Normal distribution

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Advanced High School Statistics

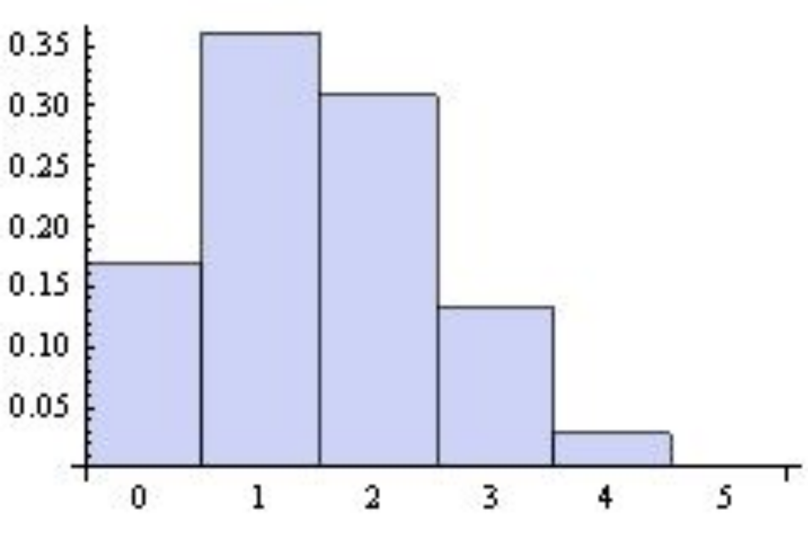
From discrete to continuous...

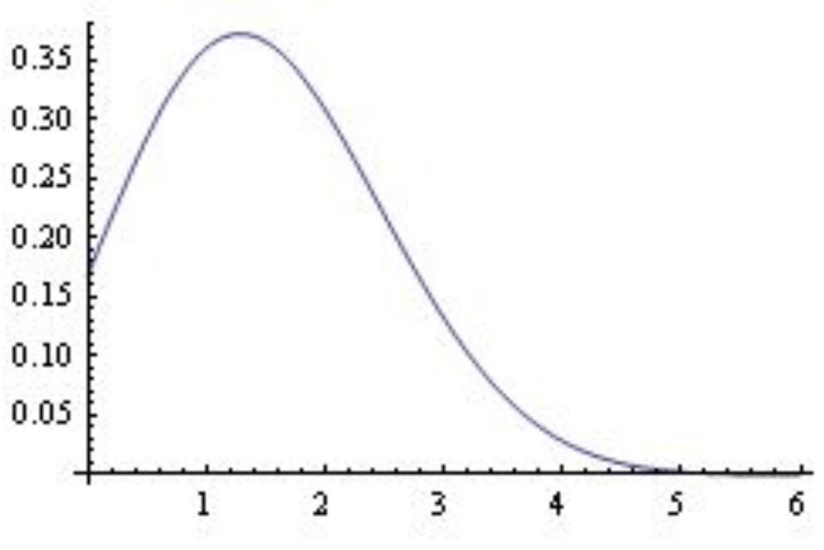
Discrete

- sum of probabilities must = 1

Continuous

- total *area* must = 1
- probability of a specific value = 0, e.g. P(X = 2) = 0
- only intervals have probability,   e.g. P(1 < X < 2) = ?

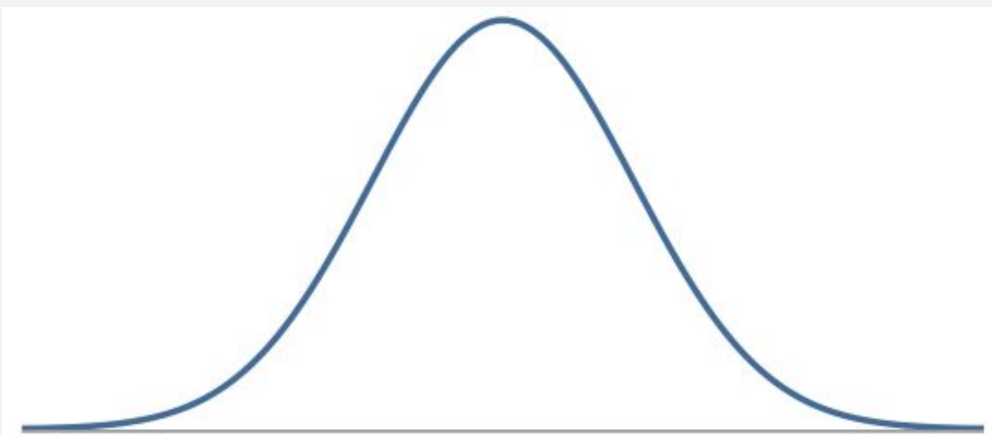




The Normal distribution...

- is the most well known continuous distribution
- Is unimodal and symmetric, bell shaped curve
- has mean μ and standard deviation σ
- has tails that extend infinitely in both directions

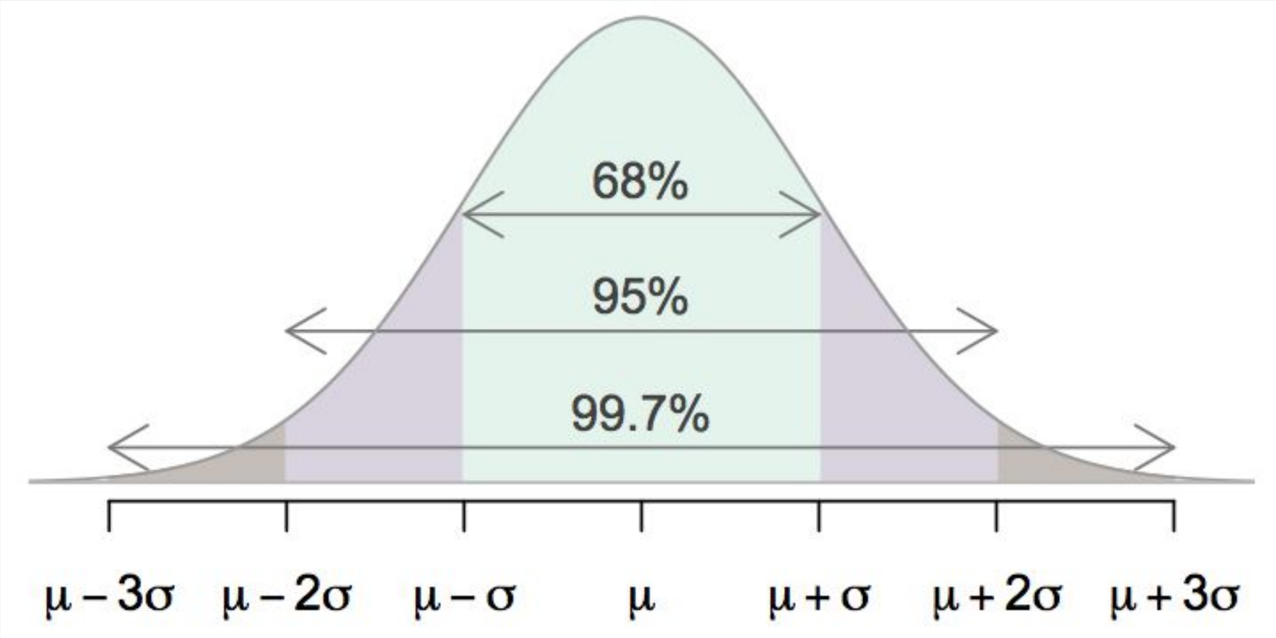
Many variables are nearly normal, but none are *exactly* normal



