

Figure 2-12a

For example, the volume of air in your lungs varies periodically with time as you breathe. A reasonable sketch of the graph of this function is shown in Figure 2-12a.

Since the graph looks like a sinusoid, a sine or cosine function would be a reasonable mathematical model. You now know how to write an equation for a sinusoidal function with any given period, amplitude, phase, and axis location. This is the technique you will use for mathematical modeling.

Objective

Given a situation from the real world in which something varies sinusoidally, derive an equation and use it as a mathematical model to make predictions and reach conclusions about the real world.

Example

Suppose that the waterwheel in Figure 2-12b rotates at 6 revolutions per minute (rpm). You start your stopwatch. Two seconds later, point P on the rim of the wheel is at its greatest height. You are to model the distance d of point P from the surface of the water in terms of the number of seconds t the stopwatch reads.

Solution: Assuming that d varies sinusoidally with t , you can sketch a graph as in Figure 2-12c. Your thought process should be as follows:

1. The sinusoidal axis is 6 units above the t -axis, because the center of the waterwheel is 6 feet above the surface of the water.
2. The amplitude is 7 units, since the point P goes 7 feet above and 7 feet below the center of the wheel.

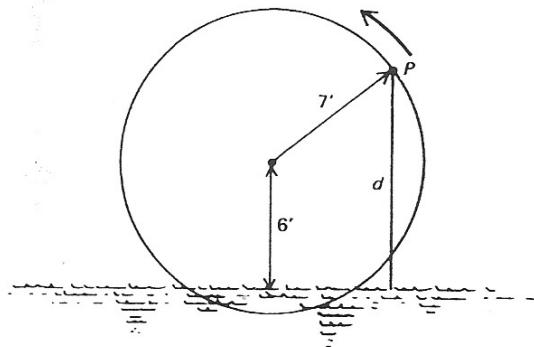


Figure 2-12b

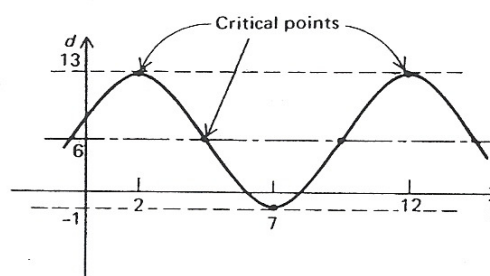


Figure 2-12c

3. Therefore, the upper and lower bounds of the graph are $6 + 7 = 13$, and $6 - 7 = -1$.
4. The point P was at its highest when the stopwatch read 2 seconds. Thus, the phase displacement (for the cosine) is 2 units.
5. The period is 10 seconds, since the waterwheel makes 6 complete revolutions every 60 seconds (1 minute).
6. Therefore, the sinusoid reaches its next high point at $2 + 10 = 12$ units on the t -axis.
7. Halfway between two high points there is a low point at $t = \frac{1}{2}(2 + 12) = 7$; halfway between each high and low point the graph crosses the sinusoidal axis.
8. With the critical points from step 7 you can sketch the graph.

Once the graph has been sketched, you have made the transition from the real world to the mathematical world. From this point on, completing and using the mathematical model requires only the mathematical techniques you have learned in the preceding sections. From the graph, the four constants in the sinusoidal equation are

$$A = 7$$

$$B = \frac{2\pi}{\text{period}} = \frac{2\pi}{10} = \frac{\pi}{5}$$

$$C = 6$$

$$D = 2$$

The equation is therefore

$$d = 6 + 7 \cos \frac{\pi}{5}(t - 2)$$

The equation can be used to make predictions of d given t or of t given d . For example, to find the time when the point P first emerges from below the water, let $d = 0$ and solve the resulting equation as in Section 2-11.

$$6 + 7 \cos \frac{\pi}{5}(t - 2) = 0 \quad \text{Setting } d = 0$$

$$\cos \frac{\pi}{5}(t - 2) = -\frac{6}{7} \quad \text{Add } -6; \text{ then divide by } 7.$$

$$\frac{\pi}{5}(t - 2) = \cos^{-1}\left(-\frac{6}{7}\right) \quad \text{Take } \cos^{-1} \text{ of each member.}$$

$$t = 2 + \frac{5}{\pi} \cos^{-1}\left(-\frac{6}{7}\right) \quad \text{Multiply by } \frac{5}{\pi}, \text{ then add } 2.$$

$$t = 2 + \frac{5}{\pi} (\pm 2.60049 \dots + 2\pi n) \quad \text{By calculator in radian mode.}$$

$$t = 2 \pm 4.1388 \dots + 10n \quad \text{Distribute } \frac{5}{\pi}.$$

$$t \approx 6.14 + 10n \text{ or } -2.14 + 10n \quad \text{Arithmetic, and rounding.}$$

$$t = 6.14, 16.14, \dots, \text{ or } -2.14, 7.86, \dots \quad \text{Selecting integers, } n.$$

