58 minutes

c. Compare the estimated time of death from part (a) and (b). Are your answers fairly close? Determine a more accurate estimated time of death.

Yes. The estimated times of death are fairly close—within 80 minutes of each other.

A more accurate estimated time of death is 2:12 AM.

I determined the time halfway between the two calculated times of death, or 40 minutes after the earliest estimated time of death.

1:32 AM + 40 minutes = 2:12 AM

d. Assuming the body remains in a room with a constant temperature of 70°F, determine the temperature of the body after 24 hours.

After 24 hours, the temperature of the body will be 72.6°F.

$$\begin{aligned} t &= -10 \ln\left(\frac{T - R}{98.6 - R}\right) \\ 24 &= -10 \ln\left(\frac{T - 70}{98.6 - 70}\right) \\ -2.4 &= \ln\left(\frac{T - 70}{28.6}\right) \\ e^{-2.4} &= \frac{T - 70}{28.6} \\ 0.091 &\approx \frac{T - 70}{28.6} \\ 2.60 &\approx T - 70 \\ 72.6 &\approx T \end{aligned}$$

e. When will the temperature of the body drop to 60°F? Explain your reasoning.

Assuming the body is not moved, the temperature of the body will not drop to 60°F, because the constant room temperature is 70°F.

$$\begin{aligned} t &= -10 \ln\left(\frac{60 - 70}{98.6 - 70}\right) \\ &= -10 \ln\left(\frac{-10}{28.6}\right) \end{aligned}$$

The argument is negative, so the solution is undefined.

13

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Guiding Questions for Share Phase, Question 2

- What equation is used to estimate the time of death in this situation?
- How is this situation different from the previous situation?
- Are the estimated times of death within an hour of each other?
- What time is halfway between the two estimated times of death?
- How is 24 hours used in the formula?
- What is the value of *R* in this situation? What does it represent in the problem situation?
- What strategy is used to solve this problem?
- Can the logarithmic equation be rewritten as an exponential equation?
- If the room temperature is 70°, can the temperature of the body ever be lower than 70°?
- What equation is used to determine the estimated time of death with respect to the 6:30 PM body temperature?
- What equation is used to determine the estimated time of death with respect to the 7:00 PM body temperature?
- How close are the two estimated times of death?
- What is halfway between the two estimated times of death?

2. At 6:00 PM, a coroner was called to the home of a person who had died. The body was found at the bottom of the stairs in a hallway, where the constant temperature is 65°F. At 6:30 PM the body's measured temperature was 95.9°F, and at 7:00 PM the body's measured temperature was 93.7°F.

A witness claims that the person fell down the stairs around 5:30 pm, and she immediately called for an ambulance. Is her statement consistent with the forensic evidence?

Explain your reasoning.

Yes. I believe her story because the average estimated time of death is 5:33 PM.

Using the 6:30 PM body temperature, the estimated time of death is approximately 50 minutes earlier, or 5:40 PM.

$t = -10 \ln \left(\frac{T - R}{98.6 - R}\right)$

$t = -10 \ln \left(\frac{95.9 - 65}{98.6 - 65}\right)$

$t \approx -10 \ln 0.920$

$t \approx 0.834$

$t \approx 50$ minutes

Using the 7:00 PM body temperature, the estimated time of death is approximately 1 hour and 35 minutes earlier, or 5:25 PM.

$t = -10 \ln \left(\frac{T - R}{98.6 - R}\right)$

$t = -10 \ln \left(\frac{93.7 - 65}{98.6 - 65}\right)$

$t \approx -10 \ln 0.854$

$t \approx 1.58$

$t \approx 1$ hour and 35 minutes

The estimated times of death are within 15 minutes of each other. To determine a more accurate time of death, I determined the time halfway between the two calculated times of death, or 7.5 minutes after the earliest estimated time of death.

5:25 PM + 7.5 minutes ≈ 5:33 PM

Be prepared to share your solutions and methods.

Check for Students' Understanding

Some items such as automobiles are worth less over time. The age of an item can be predicted using the formula $t = \frac{\log\left(\frac{V}{C}\right)}{\log(1 - r)}$, where t is the age of the item in year, V is the value of the item after t years, C is the original value of the item, and r is the yearly rate of depreciation expressed as a decimal.

A car was originally purchased for \$32,000 and is currently valued at \$20,000. The average rate of depreciation for this car is 12% per year.