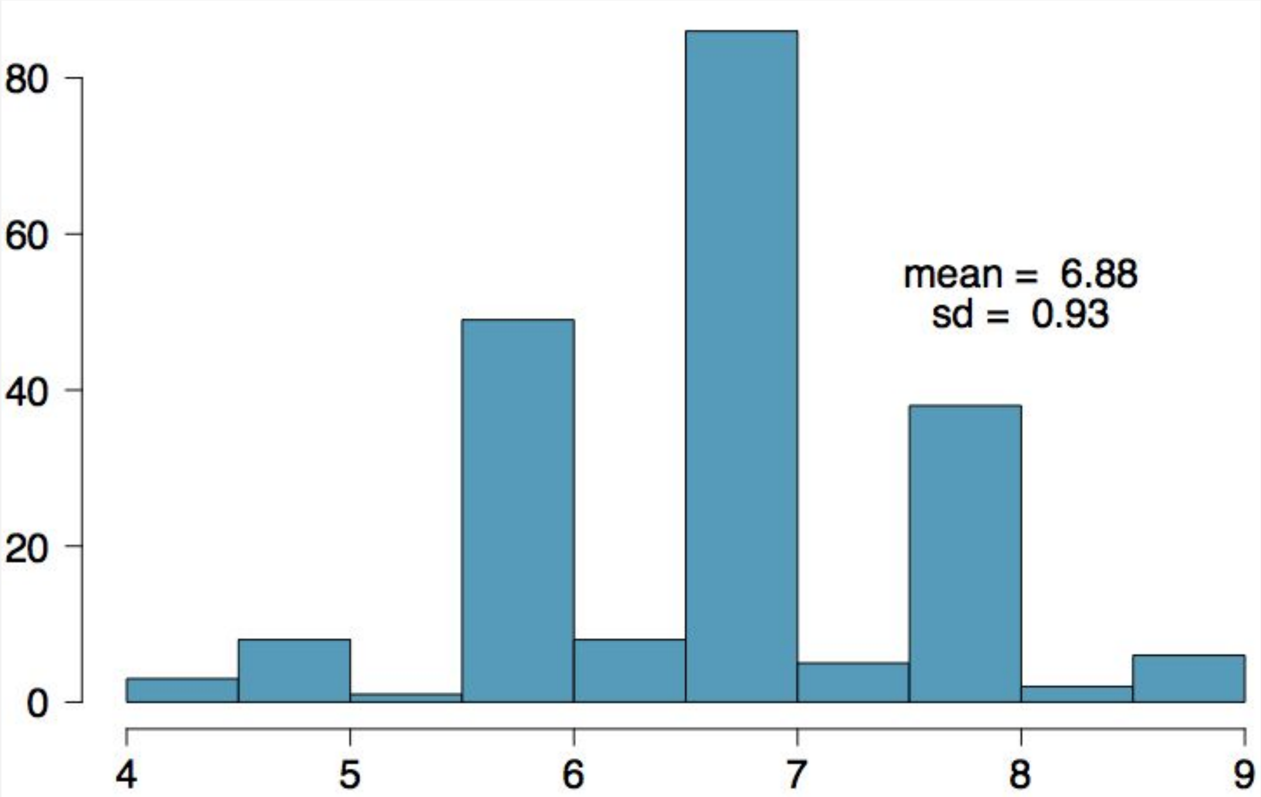
The source of the 68-95-99.7 Rule

For nearly normally distributed data,

- about 68% falls within 1 SD of the mean,
- about 95% falls within 2 SDs of the mean,
- about 99.7% falls within 3 SDs of the mean.

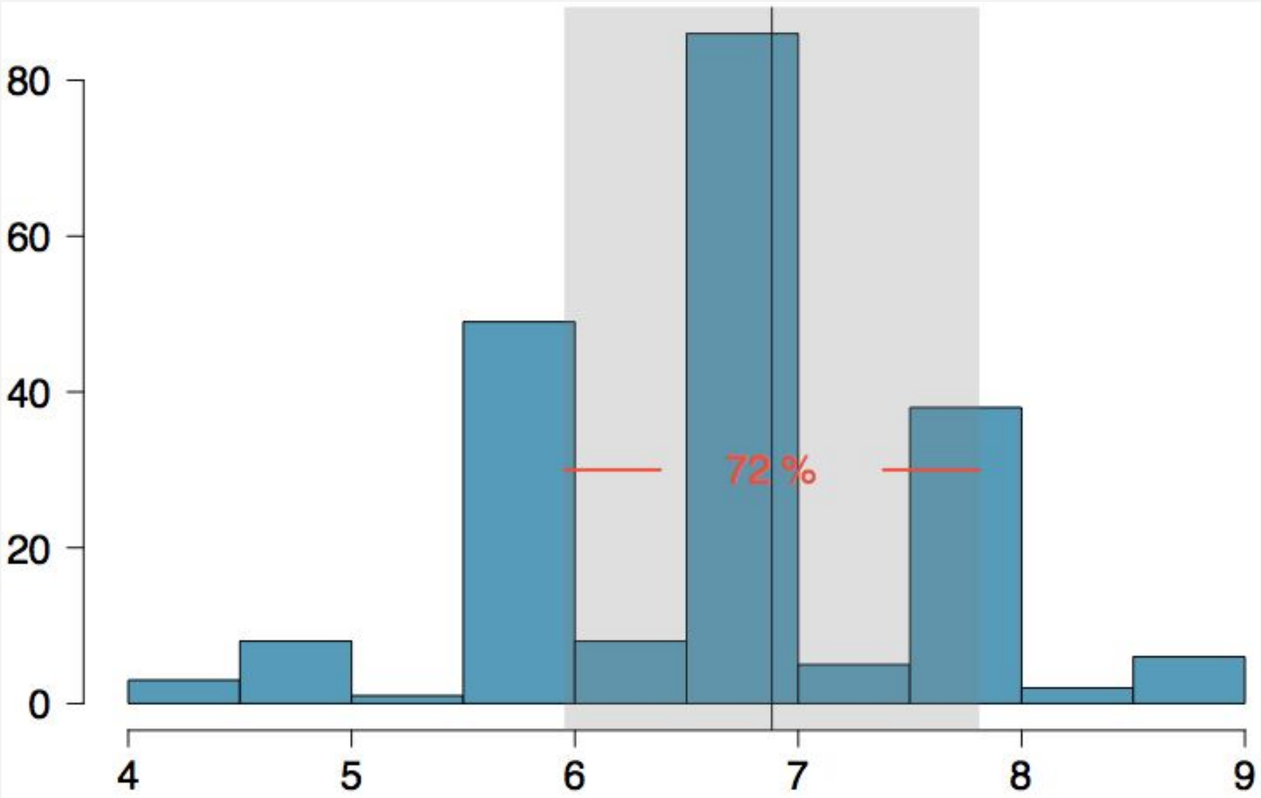


Number of hours of sleep on school nights



Mean = 6.88 hours, SD = 0.92 hrs

Number of hours of sleep on school nights



Mean = 6.88 hours, SD = 0.92 hrs

72% of the data are within 1 SD of the mean: 6.88 ± 0.93

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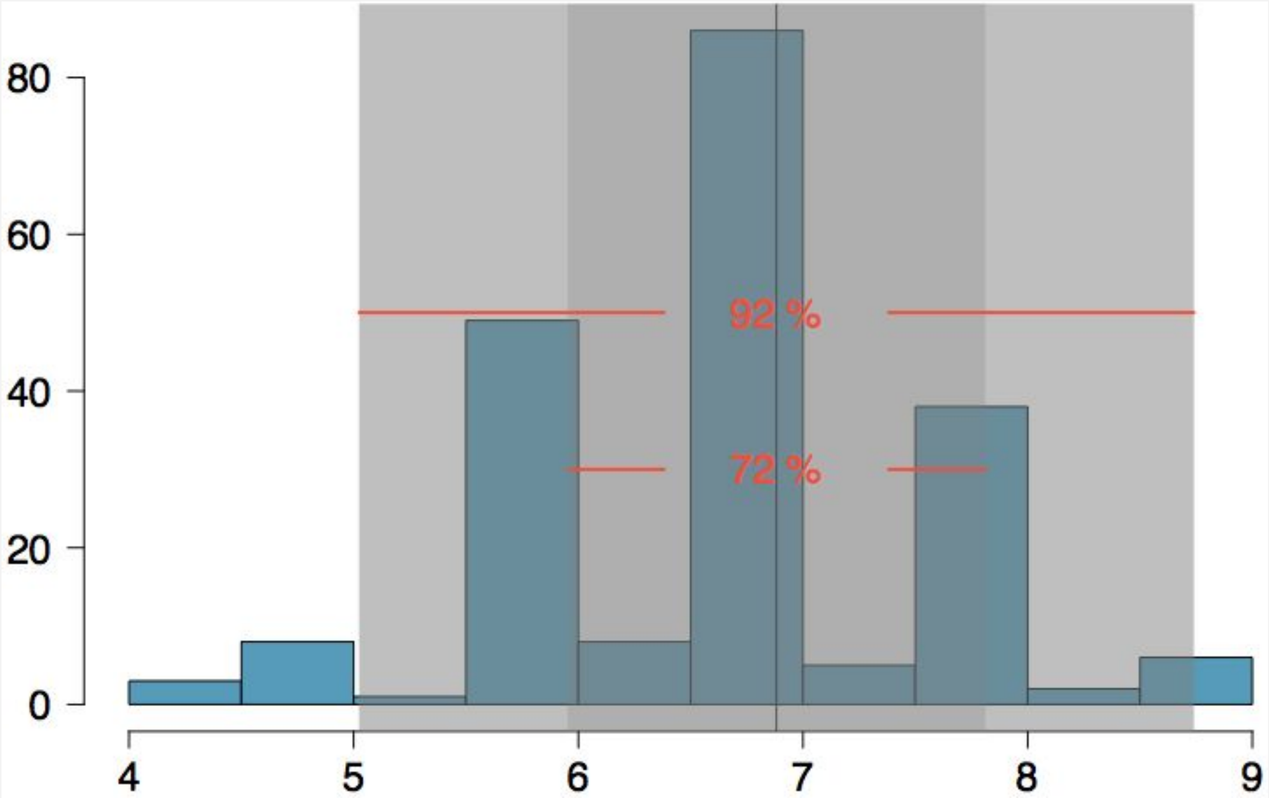