To find out which of these is the particular value desired, you return to the real-world situation. From Figure 2-12b, you notice that the point P emerges from

the water the *second* time that $d = 0$. The first time $d = 0$, P is going *into* the water. So the particular value of t required is $t = 7.86$.

The exercise that follows is designed to give you experience using sinusoidal functions as mathematical models. In doing this, you will use all of the trigonometric techniques you have learned in this chapter as well as a lot of algebraic techniques you have learned in the past.

EXERCISE 2-12

- Ferris Wheel Problem** As you ride the Ferris wheel, your distance from the ground varies sinusoidally with time. When the last seat is filled and the Ferris wheel starts, your seat is at the position shown in Figure 2-12d. Let t be the number of seconds that have elapsed since the Ferris wheel started. You find that it takes you 3 s to reach the top, 43 ft above the ground, and that the wheel makes a revolution once every 8 s. The diameter of the wheel is 40 ft.
 - Sketch a graph of this sinusoid.
 - What is the lowest you go as the Ferris wheel turns, and why is this number greater than zero?
 - Write an equation of this sinusoid.

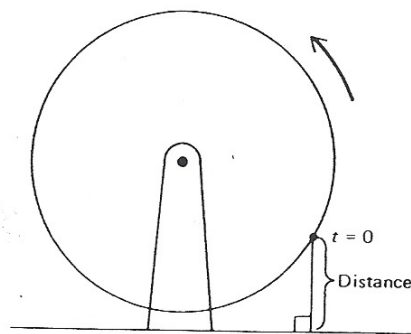


Figure 2-12d

- d. Predict your height above the ground when
- $t = 6$
 - $t = 4\frac{1}{3}$
 - $t = 9$
 - $t = 0$
- e. What is the value of t the *second* time you are 18 ft above the ground?
2. **Steamboat Problem** Mark Twain sat on the deck of a river steamboat. As the paddlewheel turned, a point on the paddle blade moved in such a way that its distance, d , from the water's surface was a sinusoidal function of time. When his stopwatch read 4 s, the point was at its highest, 16 ft above the water's surface. The wheel's diameter was 18 ft, and it completed a revolution every 10 s.
- Sketch a graph of the sinusoid.
 - Write the equation of the sinusoid.
 - How far above the surface was the point when Mark's stopwatch read:
 - 5 s
 - 17 s
 - What is the first positive value of time at which the point was at the water's surface? At that time, was it going into or coming out of the water? Explain.
3. **Extraterrestrial Being Problem** Researchers find a creature from an alien planet. Its body temperature is varying sinusoidally with time. 35 min after they start timing, it reaches a high of 120°F . 20 min after that it reaches its next low, 104°F .
- Sketch a graph of this sinusoid.
 - Write an equation expressing temperature in terms of minutes since they started timing.
 - What was its temperature when they first started timing?
 - Find the first three times after they started timing at which the temperature was 114°F .
4. **Fox Population Problem** Naturalists find that the populations of some kinds of predatory animals vary periodically. Assume that the population of foxes in a certain forest varies sinusoidally with time. Records started being kept when time $t = 0$ years. A minimum number, 200 foxes, occurred when $t = 2.9$ years. The next maximum, 800 foxes, occurred at $t = 5.1$ years.
- Sketch a graph of this sinusoid.
 - Write an equation expressing the number of foxes as a function of time, t .
 - Predict the population when $t = 7$.
 - Foxes are declared to be an endangered species when their population drops below 300. Between what two nonnegative values of t were foxes first endangered?
 - Show on your graph that your answers to part (d) are correct.
5. **Bouncing Spring Problem** A weight attached to the end of a long spring is bouncing up and down (Figure 2-12e). As it bounces, its distance from the floor varies sinusoidally with time. You start a stopwatch. When the stopwatch reads 0.3 s, the weight first reaches a high point 60 cm above the floor. The next low point, 40 cm above the floor, occurs at 1.8 s.
- Sketch a graph of this sinusoidal function.
 - Write an equation expressing distance from the floor in terms of the number of seconds the stopwatch reads.
 - Predict the distance from the floor when the stopwatch reads 17.2 s.
 - What was the distance from the floor when you started the stopwatch?