How old is the car to the nearest tenth of a year?

$$t = \frac{\log\left(\frac{V}{C}\right)}{\log(1 - r)}$$

$$t = \frac{\log\left(\frac{20{,}000}{32{,}000}\right)}{\log(1 - 0.12)}$$

$t \approx 3.7$

The car is about 3.7 years old.

Chapter 13 Summary

KEY TERMS

- logarithmic equation (13.1)
- Zero Property of Logarithms (13.2)
- Logarithm with Same Base and Argument (13.2)
- Product Rule of Logarithms (13.2)
- Quotient Rule of Logarithms (13.2)
- Power Rule of Logarithms (13.2)
- Change of Base Formula (13.3)

13.1 Writing Exponential and Logarithmic Equations

A logarithmic function is the inverse of an exponential function. The diagram below shows the relationship between the base, argument, and exponent of logarithmic and exponential equations.

$$\text{base}^{\text{exponent}} = \text{argument} \qquad \Leftrightarrow \qquad \log_{\text{base}} (\text{argument}) = \text{exponent}$$

Example

Arrange the terms 3, 4, and 81 to create a true exponential equation and a true logarithmic equation.

Exponential equation: $3^4 = 81$

Logarithmic equation: $\log_3 (81) = 4$

13.1 Solving Logarithmic Equations

To solve a logarithmic equation, begin by converting it to an exponential equation. If the argument is unknown, writing the equation in exponential form will isolate the argument and make it simpler to solve. If the exponent is unknown, you will need to rewrite both sides of the exponential equation so that they have the same base. If the base is unknown, you will need to rewrite both sides of the exponential equation so that they have the same exponent.

Example

Solve the logarithmic equation.

$$\log_n (\sqrt[3]{25}) = \frac{2}{3}$$

$$n^{\frac{2}{3}} = \sqrt[3]{25}$$

$$n^{\frac{2}{3}} = 25^{\frac{1}{3}}$$

$$n^{\frac{2}{3}} = (5^2)^{\frac{1}{3}}$$

$$n^{\frac{2}{3}} = 5^{\frac{2}{3}}$$

$$n = 5$$

13.1 Estimating Logarithms

To estimate the value of a logarithm, identify the closest logarithm whose argument is less than the given argument and the closest logarithm whose argument is greater than the given argument. If the argument is closer to the lower limit, the value of the given logarithm will also be closer to the value of the logarithm of the lower limit. Similarly, if the argument is closer to the upper limit, the value of the given logarithm will also be closer to the value of the logarithm of the upper limit.

Example

Estimate the value of $\log_6 (210)$ to the nearest tenth.

$$\begin{array}{ccccc} \log_6 (36) & < & \log_6 (210) & < & \log_6 (216) \\ 2 & < & n & < & 3 \end{array}$$

Because 210 is closer to 216 than 36, the approximation should be closer to 3 than 2. Therefore, $\log_6 (210) \approx 2.9$.

13.2 Writing Logarithmic Expressions in Expanded Form

To write a logarithmic expression in expanded form, use the properties of logarithms.

Example

Write the logarithmic expression $\log_3 \left(\frac{5x^2}{2y^5}\right)$ in expanded form.

$$\log_3 \left(\frac{5x^2}{2y^5}\right) = \log_3 (5) + 2 \log_3 (x) - \log_3 (2) - 5 \log_3 (y)$$

13.2 Writing Logarithmic Expressions as a Single Logarithm

To write a logarithmic expression as a single logarithm, use the properties of logarithms.

Example

Write the logarithmic expression $4 \log_6 (x) + 2 \log_6 (y) - 7 \log_6 (x)$ as a single logarithm.

$$\begin{aligned} 4 \log_6 (x) + 2 \log_6 (y) - 7 \log_6 (x) &= \log_6 \left(\frac{x^4 y^2}{x^7}\right) \\ &= \log_6 \left(\frac{y^2}{x^3}\right) \end{aligned}$$

13.3 Solving Exponential Equations Using the Change of Base Formula

The Change of Base Formula states that $\log_b (c) = \frac{\log_a (c)}{\log_a (b)}$, where $a, b, c > 0$ and $a, b \neq 1$. To solve an exponential equation using the Change of Base Formula, first write the equation using logarithms. Next, write the logarithm as the quotient of the common log of the argument and the common log of the base. Use your calculator to determine that quotient. Finally, solve for the variable.