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72% of the data are within 1 SD of the mean: 6.88 ± 0.93

92% of the data are within 1 SD of the mean: 6.88 ± 2 x 0.93

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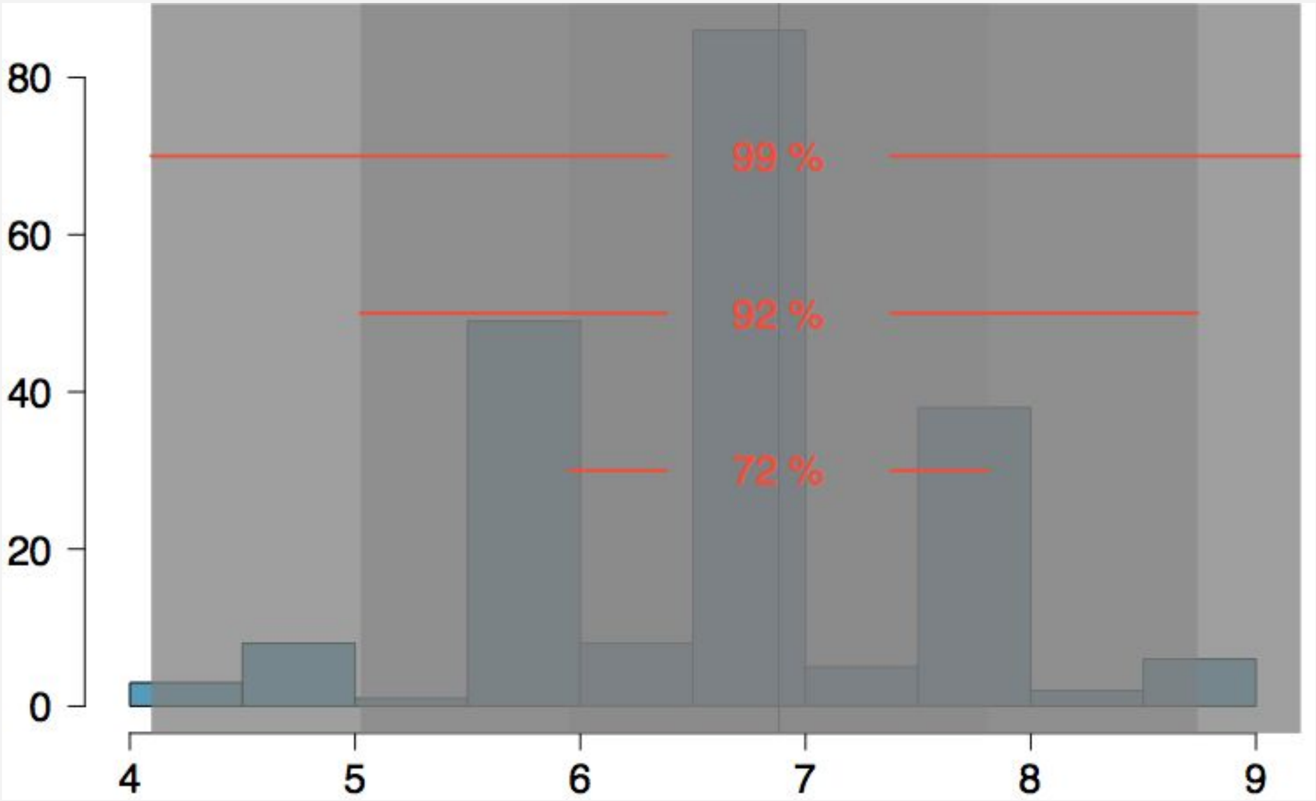


Mean = 6.88 hours, SD = 0.92 hrs

72% of the data are within 1 SD of the mean: 6.88 ± 0.92

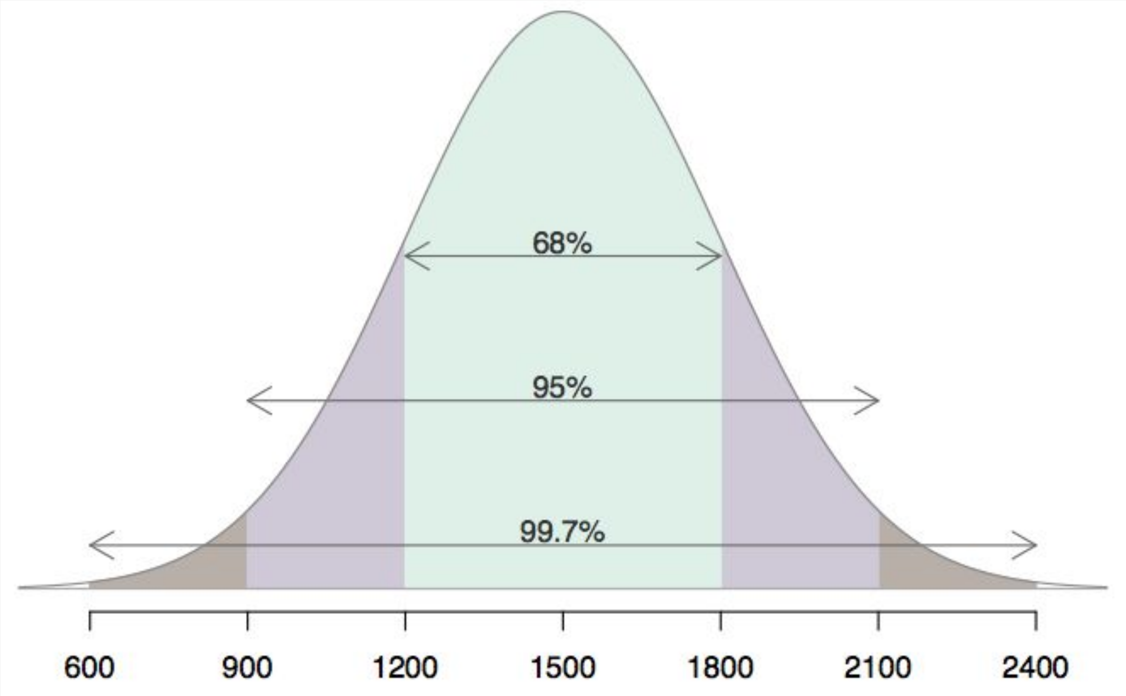
92% of the data are within 2 SD of the mean: 6.88 ± 2 x 0.92