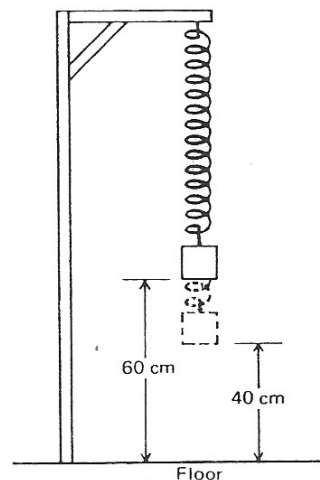


Figure 2-12e

- e. Predict the first positive value of time at which the weight is 59 cm above the floor.
6. **Tarzan Problem** Tarzan is swinging back and forth on his grapevine. As he swings, he goes back and forth across the river bank, going alternately over land and water (Figure 2-12f). Jane decides to model mathematically his motion and starts her stopwatch. Let t be the number of seconds the stopwatch reads and let y be the number of meters Tarzan is from the river bank. Assume that y varies sinusoidally with t , and that y is positive when Tarzan is over water and negative when he is over land.

Jane finds that when $t = 2$, Tarzan is at one end of his swing, where $y = -23$. She finds that when $t = 5$ he reaches the other end of his swing and $y = 17$.

- Sketch a graph of this sinusoidal function.
- Write an equation expressing Tarzan's distance from the river bank in terms of t .
- Predict y when:
 - $t = 2.8$
 - $t = 6.3$
 - $t = 15$
- Where was Tarzan when Jane started the stopwatch?

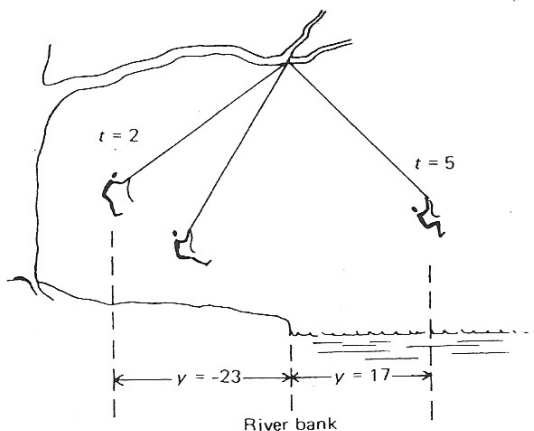


Figure 2-12f

- e. Find the least positive value of t for which Tarzan is directly over the river bank (i.e., $y = 0$).
7. **Roller Coaster Problem** A portion of a roller coaster track is to be built in the shape of a sinusoid (Figure 2-12g). You have been hired to calculate the lengths of the horizontal and vertical timer supports to be used.
- The high and low points on the track are separated by 50 m horizontally and by 30 m vertically. The low point is 3 m below the ground. Letting y be the number of meters the track is above the ground and x the number of meters horizontally from the high point, write an equation expressing y in terms of x .
 - How long is the vertical timber at the high point? At $x = 4$ m? At $x = 32$ m?
 - How long is the horizontal timber that is 25 m above the ground? 5 m above the ground?
 - Where does the track first go below ground?
 - The vertical timbers are spaced every 2 m, starting at $x = 0$ and ending where the track goes below the ground. Write a computer program that prints out the length of each timber. The program should also print the total length of all the vertical timbers, so that you will know how much to purchase. You can modify the program of Exercise 2-11, Problem 12, for this purpose.

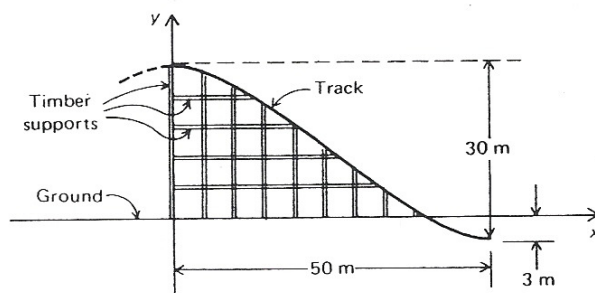


Figure 2-12g