Example

Solve the exponential equation by using the Change of Base Formula.

$$\begin{aligned} 8^{3x-4} &= 257 \\ 3x - 4 &= \frac{\log 257}{\log 8} \\ 3x - 4 &\approx 2.669 \\ 3x &\approx 6.669 \\ x &\approx 2.223 \end{aligned}$$

13.3 Solving Exponential Equations by Taking the Log of Both Sides

To solve an exponential equation by taking the log of both sides, first isolate the variable term on one side. Next, take the common log of both sides. Use your calculator to determine the common logs. Then, solve for the variable.

Example

Solve the exponential equation by taking the log of both sides.

$$\begin{aligned} 6^{x-5} + 7 &= 259 \\ 6^{x-5} &= 252 \\ \log (6^{x-5}) &= \log 252 \\ (x - 5) \log 6 &= \log 252 \\ x - 5 &= \frac{\log 252}{\log 6} \\ x - 5 &\approx 3.086 \\ x &\approx 8.086 \end{aligned}$$

13.4 Solving Logarithmic Equations

To solve a logarithmic equation, rewrite the equation using exponents or rewrite the equation using the properties of logarithms. Then, solve.

Example

Solve the logarithmic equation. Check your answer.

$$\begin{aligned} \log_6 x + \log_6 (x + 5) &= 2 \\ \log_6 (x(x + 5)) &= 2 \\ \log_6 (x^2 + 5x) &= 2 \\ x^2 + 5x &= 6^2 \\ x^2 + 5x - 36 &= 0 \\ (x + 9)(x - 4) &= 0 \\ x &= \cancel{-9}, 4 \end{aligned}$$

Check: -9 is an extraneous solution.

$$\begin{aligned} \log_6 4 + \log_6 (4 + 5) &\overset{?}{=} 2 \\ \log_6 (4(9)) &\overset{?}{=} 2 \\ \log_6 (36) &\overset{?}{=} 2 \\ 6^2 &\overset{?}{=} 36 \\ 36 &= 36 \end{aligned}$$

13.5 Using Exponential Models to Analyze Problem Situations

Some problem situations can be modeled with exponential equations. To solve these problems, identify what the question is asking and then use the given model to determine the answer.

Example

The given exponential equation models the appraised value of a vehicle over time. Use the model to answer the question.

Given $V = 47{,}000(10^{-0.0499t})$, where t represents the time in years, determine how old the vehicle is if its appraised value is $21,028.

$$\begin{aligned} V &= 47{,}000(10^{-0.0499t}) \\ 21{,}028 &= 47{,}000(10^{-0.0499t}) \\ 0.4474042553 &\approx 10^{-0.0499t} \\ \log 0.4474042553 &\approx \log 10^{-0.0499t} \\ \log 0.4474042553 &\approx -0.0499t \log 10 \\ -0.3492998896 &\approx -0.0499t \\ 6.999997787 &\approx t \end{aligned}$$

The vehicle is about 7 years old.

13.5 Using Logarithmic Models to Analyze Problem Situations

Some problem situations can be modeled with logarithmic equations. To solve these problems, identify what the question is asking and then use the given model to determine the answer.

Example

The number of monthly payments a person must make to repay a car loan or a home mortgage can be determined by the formula $n = \frac{-\log\left(1 - \frac{I \cdot P_0}{p}\right)}{\log(1 + I)}$, where P_0 represents the original value of the loan, p represents the amount of the monthly payment, and I is the interest rate per payment period in decimal form.

Calculate the number of payments of \$540 per month that would need to be made on a \$79,000 mortgage loan at 4.2% interest.

$$
\begin{aligned}
n &= \frac{-\log\left(1 - \frac{I \cdot P_0}{p}\right)}{\log(1 + I)} \\
&= \frac{-\log\left(1 - \frac{\left(\frac{0.042}{12}\right)(79{,}000)}{540}\right)}{\log\left(1 + \frac{0.042}{12}\right)} \\
&\approx \frac{-\log(1 - 0.512037037)}{\log(1 + 0.0035)} \\
&\approx \frac{-\log 0.487962963}{\log 1.0035} \\
&\approx \frac{0.3116131403}{0.0015173768} \\
&\approx 205.3630583
\end{aligned}
$$

A little more than 205 monthly payments would need to be made to pay off the loan.