99% of the data are within 3 SD of the mean: 6.88 ± 3 x 0.92

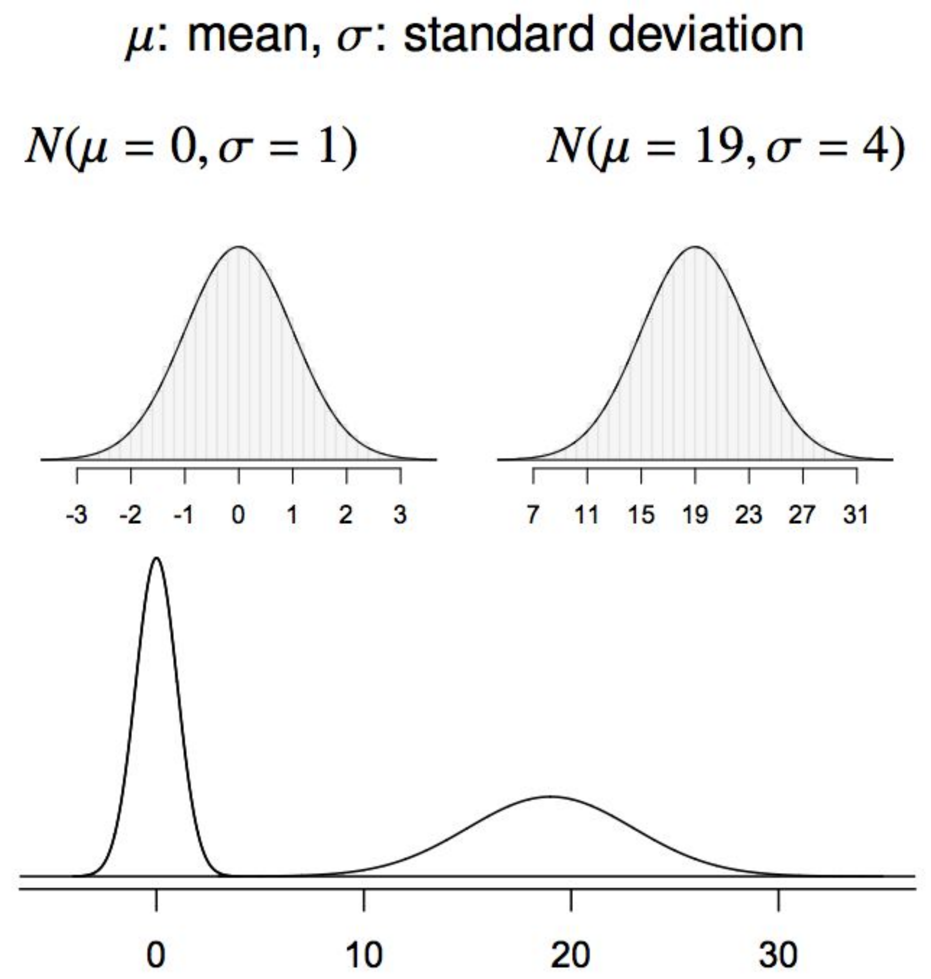
Describing variability using the 68-95-99.7 Rule

SAT scores are distributed nearly normally with mean 1500 and standard deviation 300.

- ~68% of students score between 1200 and 1800 on the SAT.
- ~95% of students score between 900 and 2100 on the SAT.
- ~99.7% of students score between 600 and 2400 on the SAT.



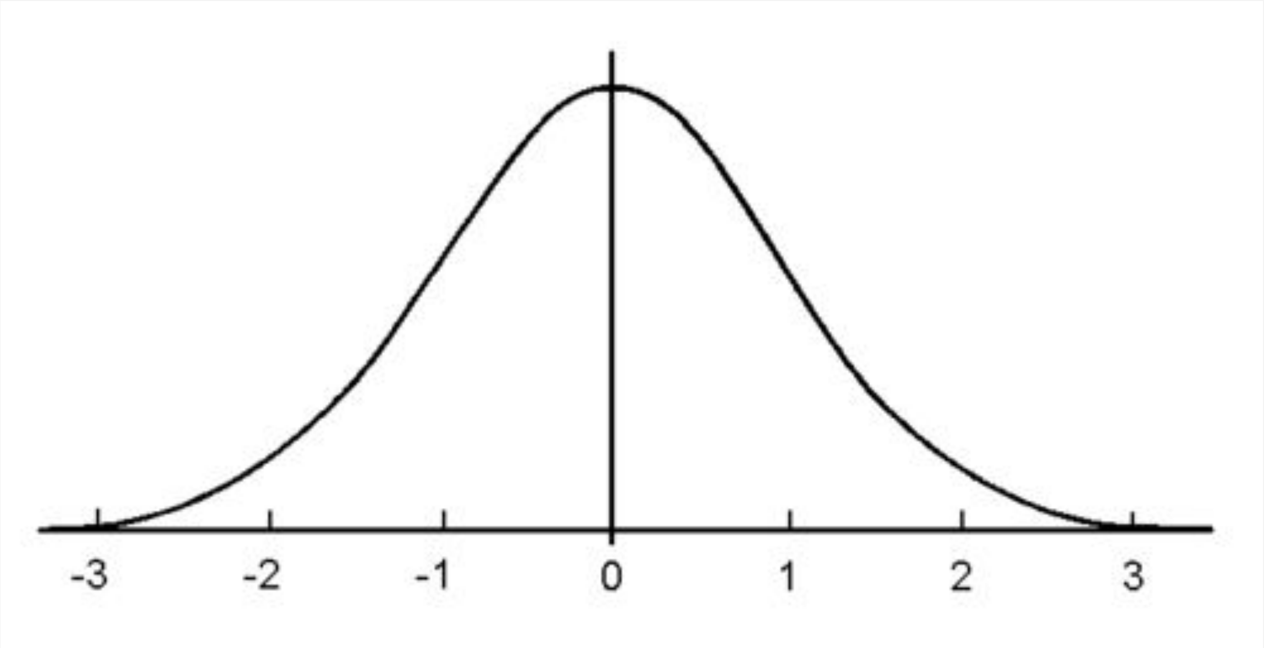
Normal distributions with different parameters



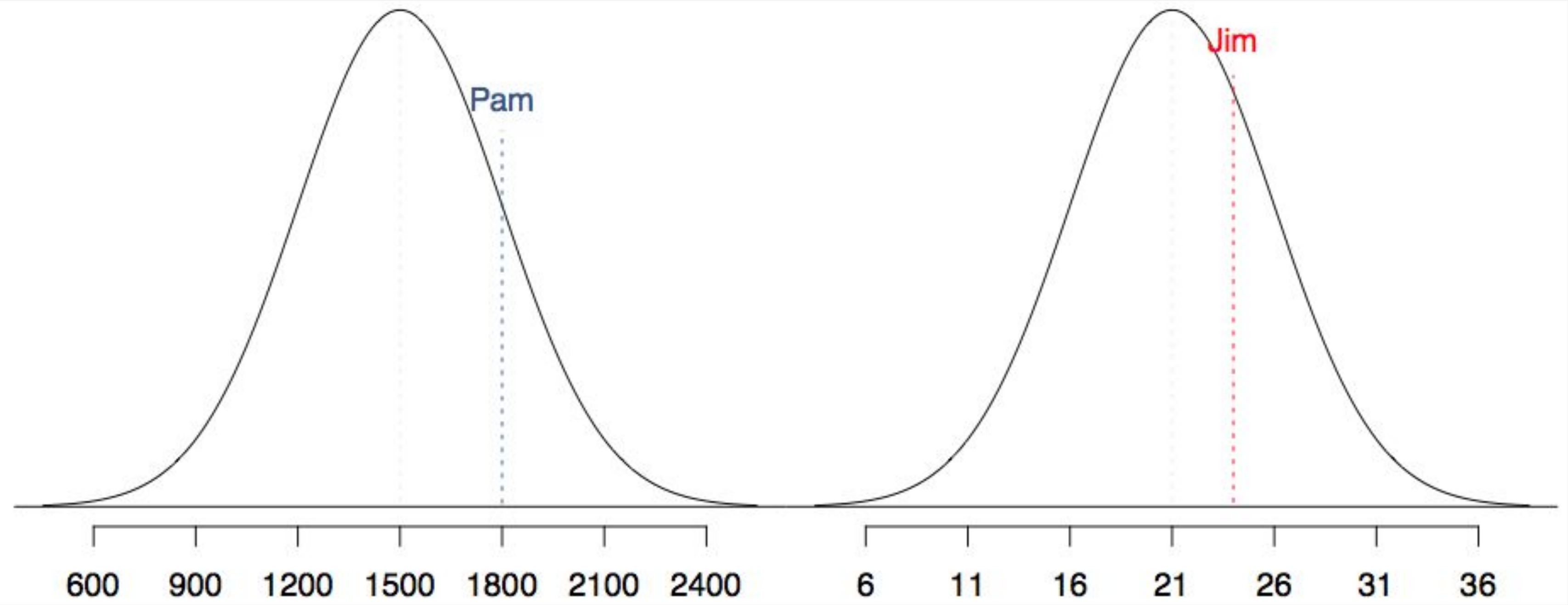
The Standard Normal Curve

What units are on the horizontal axis?

- Z-scores!



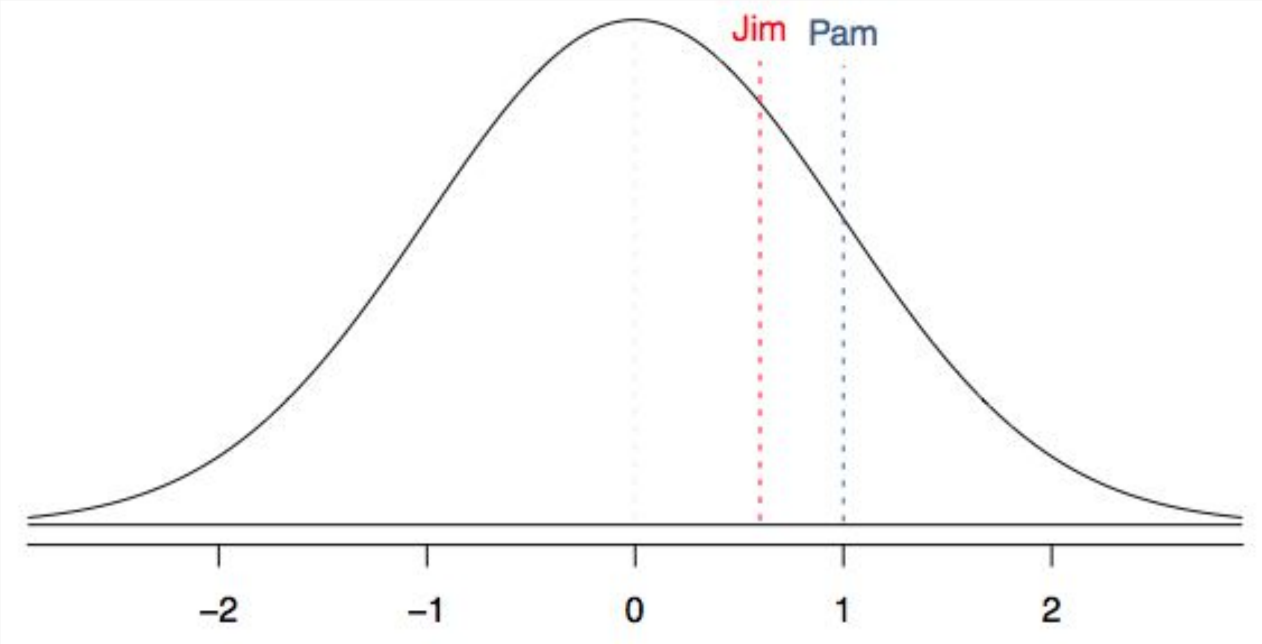
SAT scores are distributed nearly normally with mean 1500 and standard deviation 300. ACT scores are distributed nearly normally with mean 21 and standard deviation 5. A college admissions officer wants to determine which of the two applicants scored better on their standardized test with respect to the other test takers: Pam, who earned an 1800 on her SAT, or Jim, who scored a 24 on his ACT?



Standardizing with Z-scores

Since we cannot just compare these two raw scores, we instead compare their Z-scores, that is, how many SDs beyond the mean their raw score is.

- Pam's Z-score = (1800 - 1500) / 300 = 1. She is 1 standard deviation above the mean.
- Jim's Z-score = (24 - 21) / 5 = 0.6. He is 0.6 standard deviations above the mean.
- Therefore, Pam did better.



Standardizing with Z scores (cont.)

These are called standardized scores, or Z scores.

- Z score of an observation is the *number of standard deviations* it falls above or below the mean.

$$Z = \frac{(\text{observation} - \text{mean})}{\text{SD}}$$

- Z scores are defined for distributions of any shape, but only when the distribution is normal can we use Z scores to calculate percentiles.
- Observations that are more than 2 SD away from the mean ($|Z| > 2$) are generally considered unusual.

Standardizing with Z scores (cont.)

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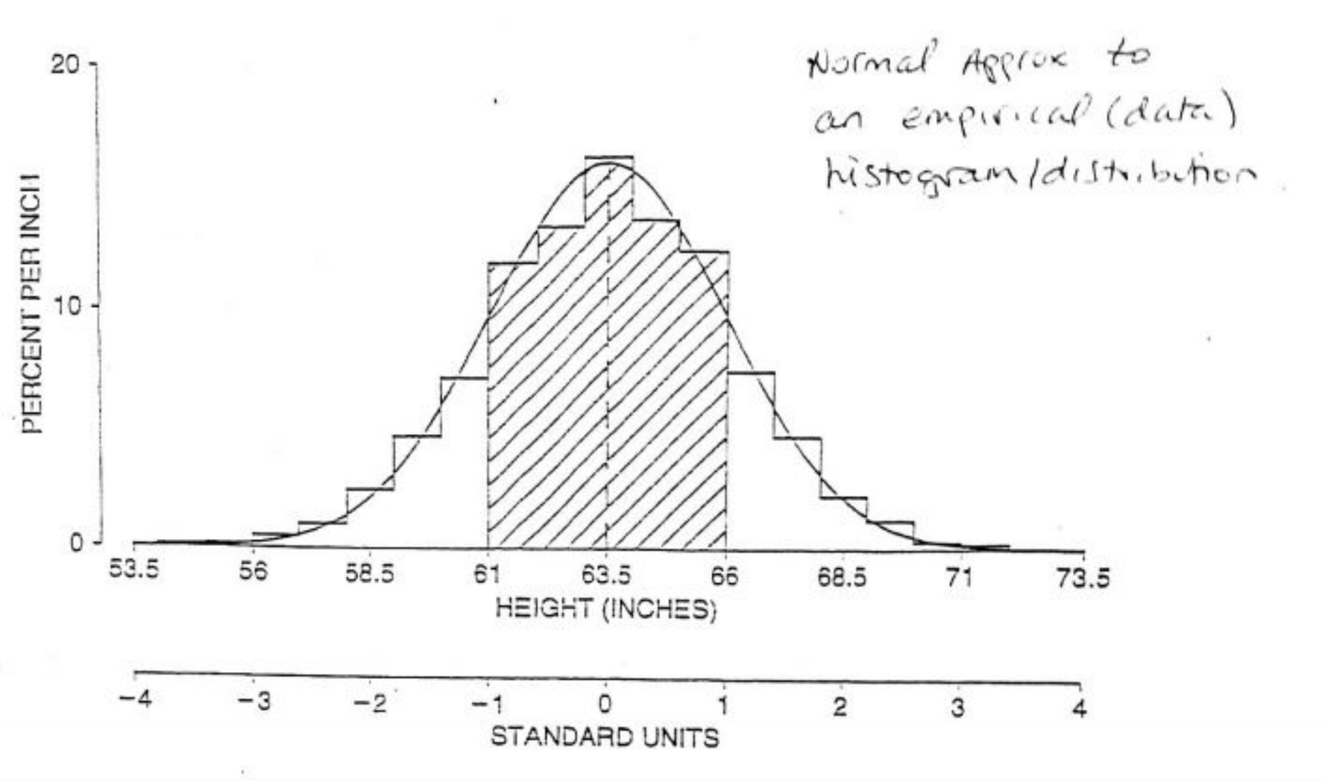
Normal approximation for data

Step 1. Convert *x* value to Z-score

Step 2: Use calculator to find corresponding area under standard normal curve



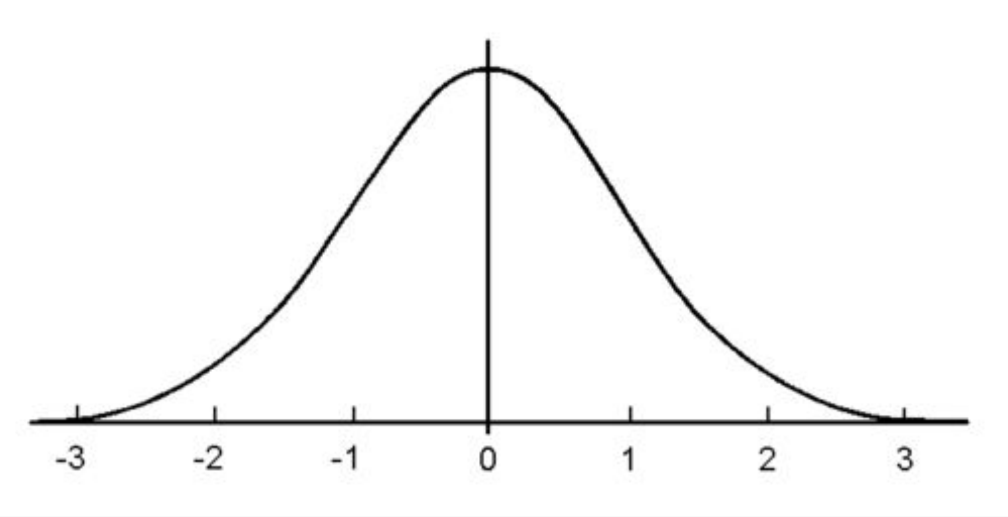
Finding areas under the standard normal curve

$P(Z < -1) =$

$P(Z < -1.5) =$

$P(Z > 2.1) =$

$P(-1.2 < Z < 2.1) =$

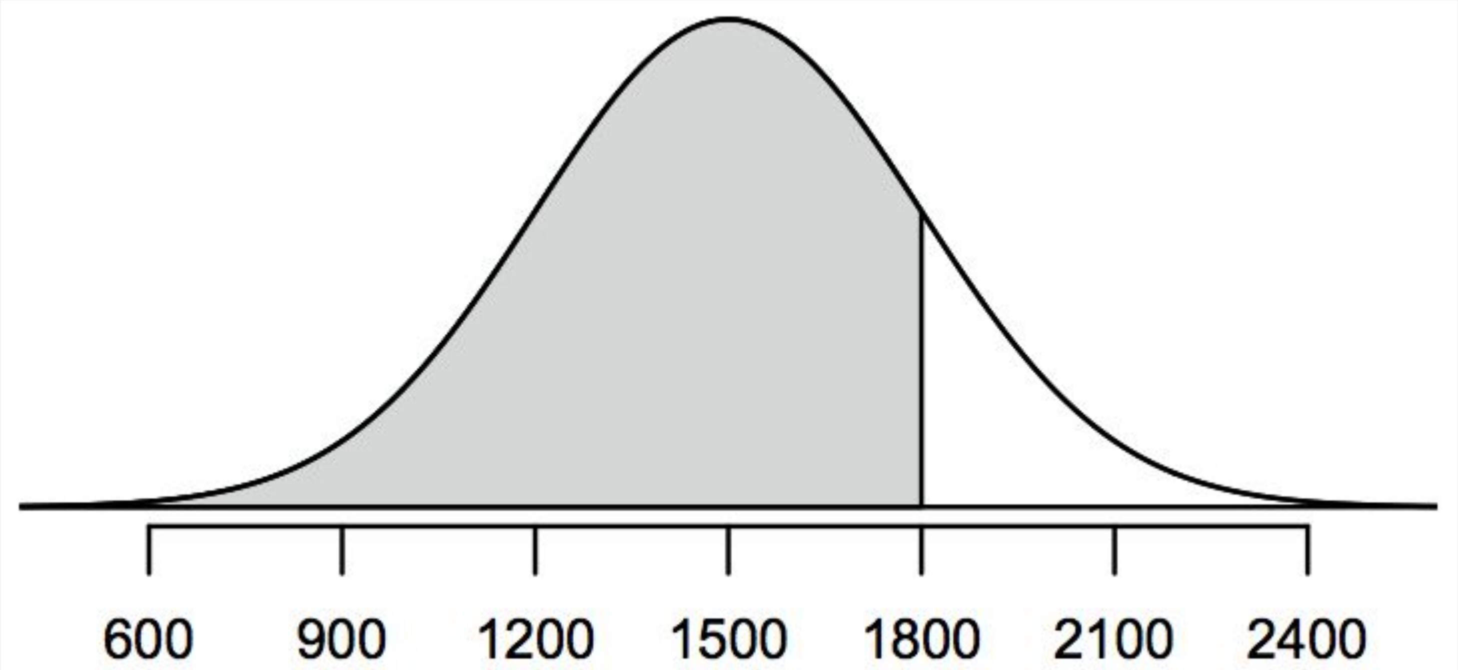


2nd VARS, 2:normcdf(
lower: ?
upper: ?
Paste

TI-83: do 2:normcdf(lower, upper)

Percentiles

- Percentile is the percentage of observations that fall below a given data point.
- Graphically, percentile is the area below the probability distribution curve to the left of that observation.



Finding percentiles from the standard normal curve

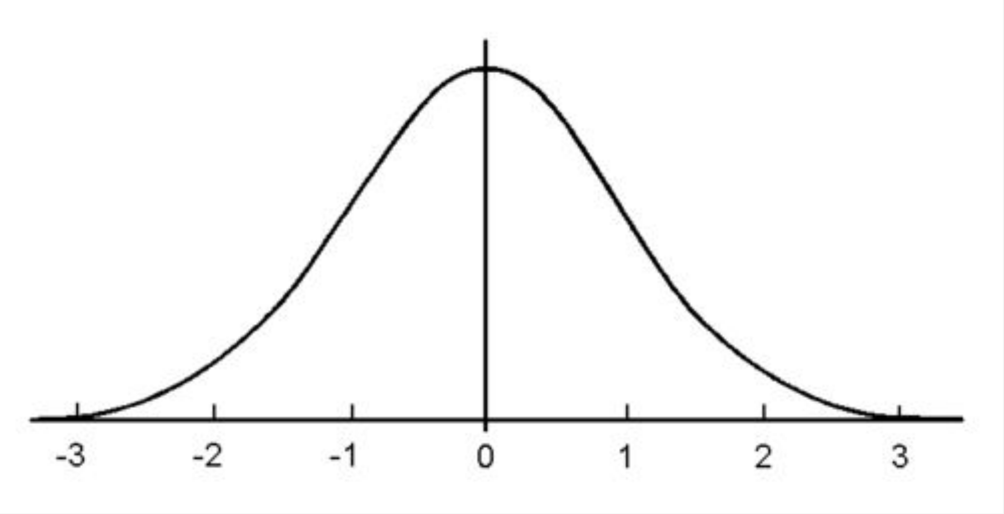
What Z-score corresponds to the 50th percentile?
i.e. $P(Z < ?) = 0.5$ Z =

What Z-score corresponds to the 20th percentile?
i.e. $P(Z < ?) = 0.2$ Z =

What Z-score has 70% of the area to the *right* of it?
i.e. $P(Z < ?) =$ **0.3**   Z =

2nd VARS: 3: invNorm(
area = ? (enter percentile as a decimal)
Paste

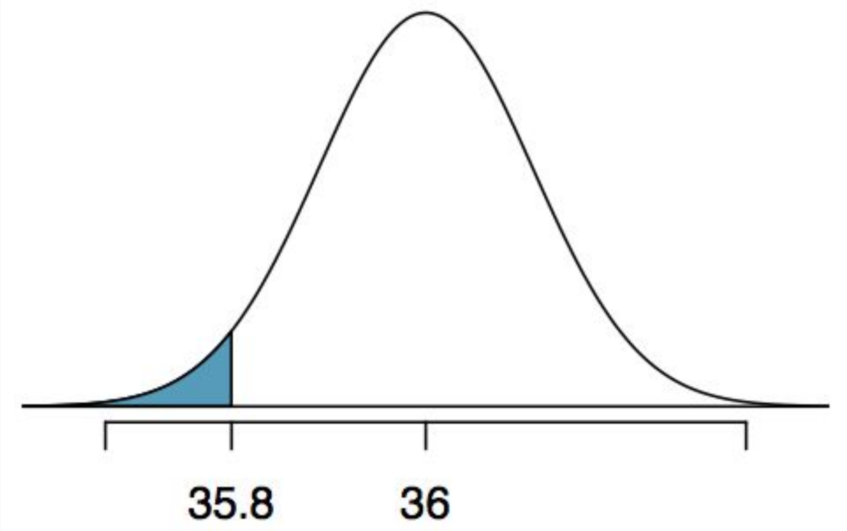
TI-83 do:
3:invNorm(percentile as a decimal)



Quality control

At Heinz ketchup factory the amounts which go into bottles of ketchup are supposed to be normally distributed with mean 36 oz. and standard deviation 0.11 oz. If the amount of ketchup in the bottle is below 35.8 oz. then the bottle fails the quality control inspection. What is the probability that a randomly selected bottle fails quality control (that is, what percent of bottles fail quality control)?

- *Let X = amount of ketchup in a bottle:* $\mu = 36$, $\sigma = 0.11$. $P(X < 35.8) = ?$



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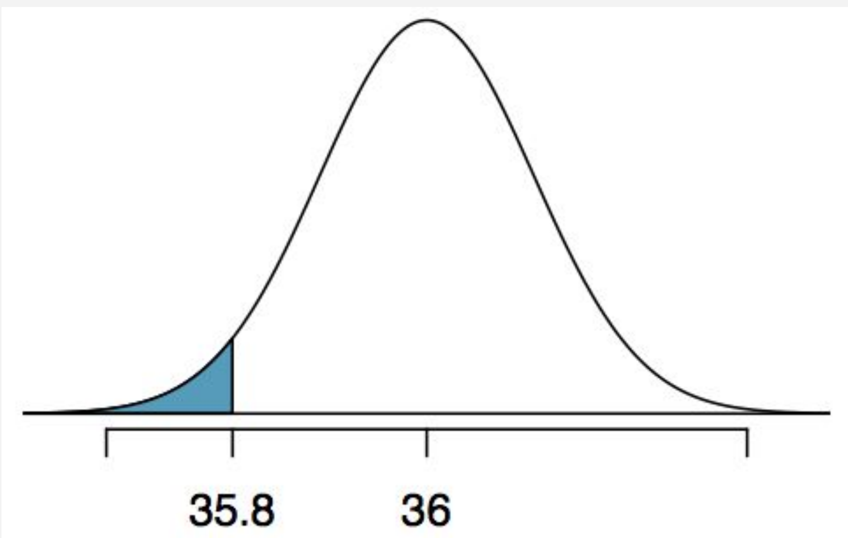


$$Z = \frac{35.8 - 36}{0.11} = -1.82$$

$P(Z < -1.82) = 0.034 = 3.4\%$

Practice

If the amount of ketchup in the bottle is below 35.8 oz. or above 36.2 oz., then the bottle *fails* the quality control inspection. Recall $\mu = 36$, $\sigma = 0.11$.

What percent of bottles pass the quality control inspection?

(a) 1.82%

(b) 3.44%

(c) 6.88%

(d) 93.12%

(e) 96.56%

Practice

If the amount of ketchup in the bottle is below 35.8 oz. or above 36.2 oz., then the bottle *fails* the quality control inspection. Recall $\mu = 36, \sigma = 0.11$.

What percent of bottles pass the quality control inspection?

(a) 1.82%

(b) 3.44%

(c) 6.88%

(d) 93.12%

(e) 96.56%

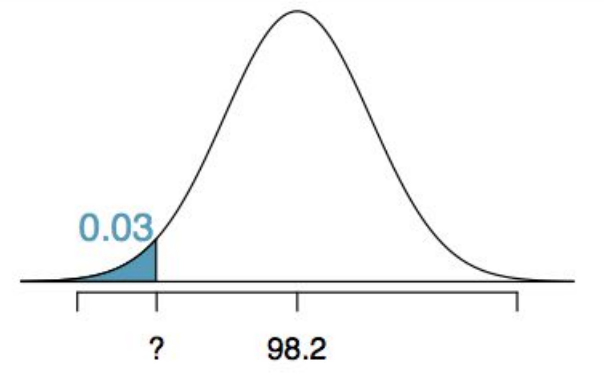
$$Z_{35.8} = \frac{35.8 - 36}{0.11} = -1.82$$

$$Z_{36.2} = \frac{36.2 - 36}{0.11} = 1.82$$

$$P(35.8 < X < 36.2) = P(-1.82 < Z < 1.82) = 0.9312$$

Finding cutoff points

Body temperatures of healthy humans are distributed nearly normally with mean 98.2ºF and standard deviation 0.73ºF. What is the cutoff for the lowest 3% of human body temperatures?



Note: On the TI you can use invNorm. Enter the percentile as a decimal and it will return the value of the item at the corresponding Z-score.

Z=invNorm(0.03)= -1.88

$$Z = \frac{obs - mean}{SD} \rightarrow \frac{x - 98.2}{0.73} = -1.88$$

$$x = (-1.88 \times 0.73) + 98.2 = 96.8^{\circ}F$$

Mackowiak, Wasserman, and Levine (1992), A Critical Appraisal of 98.6 Degrees F, the Upper Limit of the Normal Body Temperature, and Other Legacies of Carl Reinhold August Wunderlick.

Practice

Body temperatures of healthy humans are distributed nearly normally with mean 98.2°F and standard deviation 0.73°F. What is the cutoff for the *highest* 10% of human body temperatures?

(a) 97.3°F

(b) 99.1°F

(c) 99.4°F

(d) 99.6°F

Practice

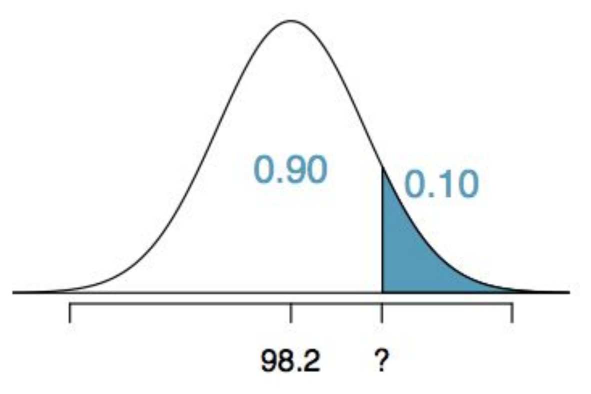
Body temperatures of healthy humans are distributed nearly normally with mean 98.2ºF and standard deviation 0.73ºF. What is the cutoff for the *highest* 10% of human body temperatures?

(a) 97.3ºF

(b) 99.1ºF

(c) 99.4ºF

(d) 99.6ºF



We are looking for the Z score that corresponds to the *90th percentile*. In this case Z = invNorm(0.9) = 1.28.

$$Z = \frac{obs - mean}{SD} \rightarrow \frac{x - 98.2}{0.73} = 1.28$$

$$x = (1.28 \times 0.73) + 98.2 = 99.1$$

Practice

Which of the following is false?

1. Majority of Z scores in a right skewed distribution are negative.
2. In skewed distributions the Z score of the mean might be different than 0.
3. For a normal distribution, IQR is less than 2 x SD.
4. Z scores are helpful for determining how unusual a data point is compared to the rest of the data in the distribution.

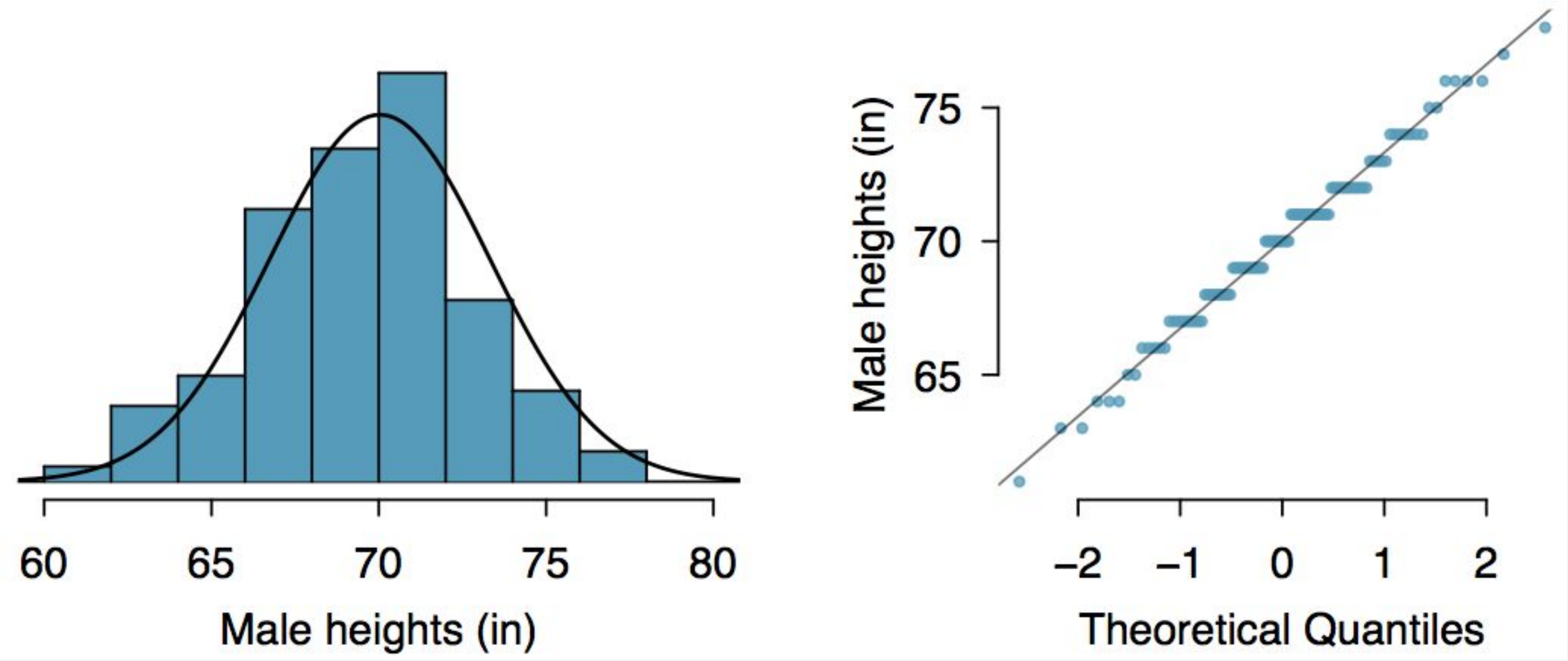
Practice

Which of the following is false?

1. Majority of Z scores in a right skewed distribution are negative.
2. *In skewed distributions the Z score of the mean might be different than 0.*
3. For a normal distribution, IQR is less than 2 x SD.
4. Z scores are helpful for determining how unusual a data point is compared to the rest of the data in the distribution.

Normal probability plot

A histogram and normal probability plot of a sample of 100 male heights.

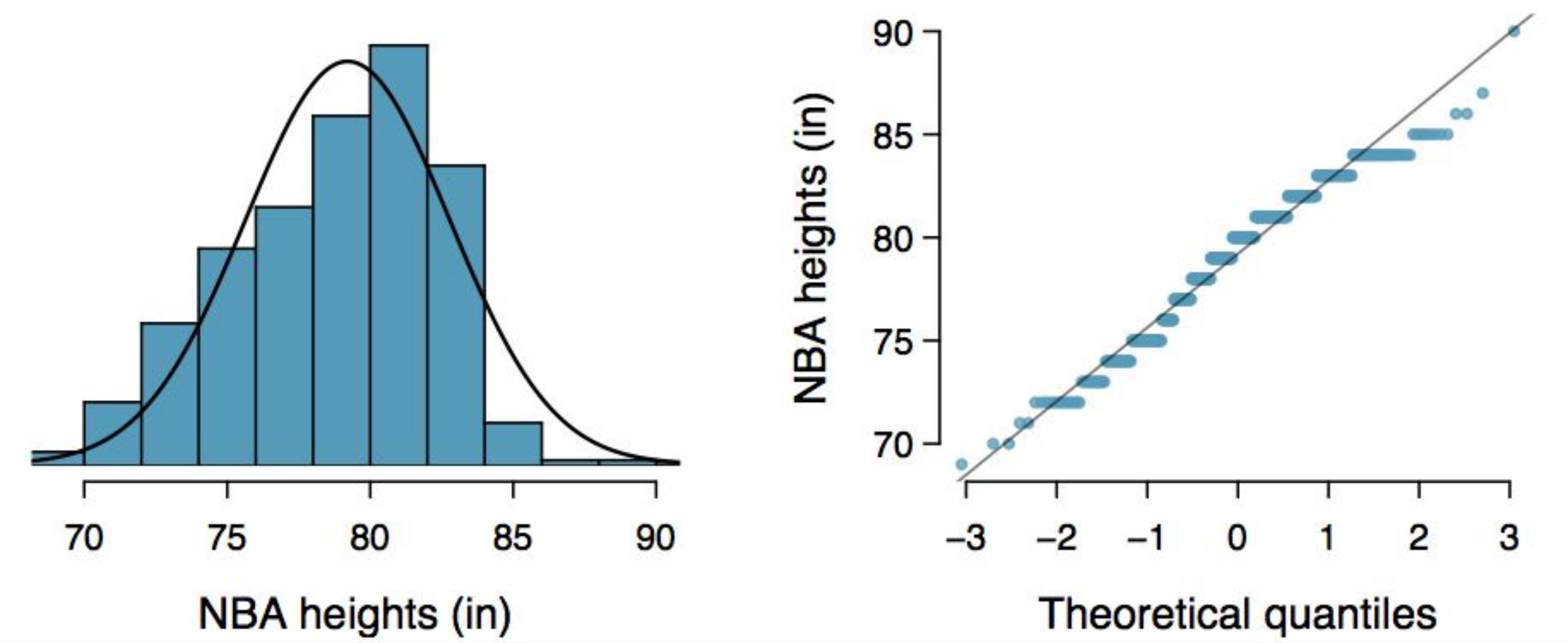


Anatomy of a normal probability plot

- Data are plotted on the y-axis of a normal probability plot, and theoretical quantiles (following a normal distribution) on the x-axis.
- If there is a linear relationship in the plot, then the data follow a nearly normal distribution.
- Constructing a normal probability plot requires calculating percentiles and corresponding z-scores for each observation, which is tedious. Therefore we generally rely on software when making these plots.

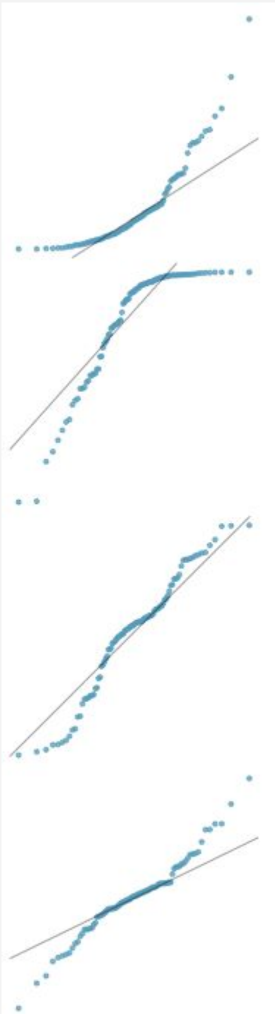
Practice

Below is a histogram and normal probability plot for the NBA heights from the 2008-2009 season. Do these data appear to follow a normal distribution?



Why do the points on the normal probability have jumps?

Normal probability plot and skewness (optional)



Right skew - Points bend up and to the left of the line.

Left skew - Points bend down and to the right of the line.

Short tails (narrower than the normal distribution) - Points follow an S shaped-curve.

Long tails (wider than the normal distribution) - Points start below the line, bend to follow it, and end above it.

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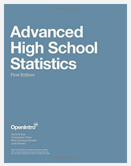

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